Desingularizations of

Conformally Kähler,

Einstein Orbifolds

Claude LeBrun
Stony Brook University

“Living on the Edge of Moduli Space”
Aarhus University Workshop
Aarhus, Denmark, May 18, 2026

Most recent results:

Joint work with

Joint work with

Tristan Ozuch

Joint work with

Tristan Ozuch
Massachusetts Institute of Technology

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Massachusetts Institute of Technology

e-print:
arXiv:2601.19215 [math.DG]

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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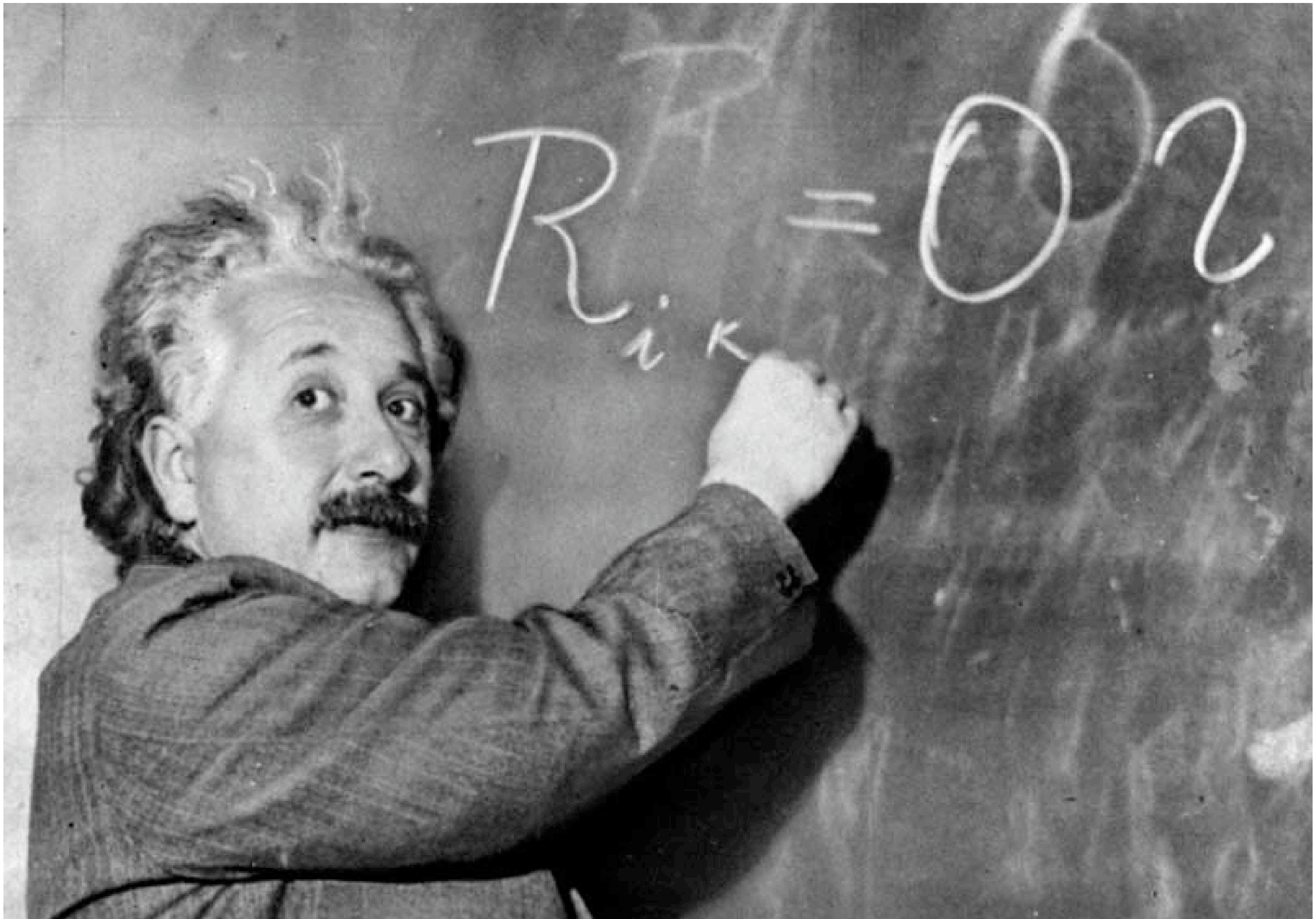

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Has same sign as the *scalar curvature*

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

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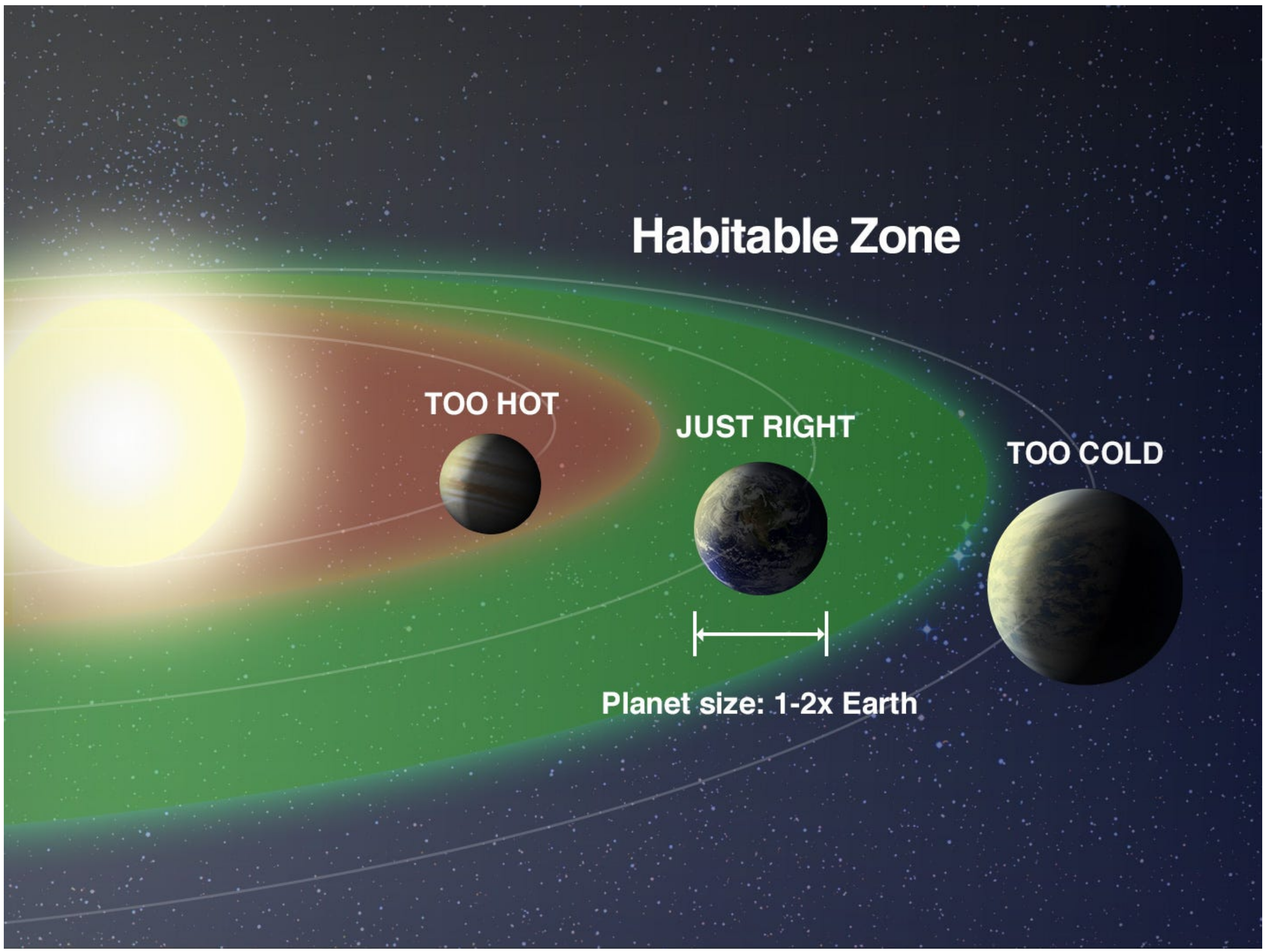

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Interplay between existence and obstruction results sometimes leads to classification statements!

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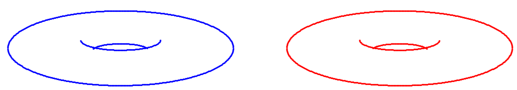

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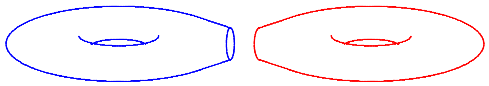

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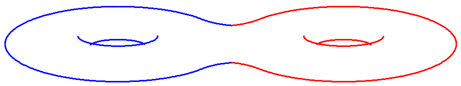

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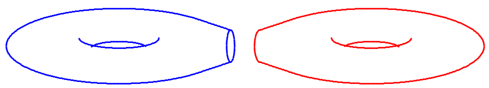

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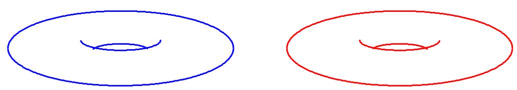

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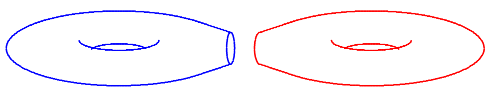

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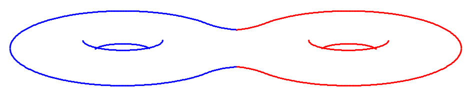

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Diffeotypes: exactly the Del Pezzo surfaces.

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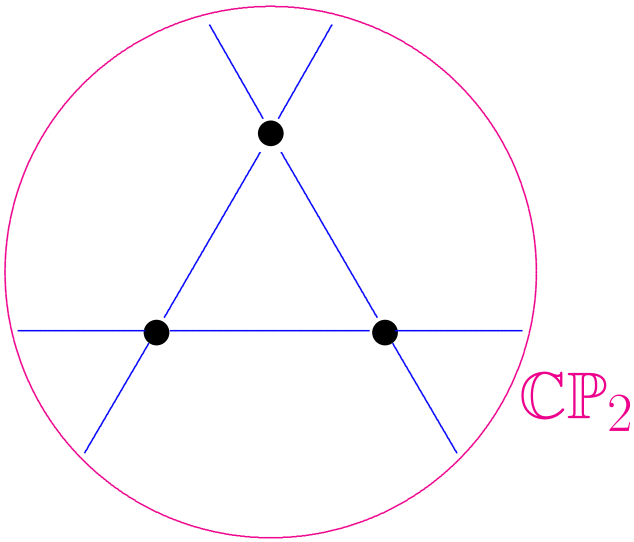

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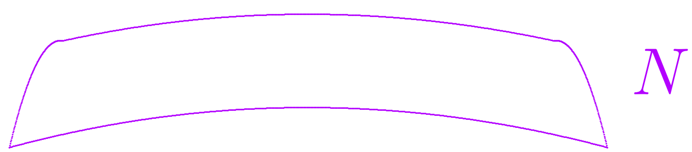

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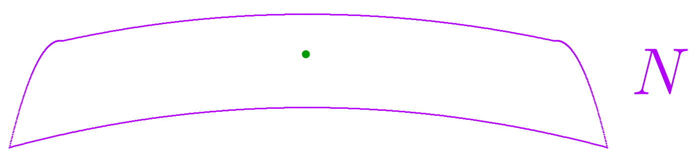

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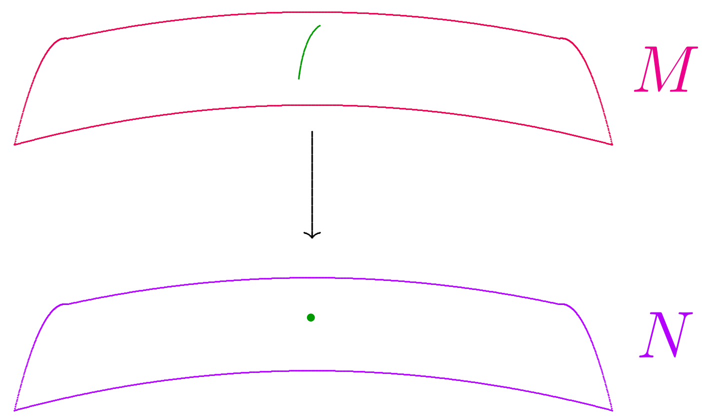
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$



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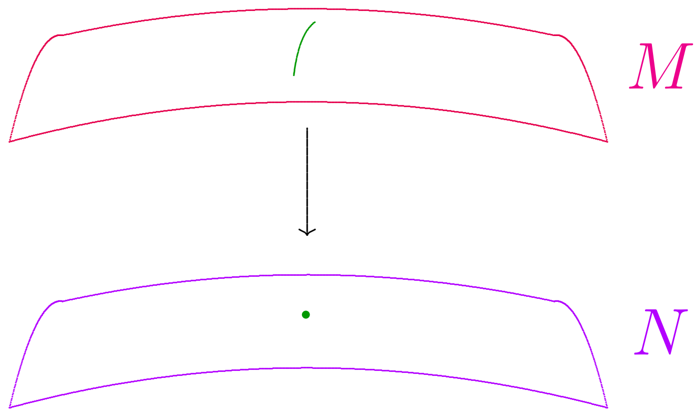

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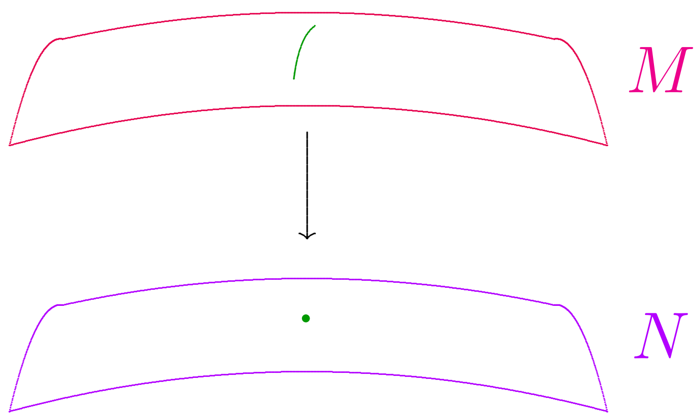

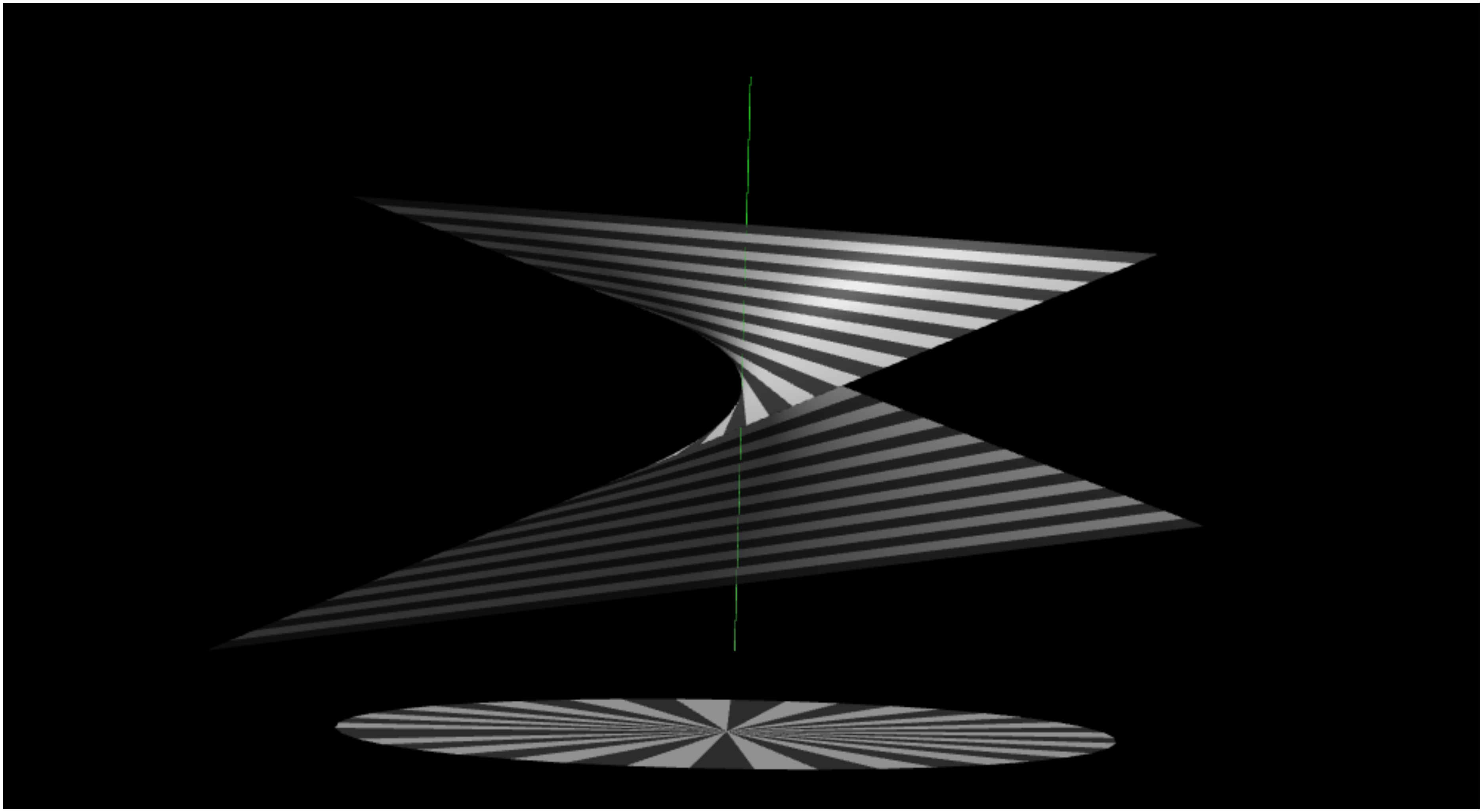

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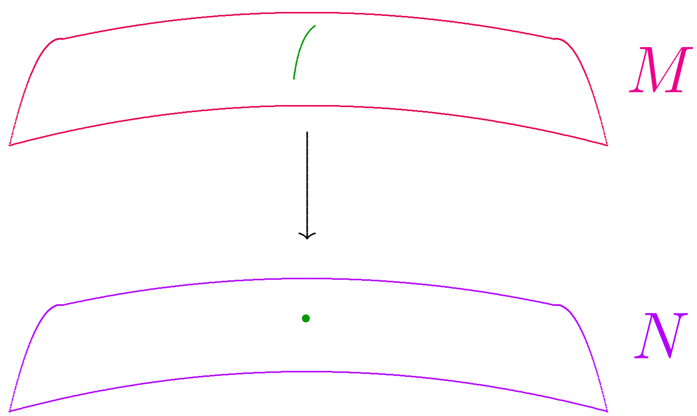

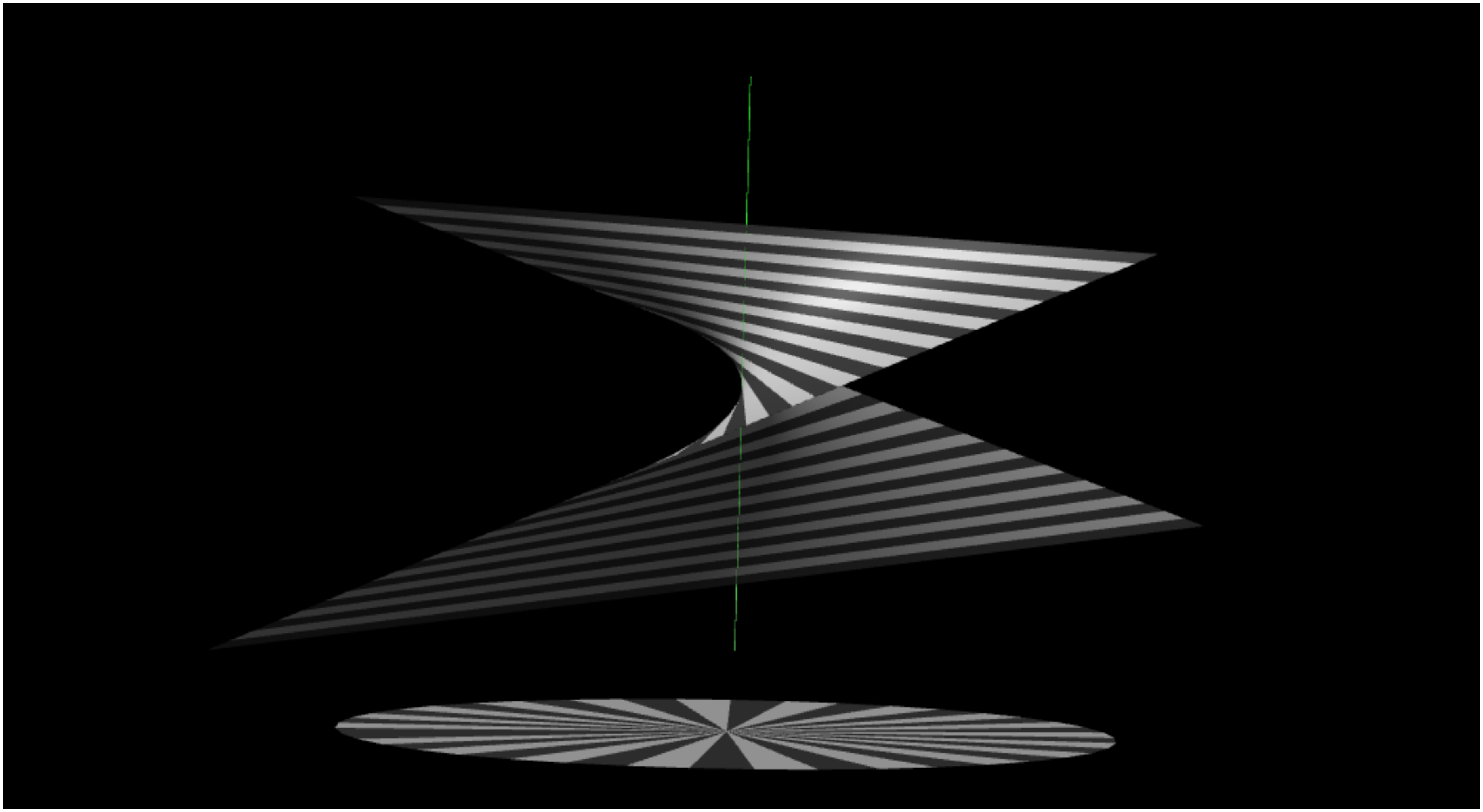

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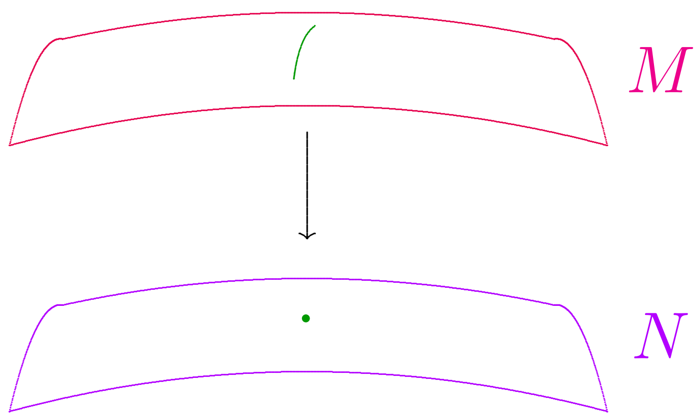

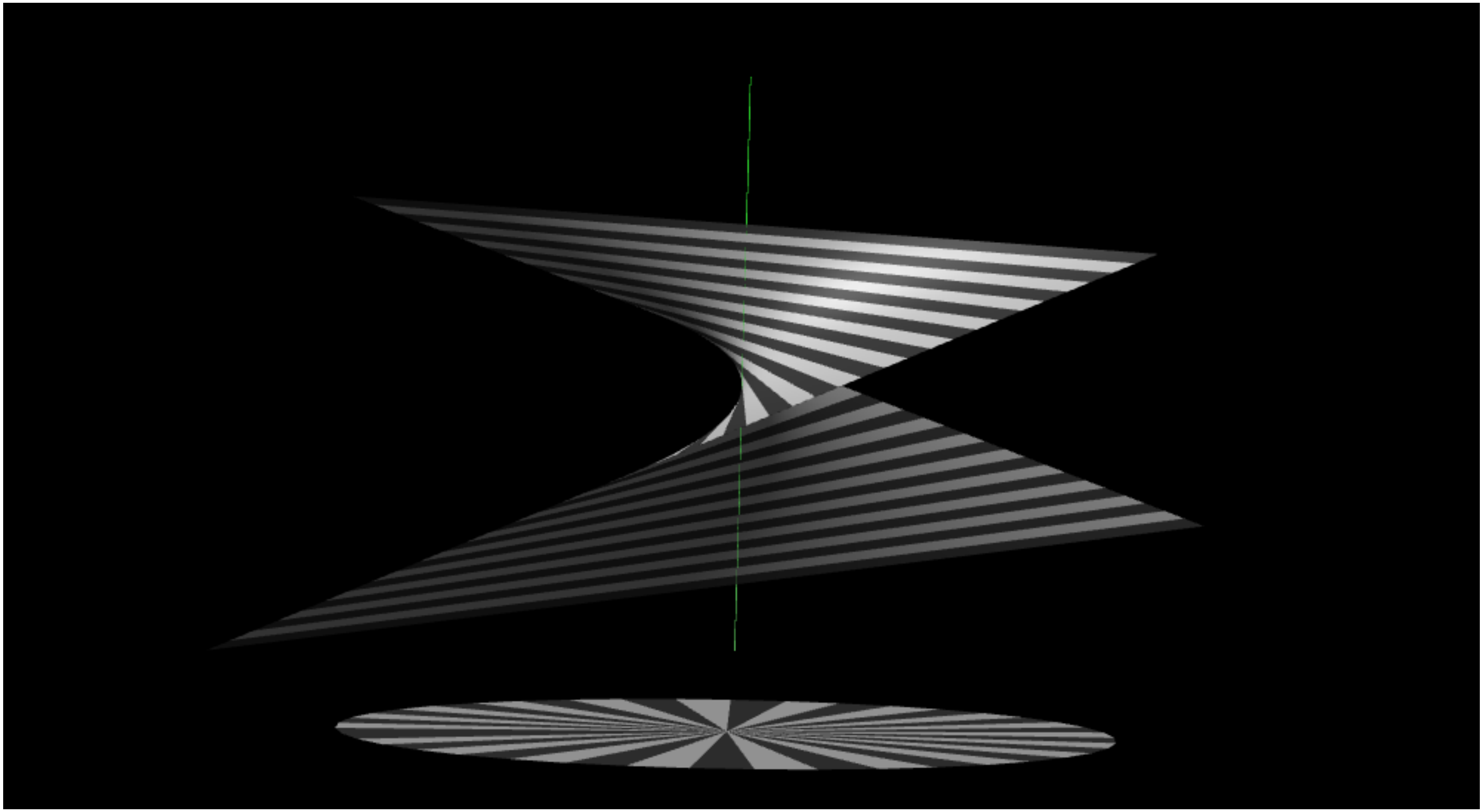

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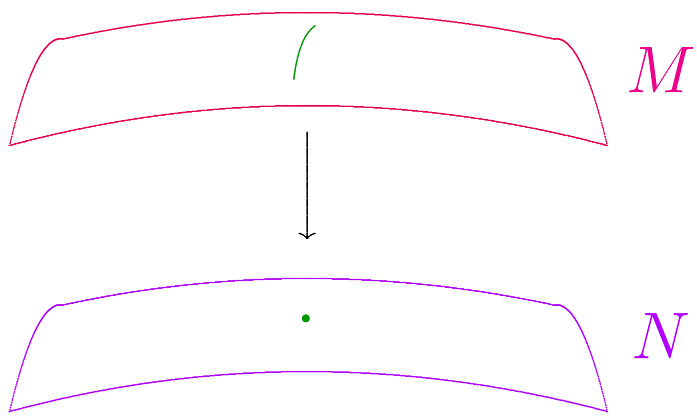

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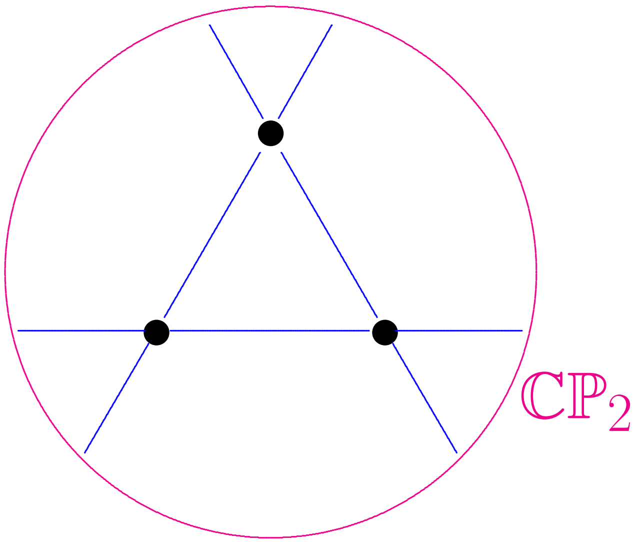

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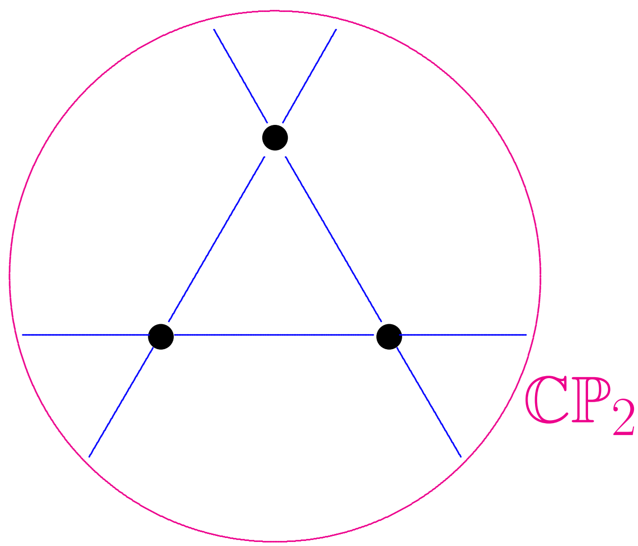

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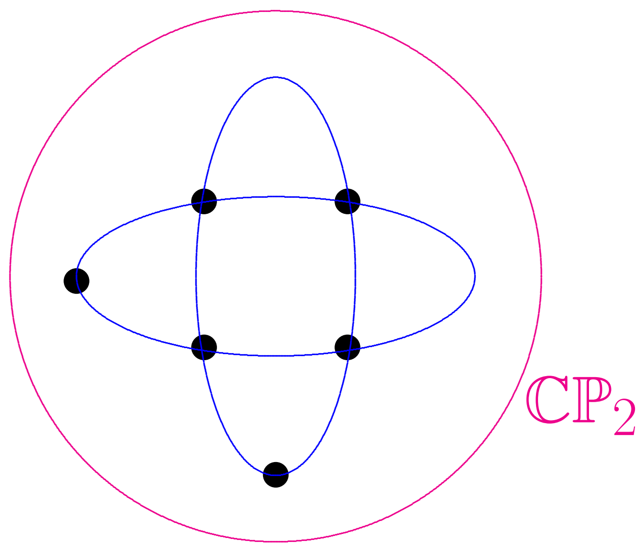

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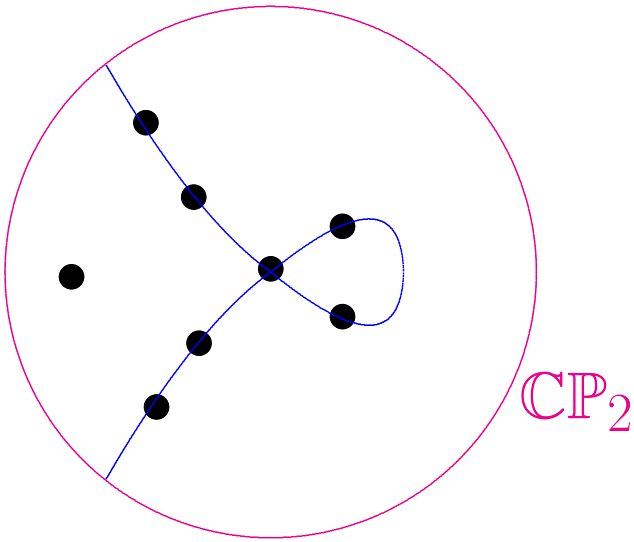

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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$$g = g(J\cdot, J\cdot)$$

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Existence: Page

1978

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

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2008

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Uniqueness: Bando-Mabuchi '87

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Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

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L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi. Most of them are actually Kähler-Einstein.

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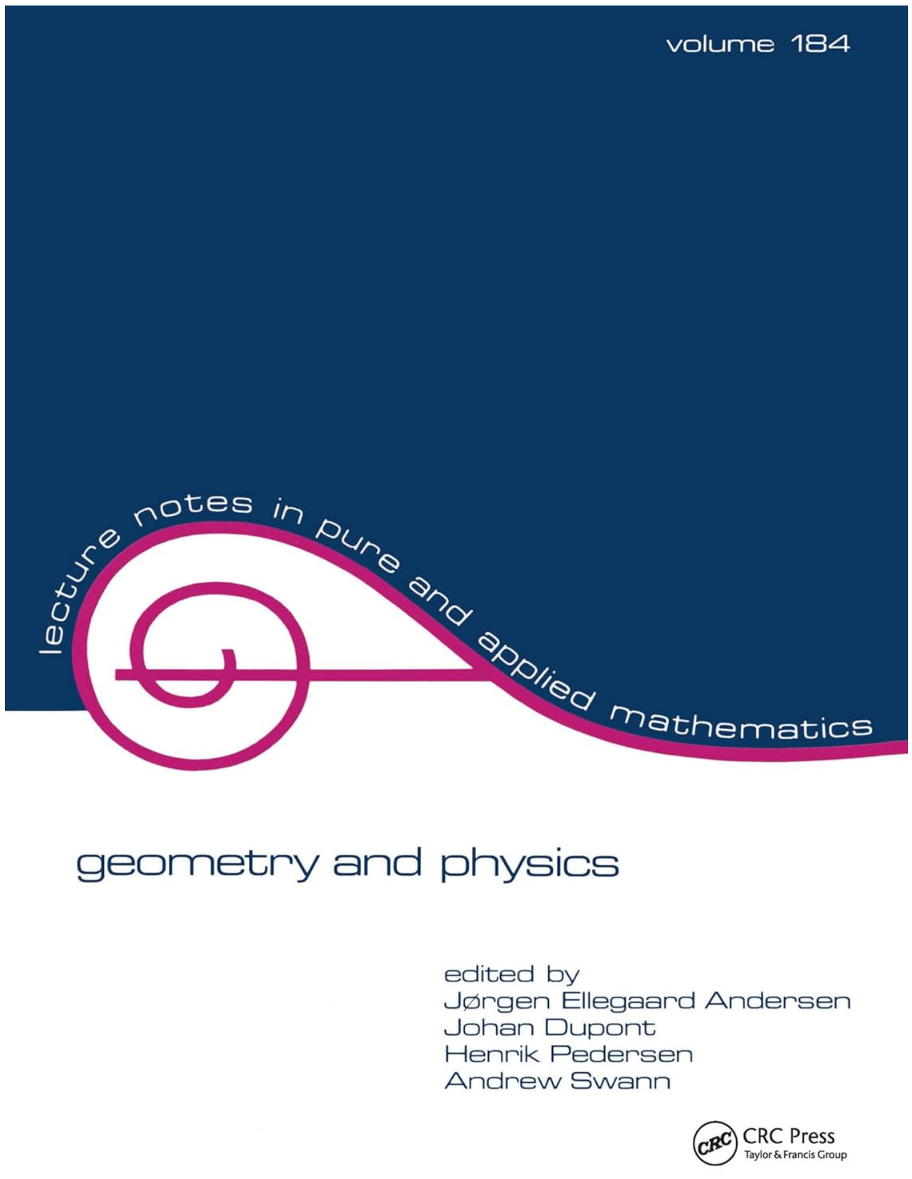

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All of these results depend on phenomenon which only occur in dimension 4. They do not generalize to higher dimensions!

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Λ^+ self-dual 2-forms.
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Top eigenspace $L \subset \Lambda^+$ of W_+ is a line bundle.

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface.*

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W_+) > 0$*, and these sweep out exactly one connected component of moduli space* $\mathscr{E}(M)$*.*

The existence of these conformally Kähler, Einstein metrics depends in part on the theory of Gromov-Hausdorff convergence.

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- Mutually diffeomorphic $M_{j_i} \approx M^4$;
- X has only isolated singularities $\approx \mathbb{R}^4/\Gamma_\ell$;
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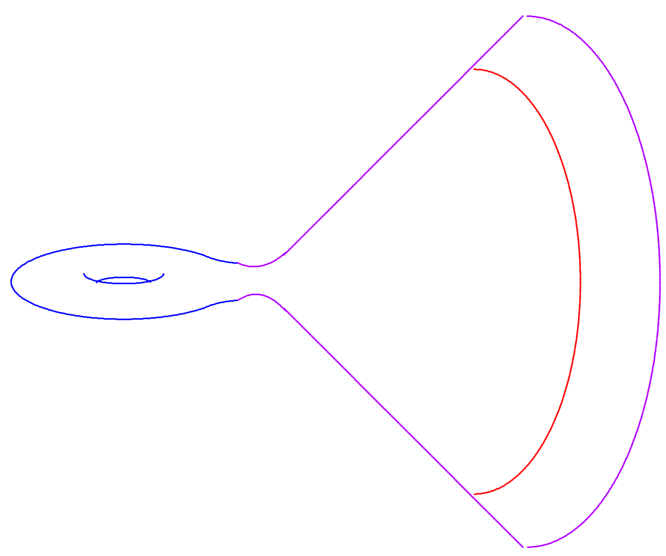

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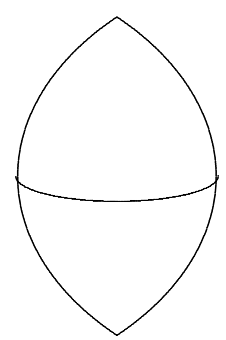

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Goal: Show that this doesn't change anything!

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We avoid this question by means of a definition!

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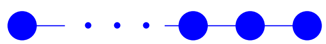
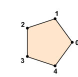
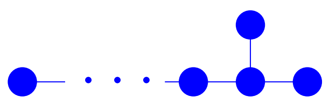
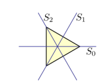
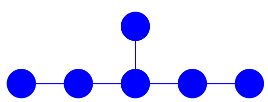
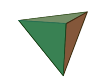
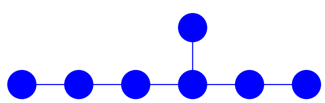
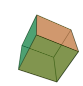
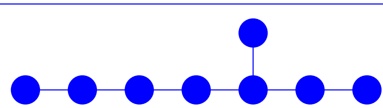
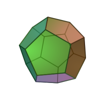
Dynkin Diagram		$\Gamma < SU(2)$	*Classical Figure*
A_k		*Cyclic*	
D_k		*Binary Dihedral*	
E_6		*Binary Tetrahedral*	
E_7		*Binary Octahedral*	
E_8		*Binary Icosahedral*	

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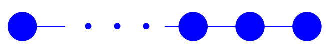
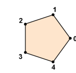
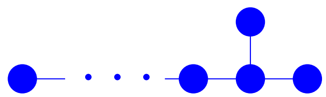
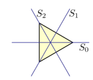
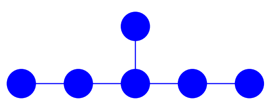
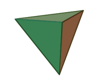
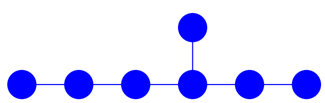
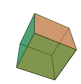
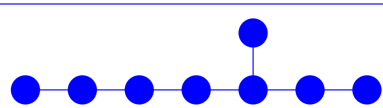
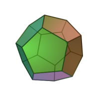
A_k 	$\mathbb{Z}_{k+1}$	
D_k 	Dih^*_{k-2}	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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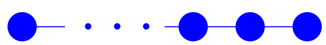
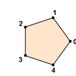
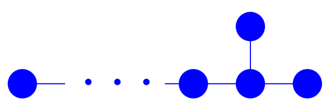
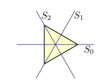
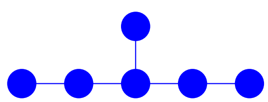
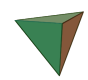
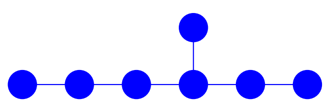
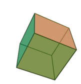
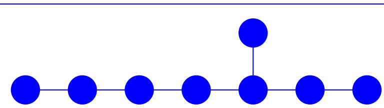
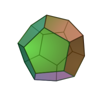
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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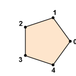
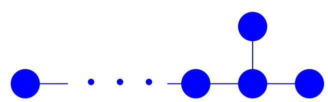
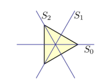
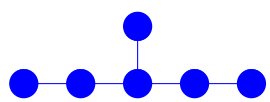
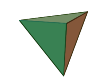
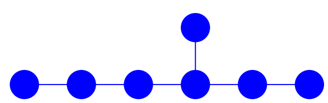
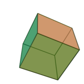
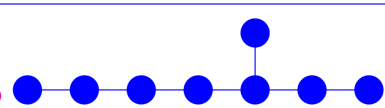
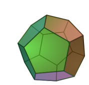
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

The corresponding gravitational instantons are exactly the hyper-Kähler ALE manifolds.

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Standard shorthand: $\frac{1}{\ell m^2}(1, \ell m n - 1)$.

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Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

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These examples form part of a systematic picture. . .

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- $\mathbb{CP}_2 \# 8 \overline{\mathbb{CP}}_2$: only (!) $A_1, \ldots, A_{10}$, D_4, $\frac{1}{4}(1,1)$, $\frac{1}{8}(1,3)$, or $\frac{1}{9}(1,2)$ singularities arise.

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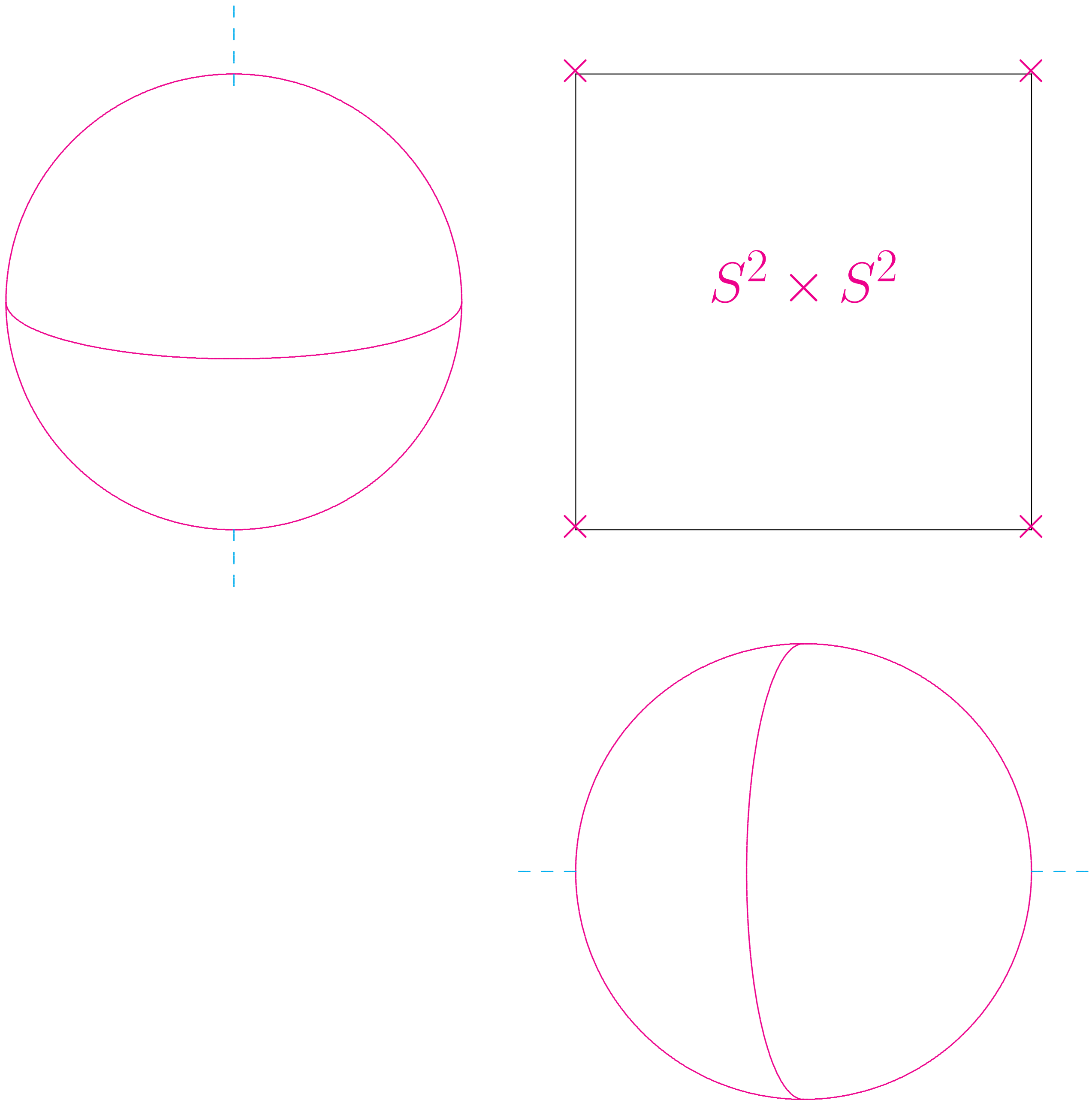

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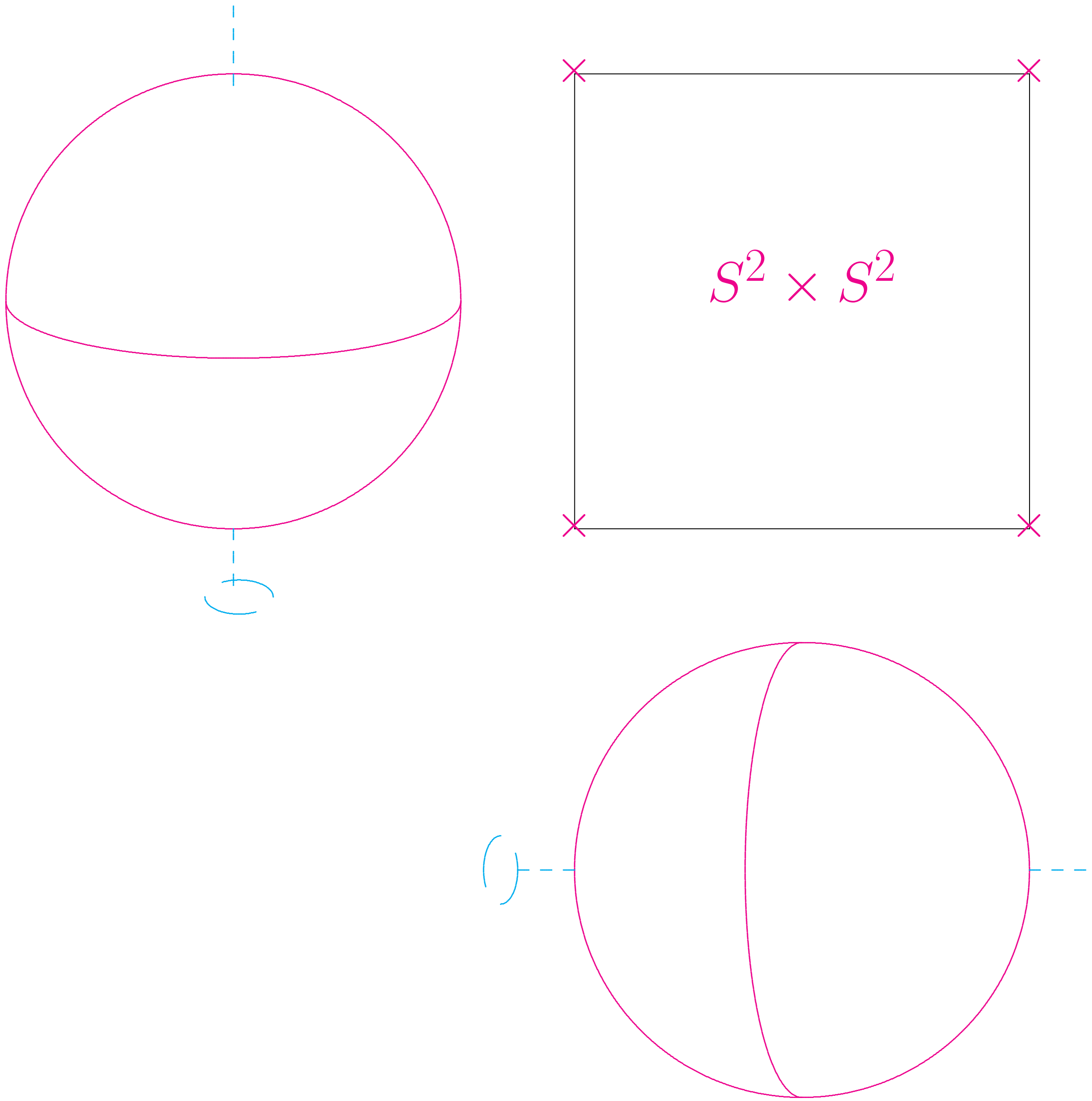

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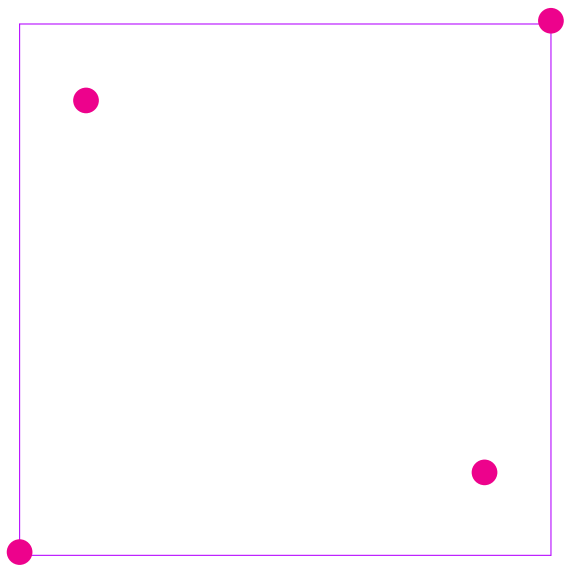

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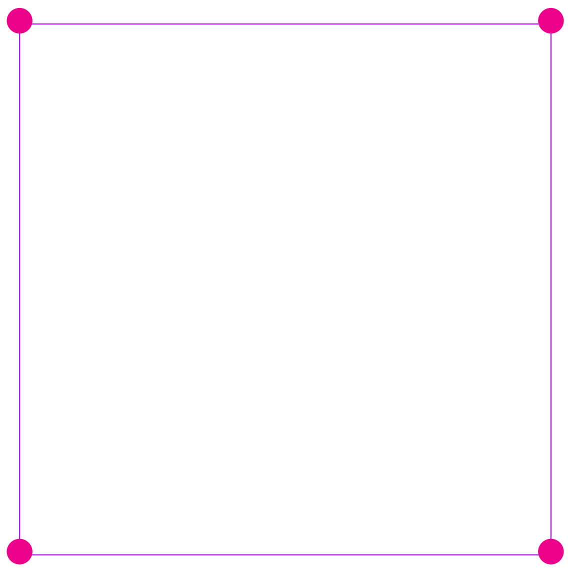

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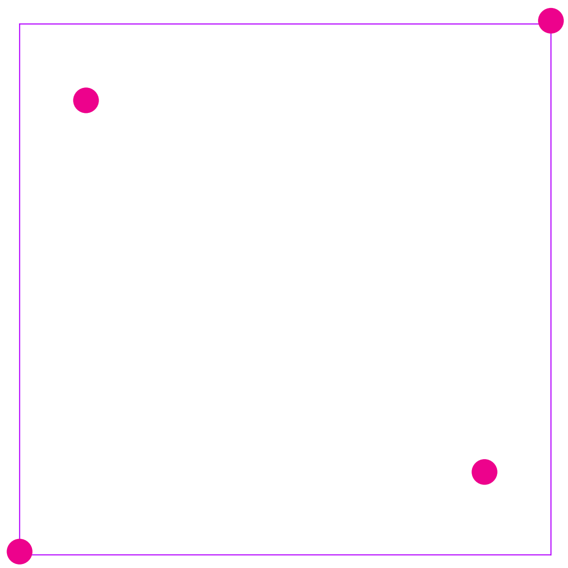

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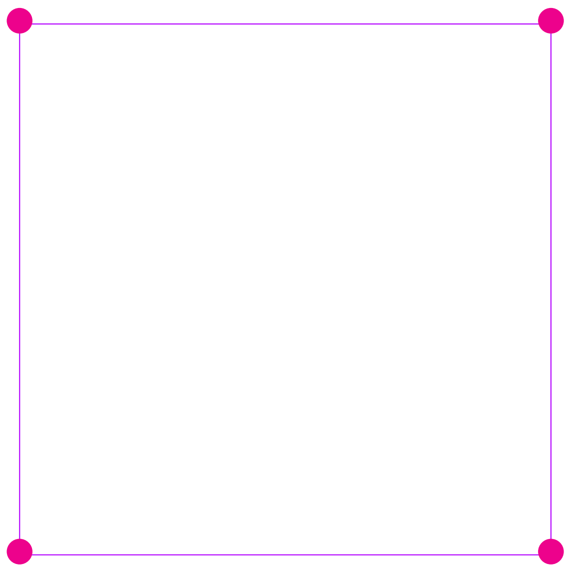

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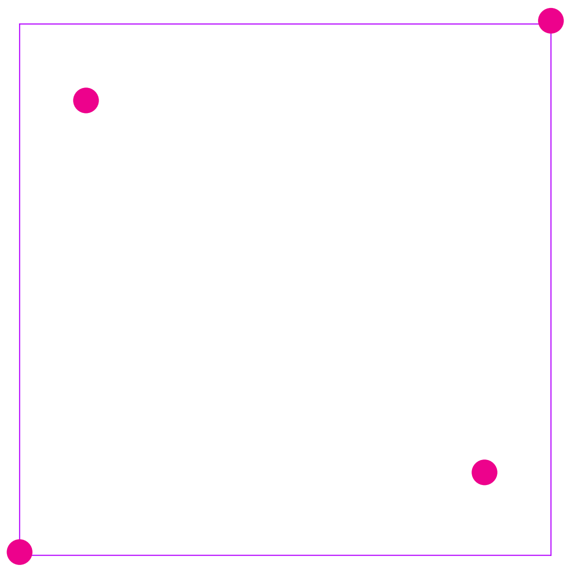

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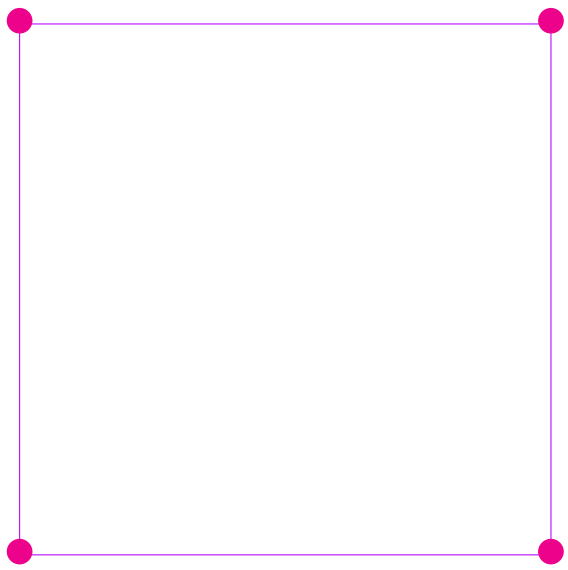

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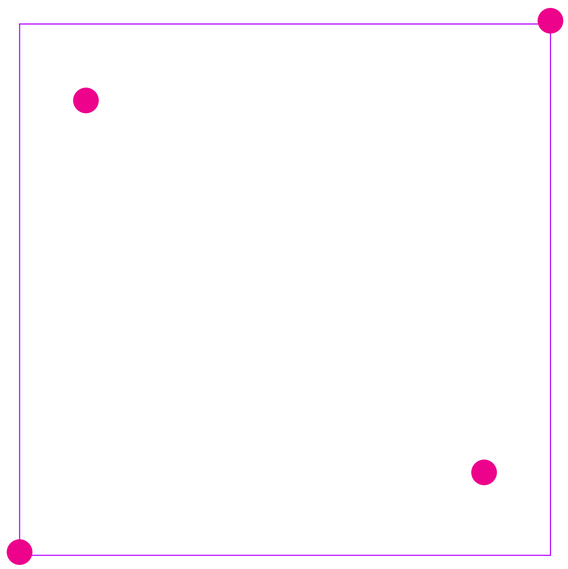

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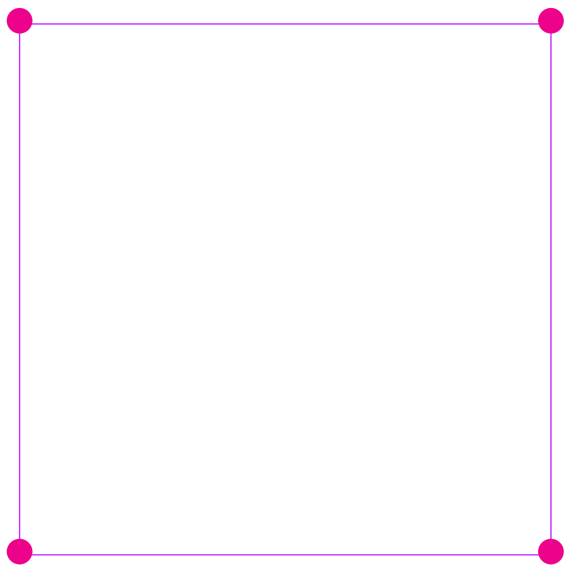

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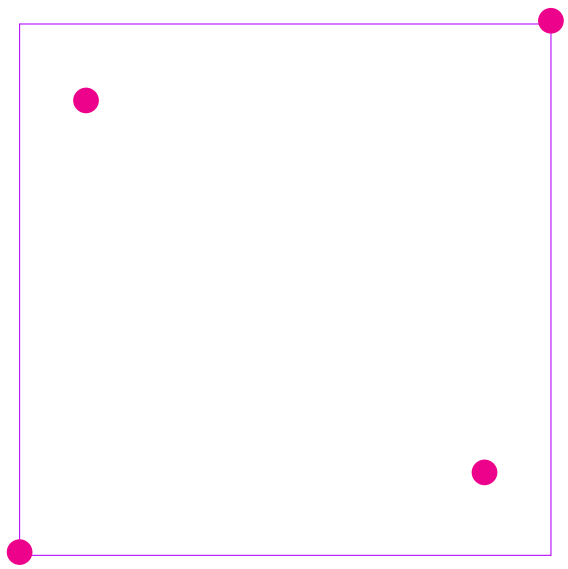

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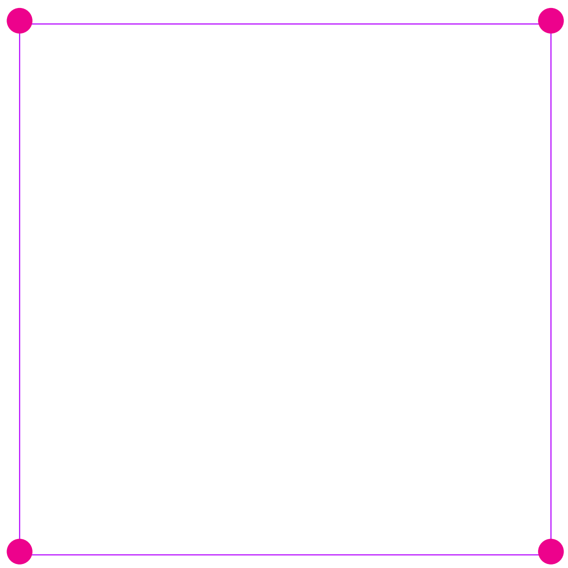

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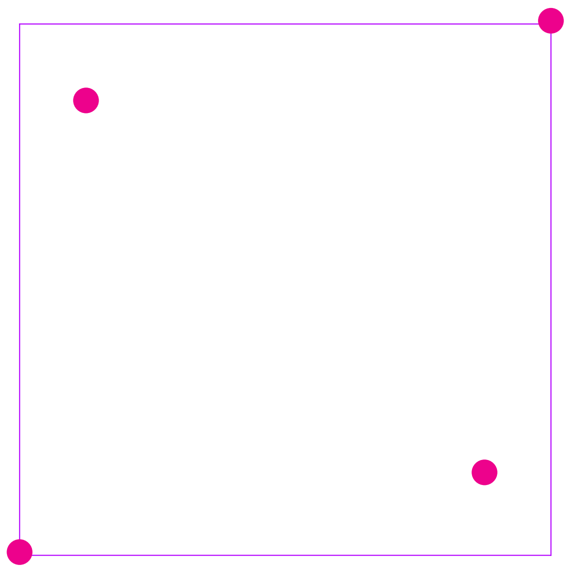

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But what about limits of general Einstein metrics?

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Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

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And hence that they are actually conformally Kähler!

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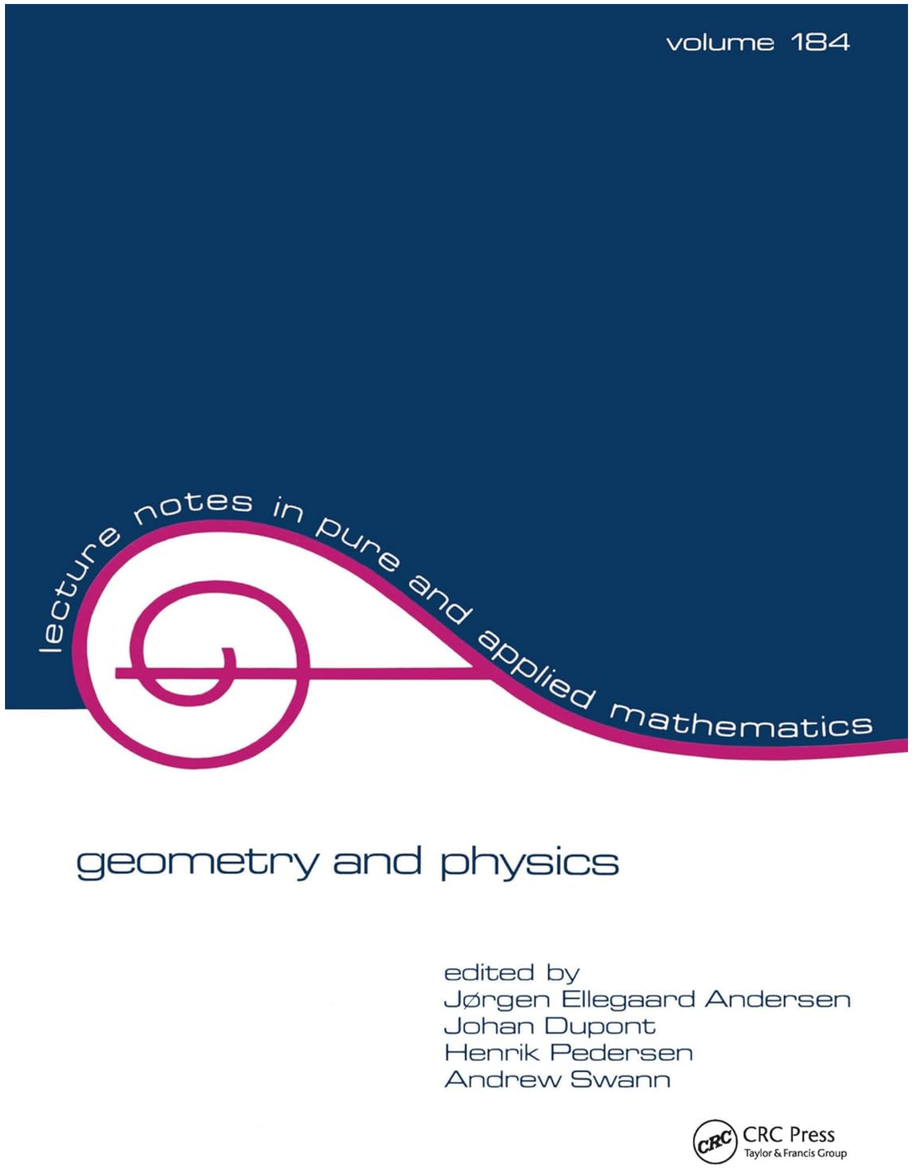

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