PLURISUBHARMONIC FUNCTIONS IN CALIBRATED GEOMETRIES

F. Reese Harvey and H. Blaine Lawson, Jr.*

ABSTRACT

In this paper we introduce and study the notion of plurisubharmonic functions in calibrated geometry. These functions generalize the classical plurisubharmonic functions from complex geometry and enjoy many of their important properties. Moreover, they exist in abundance whereas the corresponding pluriharmonics are generally quite scarce. A number of the results established in complex analysis via plurisubharmonic functions are extended to calibrated manifolds. This paper investigates, in depth, questions of: pseudoconvexity and cores, positive ϕ-currents, Duval-Sibony Duality, and boundaries of ϕ-submanifolds, all in the context of a general calibrated manifold (X, ϕ). Analogues of totally real submanifolds are used to construct enormous families of strictly ϕ-convex spaces with every topological type allowed by Morse Theory. Specific calibrations are used as examples throughout. Analogues of the Hodge Conjecture in calibrated geometry are considered.

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TABLE OF CONTENTS

0. Introduction.

Calibrated geometries, as introduced in [HL$_3$], are geometries of distinguished submanifolds determined by a fixed, closed differential form ϕ on a riemannian manifold X. The basic example is that of a Kähler manifold (or more generally a symplectic manifold, with compatible complex structure) where the distinguished submanifolds are the holomorphic curves. However, there exist many other interesting geometries, each carrying a wealth of ϕ-submanifolds, particularly on spaces with special holonomy. These have attracted particular attention in recent years due to their appearance in generalized Donaldson theories and in modern versions of string theory in Physics.

Unfortunately, analysis on these spaces (X, ϕ) has been difficult, in part because there is generally no reasonable analogue of the holomorphic functions and transformations which exist in the Kähler case. However, in complex analysis there are many important results which can be established using only the plurisubharmonic functions. It turns out that analogues of these functions exist in abundance on any calibrated manifold, and they enjoy almost all the pleasant properties of their cousins from complex analysis. The point of this paper is to introduce and study these functions.

We begin by defining our notion of a ϕ-plurisubharmonic function on any calibrated manifold (X, ϕ). In the Kähler case they are exactly the classical plurisubharmonic functions. We then study the basic properties of these functions, and subsequently use them to establish a series of results in geometry and analysis on (X, ϕ).

A fundamental result is that:

The restriction of a ϕ-plurisubharmonic function to a ϕ-submanifold M is subharmonic

in the induced metric on M.

Any convex function on the riemannnian manifold X is ϕ-plurisubharmonic. Moreover, at least locally, there exists an abundance of ϕ-plurisubharmonic functions which are not convex.

The definition of ϕ-plurisubharmonicity extends to arbitrary distributions on X. For most calibrations – including all the "classical" ones – such distributions enjoy all the nice properties of generalized subharmonic functions such as being locally Lebesgue integrable and represented by an upper-semicontinuous function taking values in $[-\infty, \infty)$. In general the maximum $F = \max\{f, g\}$ of two smooth ϕ-plurisubharmonic functions is again ϕ-plurisubharmonic and can be uniformly approximated by a decreasing sequence of smooth ϕ-plurisubharmonic functions.

To define ϕ-plurisubharmonic functions on a calibrated manifold (X, ϕ) we introduce a second order differential operator $\mathcal{H}^\phi : C^\infty(X) \to \mathcal{E}^p(X)$, the *$\phi$-Hessian*, given by

$$\mathcal{H}^\phi(f) \;=\; \lambda_\phi(\mathrm{Hess} f)$$

where $\mathrm{Hess} f$ is the riemannian hessian of f and $\lambda_\phi : \mathrm{End}(TX) \to \Lambda^p T^*X$ is the bundle map given by $\lambda_\phi(A) = D_{A^*}(\phi)$ where $D_{A^*} : \Lambda^p T^*X \longrightarrow \Lambda^p T^*X$ is the natural extension of $A^* : T^*X \to T^*X$ as a derivation.

When the calibration ϕ is parallel there is a natural factorization

$$\mathcal{H}^\phi \;=\; dd^\phi$$

where d is the de Rham differential and $d^\phi : C^\infty(X) \to \mathcal{E}^{p-1}(X)$ is given by

$$d^\phi f \equiv \nabla f \lrcorner \phi.$$

In general these operators are related by the equation: $\mathcal{H}^\phi f = dd^\phi f - \nabla_{\nabla f}(\phi)$.

Recall that a calibration ϕ of degree p is a closed p-form with the property that $\phi(\xi) \leq 1$ for all unit simple tangent p-vectors ξ on X. Those ξ for which $\phi(\xi) = 1$ are called ϕ-planes, and the set of ϕ-planes is denoted by $G(\phi)$. With this understood, a function $f \in C^\infty(X)$ is defined to be **ϕ-plurisubharmonic** if $\mathcal{H}^\phi(f)(\xi) \geq 0$ for all $\xi \in G(\phi)$. It is **strictly ϕ-plurisubharmonic** at a point $x \in X$ if $\mathcal{H}^\phi(f)(\xi) > 0$ for all ϕ-planes ξ at x. In a similar fashion, f is called **ϕ-pluriharmonic** if $\mathcal{H}^\phi(f)(\xi) = 0$ for all $\xi \in G(\phi)$. Denote by $\mathcal{PSH}(X, \phi)$ the convex cone of ϕ-plurisubharmonic functions on X.

When X is a complex manifold with a Kähler form ω, one easily computes that $d^\omega = d^c$, the conjugate differential. In this case, $\mathcal{H}^\omega = dd^\omega = dd^c$ and the ω-planes correspond to the complex lines in TX. Hence, the definitions above coincide with the classical notions of plurisubharmonic and pluriharmonic functions on X.

With this said, we must remark that in many calibrated manifolds the ϕ-pluriharmonic functions are scarce. For the calibrations on manifolds with strict G_2 or Spin_7 holonomy, for example, every pluriharmonic function is constant. For the Special Lagrangian calibration $\phi = \mathrm{Re}\{dz\}$, every ϕ-pluriharmonic function f defined locally in $\mathbf{C}^n$ is of the form $f = a+q$ where a is affine and q is a traceless Hermitian quadratic form (cf. [Fu].) Nevertheless, as we stated above, the ϕ-plurisubharmonic functions in any calibrated geometry are locally abundant.

The fundamental property of the ϕ-Hessian:

$$\left(\mathcal{H}^\phi f\right)(\xi) = \text{trace}\left\{\text{Hess} f\big|_\xi\right\} \qquad \text{for all } \phi - \text{planes } \xi$$

is established in Section 2 (Corollary 2.5).

Beginning with Section 3 the ϕ-plurisubharmonic functions are used to study geometry and analysis on calibrated manifolds. The first concept to be addressed is the analogue of pseudoconvexity in complex geometry.

Convexity.

Let (X, ϕ) be a calibrated manifold and $K \subset X$ a closed subset. By the **ϕ-convex hull of** K we mean the subset

$$\widehat{K} = \{x \in X : f(x) \leq \sup_K f \ \text{ for all } f \in \mathcal{PSH}(X, \phi)\}$$

The manifold (X, ϕ) is said to be **ϕ-convex** if $K \subset\subset X \Rightarrow \widehat{K} \subset\subset X$ for all K.

Theorem 3.3. *A calibrated manifold (X, ϕ) is ϕ-convex if and only if it admits a ϕ-plurisubharmonic proper exhaustion function $f : X \to \mathbf{R}$.*

The manifold (X, ϕ) will be called **strictly ϕ-convex** if it admits an exhaustion function f which is strictly ϕ-plurisubharmonic, and it will be called **strictly ϕ-convex**

at infinity if f is strictly ϕ-plurisubharmonic outside of a compact subset. It is shown that in the second case, f can be assumed to be ϕ-plurisubharmonic everywhere. Analogues of Theorem 3.3 are established in each of these cases.

Note that in complex geometry, ϕ-convex manifolds are Stein and manifolds which are ϕ-convex at infinity are strongly pseudoconvex.

We next consider the **core** of X which is defined to be the set of points $x \in X$ with the property that no $f \in \mathcal{PSH}(X, \phi)$ is strictly ϕ-plurisubharmonic at x. The following results are established:

1) Every compact ϕ-submanifold is contained in Core(X). In fact, every ∂-closed ϕ-positive current T is supported in the core. More generally, every ϕ-positive current T is supported in $\widehat{\mathrm{supp}(dT)} \cup \mathrm{Core}(X)$. (See [HL$_3$] or §§5 and 6 below for a discussion of ϕ-positive currents.)

2) The manifold X is strictly ϕ-convex at infinity if and only if Core(X) is compact.

3) The manifold X is strictly ϕ-convex if and only if Core(X) $= \emptyset$.

Examples of complete calibrated manifolds with compact cores are given in an appendix to §3. A very general construction of strictly ϕ-convex manifolds is presented in §8.

We next examine the analogues of pseudoconvex boundaries in calibrated geometry.

Boundary Convexity.

Let $\Omega \subset X$ be a domain with smooth boundary $\partial\Omega$, and let $\rho : X \to \mathbf{R}$ be a **defining function for** $\partial\Omega$, that is, a smooth function defined on a neighborhood of $\overline{\Omega}$ with $\Omega = \{x : \rho(x) < 0\}$, and $\nabla\rho \neq 0$ on $\partial\Omega$. Then $\partial\Omega$ is said to be ϕ**-convex** if

$$\mathcal{H}^{\phi}(\rho)(\xi) \ \geq\ 0 \qquad \text{for all } \phi - \text{planes } \xi \text{ tangential to } \partial\Omega, \tag{0.1}$$

i.e., for all $\xi \in G(\phi)$ with span$(\xi) \subset T(\partial\Omega)$. The boundary ϕ is **strictly ϕ-convex** if the inequality in (0.1) is strict everywhere on $\partial\Omega$. These conditions are independent of the choice of defining function ρ.

Theorem 4.5. *Let $\Omega \subset\subset X$ be a compact domain with strictly ϕ-convex boundary, and let $\delta = -\rho$ where ρ is an arbitrary defining function for $\partial\Omega$. Then $-\log\delta : \Omega \to \mathbf{R}$ is strictly ϕ-plurisubharmonic outside a compact subset. In particular, the domain Ω is strictly ϕ-convex at infinity.*

Elementary examples show that the converse of this theorem does not hold in general. However, there is a weak partial converse.

Proposition 4.8. *Let $\Omega \subset\subset X$ be a compact domain with smooth boundary. Suppose ϕ is parallel and consider the function $\delta = \mathrm{dist}(\bullet, \partial\Omega)$. If $-\log\delta$ is strictly ϕ-plurisubharmonic near $\partial\Omega$, then $\partial\Omega$ is ϕ-convex.*

We note that boundary convexity can be interpreted geometrically as follows. Let II denote the second fundamental form of the hypersurface $\partial\Omega$ oriented by the outward-pointing normal. Then $\partial\Omega$ is ϕ-convex if and only if trace$(II\big|_{\xi}) \leq 0$ for all ϕ-planes ξ which are tangent to $\partial\Omega$.

ϕ-Positive Currents.

Our ϕ-plurisubharmonic functions are intimately related to the study of ϕ-positive currents introduced in [HL$_3$]. We recall that a p-dimensional current T is called **ϕ-positive** if it is representable by integration and its generalized tangent p-vector

$$\overrightarrow{T} \in \text{ConvexHull}(G(\phi)) \qquad \|T\| - a.e.$$

where $\|T\|$ denotes the total variation measure of T. Examples include ϕ-submanifolds and, more generally, rectifiable ϕ-currents. By Almgren's Theorem [A] we know that rectifiable ϕ-currents T with $dT = 0$ are regular, that is, given by integration over ϕ-submanifolds with positive integer multiplicities, outside a closed subset of Hausdorff dimension $p-2$.

ϕ-Positive currents generalize the positive currents in complex geometry, and d-closed recitifiable ϕ-currents generalize positive holomorphic chains.

Recall from above that if T is a ϕ-positive current (with compact support), then

$$\operatorname{supp} T \ \subset \ \widehat{\operatorname{supp}(dT)} \cup \text{Core}(X). \tag{0.2}$$

In particular, if $dT = 0$, then supp$T \subset \text{Core}(X)$, and if X is strictly ϕ-convex ($\text{Core}(X) = \emptyset$), then there exist no d-closed ϕ-positive currents with compact support on X.

In Section 5 we review the known facts concerning ϕ-positive currents. These include compactness theorems, regularity theorems, mass-minimizing properties and dual characterizations.

Superharmonic ϕ-Currents.

Assume for now that the calibration ϕ is parallel, and consider the adjoint of the operator $dd^\phi : \mathcal{E}^0(X) \to \mathcal{D}^p(X)$ which can be written as

$$\partial_\phi \partial : \mathcal{D}'_p(X) \ \longrightarrow \ \mathcal{D}'_0(X)$$

where $\partial : \mathcal{D}'_p(X) \to \mathcal{D}'_{p-1}(X)$ denotes the usual adjoint of $d : \mathcal{E}^{p-1}(X) \to \mathcal{E}^p(X)$ and $\partial_\phi : \mathcal{D}'_{p-1}(X) \ \longrightarrow \ \mathcal{D}'_0(X)$ is the adjoint of d^ϕ, defined by $(\partial_\phi R)(f) \equiv R(d^\phi f)$.

Positive currents T with the property: $\partial_\phi \partial T \leq 0$, i.e., $\partial_\phi \partial T$ is a non-positive measure, satisfy a version of (0.2) above.

Lemma 6.2. *Suppose T is a ϕ-positive current with compact support on X which satisfies $\partial_\phi \partial T \leq 0$ outside a compact subset $K \subset X$. Then*

$$\operatorname{supp} T \ \subset \ \widehat{K} \cup \text{Core}(X).$$

In particular, if $\partial_\phi \partial T \leq 0$ on X, then $\operatorname{supp} T \subset \text{Core}(X)$.

Another consequence is the following. Suppose X is strictly ϕ-convex. If T is ϕ-positive current with $\partial_\phi \partial T \leq 0$ on $X - K$, then $\operatorname{supp} T \subset \widehat{K}$. In fact, it turns out that the points $x \in \widehat{K}$ can be characterized in terms of certain ϕ-positive currents T which satisfy $\partial_\phi \partial T = -[x]$ in $X - K$.

Duval - Sibony Duality.

Points in the ϕ-convex hull of a compact set $K \subset X$ have a useful characterization in terms of ϕ-positive currents and certain Poisson-Jensen measures. The following results generalize work of Duval and Sibony in the complex case. They remain valid (as does Lemma 6.2 above) when ϕ is not parallel if the operator $\partial_\phi \partial$ is replaced with $\mathcal{H}_\phi$.

Let $K \subset X$ be a compact subset and x a point in $X - K$. By a **Green's current for** (K, x) we mean a ϕ-positive current T which satisfies

$$\partial_\phi \partial T = \mu - [x] \tag{0.3}$$

where μ is a probability measure with support on K and $[x]$ denotes the Dirac measure at x. In this case μ is called a **Poisson-Jensen** measure for (K, x). By the remarks above we see that $x \in \widehat{K}$. In fact we have the following.

Theorem 6.8. *Suppose ϕ is parallel and X is strictly ϕ-convex. Let $K \subset X$ be a compact subset and $x \in X - K$. Then there exists a Green's current for (K, x) if and only if $x \in \widehat{K}$.*

We note that if $M \subset X$ is a compact ϕ-submanifold with boundary, and if G_x is the Greens function for the riemannian laplacian on M with singularity at $x \in M - \partial M$, then $\partial_\phi \partial (G_x[M]) = \mu - [x]$ for a probability measure μ on ∂M.

As a application we obtain the following approximation result. A domain $\Omega \subset X$ is said to be **ϕ-convex relative to** X if $K \subset\subset \Omega \ \Rightarrow \widehat{K}_X \subset\subset \Omega$.

Proposition 6.16. *Suppose ϕ is parallel and X is strictly ϕ-convex. An open subset $\Omega \subset X$ is ϕ-convex relative to X if and only if $\mathcal{PSH}(X, \phi)$ is dense in $\mathcal{PSH}(\Omega, \phi)$.*

ϕ-Free Submanifolds and Srictly ϕ-Convex Subdomains.

We next examine the analogues in calibrated geometry of the totally real submanifolds in complex analysis. Using them we show how to construct strictly ϕ-convex manifolds in enormous families with every topological type allowed by Morse theory.

Let (X, ϕ) be any fixed calibrated manifold. A closed submanifold $M \subset X$ is called **ϕ-free** if there are no ϕ-planes tangential to M, i.e., no $\xi \in G(\phi)$ with $\operatorname{span} \xi \subset TM$.

Note that M is automatically ϕ-free if it is **ϕ-isotropic**, that is, if $\phi\big|_M \equiv 0$. .

Any submanifold of dimension $< p$ is ϕ-free. Furthermore, generic local submanifolds of dimension p are ϕ-free. In some geometries this also holds for certain dimensions $> p$.

In a Kähler geometry the ω-free submanifolds are exactly those which are *totally real* (the tangent spaces contain no complex lines). In Special Lagrangian geometry, the ϕ-free submanifolds include the complex submanifolds of all dimensions.

Theorem 7.2. *Suppose M is a closed submanifold of (X, ϕ) and let $f_M(x) \equiv \operatorname{dist}(x, M)^2$ denote the square of the distance to M. Then M is ϕ-free if and only if the function f_M is strictly ϕ-plurisubharmonic at each point in M (and hence in a neighborhood of M).*

The existence of ϕ-free submanifolds insures the existence of many strictly ϕ-convex domains in (X, ϕ).

Theorem 7.4. *Suppose M is a ϕ-free submanifold of (X, ϕ). Then there exists a fundamental neighborhood system $\mathcal{F}(M)$ of M such that:*

(a) M is a deformation retract of each $U \in \mathcal{F}(M)$.

(b) Each neighborhood $U \in \mathcal{F}(M)$ is strictly ϕ-convex .

(c) $\mathcal{PSH}(V, \phi)$ is dense in $\mathcal{PSH}(U, \phi)$ if $U \subset V$ and $V, U \in \mathcal{F}(M)$.

(d) Each compact set $K \subset M$ is $\mathcal{PSH}(U, \phi)$-convex for each $U \in \mathcal{F}(M)$.

This result provides rich families of strictly convex domains. Note that neighborhoods $U \in \mathcal{F}(M)$ include the sets $\{x : \mathrm{dist}(x, M) < \epsilon(x)\}$ for positive functions $\epsilon \in C^\infty(M)$ which die arbitrarily rapidly at infinity. As noted, any submanifold of dimension $< p$ is ϕ-free. Furthermore, any submanifold of a ϕ-free submanifold is again ϕ-free.

For example if X is a Calabi-Yau manifold with Special Lagrangian calibration ϕ, then any complex submanifold $Y \subset X$ is ϕ-free, as is any smooth submanifold $A \subset Y$. The topological type of such manifolds A can be quite complicated.

This construction can be refined even further by replacing $A \subset Y$ with an arbitrary closed subset. It turns out that the following two classes of subsets of (X, ϕ):

(1) Closed subsets A of ϕ-free submanifolds.

(2) Zero sets of non-negative strictly ϕ-plurisubharmonic functions f.

are essentially the same. This is given in Propositions 7.7 and 7.8 One also has:

Proposition 7.12. *Let f be a non-negative, real analytic function on (X, ϕ) and consider the real analytic subvariety $Z \equiv \{f = 0\}$. If f is strictly ϕ-plurisubharmonic at points of Z, then each stratum of Z is ϕ-free .*

The results above generalize work of Harvey-Wells [$\mathrm{HW}_{1,2}$] in the complex case.

Boundaries of ϕ-Submanifolds.

A very natural question in calibrated geometry is the following: Given a compact oriented submanifold $\Gamma \subset X$ of dimension $p-1$, when does there exist a ϕ-submanifold M with $\partial M = \Gamma$? A companion question is: Given a compactly supported current S of dimension $p-1$ in X, when does there exist a ϕ-positive current T with $\partial T = \Gamma$? For this second question there is a complete answer when X is strictly ϕ-convex and ϕ is exact.

Theorem 9.1. *Consider a current $S \in \mathcal{E}'_{p-1}(X)$. Then $S = \partial T$ for some ϕ-positive current $T \in \mathcal{E}'_p(X)$ if and only if*

$$\int_S \alpha \ \geq \ 0 \qquad \text{for all } \alpha \in \mathcal{E}^{p-1}(X) \quad \text{such that } d\alpha \text{ is } \Lambda^+(\phi)\text{−positive}$$

Note . $\Lambda^+(\phi)$-positive means that $d\alpha(\xi) \geq 0$ for all $\xi \in G(\phi)$.

There is a similar result for compact calibrated manifolds (X, ϕ) with no condition on ϕ. The result uses ϕ-quasiplurisubharmonic functions – those which satisfy the condition that $dd^\phi f + \phi$ is $\Lambda^+(\phi)$-positive. See [HL_4] for the Kähler case.

ϕ-Flat Hypersurfaces.

In Section 10 we expand the notion of ϕ-pluriharmonic functions to include functions f which are ϕ-pluriharmonic modulo the ideal generated by df. In most interesting geometries these functions are characterized by the fact that their level sets are **ϕ-flat**, i.e., the trace of the second fundamental form on all tangential ϕ-planes is zero. These functions are important for the boundary problem. If f is such a function defined in a neighborhood of a compact ϕ-submanifold with boundary $M \subset X$, then

$$\inf_{\partial M} \ \leq f(x) \ \leq \ \sup_{\partial M} f \qquad \text{for} \ \ x \in M.$$

Generalized Hodge Manifolds.

In Section 8 we discuss analogues of Hodge manifolds in the general calibrated setting. We also examine various analogues of the Hodge Conjecture in these spaces.

The paper is organized in the order presented above. Several of the sections have appendices which contain examples or discussions of side issues. They can be skipped when reading the paper the first time.

We mention that the operator d^ϕ has been independently found by M. Verbitsky [V] who studied the generalized Kähler theory (in the sense of Chern) on G_2-manifolds. The authors would like to thank Robert Bryant for useful comments and conversations related to this paper.

1. Plurisubharmonic Functions

Suppose ϕ is a calibration on a riemannian manifold X. The ϕ-Grassmann bundle, denoted $G(\phi)$, consists of the unit simple vectors ξ with $\phi(\xi) = 1$, i.e., the ϕ-planes. An oriented submanifold M is a *ϕ-submanifold*, or is *calibrated by* ϕ, if the oriented unit tangent space $\vec{T}_x M$ lies in $G_x(\phi)$ for each $x \in M$, or equivalently, if ϕ restricts to M to be the volume form on M. Let $n = \dim X$ and $p = \text{degree}(\phi)$.

Definition 1.1. The **d^ϕ-operator** is defined by

$$d^\phi f \;=\; \nabla f \lrcorner \phi$$

for all smooth functions f on X.

Hence

$$d^\phi : \mathcal{E}^0(X) \longrightarrow \mathcal{E}^{p-1}(X) \qquad \text{and} \qquad dd^\phi : \mathcal{E}^0(X) \longrightarrow \mathcal{E}^p(X)$$

where $\mathcal{E}^p(X)$ denotes the space of C^∞ p-forms on X. This dd^ϕ operator provides a way of defining plurisubharmonic functions in calibrated geometry when the calibration ϕ is parallel.

If ω is a Kähler form on a complex manifold, then $d^\omega = d^c = -J \circ d$ is the conjugate differential. Thus, the dd^ϕ-operator generalizes the dd^c-operator in complex geometry.

Definition 1.2. Suppose $\nabla\phi = 0$. A function $f \in C^\infty(X)$ is **ϕ-plurisubharmonic** if

$$(dd^\phi f)(\xi) \;\geq 0 \qquad \text{for all} \;\; \xi \in G(\phi).$$

The set of such functions will be denoted $\mathcal{PSH}(X, \phi)$. If $(dd^\phi f)(\xi) > 0$ for all $\xi \in G(\phi)$, then f is **strictly ϕ-plurisubharmonic**. If $(dd^\phi f)(\xi) = 0$ for all $\xi \in G(\phi)$, then f is **ϕ-pluriharmonic**.

Remark 1.3. If ϕ is not parallel, we define ϕ-plurisubharmonic functions by replacing $dd^\phi f$, in Definition 1.2, with $dd^\phi f - \nabla_{\nabla f}\phi$. This modified dd^ϕ-operator is discussed in detail in Section 2. Note that the difference $\nabla_{\nabla f}\phi$ is a first order operator.

The next result justifies the use of the word plurisubharmonic in the context of a ϕ-geometry. A calibration ϕ is **integrable** if for each point $x \in X$ and each $\xi \in G_x(\phi)$ there exists a ϕ-submanifold M through x with $\vec{T}_x M = \xi$.

Theorem 1.4. *Let (X, ϕ) be any calibrated manifold. If a function $f \in C^\infty(X)$ is ϕ-plurisubharmonic, then the restriction of f to any ϕ-submanifold $M \subset X$ is subharmonic in the induced metric. If ϕ is integrable, then the converse holds.*

Theorem 1.4 is an immediate consequence of the formula

$$\left(dd^\phi f - \nabla_{\nabla f}\phi\right)\big|_M \;=\; (\Delta_M f)\,\mathrm{vol}_M \tag{1.1}$$

This formula follows from the three equations (2.7), (2.12), and (2.15), proved below, and the fact that ϕ-submanifolds are minimal submanifolds. We continue for the moment to present results whose proofs will be given in Section 2.

The ϕ-plurisubharmonic functions enjoy many of the useful properties of their classical cousins in complex analysis. Here are some basic examples.

PROPOSITION 1.5. *Let $f, g \in C^\infty(X)$ be ϕ-plurisubharmonic.*

(i) If $\psi \in C^\infty(\mathbf{R})$ is convex and increasing, then $\psi \circ f$ is ϕ-plurisubharmonic.

(ii) The function $\log(e^f + e^g)$ is ϕ-plurisubharmonic.

(iii) The decreasing sequence of functions $h_n \equiv \frac{1}{n}\log\left(e^{nf} + e^{ng}\right)$ of ϕ-plurisubharmonic functions approximates $\max\{f, g\}$. More precisely, $h_n - \frac{1}{n}\log 2 \ \le\ \max\{f, g\} \ \le\ h_n$.

Another important elementary property is given in the next proposition. A p-form ϕ is said to **involve all the variables** if $\zeta \lrcorner \phi \neq 0$ for all non-zero tangent vectors ζ.

PROPOSITION 1.6. *The dd^ϕ-operator is (overdetermined) elliptic if and only if the calibration involves all the variables. Its symbol is*

$$\sigma_\zeta(dd^\phi) \ =\ \zeta \wedge (\zeta \lrcorner \phi) \qquad \text{for } \ \zeta \in T_x^*X \cong T_xX. \tag{1.2}$$

Proof. The computation of (1.2) is straightforward. By definition the operator dd^ϕ is (overdetermined) elliptic if $\sigma_\zeta(dd^\phi)$ is injective (i.e., $\neq 0$) for all $\zeta \neq 0$. Observe that $\zeta \wedge (\zeta \lrcorner \phi) = 0$ if and only if $\zeta \lrcorner \phi = 0$ since $\langle \zeta \wedge (\zeta \lrcorner \phi), \phi \rangle = |\zeta \lrcorner \phi|^2$. Hence, $\zeta \wedge (\zeta \lrcorner \phi) \neq 0$ for all $\zeta \neq 0$ if and only if $\zeta \lrcorner \phi \neq 0$ for all $\zeta \neq 0$. ■

DEFINITION 1.7. The **ϕ-Laplacian**, Δ_ϕ, on a function $f \in C^\infty(X)$ is defined by

$$\Delta_\phi f \ \equiv\ \langle dd^\phi f, \phi \rangle$$

The symbol of Δ_ϕ is the quadratic form $|\zeta \lrcorner \phi|^2$ for $\zeta \in T_x^*X \cong T_xX$. In particular, the ϕ-Laplacian is (determined) elliptic if and only if the calibration involves all the variables.

Definition 1.7 is easily extended to include arbitrary distributions f on X by requiring $(dd^\phi f, \alpha \otimes *1) \geq 0$ for all smooth, compactly supported sections α of the bundle $\Lambda_p TX$ which are positive linear combinations of elements in $G_x(\phi)$ at each point x.

PROPOSITION 1.8. *Suppose ϕ is a calibration which can be written as a positive linear combination of ϕ-planes at each point $x \in X$. If a distribution f is ϕ-plurisubharmonic, then f is Δ_ϕ-subharmonic, i.e., $\Delta_\phi f$ is a non-negative measure.*

REMARK . If ϕ is a calibration which satisfies the hypotheses of both Propositions 1.6 and 1.8, then all the classical results concerning subharmonic functions with respect to the scalar elliptic operator Δ_ϕ can be brought to bear on ϕ-plurisubharmonic distributions. For example, each ϕ-plurisubharmonic distribution is in L^1_{loc} (locally Lebesgue integrable) and represented by an upper-semicontinuous function taking values in $[-\infty, \infty)$. However, because of the geometric emphasis in this paper and the desire to keep technical considerations to a minimum, only C^∞ ϕ-plurisubharmonic functions will be considered. Exceptions will occur as remarks.

Suppose ϕ is a calibration for which $\Lambda(\phi) \equiv \operatorname{span} G(\phi) \subset \Lambda^p T^*X$ is a subbundle. Then *for any ϕ-plurisubharmonic distribution f, the $\Lambda(\phi)$-component of $dd^\phi f$ has measure-coefficients.* This is proved in Proposition 5.19 below.

Remark **(The Abundance of ϕ-plurisubharmonic Functions)**. We shall see in the next section that any convex function on the riemannian manifold X is automatically ϕ-plurisubharmonic . However, in the euclidean case $(\mathbf{R}^n, \phi)$ with ϕ parallel, there are many strictly ϕ-plurisubharmonic quadratic functions which are not convex. (See Remark 2.9.) It follows that in small neighborhoods of a point on any calibrated manifold, such functions exist. Of course the strictly ϕ-plurisubharmonic functions form an open cone in $C^\infty(X)$ for any calibrated manifold (X, ϕ).

Appendix: Pluriharmonic Functions

While ϕ-plurisubharmonic functions are abundant, the ϕ-pluriharmonic functions are often quite scarce. To illustrate this phenomenon we shall sketch some of the basic facts in the "classical" cases.

To begin we note that for some calibrations ϕ, one has that:

$$dd^\phi f \;=\; 0 \qquad \text{if and only if} \qquad (dd^\phi f)(\xi) \;=\; 0 \quad \text{for all } \xi \in G(\phi) \tag{1.3}$$

while for others this is not true. It is the right hand side that defines pluriharmonicity. If (1.3) holds and the map λ_ϕ is everywhere injective (as in Example 1.11), then the only pluriharmonic functions are the affine functions, i.e., the functions with parallel gradient. Note that if f is affine, then ∇f splits the manifold locally as a riemannian product $X = \mathbf{R} \times X_0$.

Example 1.8. **(Complex geometry)**. Let ω be a Kähler form on a complex manifold X. Then $d^\omega = d^c$ is the conjugate differential, $dd^c f$ is the complex hermitian Hessian of f, $G(\omega)$ is the grassmannian of complex lines, and the statement (1.3) is valid. In particular, the ω-pluriharmonic functions are just the classical pluriharmonic functions on X.

Example 1.9. **(Special Lagrangian geometry)**. Consider the special Lagrangian calibration $\phi = \mathrm{Re}(dz)$ on $\mathbf{C}^n$. Let Z_{ij} denote the bidegree $(n-1, 1)$ form obtained from $dz = dz_1 \wedge \cdots \wedge dz_n$ by replacing dz_i with $d\bar{z}_j$ (in the ith position). A short calculation shows that

$$dd^\phi f \;=\; 2\mathrm{Re}\left\{\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} Z_{ij}\right\} + (\Delta f)\mathrm{Re}(dz) \tag{1.4}$$

For this calibration one can show that (1.3) is valid. Consequently, Lei Fu [Fu] has described all the ϕ-pluriharmonic functions.

Proposition 1.10. *Let f be a special Lagrangian pluriharmonic function defined locally on $\mathbf{C}^n$, $n \geq 3$. Then $f = A + Q$ where A is affine and Q is a traceless hermitian quadratic function.*

Proof. If $dd^\phi f = 0$ and $n \geq 3$ (so that Z_{ij} and $\overline{Z}_{ij}$ are of different bi-degrees), then(1.4) implies that $\frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} = 0$ for all i, j. Therefore, all third partial derivatives if f are zero. For polynomials of degree ≤ 2 the result is transparent from (1.4). ∎

Example 1.11. **(Associative, Coassociative and Cayley geometry)**. Consider one of the calibrations:

1. (Associative) $\phi(x \wedge y \wedge z) = \langle x, yz \rangle$ for $x, y, z \in \mathrm{Im}\mathbf{O}$
2. (Coassociative) $\psi = *\phi$ on $\mathrm{Im}\mathbf{O}$
3. (Cayley) $\Phi(x \wedge y \wedge z \wedge w) = \langle x, y \times z \times w \rangle$ for $x, y, z, w \in \mathbf{O}$

where $\mathbf{O}$ denotes the octonions. As in the special Lagrangian case one can show that (1.3) is valid for each of these calibrations. Furthermore, an application of representation theory shows the maps $\lambda_\phi, \lambda_\psi$ and λ_Φ are injective. These calculations carry over to manifolds with G_2 or Spin_7-holonomy to establish the following.

Proposition 1.12. *Let X be a manifold with holonomy contained in G_2 or $Spin_7$ and having dimension 7 or 8 respectively. Suppose ϕ is a parallel calibration on X of one of the three types above. Then every ϕ-pluriharmonic function on X is affine. Moreover, if the holonomy is exactly G_2 or $Spin_7$, every ϕ-pluriharmonic function is constant.*

Proof. The first assertion follows because (1.3) is valid and the λ-maps are injective. The second follows because any non-constant affine function on X would reduce its holonomy to a subgroup of $1 \times SO_{n-1}$. ∎

Example 1.13. **(Quaternionic-Kähler geometry)**. Let $\mathbf{H}$ denote the quaternions and consider $\mathbf{H}^n$ as a right-$\mathbf{H}$ vector space. Each of the complex structures I, J, K (right multiplication by i, j, k) determines a Kähler form $\omega_I, \omega_J, \omega_K$ respectively. The 4-form

$$\Psi \equiv \frac{1}{6}(\omega_I^2 + \omega_J^2 + \omega_K^2) \tag{1.5}$$

on $\mathbf{H}^n \equiv \mathbf{R}^{4n}$ is a calibration with $G(\Psi)$ consisting of the oriented quaternion lines in $\mathbf{H}^n$. In this case, $dd^\Psi f \equiv 0$ if and only if $\mathrm{Hess} f \equiv 0$. However, the assertion (1.3) is not valid in this case, and in fact there is a rich family of Ψ-pluriharmonic functions. For example, if f is ω_I-pluriharmonic, then f is Ψ-pluriharmonic. Hence, so is any ω-pluriharmonic f where $\omega = a\omega_I + b\omega_J + c\omega_K$ with $a^2 + b^2 + c^2 = 1$.

It is well known that the only Ψ-submanifolds in $\mathbf{H}^n$ are the affine quaternion lines.

Of course the calibration (1.5) exists on any quaternionic Kähler manifold, i.e., one with $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$-holonomy. (See [GL] for examples.) With this full holonomy group it seems unlikely that there are many Ψ-pluriharmonic functions. However, if the holonomy is contained in SP_n, they exist in abundance as seen in the next example.

Example 1.14. **(Hyper-Kähler manifolds)**. Let $(X, \omega_I, \omega_J, \omega_K)$ be a hyper-Kähler manifold. Then X carries several parallel calibrations. There are, of course, the Kähler forms $\omega = a\omega_I + b\omega_J + c\omega_K$ with $a^2 + b^2 + c^2 = 1$, and two others of particular interest.

(1) Let $\Psi = \frac{1}{6}(\omega_I^2 + \omega_J^2 + \omega_K^2)$. Then as in Example 1.13, *any ω-pluriharmonic function is Ψ-pluriharmonic*. Hence, the sheaf of Ψ-pluriharmonic functions is quite rich. On the other hand there are precious few Ψ-submanifolds.

(2) Consider the generalized Cayley form $\Xi \equiv \frac{1}{2}(\omega_I^2 - \omega_J^2 - \omega_K^2)$. For this calibration there exist no interesting pluriharmonic functions, at least in dimension 8, but there are many Ξ-submanifolds (cf. [BH]).

Example 1.15. **(Lie group geometry)**. Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$, defined as the set of left-invariant vector fields on G. Consider the fundamental 3-form ϕ on G defined by $\langle x, [y, z] \rangle$ and normalized to have comass one. Calculations indicated that in all but a finite number of cases non-constant pluriharmonic functions do not exists, however there are ϕ-submanifolds, namely the "minimal" SU_2-subgroups (cf. [B]).

Example 1.16. **(Double point geometry)**. Let $\phi = dx_1 \wedge \cdots \wedge dx_n + dy_1 \wedge \cdots \wedge dy_n$ in $\mathbf{R}^{2n}$ for $n \geq 3$. The only ϕ-planes are those parallel to the x or y axes. An easy calculation shows that $dd^\phi f = 0$ if and only if $f(x, y) = g(x) + h(y)$ for harmonic functions g and h. However, a function $f(x, y)$ is ϕ-pluriharmonic if and only if it is harmonic in x and y separately.

2. The ϕ-Hessian.

In this section we prove the assertions made in §1. The arguments will involve ideas and notation important for the rest of the paper. We will end the section with a generalization of Theorem 1.4 to submanifolds which are *ϕ-critical.*

Recall that the **Hessian** (or second covariant derivative) of a smooth function f on a riemannian manifold X is defined on tangent vector fields V, W by

$$\mathrm{Hess}(f)(V,W) \;\equiv\; V(Wf) - (\nabla_V W)f \tag{2.1}$$

where ∇ denotes the riemannian connection. Note that $V(Wf)-(\nabla_V W)f = V(\langle W, \nabla f\rangle) - \langle \nabla_V W, \nabla f\rangle = \langle W, \nabla_V(\nabla f)\rangle$ so that at a point $x \in X$, the Hessian is the symmetric 2-tensor, or the symmetric linear map of T_xX given by

$$\mathrm{Hess}(f)(V) \;=\; \nabla_V(\nabla f) \tag{2.2}$$

Let V be a real inner product space Given an element $\phi \in \Lambda^p V^*$, we define a linear map, central to this paper,

$$\lambda_\phi : \mathrm{End}(V) \;\longrightarrow\; \Lambda^p V^* \tag{2.3}$$

by

$$\lambda_\phi(A) \;\equiv\; D_{A^t}(\phi)$$

where D_{A^t} denotes the extension of the transpose $A^t : V^* \to V^*$ to $D_{A^t} : \Lambda^p V^* \to \Lambda^p V^*$ as a derivation.

Note . Recall that the natural inner product on $\mathrm{End}(V)$ is given by:

$$\langle A, B\rangle \;=\; \mathrm{tr} AB^* \qquad \text{for } A, B \in \mathrm{End}(V)$$

Using this inner product we have the adjoint map

$$\lambda_\phi^* : \Lambda^p V^* \;\longrightarrow\; \mathrm{End}(V) \tag{2.4}$$

which will also be important.

Definition 2.1. The **ϕ-Hessian** of a function $f \in C^\infty(X)$ is the p-form $\mathcal{H}^\phi(f)$ defined by letting the symmetric endomorphism $\mathrm{Hess}f$ act on ϕ as a derivation, i.e.,

$$\mathcal{H}^\phi(f) \;\equiv\; D_{\mathrm{Hess}^t f}(\phi). \tag{2.5}$$

In terms of the bundle map $\lambda_\phi : \mathrm{End}(TX) \to \Lambda^p T^* X$,

$$\mathcal{H}^\phi(f) \;\equiv\; \lambda_\phi(\mathrm{Hess} f) \tag{2.6}$$

is the image of the Hessian of f.

The second order differential operators dd^ϕ and $\mathcal{H}^\phi$ differ by a pure first order operator. This is the first of the three equations needed to prove Theorem 1.4.

THEOREM 2.2. *If ϕ is a closed form on X, then*

$$\mathcal{H}^{\phi}(f) \;=\; dd^{\phi} f - \nabla_{\nabla f}\phi \tag{2.7}$$

Proof. By (2.2) we have $(\mathrm{Hess} f)(V) = \nabla_V \nabla f = [V, \nabla f] + \nabla_{\nabla f} V$, i.e.,

$$\mathrm{Hess} f \;=\; -\mathcal{L}_{\nabla f} + \nabla_{\nabla f}$$

as operators on vector fields ($\mathcal{L}$ = the Lie derivative). The right hand side of this formula has a standard extension to all tensor fields as a derivation that commutes with contractions. It is zero on functions, that is, it is a bundle endomorphism whose value on T^*X is minus the transpose of its value on TX. In particular, we find that $D_{\mathrm{Hess}^t f} = \mathcal{L}_{\nabla f} - \nabla_{\nabla f}$ on p-forms, i.e.,

$$\mathcal{H}^{\phi}(f) \;=\; \mathcal{L}_{\nabla f}(\phi) - \nabla_{\nabla f}\phi \tag{2.8}$$

Finally, since $d\phi = 0$, the classical formula relating $\mathcal{L}, d$ and $\lrcorner$ gives

$$dd^{\phi} f \;=\; d(\nabla f \lrcorner \phi) \;=\; \mathcal{L}_{\nabla f}(\phi) \qquad \blacksquare$$

Many of the nice results for the dd^{ϕ}-operator continue to hold in the non-parallel case after replacing it with the ϕ-Hessian. Perhaps even more importantly, many properties of the dd^{ϕ}-operator in the parallel case can best be understood by considering the ϕ-Hessian.

The second formula needed for the proof of Theorem 1.4 is algebraic in nature, involving the bundle map $\lambda_{\phi} : \mathrm{End}(TX) \to \Lambda^p T^*X$. Consequently, as before, we replace T_xX by a general inner product space V. If ξ is a p-plane in V (not necessarily oriented), let $P_{\xi} : V \to \xi$ denote orthogonal projection. The following, along with its reinterpretations $(2.9)'$ and (2.12), is a central result of this paper.

THEOREM 2.3. *Suppose ϕ has comass one. For each $A \in \mathrm{End}(V)$,*

$$(\lambda_{\phi} A)(\xi) \;=\; \langle A, P_{\xi} \rangle \qquad \text{if } \xi \in G(\phi). \tag{2.9}$$

Equivalently,

$$(\lambda_{\phi}^*)(\xi) \;=\; P_{\xi} \qquad \text{if } \xi \in G(\phi). \tag{2.10}$$

Note that if $e_1, ..., e_p$ is an orthonormal basis for the p-plane ξ, then

$$\langle A, P_{\xi} \rangle \;=\; \sum_{j=1}^{p} \langle e_j, A e_j \rangle$$

Consequently, it is natural to refer to $\langle A, P_{\xi} \rangle$ as the **ξ-trace of** A and to use the notation

$$\mathrm{tr}_{\xi} A \;\equiv\; \langle A, P_{\xi} \rangle.$$

In particular, for each $A \in \mathrm{End}(V)$,

$$(\lambda_\phi A)(\xi) \;=\; \mathrm{tr}_\xi A \qquad \text{if } \xi \in G(\phi). \tag{2.9)$'$}$$

Suppose $\xi \in G(p, V) \subset \Lambda_p V$ is a unit simple vector. If a, b are unit vectors in V with $a \in \mathrm{span}\,\xi$ and $b \perp \mathrm{span}\,\xi$, then

$$b \wedge (a \lrcorner \xi)$$

is called a **first cousin** of ξ. The first cousins of ξ form a basis for the tangent space to the Grassmannian $G(p, V) \subset \Lambda_p V$ at the point ξ. Since ϕ restricted to $G(p, V)$ is a maximum on $G(\phi)$, this fact implies the following result, which we shall use frequently.

LEMMA 2.4. **(The First Cousin Principle)**. *If $\phi \in \Lambda^p V^*$ has comass one and $\xi \in G(\phi)$, then*

$$\phi(\eta) \;=\; 0$$

for all first cousins $\eta = b \wedge (a \lrcorner \xi)$ of ξ.

Note that $D_{(b\otimes a)^t}\phi = D_{a\otimes b}\phi = a \wedge (b \lrcorner \phi)$ and $D_{b\otimes a}\xi = b \wedge (a \lrcorner \xi)$ so that if $A = b \otimes a$ is rank one, then

$$\lambda_\phi(a \otimes b)(\xi) \;=\; (D_{a\otimes b}\phi)(\xi) \;=\; \phi(D_{b\otimes a}\xi) \;=\; \phi(b \wedge (a \lrcorner \xi)) \tag{2.11}$$

Proof of Theorem 2.3. Pick an orthonormal basis for ξ and extend to an orthornormal basis of V. It suffices to prove (2.9) when $A = b\otimes a$ with a and b elements of this basis. It is easy to see that $\langle b\otimes a, P_\xi\rangle = 0$ unless $a = b \in \xi$, in which case $\langle a\otimes a, P_\xi\rangle = 1$. By equation (2.11) we have $\lambda_\phi(b\otimes a)(\xi) = \phi(b\wedge(a\lrcorner\xi))$ and $b\wedge(a\lrcorner\xi) = 0$ unless $a \in \xi$ and either $b \in \xi^\perp$ or $b = a$. If $b \in \xi^\perp$, then $(b \wedge (a\lrcorner\xi)$ is a first cousin of ξ and $\phi((b \wedge (a\lrcorner\xi)) = 0$ by the First Cousin Principle. If $a = b \in \xi$, then $b \wedge (a\lrcorner\xi) = \xi$ and therefore $\phi((b \wedge (a\lrcorner\xi)) = \phi(\xi) = 1$. ∎

Theorem 2.3 has many consequences. We mention several. From (2.9)$'$ we have:

COROLLARY 2.5. *Suppose (X, ϕ) is a calibrated manifold. For each function $f \in C^\infty(X)$,*

$$\mathcal{H}^\phi(f)(\xi) \;=\; \mathrm{tr}_\xi(\mathrm{Hess} f) \qquad \text{if } \xi \in G(\phi). \tag{2.12}$$

This equation (2.12) is the second equation needed in the proof of Theorem 1.4.

REMARK . Equation (2.12) provides an alternative definition of ϕ-plurisubharmonic (as well as strict ϕ-plurisubharmonic and ϕ-pluriharmonic) functions, which bypasses the bundle map λ_ϕ and uses only the trace of the Hessian of f on ϕ-planes ξ.

Another application of Theorem 2.3 is given by:

COROLLARY 2.6. *If $A \in \mathrm{End}(V)$ is skew, then the p-form $\lambda_\phi A$ vanishes on $G(\phi)$.*

Theorem 2.3 can be used to prove Proposition 1.5. Note that for $A, B \in \mathrm{Sym}^2(\mathrm{V}) \subset \mathrm{End}(\mathrm{V})$, if $A \geq 0$, $B \geq 0$, then $\langle A, B\rangle = \mathrm{tr}AB \geq 0$. Hence for all $\xi \in G_p(V)$ one has

$\langle e\otimes e, P_\xi\rangle \geq 0$, and more generally $\langle A, P_\xi\rangle \geq 0$ whenever $A \geq 0$. Since df and ∇f are metrically equivalent,

$$\lambda_\phi(\nabla f\otimes\nabla f) = df\wedge(\nabla f \lrcorner \phi) = df\wedge d^\phi f. \tag{2.13}$$

Therefore, Theorem 2.3 has the following consequence.

COROLLARY 2.7. *For any $f\in C^\infty(X)$,*

$$(df\wedge d^\phi f)(\xi) \;=\; |\nabla f\lrcorner\xi|^2 \;\geq\; 0 \qquad \text{for all} \;\; \xi\in G(\phi).$$

Proof of Proposition 1.5. We will use Corollary 2.7. For (i) note that

$$dd^\phi(\psi\circ f) \;=\; (\psi'\circ f)\, dd^\phi f + (\psi''\circ f)\, df\wedge d^\phi f,$$

which combined with (2.7) shows that

$$\mathcal{H}^\phi(\psi\circ f) \;=\; (\psi'\circ f)\,\mathcal{H}^\phi f + (\psi''\circ f)\, df\wedge d^\phi f. \tag{2.14}$$

For (ii) compute that:

$$\mathcal{H}^\phi \log(e^f+e^g) \;=\; \left(\frac{e^f}{e^f+e^g}\right)\mathcal{H}^\phi f + \left(\frac{e^g}{e^f+e^g}\right)\mathcal{H}^\phi g + \frac{e^f e^g}{(e^f+e^g)^2} d(f-g)\wedge d^\phi(f-g).$$

For (iii) set $a=e^f$ and $b=e^g$ in the inequalities: $\max\{a,b\} \;\leq\; \sqrt[n]{a^n+b^n} \;\leq\; 2^{\frac{1}{n}}\max\{a,b\}$ and take the log. ∎

Theorem 2.3 can also be used to understand the relationship between convex functions and ϕ-plurisubharmonic functions. A function $f\in C^\infty(X)$ is called **convex** if $\mathrm{Hess} f\geq 0$ at each point, and it is called **affine** if $\mathrm{Hess} f\equiv 0$ on X. (If f is affine, ∇f splits X locally as a riemannian product $\mathbf{R}\times X_0$.)

COROLLARY 2.8. *Every convex function is ϕ-plurisubharmonic , and every strictly convex function is strictly ϕ-plurisubharmonic (and every affine function is ϕ-pluriharmonic).*

REMARK 2.9. The converse always fails; there are always ϕ-plurisubharmonic functions which are not convex. To see this, consider first the euclidean case with $X=V$ and ϕ parallel. Recall that the orthogonal projections P_e onto lines in V generate the extreme rays of the convex cone of convex functions (positive semi-definite quadratic forms) in $\mathrm{Sym}^2\mathrm{V}\subset\mathrm{End(V)}$. This cone is self-dual. The projections $P_\xi = \lambda_\phi^*(\xi)$ for $\xi\in G(\phi)$ generate a proper convex subcone (in fact a proper convex subcone of the cone generated by orthogonal projections onto p-planes). Hence, by the Bipolar Theorem there must exist a non-convex quadratic function $Q\in\mathrm{Sym}^2\mathrm{V}$ with $\langle Q, P_\xi\rangle\geq 0$ for all $\xi\in G(\phi)$. By (2.9), Q is ϕ-plurisubharmonic .

We now recall some elementary facts about submanifolds. Given a submanifold $\overline{X}\subset X$, let $(\bullet)^T$ and $(\bullet)^N$ denote orthogonal projection of T_xX onto the tangent and normal spaces of $\overline{X}$ respectively. Then the canonical riemannian connection $\overline{\nabla}$ of the induced metric on $\overline{X}$ is given by $\overline{\nabla}_V W = (\nabla_V W)^T$ for tangent vector fields V, W on $\overline{X}$. The **second fundamental form** is defined by

$$B_{V,W} \equiv (\nabla_V W)^N = \nabla_V W - \overline{\nabla}_V W.$$

This is a symmetric bilinear form on $T\overline{X}$ with values in the normal space. Its trace $H = \operatorname{trace} B$ is the **mean curvature vector field** of $\overline{X}$, and $\overline{X}$ is called a **minimal submanifold** if $H \equiv 0$. Finally, let $\overline{\Delta}$ denote the Laplace-Beltrami operator on $\overline{X}$ and $\overline{\mathrm{Hess}}$ denote the Hessian operator on $\overline{X}$. The proof of the following is straightforward.

$$\overline{\mathrm{Hess}}(f)(V,W) = \mathrm{Hess}(f)(V,W) - B_{V,W} \cdot f$$

for tangent vectors $V, W \in T\overline{X}$. Taking the $T\overline{X}$-trace yields:

$$\overline{\Delta} = \mathrm{tr}_{T\overline{X}} \mathrm{Hess} f - H(f).$$

With a change of notation, this is the final formula needed to prove Theorem 1.4.

PROPOSITION 2.10. *Suppose M is a p-dimensional submanifold of X with mean curvature vector field H. Then for each $f \in C^\infty(X)$,*

$$\Delta_M f = \mathrm{tr}_{TM} \mathrm{Hess} f - H(f) \qquad \text{on } M \tag{2.15}$$

COROLLARY 2.11. *Suppose M is a ϕ-submanifold of X. Then*

$$\mathcal{H}^\phi(f)\big|_M = (\Delta_M f)\mathrm{vol}_M \tag{2.16}$$

Proof. Combine (2.12) and (2.15) with the fact that $H = 0$. ∎

Combining this with (2.7) gives equation (1.1) and proves Theorem 1.4. This completes the proof of all the results in §1 except Proposition 1.8.

Proof of Proposition 1.8. By definition, a distributional section of a vector bundle $E \to X$ is a continuous linear functional on the space of smooth compactly supported sections of $E^* \otimes \Lambda^n T^* X$, or equivalently, on the space of $\widetilde{s} \equiv s \otimes *1$ for $s \in \Gamma_{\mathrm{cpt}} E^*$.

Suppose f is a ϕ-plurisubharmonic distribution, that is, $(dd^\phi f, \widetilde{\alpha}) \geq 0$ for all smooth compactly supported sections α of the bundle $\Lambda_p TX$ which are positive linear combinations of ϕ-planes at each point. Let $\widetilde{\phi}$ denote the section of $\Lambda_p TX$ corresponding to ϕ under the metric equivalence $\Lambda_p \cong \Lambda^p$. Set $\widetilde{\alpha} = g\widetilde{\phi}$ with $g \geq 0$ a smooth compactly supported function. By hypothesis $0 \leq (dd^\phi f, g\widetilde{\phi})$, and we claim that $(dd^\phi f, g\widetilde{\phi}) = (\Delta_\phi f, \widetilde{g})$ where $\widetilde{g} \equiv g(*1)$. To see this it suffices to consider the case where f is smooth, where one has $(dd^\phi f, g\widetilde{\phi}) = (\langle dd^\phi f, \phi\rangle, \widetilde{g}) = (\Delta_\phi f, \widetilde{g})$. Thus we conclude that $0 \leq (\Delta_\phi f, \widetilde{g})$ for all compactly supported functions g with $g \geq 0$. ∎

For future reference we add a remark.

Remark 2.12. The operator d^ϕ can be expressed in terms of the Hodge d^*-operator as

$$d^\phi f \;=\; d^*(f\phi)$$

and therefore

$$dd^\phi f \;=\; dd^*(f\phi).$$

To prove this, first note that if $v \in T_xX$ and $\alpha \in T_x^*X$ are metrically equivalent, then $v \lrcorner \phi = *(\alpha \wedge *\phi)$. Hence, $d^\phi f = \nabla f \lrcorner \phi = *(df \wedge *\phi) = *(d(f * \phi) - f(d * \phi)) = *d*(f\phi) - f * d * \phi$, that is,

$$d^\phi f \;=\; d^*(f\phi) - fd^*\phi$$

so that the first equation holds if ϕ is a harmonic form, and in particular if ϕ is parallel. Note also that for $\psi = *\phi$

$$d^\psi f \;=\; \pm * d(f\phi) \qquad \text{and} \qquad dd^\psi f \;=\; \pm * d^* d(f\phi).$$

Appendix A. Submanifolds which are ϕ-critical.

This appendix is not important for the remainder of the paper and can be skipped. Here we establish a useful extension of Theorem 2.3 to certain ξ which are not ϕ-planes. Let $G \equiv G(p, V)$ denote the Grassmannian of oriented p-planes in the inner product space V, considered as the subset $G \subset \Lambda_p V$ of unit simple vectors.

Definition 2.A.1. Given $\phi \in \Lambda^p V^*$ an element $\xi \in G$ is said to be a **ϕ-critical point** if $\xi \in G$ is a critical point of the function $\phi\big|_G$. Equivalently, ϕ must vanish on $T_\xi G \subset \Lambda_p V$. Let $G^{\rm cr}(\phi)$ denote the set of ϕ critical points.

Note that if ϕ is a calibration on G, i.e., $\sup \phi\big|_G = 1$, then

$$G(\phi) \;\subset\; G^{\rm cr}(\phi)$$

since for each $\xi \in G(\phi)$ the form ϕ attains its maximum value 1 at ξ. Equation $(2.9)'$ extends from $G(\phi)$ to $G^{\rm cr}(\phi)$ as follows

Proposition 2.A.2. *Suppose $\phi \in \Lambda^p V^*$ and $A \in \mathrm{End}(V)$. Then for all $\xi \in G^{\rm cr}(\phi)$*

$$\lambda_\phi(A)(\xi) \;=\; (\mathrm{tr}_\xi A)\phi(\xi)$$

Proof. This is an immediate consequence of the more general Proposition 2.A.4 below. ■

Recall that at a point $\xi \in G$ there is a canonical isomorphism:

$$T_\xi G \;\cong\; \mathrm{Hom}\,(\mathrm{span}\,\xi, (\mathrm{span}\,\xi)^\perp). \tag{2.A.1}$$

On the other hand, $T_\xi G$ is canonically a subspace of $\Lambda_p V$. It is exactly the subspace spanned by the first cousins of ξ. More specifically, the isomorphism (2.A.1) associates to $L : \mathrm{span}\,\xi \to (\mathrm{span}\,\xi)^\perp$ the p-vector $D_L\xi$.

DEFINITION 2.A.3. Let $A \in \mathrm{End}(V)$ be a linear map. At each point $\xi \in G$ we define a tangent vector

$$D_{\widetilde{A}}\xi \ \in \ T_\xi G$$

where $\widetilde{A} = P_{\xi^\perp} \circ A \circ P_\xi$. This vector field $\xi \to D_{\widetilde{A}}\xi$ on G is called the A**-vector field**.

REMARK . A straightforward calculation shows that this A-vector field on G is the gradient vector field ∇F_A of the height function $F_A : G \to \mathbf{R}$ given by $F_A(\xi) = \langle A, P_\xi\rangle$.

PROPOSITION 2.A.4. *Suppose $\phi \in \Lambda^p V^*$ and $A \in \mathrm{End}(V)$. Then for all p-planes $\xi \in G(p, V)$,*

$$\lambda_\phi(A)(\xi) \ = \ (\mathrm{tr}_\xi A)\phi(\xi) + \phi(D_{\widetilde{A}}\xi) \tag{2.A.2}$$

Proof. Pick an orthonormal basis for ξ and extend to an orthornormal basis of V. It suffices to prove (2.A.2) when $A = b \otimes a$ with a and b elements of this basis. Using formula (2.11) we see the following.

(1) If $a \in \xi^\perp$, then all terms in (2.A.2) are zero.

(2) If $a \in \xi$ and $b \in \xi^\perp$, then $\widetilde{A} = A = b \otimes a$, $\mathrm{tr}_\xi A = 0$, and $\lambda_\phi(b\otimes a)(\xi) = (a \wedge (b \lrcorner \phi)) = \phi(b \wedge (a \lrcorner \xi)) = \phi(D_A\xi)$

(3) If $a = b \in \xi$, then $\lambda_\phi(A) = \phi(a \wedge (a \lrcorner \xi) = \phi(\xi)$ and $\mathrm{tr}_\xi(A) = 1$. Since $\widetilde{A} = 0$, equation (2.A.2) holds in this case.

(4) If $a, b \in \xi$ and $a \perp b$, then $b \wedge (a \lrcorner \xi) = 0$, and one sees easily that all three terms in (2.A.2) are zero. ■

REMARK 2.A.5. Proposition 2.A.2 can be restated as

$$\lambda^*_\phi(\xi) \ = \ \phi(\xi)P_\xi \qquad \text{for all} \quad \xi \in G^{\mathrm{cr}}(\phi). \tag{2.A.3}$$

Conversely, if $\lambda^*_\phi(\xi) \ = \ cP_\xi$ for some $\xi \in G^{\mathrm{cr}}(\phi)$, then $c = \phi(\xi)$ and ξ is ϕ-critical.

Proof. For all $\xi \in G(p, V)$ we have $\langle P_\xi, \lambda^*_\phi(\xi)\rangle = (\lambda_\phi P_\xi)(\xi) = (D_{P^t_\xi}\phi)(\xi) = \phi(D_{P_\xi}\xi) = p\phi(\xi)$ since $D_{P_\xi}\xi = p\xi$. Therefore, $\lambda^*_\phi(\xi) = cP_\xi$ implies that $pc = p\phi(\xi)$ and equation (2.A.3) holds. Equation (2.A.2) now implies that $\phi(D_{\widetilde{A}}\xi) = 0$ for all $A \in \mathrm{End}(V)$ and, in particular, $\phi(D_L\xi) = 0$ for all $L : \xi \to \xi^\perp$. That is, ϕ vanishes on $T_\xi G \subset \Lambda_p V$, i.e., $\xi \in G^{\mathrm{cr}}(\phi)$. ■

We now define an oriented submanifold M of X to be ϕ-**critical** if $\overrightarrow{T}_x M \in G^{\mathrm{cr}}(\phi)$ for all $x \in M$. We leave it to the reader to use Proposition 2.A.2 to establish the following extension of the previous results.

THEOREM 2.A.6. *Suppose ϕ is a p-form on a riemannian manifold X and $M \subset X$ is a ϕ-critical submanifold with mean curvature vector field H. Then for all $f \in C^\infty(X)$,*

$$\lambda_\phi(\mathrm{Hess} f) \ = \ [\Delta_M(f) + H(f)]\phi$$

when restricted to M. In particular, if M is minimal, then on M

$$\lambda_\phi(\mathrm{Hess} f) \ = \ (\Delta_M f)\phi$$

EXAMPLE . Let $\phi = \frac{1}{6}\{\omega_I^2 + \omega_J^2 + \omega_K^2\}$ be the quaternion calibration on $\mathbf{H}^n$. Then $\pm\frac{1}{3}$ are critical values and the ϕ-critical submanifolds with critical value $\pm\frac{1}{3}$ include all complex Lagrangian submanifolds for any complex structure defined by right multiplication by a unit imaginary quaternion (cf. [U]).

Appendix B. Constructing ϕ-plurisubharmonic functions.

Straightforward calculation shows that if $F(x) = g(u_1(x), ..., u_m(x))$, then

$$dd^\phi F \;=\; \sum_{j=1}^m \frac{\partial g}{\partial t_j} dd^\phi u_j + \sum_{i,j=1}^m \frac{\partial^2 g}{\partial t_i \partial t_j} \lambda_\phi(\nabla u_i \circ \nabla u_j) \tag{2.B.1}$$

PROPOSITION 2.B.1. *If $u_1, ..., u_m$ are ϕ-pluriharmonic and $g(t_1, ..., t_m)$ is convex, then $F = g(u_1, ..., u_m)$ is ϕ-plurisubharmonic . More generally, if $\frac{\partial g}{\partial t_j} \geq 0$ for $j = 1, ..., m$ and g is convex, then $F = g(u_1, ..., u_m)$ is ϕ-plurisubharmonic whenever each u_j is ϕ-plurisubharmonic.*

Proof. Under our assumptions the first term in equation (2.B.1) is ≥ 0 on any $\xi \in G(\phi)$. To show that the second term is ≥ 0 is suffices to consider the case where the matrix $((\frac{\partial g}{\partial t_j}))$ is rank one, i.e., equal to $((x_i x_j))$ for some vector $x \in \mathbf{R}^n$. Then the second term equals $\lambda_\phi\{(\sum_i x_i \nabla u_i) \circ (\sum_j x_j \nabla u_j)\}$ which is ≥ 0 on $\xi \in G(\phi)$ by (2.13) and Corollary 2.7. ∎

We now analyze the case where $m = 2$ and determine necessary and sufficient conditions for $F = g(u_1, u_2)$ to be ϕ-plurisubharmonic.

LEMMA 2.B.2. *Fix $v, w \in \mathbf{R}^n$ and $\xi \in G(\phi)$. Let v_0 and w_0 denote the orthogonal projections of v and w respectively onto ξ (considered as a p-plane in $\mathbf{R}^n$). Then*

$$\lambda_\phi(v \circ w)(\xi) \;=\; \langle v_0, w_0 \rangle.$$

Proof. Write $v = v_0 + v_1$ and $w = w_0 + w_1$ with respect to the decomposition $\mathbf{R}^n = \mathrm{span}\,\xi \oplus (\mathrm{span}\,\xi)^\perp$. Then for $\xi \in G(\phi)$ we have

$$\lambda_\phi(v \circ w)(\xi) \;=\; \phi\{(v_0 + v_1) \wedge ((w_0 + w_1) \lrcorner \xi))\} \;=\; \phi\{(v_0 + v_1) \wedge (w_0 \lrcorner \xi)\}$$

$$=\; \phi(v_0 \wedge (w_0 \lrcorner \xi)) \;=\; \langle v_0, w_0 \rangle \phi(\xi) \;=\; \langle v_0, w_0 \rangle.$$

where the third equality follows from the First Cousin Priinciple. ∎

By Lemma 2.B.2 we have that for $\xi \in G(\phi)$,

$$\begin{aligned} \lambda_\phi\{a\, v \circ v + &2b\, v \circ w + c\, w \circ w\}(\xi) \\ &= a\|v_0\|^2 + 2b\langle v_0, w_0 \rangle + c\|w_0\|^2 \\ &= \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} \|v_0\|^2 & \langle v_0, w_0 \rangle \\ \langle v_0, w_0 \rangle & \|w_0\|^2 \end{pmatrix} \right\rangle \end{aligned} \tag{2.B.2}$$

REMARK 2.B.3. A symmetric $n \times n$-matrix A is ≥ 0 iff $\langle A, P\rangle \geq 0$ for all rank-one symmetric $n \times n$-matrices P.

REMARK 2.B.4. The matrix $\begin{pmatrix} \|v_0\|^2 & \langle v_0, w_0\rangle \\ \langle v_0, w_0\rangle & \|w_0\|^2 \end{pmatrix}$ is rank-one iff v_0 and w_0 are linearly dependent.

LEMMA 2.B.5. *Let $v, w \in \mathbf{R}^n$ be linearly independent. Suppose that for every line*

$$\ell \ \subset \ \text{span}\,\{v, w\}$$

there exists a $(p-1)$-plane $\xi_0 \subset \text{span}\,\{v, w\}^\perp$ such that $\ell \oplus \xi_0$ (when properly oriented) is a ϕ-plane. Then $\lambda_\phi\{av \circ v + 2bv \circ w + cw \circ w\}$ is ϕ-positive if and only if $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \geq 0$.

Proof. Necessity is already done. For sufficiency fix a, b, c. For each $\ell \subset \text{span}\,\{v, w\}$ let $\xi \in G(\phi)$ be the oriented p-plane $\ell \oplus \xi_0$ given in the hypothesis, and note that by equation (2.B.2)

$$\lambda_\phi\{av \circ v + 2bv \circ w + cw \circ w\}(\xi) \ = \ \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} v_\ell^2 & v_\ell w_\ell \\ v_\ell w_\ell & w_\ell^2 \end{pmatrix} \right\rangle \ \geq \ 0$$

where $v_\ell = \langle v, e\rangle$, $w_\ell = \langle w, e\rangle e$, and $\ell = \text{span}\,\{e\}$. Now the matrix $\begin{pmatrix} v_\ell^2 & v_\ell w_\ell \\ v_\ell w_\ell & w_\ell^2 \end{pmatrix}$ is rank-one, and every rank-one 2×2 matrix, up to positive scalars, occurs in this family. The result follows from Remark 2.B.3. ∎

DEFINITION 2.B.6. A calibration ϕ on a manifold X is called **rich** (or **2-rich**) if for any 2-plane $P \subset T_xX$ at any point x, and for any line $\ell \subset P$, there exists a $(p-1)$-plane $\xi_0 \subset P^\perp$ so that $\pm\ell \oplus \xi_0$ is a ϕ-plane.

PROPOSITION 2.B.7. *Let (X, ϕ) be a rich calibrated manifold. Suppose u_1, u_2 are ϕ-pluriharmonic functions on X with $\nabla u_1 \wedge \nabla u_2 \neq 0$ on a dense set. Then for any C^2-function $g(t_1, t_2)$*

$$F \ = \ g(u_1, u_2) \in \mathcal{PSH}(X, \phi) \quad \Leftrightarrow \quad g \ \text{ is convex}$$

Proof. Apply Proposition 2.B.1, equation (2.B.2) and Lemma 2.B.5. ∎

PROPOSITION 2.B.8. *The Special Lagrangian calibration on a Calabi-Yau n-fold, $n \geq 3$, and the associative and coassociative calibrations on a G_2-manifold are rich calibrations.*

Proof. For the Special Lagrangian case it suffices to consider $\phi = \text{Re}(dz)$ on $\mathbf{C}^n$, $n \geq 3$. Let $e_1, Je_1, ...e_n, Je_n$ be the standard hermitian basis of $\mathbf{C}^n$. By unitary invariance we may assume that $\ell = \text{span}\,\{e_1\}$ and $P = \text{span}\,\{e_1, \alpha Je_1 + \beta e_2\}$. Then the $(p-1)$-plane $\xi_0 = -Je_2 \wedge Je_3 \wedge e_4 \wedge \cdots \wedge e_n$ does the job.

Consider now the associative calibration $\phi(x, y, z) = \langle x \cdot y, z\rangle$ on the imaginary octonians $\text{Im}(\mathbf{O}) = \text{Im}(\mathbf{H}) \oplus \mathbf{H} \cdot \epsilon$ where $\mathbf{H}$ denotes the quaternions. By the transitivity of the group G_2 on $S^6 = G_2/\text{SU}(3)$ and the transitivity of SU(3) on the tangent space, we may assume $\ell = \text{span}\,\{i\}$ and $P = \text{span}\,\{i, j\}$ in $\text{Im}(\mathbf{H})$. We now choose $\xi_0 = \epsilon \wedge (i \cdot \epsilon)$. For the coassociative calibration we choose $\xi_0 = k \wedge (i\epsilon) \wedge (k\epsilon)$ and note that $i \wedge \xi_0 = i \wedge k \wedge (i\epsilon) \wedge (k\epsilon)$ is coassociative because its orthogonal complement is $j \wedge \epsilon \wedge (j\epsilon)$ which is associative. ■

We now give some examples and applications of the material above. We start with Special Lagrangian geometry where the ϕ-pluriharmonic functions are given by Proposition 1.10. Hence, we may apply Proposition 2.B.7 to conclude the following. Let $u_1(z)$ and $u_2(z)$ be two traceless hermitian quadratic forms on $\mathbf{C}^n$. (For example, take $u_1(z) = |z_1|^2 - |z_2|^2$ and $u_2(z) = (n-2)|z_1|^2 - |z_3|^2 - \cdots - |z_n|^2$.) Then $g(u_1(z), u_2(z))$ is ϕ-plurisubharmonic if and only if g is convex.

Formula (2.B.1) can be usefully appled to more general functions u_j. For example, in the Special Lagrangian case on $\mathbf{C}^n$ with $\phi = \mathrm{Re}(dz)$, one has that $dd^\phi(\frac{1}{2}|z_k|^2) = \phi$, for any complex coordinate z_k in any unitary coordinate system on $\mathbf{C}^n$. Hence a linear combination of these functions have the property that $dd^\phi u = c\phi$ for some constant c.

Proposition 2.B.9. *Let (X, ϕ) be a rich calibrated manifold. Suppose $u_1, ..., u_n \in C^\infty(X)$ satisfy the equations $dd^\phi u_i = c_i\phi$ for constants $c_1, ..., c_n$. Then for any C^2-function $g(t_1, ..., t_n)$*

$$F \;=\; g(u_1, ..., u_n) \in \mathcal{PSH}(X, \phi) \;\;\Leftrightarrow\;\; \left\{\sum_{i=1}^n c_i \frac{\partial g}{\partial t_i}\right\}\mathbf{Id} + \left\langle \mathrm{Hess}_g, (((\langle(\nabla u_i)^\xi, (\nabla u_j)^\xi\rangle))\right\rangle \;\geq\; 0$$

for all ϕ-planes ξ at all points of X.

3. Convexity in Calibrated Geometries

We suppose throughout this section that (X, ϕ) is a non-compact, connected calibrated manifold.

Definition 3.1. If K is a compact subset of X, we define the (X,ϕ)**-convex hull of** K by

$$\widehat{K} \ \equiv \ \{x \in X : f(x) \leq \sup_K f \ \ \text{for all } f \in \mathcal{PSH}(X,\phi)\}$$

If $\widehat{K} = K$, then K is called (X,ϕ)**-convex.**

Lemma 3.2. *Suppose K is a compact subset of X. Then $x \notin \widehat{K}$ if and only if there exists a smooth non-negative ϕ-plurisubharmonic function f on X which is identically zero on a neighborhood of K and has $f(x) > 0$. Furthermore, if there exists a ϕ-plurisubharmonic function on X which is strict at x, then f can be chosen to be strict at x.*

Proof. Suppose $x \notin \widehat{K}$. Then there exists $g \in \mathcal{PSH}(X,\phi)$ with $\sup_K g < 0 < g(x)$. Pick $\varphi \in C^\infty(\mathbf{R})$ with $\varphi \equiv 0$ on $(-\infty, 0]$ and with $\varphi > 0$ and convex increasing on $(0,\infty)$. Then $f = \varphi \circ g$ satisfies the required conditions. Furthermore, assume $h \in \mathcal{PSH}(X,\phi)$ is strict at x. Then take $\overline{g} = g + \epsilon h$. For small enough ϵ, $\sup_K \overline{g} < 0 < \overline{g}(x)$. If φ is also strictly increasing on $(0,\infty)$, then $f = \varphi \circ \overline{g}$ is strict at x. ∎

Theorem 3.3. *The following two conditions are equivalent.*

1) *If $K \subset\subset X$, then $\widehat{K} \subset\subset X$.*
2) *There exists a ϕ-plurisubharmonic proper exhaustion function f on X.*

Definition 3.4. If the equivalent conditions of Theorem 3.3 are satisfied, then (X,ϕ) is a **convex calibrated manifold** and X **is ϕ-convex .**

Proof that $2) \Rightarrow 1)$**:** If K is compact, then $c \equiv \sup_K f$ is finite and $\widehat{K}$ is contained in the compact prelevel set $\{x \in X : f(x) \leq c\}$.

Proof that $1) \Rightarrow 2)$**:** A ϕ-plurisubharmonic proper exhaustion function on X is constructed as follows. Choose an exhaustion of X by compact (X,ϕ)-convex subsets $K_1 \subset K_2 \subset K_3 \subset \cdots$ with $K_m \subset K^0_{m+1}$ for all m. By Lemma 3.2 and the compactness of $K_{m+2} - K^0_{m+1}$, there exists a ϕ-plurisubharmonic function $f_m \geq 0$ on X with f_m identically zero on a neighborhood of K_m and $f_m > 0$ on $K_{m+2} - K^0_{m+1}$. By rescaling we may assume $f_m > m$ on $K_{m+2} - K^0_{m+1}$. The locally finite sum $f = \sum_{m=1}^\infty f_m$ satisfies 2). ∎

Lemma 3.5. *Condition 2) in Theorem 3.3 is equivalent to the* a priori *weaker condition:*

$2)'$ *There exists a continuous proper exhaustion function f on X which is smooth and ϕ-plurisubharmonic outside a compact subset of X.*

In fact if f satisfies $2)'$, then f can be modified on a compact subset to be ϕ-plurisubharmonic on all of X. Consequently, if f satisfies $2)'$ and is strict outside a compact set, then its modification $\varphi \circ f$ is also strict outside a compact set.

Proof. For large enough c, f is smooth and ϕ-plurisubharmonic outside the compact set $\{x \in X : f(x) \leq c - 1\}$. Pick a convex increasing function $\varphi \in C^\infty(\mathbf{R})$ with $\varphi \equiv c$ on a neighborhood of $(-\infty, c-1]$ and $\varphi(t) = t$ on $(c+1, \infty)$. Then $\varphi \circ f$ is ϕ-plurisubharmonic on all of X (in particular smooth) and equal to f outside of the compact set $\{x \in X : f(x) \leq c+1\}$. ■

THEOREM 3.6. *The following two conditions are equivalent:*

1) $K \subset\subset X \Rightarrow \widehat{K} \subset\subset X$, and X carries a strictly ϕ-plurisubharmonic function.

2) There exists a strictly ϕ-plurisubharmonic proper exhaustion function for X.

DEFINITION 3.7. If the equivalent conditions of Theorem 3.6 are satisfied, then (X, ϕ) is a **strictly convex calibrated manifold** or X is **strictly ϕ-convex**.

Proof of Theorem 3.6. Suppose that X is equipped with both a ϕ-plurisubharmonic proper exhaustion function f and a strictly ϕ-plurisubharmonic function g. Then the sum $f + e^g$ is a strictly ϕ-plurisubharmonic exhaustion function. Now Theorem 3.6 follows immediately from Theorem 3.3. ■

We shall construct many ϕ-convex manifolds in the course of our discussion (See, in particular, §7). However, we present some elementary examples here.

EXAMPLE 1. Suppose $\phi \in \Lambda^p \mathbf{R}^n$ is a parallel calibration on $\mathbf{R}^n$. Let $f(x) = \frac{1}{2}\|x\|^2$. Then $dd^\phi f = p\phi$ and hence f is a strictly ϕ-plurisubharmonic exhaustion. That is, $(\mathbf{R}^n, \phi)$ is a strictly convex calibrated manifold.

EXAMPLE 2. Suppose $\phi = dx_1 \wedge \cdots \wedge dx_n$ on a domain X in $\mathbf{R}^n$. Then $dd^\phi f = (\Delta f)\phi$ and f is ϕ-plurisubharmonic if and only if f is subharmonic. Recall that if $K \subset\subset X$, then $\widehat{K} = K \cup \{$all the "holes" in K relative to $X\}$, (connected components of $X - K$ which are relatively compact in X. Thus (X, ϕ) is strictly convex for any open set $X \subset \mathbf{R}^n$.

It is instructive to extend this elementary example.

EXAMPLE 3. Suppose $\phi = dx_1 \wedge \cdots \wedge dx_p$ on a domain X in $\mathbf{R}^n$. A function $f \in C^\infty(X)$ is ϕ-plurisubharmonic if and only if $\Delta_x f \geq 0$ on X. For a set $K \subset \mathbf{R}^n$, let K_y denote the horizontal slice $\{x \in \mathbf{R}^p : (x, y) \in K\}$ of K. Suppose that for each $y \in \mathbf{R}^{n-p}$, the horizontal slice X_y has no holes in $\mathbf{R}^p$. Then (X, ϕ) is strictly convex. To prove this fact, it suffices to exhaust X by compact sets K with the same property and show that each such K is equal to its (X, ϕ)-hull. Suppose $z_0 = (x_0, y_0) \in X - K$. Since x_0 is not in a hole of K_{y_0} in $\mathbf{R}^p$, we may choose (by Example 2) an entire subharmonic function $g(x)$ with $g(x_0) >> 0$ and $\sup_{K_{y_0}} g << 0$. Now pick $\psi \in C^\infty_{\rm cpt}(\{y : |y - y_0| < \epsilon\})$ with $0 \leq \psi \leq 1$ and $\psi(y_0) = 1$. Then $f(x, y) = g(x)\psi(y)$ is ϕ-plurisubharmonic and $f(z_0) = g(x_0) >> 0$. For ϵ sufficiently small, $\sup_K f \leq 0$. This proves z_0 does not belong to the (X, ϕ)-hull of K.

EXAMPLE 4. Let $\phi = dx$ in $\mathbf{R}^2$ and set $X = \{(x, y) : x^2 - c < y < x^2,\ |x| < 1\}$. Then X is not ϕ-convex. The closure of the hull of the compact subset $K = ([-\epsilon, \epsilon] \times \{-\epsilon\}) \cup (\{\pm\epsilon\} \times [-\epsilon, 0])$ of X is easily seen to contain the origin. Similarly, a domain of "U"-shape, whose upper boundary along the bottom has a flat segment, is not ϕ-convex even though it is locally ϕ-convex (by Example 3).

It is important to "weaken" this notion of strict convexity.

THEOREM 3.8. *The following two conditions are equivalent:*

1) $K \subset\subset X \;\Rightarrow\; \widehat{K} \subset\subset X$*, and there exists a strictly* ϕ*-plurisubharmonic function defined outside a compact subset of* X

2) There exists a ϕ*-plurisubharmonic proper exhaustion function on* X *which is strict outside a compact subset of* X*.*

DEFINITION 3.9. If the equivalent conditions of Theorem 3.8 are satisfied, then the calibrated manifold (X, ϕ) is **strictly convex at** ∞ or X is **strictly** ϕ**-convex at** ∞**.**

REMARK . This is not the standard terminology used in complex geometry where such spaces are called "strongly (pseudo) convex".

Proof of Theorem 3.8. Obviously 2) implies 1). We will prove that 1) implies the following weakening of 2).

2)$'$ There exists a continuous proper exhaustion function f on X which is smooth and strictly ϕ-plurisubharmonic outside a compact subset of X.

By Lemma 3.5, Condition 2)$'$ implies Condition 2).

Now assume 1). Since $K \subset\subset X$ implies $\widehat{K} \subset\subset X$, we know from Theorem 3.3 that there exists a ϕ-plurisubharmonic exhaustion function f for X. Let g denote the strictly ϕ-plurisubharmonic function which is only defined outside of a compact set. We can assume this compact set is $\{x \in X : f(x) \leq c\}$ for some large c. Then $h \equiv \max\{f + e^g, c\}$ is a continuous proper exhaustion function which, outside the compact set $\{x \in X : f(x) \leq c\}$, is strictly ϕ-plurisubharmonic (in fact, equal to$f + e^g$). This proves 2)$'$ and completes the proof of the theorem. ■

COROLLARY 3.10. (X, ϕ) *is strictly convex at* ∞ *if and only if Condition* 2)$'$ *holds.*

Cores.

In each non-compact calibrated manifold (X, ϕ) there are certain distinguished subsets which play an important role in the ϕ-geometry of the space. (In complex manifolds which are strongly pseudoconvex, these sets correspond to the compact exceptional subvarieties.) The remainder of this section is devoted to a discussion of these subsets.

Given a function $f \in \mathcal{PSH}(X, \phi)$, consider the open set

$$S(f) \;\equiv\; \{x \in X : f \;\text{ is strictly } \phi - \text{plurisubharmonic at } \; x\}$$

and the closed set

$$W(f) \;\equiv\; X - S(f).$$

Note that

$$W(\lambda f + \mu g) \;=\; W(f) \cap W(g)$$

for $f, g \in \mathcal{PSH}(X, \phi)$ and $\lambda, \mu > 0$.

DEFINITION 3.11. The **core** of X is defined to be the intersection

$$\mathrm{Core}(X) \;\equiv\; \bigcap W(f)$$

over all $f \in \mathcal{PSH}(X, \phi)$. The **inner core** of X is defined to be the set InnerCore(X) of points x for which there exists $y \neq x$ with the property that $f(x) = f(y)$ for all $f \in \mathcal{PSH}(X, \phi)$.

PROPOSITION 3.12. $\quad \text{InnerCore}(C) \ \subset \ \text{Core}(X).$

Proof. If $x \notin \text{Core}(X)$, then there exists $g \in \mathcal{PSH}(X, \phi)$ with g strict at x. Suppose $y \neq x$. Then if ψ is compactly supported in a small neighborhood of x missing y, and ψ has sufficiently small second derivatives, one has $f = g + \psi \in \mathcal{PSH}(X, \phi)$. Obviously for such f, the values $f(x)$ and $f(y)$ can be made to differ, so therefore $x \notin \text{InnerCore}(X)$. ∎

PROPOSITION 3.13. *Every compact ϕ-submanifold $M \subset X$ is contained in the inner core.*

Proof. Each $f \in \mathcal{PSH}(X, \phi)$ is subharmonic on M by Theorem 1.4. Hence, f is constant on M. ∎

PROPOSITION 3.14. *Suppose X is ϕ-convex. Then* Core(X) *is compact if and only if X is strictly ϕ-convex at ∞, and* Core$(X) = \emptyset$ *if and only if X is strictly ϕ-convex .*

Proof. If X is strictly ϕ-convex at ∞, then choosing f to satisfy 2) in Theorem 3.8, we see that the Core$(X) \subset W(f)$ is compact. Obviously, strict ϕ-convexity implies that Core$(X) = \emptyset$.

Conversely, if Core(X) is compact, then in the construction of the ϕ-plurisubharmonic exhaustion function in the proof of Theorem 3.3 we may choose

$$K_1 \ = \ \widehat{\text{Core}(X)}$$

Then by the definition of Core(X) and Lemma 3.2, each of the functions f_m in that proof can be chosen to be strictly ϕ-plurisubharmonic on $K_{m+2} - K_{m+1}^0$. Hence the exhaustion $f = \sum_m f_m$ is strictly ϕ-plurisubharmonic outside a compact set containing the core. ∎

A slight modification of this construction gives the following general result.

PROPOSITION 3.15. *Suppose X is strictly ϕ-convex at ∞, and $K \subset X$ is a compact, ϕ-convex subset containing the core of X. Let U be any neighborhood of K. Then there exists a proper ϕ-plurisubharmonic exhaustion function $f : X \to \mathbf{R}^+$ which is strictly ϕ-plurisubharmonic on $X - U$, and identically zero on a neighborhood of K.*

Proof. Choose $K_1 = K$ in the construction of the ϕ-plurisubharmonic exhaustion function given in the proof of Theorem 3.3. Let K_ϵ denote the compact ϵ-neighborhood of K. Then

$$K \ = \ \bigcap_{\epsilon > 0} \widehat{K}_\epsilon. \tag{3.1}$$

If $x \in \bigcap_{\epsilon>0} \widehat{K}_\epsilon$, then for each $f \in \mathcal{PSH}(X, \phi)$, we have $f(x) \leq \sup_{K_\epsilon} f$. However, $\inf_\epsilon \sup_{K_\epsilon} f = \sup_K f$, and we conclude that $x \in \widehat{K}$. Thus we can choose $K_2 \equiv \widehat{K}_\epsilon$ in our construction of f, and for small enough ϵ we have $K_2 \subset U$ as well as $K_1 \subset K_2^0$. The proof is now completed as in the proof of Proposition 3.14. ∎

Obviously, many question concerning

$$\text{InnerCore}(X) \ \subset \ \text{Core}(C) \ \subset \ \widehat{\text{Core}(X)}$$

remain to be answered.

Appendix A. Structure of the Core.

Let (X,ϕ) be a calibrated manifold and consider the set

$$\mathcal{N} \;\equiv\; \{\xi \in G(\phi) : (\mathcal{H}^\phi f)(\xi) \;=\; 0 \;\text{ for all } f \in \mathcal{PSH}(X,\phi)\}.$$

PROPOSITION 3.A.1. *Let $\pi : G(\phi) \to X$ denote the projection. Then*

$$\pi(\mathcal{N}) \;=\; \mathrm{Core}(X).$$

Proof. Suppose $x \notin \mathrm{Core}(X)$. Then by definition there exists $f \in \mathcal{PSH}(X,\phi)$ with $(\mathcal{H}^\phi f)(\xi) > 0$ for all $\xi \in \pi^{-1}(x)$. Hence, $x \notin \pi(\mathcal{N})$.

Conversely, suppose $x \notin \pi(\mathcal{N})$. Then for each $\xi \in \pi^{-1}(x)$ there exists $f_\xi \in \mathcal{PSH}(X,\phi)$ with $f(\xi) > 0$. Let $W_\xi = \{\eta \in \pi^{-1}(x) : f_\xi(\eta) > 0\}$ and choose a finite cover $W_{\xi_1}, ..., W_{\xi_\ell}$ of $\pi^{-1}(x)$. Then $f \equiv f_{\xi_1} + \cdots + f_{\xi_\ell}$ is strictly ϕ-plurisubharmonic at x, and so $x \notin \mathrm{Core}(X)$. ■

PROPOSITION 3.A.2. *If $\xi \in \mathcal{N}$, then for each vector $v \in \mathrm{span}\,\xi$,*

$$df(v) \;=\; 0 \qquad \text{for all} \;\; f \in \mathcal{PSH}(X,\phi) \tag{3.A.1}$$

Proof. Suppose $f \in \mathcal{PSH}(X,\phi)$ and set $F = e^f$. Then $F \in \mathcal{PSH}(X,\phi)$, and by equation (2.14) and Corollary 2.7 we see that $0 \;=\; (\mathcal{H}^\phi F)(\xi) \;=\; e^f\{df \wedge d^\phi f + \mathcal{H}^\phi f\}(\xi) \;=\; e^f\{df \wedge d^\phi f\}(\xi) \;=\; e^f|\nabla f \lrcorner \xi|^2$. ■

DEFINITION 3.A.3. The **tangential core** of X is the set

$$T\mathrm{Core}(X) \;\equiv\; \{v \in TX : v \neq 0 \text{ and satisfies condition (3.A.1)}\}$$

Thus $T\mathrm{Core}(X) \subset TX$ is a subset defined by the vanishing of the family of smooth functions $df : TX \to \mathbf{R}$ for $f \in \mathcal{PSH}(X,\phi)$. Propositions 3.A.1 and 3.A.2 show that the restriction of the bundle map $p : TX \to X$ gives a surjective mapping

$$p : T\mathrm{Core}(X) \to \mathrm{Core}(X)$$

and for each $x \in X$, The vector space $T_x\mathrm{Core}(C) \equiv p^{-1}(x)$ contains the non-empty space generated by all $v \in \mathrm{span}\,\xi$ for $\xi \in \mathcal{N}_x$.

Consider a point $v \in T\mathrm{Core}(X)$ and suppose we have functions $f_1, ..., f_\ell \in \mathcal{PSH}(X,\phi)$ such that $\nabla df_1, ..., \nabla df_\ell$ are linearly independent at v. Then $T\mathrm{Core}(C)$ is locally contained in the codimension-ℓ submanifold $\{df_1 = \cdots = df_\ell = 0\}$. If additionally we assume that $df_1, ..., df_\ell$ are linearly independent vectors at $x = p(v)$, then $\mathrm{Core}(X)$ is locally contained in the subset $\{f_1 = \text{constant}\} \cap \cdots \cap \{f_\ell = \text{constant}\}$. NO

Appendix B. Examples of Complete Convex Manifolds and Cores

In §7 (Theorem 7.4) we shall show that there are many strictly ϕ-convex domains in any calibrated manifold (X, ϕ). They can have quite arbitrary topological type within the strictures imposed by Morse Theory and ϕ-positivity of the Hessian. However, it is also interesting geometrically to ask for convex manifolds which are complete.

In fact, there exist enormous families of complete calibrated manifolds (X, ϕ) with $\nabla\phi = 0$ which are strictly ϕ-convex at infinity. For example any asymptotically locally euclidean (ALE) manifold with SU(n), Sp(n), G_2, or Spin_7 holonomy is such a creature, since the radial function on the asymptotic chart at infinity is strictly convex. For the general construction of such spaces the reader is referred to the book of Joyce [J].

However, some manifolds of this type have been explicitly constructed, and in these cases one can explicitly construct ϕ-plurisubharmonic exhaustion functions and identify the cores. We indicate how to do this below.

We begin however with an observation in dimension 4. Every crepant resolution of singularities of $\mathbf{C}^2/\Gamma$ admits Ricci-flat ALE Kähler metric. On each such manifold there exists an S^2-family of parallel calibrations

$$\mathcal{C} \;=\; \{u\omega + v\varphi + \; w\psi : u^2 + v^2 + w^2 = 1\}$$

where ω is the given Kähler form, $\varphi = \mathrm{Re}\{\Phi\}$ and $\psi = \mathrm{Im}\{\Phi\}$ and Φ is a parallel section of the canonical bundle κ_X. Let $E = \pi^{-1}(0)$ be the exceptional locus of the resolution. Then for any $\phi \in \mathcal{C}$ we have

$$\mathrm{Core}(X, \phi) \;=\; \begin{cases} E & \text{if } \phi = \omega \\ \emptyset & \text{otherwise} \end{cases}$$

This follows from the fact that each $\phi \in \mathcal{C}$ is in fact the Kähler form for a complex structure on X compatible with the given metric. With this complex structure X is pseudo-convex, and by the Stein Reduction Theorem (cf. [GR, p. 221]) we know its core is the union of its compact complex subvarieties. For $\phi \neq \omega$ there are no such subvarieties since by the Wirtinger inequality (cf. [$\mathrm{L}_{1,2}$]), applied to ϕ, they would necessarily be homologically mass-minimizing, and by the same result applied to ω any such subvariety is ω-complex (and therefore a component of E).

Example 1. (Calabi Spaces). Let $X \to \mathbf{C}^n/\mathbf{Z}_n$ be a crepant resolution of $\mathbf{C}^n/\mathbf{Z}_n$ where the action on $\mathbf{C}^n$ is generated by scalar multiplication by $\tau = e^{2\pi i/n}$. Following Calabi [C] we define the function $F : \mathbf{C}^n/\mathbf{Z}_n \to \mathbf{R}$ by

$$F(\rho) \;=\; \sqrt[n]{\rho^n + 1} + \frac{1}{n}\sum_{k=0}^{n-1} \tau^k \log\left(\sqrt[n]{\rho^n + 1} - \tau^k\right)$$

where $\rho \equiv \|z\|^2$ (pushed down to $\mathbf{C}^n/\mathbf{Z}_n$), and the log is defined by choosing $\arg\zeta \in (-\pi, \pi)$. We then define a Kahler metric on $\mathbf{C}^n/\mathbf{Z}_n - \{0\}$ by setting

$$\omega \;=\; \frac{1}{4} dd^c F$$

Calabi shows that *this metric is Ricci-flat and (when pulled back) extends to a Ricci flat metric on* X. *The parallel form* $\Phi = dz_1 \wedge \cdots \wedge dz_n$ *extends to a parallel section of* κ_X. This metric is given explicitly on $\mathbf{R}^{2n}/\mathbf{Z}_n$ by

$$ds^2 \ = \ F'(\rho)|dx|^2 + \rho F''(\rho) dr \circ d^c r$$

where $r = \|x\|$. Define $G(\rho)$ by settiing $G'(\rho) = F'(\rho) + \rho F''(\rho)$ and $G(0) = 0$. Then direct calculation shows that

$$dd^\phi G \ = \ 2n\phi$$

where $\phi = \mathrm{Re}\{\Phi\}$. Hence, *$X$ is a complete, strictly ϕ-convex manifold.*

EXAMPLE 2. (Bryant-Salamon Spaces). Let P denote the principal Spin_3-bundle of S^3 and

$$S \ \equiv \ P \times_{\mathrm{Sp}_1} \mathbf{H}$$

the associated spinor bundle, where $\mathbf{H}$ denotes the quaternions. Bryant and Salamon have explicitly constructed a complete riemannian metric with G_2-holonomy on the total space of S. (See [BS, page 838, Case ii].) Let $\rho = |a|$ for $a \in \mathbf{H}$ (pushed-down to S) and let $Z \subset S$ denote the zero section. Then a direct calculation shows that the function

$$F(\rho) \ = \ (1+\rho)^{\frac{5}{6}} \ \ \textit{is strictly } \varphi - \textit{plurisubharmonic on } S - Z$$

where φ denotes the associative calibration on S. Since Z is an associative submanifold we conclude that

$$\mathrm{Core}(S) \ = \ Z$$

In an analogous fashion the authors construct a complete riemannian metric with Spin_7-holonomy on the total space $\widetilde{S}$ of a spinor bundle over S^4. (See [BS, page 847, Case ii].) A similar calculation shows that there exists an exhaustion function which is strictly Φ-plurisubharmonic on $\widetilde{S} - \widetilde{Z}$ where Φ denotes the Cayley calibration $\widetilde{Z}$ the zero-section of $\widetilde{S}$. Since $\widetilde{Z}$ is a Cayley submanifold, we conclude that

$$\mathrm{Core}(\widetilde{S}) \ = \ \widetilde{Z}$$

4. Boundary Convexity.

Suppose $\Omega \subset\subset X$ is an open set with smooth boundary $\partial\Omega$, where (X, ϕ) is a non-compact calibrated manifold. A p-plane $\xi \in G(\phi)$ at a point $x \in \partial\Omega$ will be called **tangential** if $\operatorname{span}\xi \subset T_x\partial\Omega$.

DEFINITION 4.1. Suppose that ρ is a **defining function** for $\partial\Omega$, that is, ρ is a smooth function defined on a neighborhood of $\overline{\Omega}$ with $\Omega = \{x : \rho(x) < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$. If

$$dd^\phi \rho(\xi) \ \geq \ 0 \quad \text{for all tangential } \xi \in G_x(\phi), \ x \in \partial\Omega, \tag{4.1}$$

then $\partial\Omega$ is called **ϕ-convex**. If the inequality in (4.1) is strict for all ξ, then $\partial\Omega$ is called **strictly ϕ-convex**. If $dd^\phi(\xi) = 0$ for all ξ as in (4.1), then $\partial\Omega$ is **ϕ-flat**.

Each of these conditions is a local condition on $\partial\Omega$. In fact:

LEMMA 4.2. *Each of the three conditions in Definition 4.1 is independent of the choice of defining function ρ. In fact, if $\overline{\rho} = u\rho$ is another choice with $u > 0$ on $\partial\Omega$, then on $\partial\Omega$*

$$dd^\phi(\overline{\rho})(\xi) \ = \ udd^\phi(\rho)(\xi) \quad \text{for all tangential} \quad \xi \in G(\phi) \tag{4.2}$$

Proof. Note that $dd^\phi(\overline{\rho})(\xi) = \ udd^\phi(\rho)(\xi) + \rho dd^\phi(u)(\xi) + \lambda_\phi(\nabla u \circ \nabla\rho)(\xi)$. Now the middle term drops because $\rho = 0$ on $\partial\Omega$. Furthermore, $\lambda_\phi(\nabla u \circ \nabla\rho)(\xi) = \frac{1}{2}\nabla\rho \wedge (\nabla u \lrcorner \phi)(\xi) + \frac{1}{2}\nabla u \wedge (\nabla\rho \lrcorner \phi)(\xi) = \frac{1}{2}\phi(\nabla u \wedge (\nabla\rho \lrcorner \xi)) + \frac{1}{2}\phi(\nabla\rho \wedge (\nabla u \lrcorner \xi))$. Now $\nabla\rho \perp \operatorname{span}(\xi)$ implies $\nabla\rho \lrcorner \xi = 0$ and, by the First Cousin Principle, that ϕ vanishes on $\nabla\rho \wedge (\nabla u \lrcorner \xi)$. ∎

COROLLARY 4.3. *Assume $\phi \in \Lambda^p\mathbf{R}^n$ is a calibration. Suppose $\partial\Omega$ is (strictly) ϕ-convex in $\mathbf{R}^n$, and locally near a point $p \in \partial\Omega$, let $\partial\Omega$ be graphed over its tangent space by a function $x_n = u(x')$ for linear coordinates (x', x_n) on $\mathbf{R}^n$. Then each nearby hypersurface: $x_n = u(x') + c$ is also (strictly) ϕ-convex.*

The next lemma will be used to establish the main result of this section.

LEMMA 4.4. *Suppose ρ is a smooth real-valued function defined near a point x in a calibrated manifold (X, ϕ), and that $\psi : \mathbf{R} \to \mathbf{R}$ is smooth near $\rho(x)$. Then:*

$$\lambda_\phi(\mathrm{Hess}_x\psi(\rho))(\xi) \ = \ \psi'(\rho)\lambda_\phi(\mathrm{Hess}_x\rho)(\xi) + |\nabla\rho|^2\psi''(\rho)\cos^2\theta(\xi) \tag{4.3}$$

where $\cos^2\theta(\xi) = \langle P_{\operatorname{span}\nabla\rho}, P_{\operatorname{span}\xi}\rangle$.

Proof. First note that

$$\mathrm{Hess}\psi(\rho) \ = \ \psi'(\rho)\mathrm{Hess}\rho + \psi''(\rho)\nabla\rho \circ \nabla\rho, \tag{4.4}$$

and then set $n = \nabla\rho/|\nabla\rho|$ so that $\nabla\rho \circ \nabla\rho = |\nabla\rho|^2 n \circ n = |\nabla\rho|^2 P_{\operatorname{span}\nabla\rho}$. For each $\xi \in G_x(\phi)$, taking the inner product of (4.4) with $P_{\operatorname{span}\xi}$ (orthogonal projection onto $\operatorname{span}\xi$) yields (4.3) because of Lemma 1.15b. ∎

We now come to the main result of this section.

THEOREM 4.5. *Let $\Omega \subset\subset X$ be a compact domain with strictly ϕ-convex boundary. Suppose $\delta = -\rho$ is an arbitrary "distance function" for $\partial\Omega$, i.e., ρ is an arbitrary defining function for $\partial\Omega$. Then $-\log\delta$ is strictly ϕ-plurisubharmonic outside a compact subset of Ω. Thus, in particular, the domain Ω is strictly ϕ-convex at ∞.*

Proof. Set $\psi(t) = -\log(-t)$ for $t < 0$. Note that $\psi'(t) = -1/t$ and $\psi''(t) = -1/t^2$, so that $\psi'(\rho) = 1/\delta$ and $\psi''(\rho) = 1/\delta^2$. Consequently, by Lemma 3.4, at each point $x \in \Omega$ near $\partial\Omega$, we have

$$dd^{\phi}(-\log\delta)(\xi) \;=\; \frac{1}{\delta}dd^{\phi}(\rho)(\xi) + \frac{|\nabla\rho|^2}{\delta^2}\cos^2\theta(\xi) \tag{4.4}$$

for all $\xi \in G(\phi)$. Note that at $x \in \partial\Omega$, $\cos^2\theta(\xi) = \langle P_{\mathrm{span}\,\nabla\rho}, P_{\mathrm{span}\,\xi}\rangle$ vanishes if and only if ξ is tangential to $\partial\Omega$. Consequently, the inequality $|\cos\theta| < \epsilon$ defines a fundamental neighborhood system for $G(p, T\partial\Omega) \subset G(p, TX)$. By restriction $|\cos\theta| < \epsilon$ defines a fundamental neighborhood system for $G(\phi) \cap G(p, T\partial\Omega) \subset G(\phi)$. The hypothesis of strict ϕ-convexity for $\partial\Omega$ implies that there exists $\overline{\epsilon} > 0$ so that $(dd^{\phi}\rho)(\xi) \geq \overline{\epsilon}$ for all ϕ-planes ξ at points of $\partial\Omega$ with $|\cos\theta| < \epsilon$ for some $\epsilon > 0$. (Note that if there are no ϕ-planes tangent to $\partial\Omega$ at a point x, then there are no ϕ-planes with $|\cos\theta| < \epsilon$ for sufficiently small ϵ in a neighborhood of x.) Consequently, we have by equation (4.4) that

$$dd^{\phi}(-\log\delta)(\xi) \;\geq\; \frac{\overline{\epsilon}}{2\delta}$$

near $\partial\Omega$ for all ϕ-planes ξ with $|\cos\theta| < \epsilon$, where θ is defined as above with $\partial\Omega$ replaced by the nearby level sets of ρ.

Now choose $M >> 0$ so that $dd^{\phi}(\rho)(\xi) \geq -M$ in a neighborhood of $\partial\Omega$ for all ξ. Then, by (4.4)

$$dd^{\phi}(-\log\delta)(\xi) \;\geq\; -\frac{M}{\delta} + \frac{1}{\delta^2}|\nabla\rho|^2\cos^2\theta.$$

If $|\cos\theta| \geq \epsilon$, this is positive in a neighborhood of $\partial\Omega$ in Ω. This proves that $-\log\delta$ is strictly ϕ-plurisubharmonic near $\partial\Omega$. By Corollary 3.10 the domain Ω is strongly ϕ-convex. ∎

Although a defining function for a strictly ϕ-convex boundary may not be ϕ-plurisubharmonic, for some applications the following may prove useful.

PROPOSITION 4.6. *Suppose $\Omega \subset\subset X$ has strictly ϕ-convex boundary $\partial\Omega$ with defining function ρ. Then, for A sufficiently large, the function $\overline{\rho} \equiv \rho + A\rho^2$ is strictly ϕ-convex in a neighborhood of $\partial\Omega$ and also a defining function for $\partial\Omega$.*

Proof. By Lemma 4.4

$$\lambda_{\phi}(\mathrm{Hess}\overline{\rho})(\xi) \;=\; (1 + 2A\rho)\lambda_{\phi}(\mathrm{Hess}\rho)(\xi) + 2|\nabla\rho|^2 A\cos^2\theta(\xi) \qquad \text{for all } \xi \in G(\phi) \tag{4.5}$$

where $\cos^2\theta(\xi) = \langle P_{\mathrm{span}\,\nabla\rho}, P_{\mathrm{span}\,\xi}\rangle$. As noted in the proof of Theorem 4.5, there exist $\epsilon, \overline{\epsilon} > 0$ so that $\lambda_{\phi}(\mathrm{Hess}\rho)(\xi) \geq \overline{\epsilon}$ if $|\cos\theta(\xi)| < \epsilon$, because of the strict boundary convexity. Therefore $\lambda_{\phi}(\mathrm{Hess}\overline{\rho})(\xi) \geq (1 + 2A\rho)\overline{\epsilon}$ if $\xi \in G(\phi)$ with $|\cos\theta(\xi)| < \epsilon$. Choose a lower bound $-M$ for $\lambda_{\phi}(\mathrm{Hess}\rho)(\xi)$ over all $\xi \in G(\phi)$ for a neighborhood of $\partial\Omega$. Then by (4.5),

$\lambda_\phi(\mathrm{Hess}\rho)(\xi) \ \geq\ -(1+2A\rho)M + 2|\nabla\rho|^2 A\epsilon^2$ for $\xi \in G(\phi)$ with $|\cos\theta(\xi)| \geq \epsilon$. For A sufficiently large, the right hand side is > 0 in some neighborhood of $\partial\Omega$. ∎

One might hope for a converse to Theorem 4.5, e.g., if the domain Ω is ϕ-convex then the boundary is ϕ-convex . However, elementary examples show that this is false.

Example . Let $\phi \equiv dx \wedge dy$ in $\mathbf{R}^3$ as in Example 3 of section 3. Let X denote the solid torus obtained by rotating the disk $\{(y,z) : y^2 + (z-R)^2 < r^2\}$ about the y-axis. Since each slice X_z has no holes in $\mathbf{R}^3$, the domain X is ϕ-convex (cf. Example 3 of §3). However, the boundary torus ∂X is ϕ-convex if and only if $2r \leq R$. This follows from an elementary calculation which uses the obvious defining function and Definition 4.1 (or by using Proposition 4.12 below)

Question 4.7. For which strictly convex calibrated manifolds is it true that ϕ-convex subdomains have ϕ-convex boundaries? More generally, when is the ϕ-convexity of a domain a local condition at the boundary?

A weak partial converse to Theorem 4.4 is given by the following.

Proposition 4.8. *Suppose the calibration is parallel, and set $\delta = \mathrm{dist}(\bullet, \partial\Omega)$. If $-\log\delta$ is strictly ϕ-plurisubharmonic near $\partial\Omega$, then $\partial\Omega$ is ϕ-convex .*

Note 4.9. Examples show that the strict convexity of $-\log\delta$ near $\partial\Omega$ is stronger than ϕ-convexity for $\partial\Omega$.

Proof. Set $\rho = -\delta$ on $\overline{\Omega}$ near $\partial\Omega$. Suppose that $\partial\Omega$ is not ϕ-convex. Then there exist $x \in \partial\Omega$ and $\xi_x \in G_x(\phi)$ with $\mathrm{span}\,(\xi) \subset T_x(\partial\Omega)$ and $(dd^\phi\rho)(\xi_x) < 0$. Let γ denote the geodesic segment in Ω which emanates orthogonally from $\partial\Omega$ at x. Since δ is the distance function, γ is an integral curve of $\nabla\delta$. Let ξ_y, $y \in \gamma$ denote the parallel translation of ξ_x along γ. Then ξ_y is a ϕ-plane with $\mathrm{span}\,(\xi_y) \perp \nabla\rho$ for all y. By formula (4.4), siince $\cos\theta(\xi_y) = 0$, we have

$$dd^\phi(-\log\delta)(\xi_y) \ = \ \frac{1}{\delta} dd^\phi(\rho)(\xi_y) \ < \ 0$$

for all y sufficiently close to x. Hence, $-\log\delta$ is not ϕ-plurisubharmonic near $\partial\Omega$. ∎

The ϕ-convexity of a boundary can be equivalently defined in terms of its second fundamental form. Note that if $M \subset X$ is a smooth hypersurface with a chosen unit normal field n we have a quadratic form II defined on TM by

$$II(V,W) \ = \ \langle B_{V,W}, n\rangle$$

where B denotes the second fundamental form of M discussed in §1. For example, when $H = S^{n-1}(r) \subset \mathbf{R}^n$ is the euclidean sphere of radius r, oriented by the outward-pointing unit normal, we find that $II(V,W) = -\frac{1}{r}\langle V,W\rangle$.

For the sake of completeness we include a proof of the following standard fact.

Lemma 4.10. *Suppose ρ is a defining function for Ω and let II denote the second fundamental form of the hypersurface $\partial\Omega$ oriented by the outward-pointing normal. Then*

$$\mathrm{Hess}\,\rho\,\big|_{T\partial\Omega} = \ -|\nabla\rho|\, II$$

Proof. Suppose e is a tangent field on $\partial\Omega$. Extend e to a vector field tangent to the level sets of ρ. By definition $II(e,e) = \langle \nabla_e e, n\rangle$ where $n = \nabla\rho/|\nabla\rho|$ is the outward normal. Then $(\mathrm{Hess}\,\rho)(e,e) = e(e\rho) - (\nabla_e e)\rho = -(\nabla_e e)\rho = -\langle \nabla_e e, \nabla\rho\rangle = -|\nabla\rho|\langle \nabla_e e, n\rangle$. ∎

REMARK . Recall that a defining function ρ for Ω satisfies $\|\nabla\rho\| \equiv 1$ in a neighborhood of $\partial\Omega$ if and only if ρ is the signed distance to $\partial\Omega$ (< 0 in Ω and > 0 outside of Ω). In fact any function ρ with $\|\nabla\rho\| \equiv 1$ in a riemannian manifold is, up to an additive constant, the distance function to (any) one of its level sets. In this case it is easy to see that

$$\mathrm{Hess}_\rho \;=\; \begin{pmatrix} 0 & 0 \\ 0 & -II \end{pmatrix} \tag{4.6}$$

where II denotes the second fundamental form of the hypersurface $H = \{\rho = \rho(x)\}$ with respect to the normal $n = \nabla\rho$ and the blocking in (3) is with respect to the splitting $T_xX = \mathrm{span}\,(n_x) \oplus T_xH$. For example let $\rho(x) = \|x\| \equiv r$ in $\mathbf{R}^n$. Then direct calculation shows that $\mathrm{Hess}_r = \frac{1}{r}(I - \hat{x}\circ\hat{x})$ where $\hat{x} = x/r$.

COROLLARY 4.11. *For all tangential $\xi \in G(\phi)$*

$$(dd^\phi\rho)(\xi) \;=\; -\|\nabla\rho\|\mathrm{tr}_\xi II.$$

Proof. Apply Lemma 1.10. ∎

As an immediate consequence we have

PROPOSITION 4.12. *Let $\Omega \subset X$ be a domain with smooth boundary $\partial\Omega$ oriented by the outward-pointing normal. Then $\partial\Omega$ is ϕ-convex if and only if its second fundamental form satisfies*

$$\mathrm{tr}_\xi II \;\le\; 0$$

for all ϕ-planes ξ which are tangent to $\partial\Omega$. This can be expressed more geometrically by saying that

$$\mathrm{tr}\left\{ B\big|_\xi \right\} \quad \text{must be inward} - \text{pointing}$$

for all tangential ϕ-planes ξ.

5. Positive Currents in Calibrated Geometries.

The important classical notion of a positive current on a complex manifold has an analogue on any calibrated manifold. This concept was introduced in section II of [HL_3]. We begin this Section by reviewing that material with some of the terminology and notation updated.

On a calibrated manifold (X, ϕ) we have:

a) ϕ-submanifolds,

b) rectifiable ϕ-currents, and

c) ϕ-positive (or $\Lambda_+(\phi)$-positive) currents.

A ϕ-submanifold is, of course, a smooth oriented submanifold M whose oriented tangent space is a ϕ-plane at every point, i.e., $\overrightarrow{M}_x \equiv \overrightarrow{T}_x M \in G(\phi)$ for all $x \in M$.

Suppose T is a locally rectifiable p-dimensional current ([F]) on X. Then its generalized tangent space is a unit simple vector $\overrightarrow{T} \in G(p, TX)$ at $\|T\|$ almost every point, where $\|T\|$ denotes the generalized volume measure associated with T.

Definition 5.1. A **rectifiable ϕ-current** is a locally recitifiable current T with $\overrightarrow{T} \in G(\phi)$ for $\|T\|$- a.a. points in X. A **ϕ-cycle** is a rectifiable ϕ-current which is d-closed.

Remark 5.2. We shall see below (Theorem 5.9) that ϕ-cycles always have a particularly nice local structure. The strongest result of this kind occurs in the Kähler case (where $\phi = \omega^p/p!$) where a theorem of J. King [K] states that each ϕ-cycle is a positive holomorphic cycle, i.e., a locally finite sum of p-dimensional complex analytic subvarieties with positive integer coefficients. On a general calibrated manifold (X, ϕ) one can also consider d-closed recitifiable currents T with $\pm\overrightarrow{T} \in G(\phi)$ for $\|T\|$-a.a. points. In the Kähler case T must be a holomorphic chain by a theorem of Harvey-Shiffman [HS], [S]. However, nothing is known about the structure of such currents for any of the other standard calibrations.

An understanding of the definition of a ϕ-positive current is a little more complicated.

Recall(Federer [F]) that a current T is **representable by integration** if T has measure coefficients when expressed as a generalized differential form. Equivalently, the mass norm $M_K(T)$ of T on each compact set K, is finite. Associated with such a current T is a Radon measure $\|T\|$ and a generalized tangent space $\overrightarrow{T}_x \in \Lambda_p T_x X$ defined for $\|T\|$ a. a. points x. Recall that each $\overrightarrow{T}_x$ has mass norm one. For any p-form α with compact support

$$T(\alpha) \;=\; \int \alpha(\overrightarrow{T})\, d\|T\| \tag{5.1}$$

Definition 5.3. At each point $x \in X$ let $\Lambda(\phi)$ denote the span of $G(\phi) \subset \Lambda_p TX$, and let

$$\Lambda_+(\phi) \;\subset\; \Lambda(\phi)$$

denote the convex cone on $G(\phi)$ with vertex the origin. The p-vectors $\xi \in \Lambda_+(\phi)$ will be called **$\Lambda_+(\phi)$-positive**.

Note that $\Lambda_+(\phi)$ is just the cone on $\mathrm{ch}\, G(\phi)$, the convex hull of the Grassmannian.

The following Lemma is needed for a robust understanding of the definition of a ϕ-positive current.

LEMMA 5.4. *The following conditions are equivalent:*

1) $\overrightarrow{T} \in \Lambda_+(\phi)$ $\qquad \|T\|$-a.e.

2) $\overrightarrow{T} \in \mathrm{ch}\, G(\phi)$ $\qquad \|T\|$-a.e.

3) $\phi(\overrightarrow{T}) = 1$ $\qquad \|T\|$-a.e.

The proof is provided later.

DEFINITION 5.5. A **ϕ-positive current** is a p-dimensional current T which is representable by integration and for which the equivalent conditions of Lemma 5.4 are satisfied.

PROPOSITION 5.6. *Suppose T is a compactly supported p-dimensional current which is representable by integration. Then*

$$T(\phi) \ \le \ M(T)$$

with equality if and only if T is a ϕ-positive current.

Consequently, any ϕ-positive current T_0 with compact support is homologically mass-minimizing, i.e.,

$$M(T_0) \ \le \ M(T) \tag{5.2}$$

for any $T = T_0 + dS$ where S is a $(p+1)$dimension current with compact support. Furthermore, equality holds in (5.2) if and only if T is also ϕ-positive.

Proof. Note that $T(\phi) = \int \phi(\overrightarrow{T}) d\|T\| \le \int \|T\| = M(T)$ since $\phi(\overrightarrow{T}) \le \|\phi\|^* \|T\| = 1$. Equality occurs if and only if $\phi(\overrightarrow{T}) = 1$ ($\|T\|$-a.e.). This is Condition 3) in Lemma 5.4. The second assertion follows from the fact that $T_0(\phi) = T(\phi)$. ■

The reader may note that only Condition 3) of 5.4 was used in this proof. However, it is Conditions 1) and 2) which give a genuine understanding of ϕ-positive currents.

The fact that

$$M(T) \ = \ T(\phi) \ = \ \int \phi(\overrightarrow{T})\, d\|T\| \tag{5.3}$$

for all ϕ-positive currents T, has important implications.

Deep results in geometric measure theory have important applications here.

THEOREM 5.7. *Fix a compact set $K \subset X$ and a constant $c > 0$. Then the set $\mathcal{P}(\phi, K, c)$ of ϕ-positive currents T with $M(T) \le c$ and $\mathrm{supp}(T) \subseteq K$ is compact in the weak topology.*

Proof. Proposition 5.6 easily implies that a weak limit of ϕ-positive currents is ϕ-positive. The result then follows from standard compactness theorems for measures. ■

THEOREM 5.8. *Fix a compact set $K \subset X$ and a constant $c > 0$. Then the set $\mathcal{R}(\phi, K, c)$ of rectifiable ϕ-currents T with $M(T) \le c$ and $\mathrm{supp}(T) \subseteq K$ is compact in the weak topology.*

Proof. This follows from Proposition 5.6 and the Federer-Fleming weak compactness theorem for rectifiable currents [FF], [F]. ∎

THEOREM 5.9. *Let T be a ϕ-cycle on X. Then there is a closed subset $\Sigma \subset \mathrm{supp}(T)$ of Hausdorff dimension $p-2$ such that $M \equiv \mathrm{supp}(T) - \Sigma$ is a ϕ-submanifold of X and*

$$T \;=\; \sum_k n_k[M_k]$$

where the n_k's are positive integers and the M_k's are the connected components of M.

Proof. This is a direct consequence of Almgren's big regularity theorem [A]. ∎

We now present a dual characterization of ϕ-positive currents which will prove useful in the next two sections.

Let $\Lambda^+(\phi) \subset \Lambda^p V$ denote the polar cone of $\Lambda_+(\phi) \subset \Lambda_p V$. By definition this is the set of $\alpha \in \Lambda^p V$ such that $\alpha(\xi) \geq 0$ for all $\xi \in \Lambda_+(\phi)$, or equivalently,

$$\Lambda^+(\phi) = \{\alpha \in \Lambda^p V : \alpha(\xi) \geq 0 \;\; \text{for all } \xi \in G(\phi)\}.$$

A p-form $\alpha \in \Lambda^p V$ is said to be $\Lambda^+(\phi)$**-positive** if $\alpha \in \Lambda^+(\phi)$, and **strictly** $\Lambda^+(\phi)$**-positive** if $\alpha(\xi) > 0$ for all $\xi \in G(\phi)$ (or equivalently, α belongs to the interior of $\Lambda^+(\phi)$).

REMARK 5.10. Note that ϕ itself is strictly $\Lambda^+(\phi)$-positive, i.e., an interior point of the cone $\Lambda^+(\phi) \subset \Lambda^p V$. If a closed convex cone has one interior point, then there exists a basis for the vector space consisting of interior points. Consequently, $\Lambda^p V$ has a basis of strictly $\Lambda^+(\phi)$-positive p-forms.

If (X, ϕ) is a calibrated manifold, the considerations and definitions above apply to the tangent space $V = T_x X$ at each point $x \in X$.

DEFINITION 5.11. A smooth p-form α on X is $\Lambda^+(\phi)$**-positive (strictly** $\Lambda^+(\phi)$**-positive)** if α is $\Lambda^+(\phi)$-positive (strictly $\Lambda^+(\phi)$-positive) at each point $x \in X$.

DEFINITION 5.12. A (twisted) current T of dimension p is said to be $\Lambda_+(\phi)$**-positive** if

$$T(\alpha) \geq 0$$

for all $\Lambda^+(\phi)$-positive p-forms α with compact support.

THEOREM 5.13. *A current T is $\Lambda_+(\phi)$-positive if and only if it is ϕ-positive.*

This result is proven in [HL_3, Prop. A.2 and Remark on page 83]. However, for the sake of completeness we include a proof.

Proof. First assume that T is representable by integration. Then employing (5.1) T is $\Lambda_+(\phi)$-positive if and only if

$$T(g\alpha) \;=\; \int g\alpha(\overrightarrow{T})d\|T\| \;\geq\; 0$$

for all functions $g \geq 0$ and all compactly supported $\Lambda^+(\phi)$-positive p-forms α. Equivalently, each measure $\alpha(\overrightarrow{T})\|T\|$ is ≥ 0 for the same set of p-forms. In turn, this is equivalent to

$$\alpha(\overrightarrow{T}) \ \geq \ 0 \qquad \|T\|-\text{a.e.}.$$

for all compactly supported $\Lambda^+(\phi)$-positive p-forms. Finally, by the Bipolar Theorem [S] this last condition is equivalent to the Condition 1) in the Lemma 5.4.

It remains to prove that if T is $\Lambda_+(\phi)$-positive, then T is representable by integration. For this we may assume that T has compact support in a small neighborhood U of X, and by Remark 5.10, we may choose a frame $\alpha_1, ..., \alpha_N$ of smooth p-forms which are strictly $\Lambda^+(\phi)$-positive on U. Let $\xi_1, ..., \xi_N$ denote the dual frame of p-vector fields, i.e., $(\alpha_i, \xi_j) \equiv \delta_{ij}$ on U. Every such current T has a unique representation as $T = \sum_{j=1}^N u_j \xi_j$ with $u_j \in \mathcal{D}'$ a distribution defined by $u_j(f) \equiv T(f\alpha_j)$ for all test functions f. (Note that $\alpha = \sum_j f_j \alpha_j$ implies that $T(\alpha) = \sum_j T(f_j \alpha_j) = \sum_j u_j(f_j) = (\sum_j u_j \xi_j)(\sum_i f_i \alpha_i) = (\sum_j u_j \xi_j)(\alpha)$.) Since T is $\Lambda_+(\phi)$-positive, each u_j satisfies

$$u_j(f) \ \geq \ 0 \qquad \text{for all} \quad f \geq 0.$$

By the Riesz Representation Theorem this proves that each u_j is a measure. Therefore $T = \sum_j u_j \xi_j$ is representable by integration. ∎

Now we give the proof of Lemma 5.4. As before $\phi \in \Lambda^p V$ is a calibration. Let K denote the unit mass ball in $\Lambda_p V$, that is, the convex hull of the Grassmannian $G(p, V) \subset \Lambda_p V$.

Lemma 5.14.

$$\operatorname{ch} G(\phi) \ = \ \{\phi = 1\} \cap \partial K \ = \ \Lambda_+(\phi) \cap \partial K$$

Proof. Note that:

a) $\operatorname{ch} G(\phi) \ \subset \ \{\phi = 1\}$ since $G(\phi) \subset \{\phi = 1\}$.

b) $\operatorname{ch} G(\phi) \ \subset \ K$ since $G(\phi) \subset G(p, V)$.

c) $K \cap \{\phi = 1\} \ = \ \partial K \cap \{\phi = 1\}$ since $K \subset \{\phi \leq 1\}$.

Hence, $\operatorname{ch} G(\phi) \subset \{\phi = 1\} \cap \partial K$.

Conversely, suppose $\phi(\xi) = 1$ and $\|\xi\| = 1$. Since $\xi \in K$,

$$\xi \ = \ \sum_j \lambda_j \xi_j \qquad \text{with each} \ \ \xi_j \in G(p, V), \ \text{each} \ \lambda > 0, \ \text{and} \ \sum_j \lambda_j = 1.$$

Hence, $1 = \phi(\xi) = \sum \lambda_j \phi(\xi_j) \leq \sum \lambda_j = 1$ forcing each $\phi(\xi_j) = 1$ and therefore each $\xi_j \in G(\phi)$.

We have shown $\operatorname{ch} G(\phi) \subset \partial K$, and by definition, $\operatorname{ch} G(\phi) \subset \Lambda_+(\phi)$. Suppose $\xi \in \partial K \cap \Lambda_+(\phi)$, i.e., $\|\xi\| = 1$ and there exists some $\lambda > 0$ such that $\lambda\xi \in \operatorname{ch} G(\phi)$. We have already shown that $\operatorname{ch} G(\phi) \subset \partial K$, therefore $\|\lambda\xi\| = 1$, and hence $\lambda = 1$ proving that $\xi \in \operatorname{ch} G(\phi)$. ∎

Corollary 5.15. *Suppose $\xi \in \Lambda_p V$ has mass norm $\|\xi\| = 1$. Then $\xi \in \Lambda_+(\phi)$ if and only if $\phi(\xi) = 1$ if and only if $\xi \in \operatorname{ch} G(\phi)$.*

This is the required restatement of Lemma 5.4

REMARK . Also note that the equation

$$G(\phi) \ = \ G(p, V) \cap \Lambda_+(\phi) \tag{5.4}$$

follows easily from Lemma 5.14. This clarifies the notion of a rectifiable ϕ-current. Namely, this proves that a rectifiable current is $\Lambda_+(\phi)$-positive if and only if it is a rectifiable ϕ-current, and eliminates a potential conflict in terminology.

We finish this section with a lemma and corollary that are often useful.

A form $\alpha \in \Lambda^+(\phi)$ lies on the boundary of $\Lambda^+(\phi)$ if and only if there exists some $\xi \in G(\phi)$ with $\alpha(\xi) = 0$.

LEMMA 5.16. *For any $\psi \in \Lambda^p V$*

$$\phi - \psi \in \operatorname{bdy}\left\{\Lambda^+(\phi)\right\} \quad \Leftrightarrow \quad \psi \leq 1 \text{ on } G(\phi) \ \text{ and } \psi(\xi) = 1 \text{ for some } \xi \in G(\phi)$$

Proof. By definition $\phi-\psi \in \Lambda^+(\phi)$ if and only if $\phi(\xi)-\psi(\xi) = 1-\psi(\xi) \geq 0$ for all $\xi \in G(\phi)$. As remarked above $\phi - \psi$ lies in the boundary of $\Lambda^+(\phi)$ iff $\phi(\xi) - \psi(\xi) = 1 - \psi(\xi) = 0$ at some point ξ. ∎

COROLLARY 5.17. *For each unit vector $e \in V$, let $\phi_e = e \lrcorner (e \wedge \phi) = \phi\big|_W$ where $W \equiv (\operatorname{span} e)^\perp$. Then:*

$$\phi_e \in \partial\Lambda^+(\phi) \qquad \text{if and only if} \qquad e \in \operatorname{span}\xi \ \text{ for some } \ \xi \in G(\phi).$$

Proof. Note that $\phi_e = e \lrcorner (e \wedge \phi) = \phi - e \wedge (e \lrcorner \phi)$ and $(e \wedge (e \lrcorner \phi))(\xi) = |a|^2$ where $e = a + b$ with $a \in \operatorname{span}\xi$ and $b \perp \operatorname{span}\xi$. Now $\phi_e \in \partial\Lambda^+(\phi)$ if and only if there exists $\xi \in G(\phi)$ with $|a| = 1$, that is, with $e = a \in \operatorname{span}\xi$. ∎

REMARK . Both $df \wedge (\nabla f \lrcorner \phi) = df \wedge d^\phi f$ and $\nabla f \lrcorner (df \wedge \phi) = \|\nabla f\|^2 \phi - df \wedge d^\phi f$ take values $\Lambda^+(\phi) \subset \Lambda^p T^* X$. Furthermore,

1) $df \wedge d^\phi f \ \in \ \operatorname{bdy}\{\Lambda^+(\phi)\} \qquad \Leftrightarrow \qquad \exists\, \xi \in G(\phi)$ tangential to the level sets of f.

2) $\nabla f \lrcorner (df \wedge \phi) \ \in \ \operatorname{bdy}\{\Lambda^+(\phi)\} \qquad \Leftrightarrow \qquad \exists\, \xi \in G(\phi)$ with $\nabla f \in \operatorname{span}\xi$.

Note that for some calibrations, condition 2) is true for all f, i.e., given a vector $n \in V$, there always exists a p-vector $\xi \in G(\phi)$ with $n \in \operatorname{span}\xi$.

Appendix: The reduced ϕ-Hessian.

We assume throughout this section that $\Lambda(\phi)$ is a vector subbundle of $\Lambda_p TX$, and we let $\Lambda(\phi) \subset \Lambda^p T^* X$ denote the corresponding bundle under the metric equivalence $\Lambda_p TX \cong \Lambda^p T^* X$.

DEFINITION 5.18. The **reduced ϕ-hessian** $\overline{\mathcal{H}}^\phi : C^\infty(X) \to \Gamma(X, \Lambda(\phi))$ is defined to be $\mathcal{H}^\phi$ followed by orthogonal projection onto the subbundle $\Lambda(\phi) \subset \Lambda^p T^* X$.

Note that a function f is ϕ-pluriharmonic if and only if $\overline{\mathcal{H}}^{\phi}(f) = 0$.

Note also that if ϕ is parallel, then $\overline{\mathcal{H}}^{\phi} = \overline{d}d^{\phi}$ where $\overline{d}$ denotes the exterior derivative followed by orthogonal projection onto $\Lambda(\phi)$.

For most of the calibrations considered as examples in this paper, the image of the map $\lambda_{\phi} : \mathrm{Sym}^2(\mathrm{TX}) \to \Lambda^{\mathrm{p}}\mathrm{T}^*\mathrm{X}$ is contained in $\Lambda(\phi)$, or equivalently, $\overline{\mathcal{H}}^{\phi} = \mathcal{H}^{\phi}$. For reference, $\overline{\mathcal{H}}^{\phi} = \mathcal{H}^{\phi}$ in the following cases.

(1) $\phi = \frac{1}{p!}\omega^p$, the pth power of the Kähler form,
(2) ϕ Special Lagrangian
(3) ϕ Associative, Coassociative or Cayley
(4) ϕ the fundamental 3-form on a simple Lie group.

Exceptions will be discussed at the end of this appendix.

Even when $\overline{\mathcal{H}}^{\phi} = \mathcal{H}^{\phi}$ the following proposition is important.

PROPOSITION 5.19. *Suppose f is a distribution on X. Then f is ϕ-plurisubharmonic if and only if $\overline{\mathcal{H}}^{\phi}(f) \equiv R$ is representable by integration and $\overrightarrow{R} \in \Lambda^+(\phi)$ $\|R\|$-a.e., that is, if and only if $\overline{\mathcal{H}}^{\phi}(f)$ is a ϕ-positive current.*

The proof is similar to the proof of Theorem 5.13 and is omitted.

DEFINITION 5.20. The ϕ-Grassmannian $G(\phi)$ **involves all the variables** if, for $u \in TX$, the condition $u \lrcorner \xi = 0$ for all $\xi \in G(\phi)$ implies $u = 0$

EXAMPLE . The 2-form $\phi \equiv dx_1 \wedge dx_2 + \lambda dx_3 \wedge dx_4$ with $|\lambda| < 1$ is a calibration on $\mathbf{R}^4$ which involves all the variables (See Section 1), but the only $\xi \in G(\phi)$ is the x_1, x_2 plane so that $G(\phi)$ does not involve all the variables.

PROPOSITION 5.21. *The operator $\overline{\mathcal{H}}^{\phi}$ is overdetermined elliptic if and only if $G(\phi)$ involves all the variables.*

Proof. We need only consider the case $\phi \in \Lambda^p V$, where V is an inner product space. The symbol of $\mathcal{H}^{\phi}$ at $u \in V$ is $u \wedge (u \lrcorner \phi)$. Hence, the reduced operator $\overline{\mathcal{H}}^{\phi}$ is elliptic if and only if

$$(u \wedge (u \lrcorner \phi))(\xi) = 0 \ \ \forall \ \ \xi \in G(\phi) \qquad \Rightarrow \qquad u = 0.$$

For $\xi \in G(p, V)$ and $u \in V$, let $u = a + b$ with $a \in \mathrm{span}\,\xi$ and $b \perp \mathrm{span}\,\xi$. Then $(u \wedge (u \lrcorner \phi))(\xi) = \phi(u \wedge (u \lrcorner \xi)) = \phi((a + b) \wedge (a \lrcorner \xi)) = |a|^2\phi(\xi) + \phi(b \wedge (a \lrcorner \xi))$. If $\xi \in G(\phi)$, then $\phi(b \wedge (a \lrcorner \xi)) = 0$ by the First Cousin Principle, and $\phi(\xi) = 1$. Hence, $(u \wedge (u \lrcorner \phi))(\xi) = |a|^2 = 0$ if and only if $u \lrcorner \xi = 0$. ∎

One can easily reduce a calibration to the elliptic case.

PROPOSITION 5.22. *Suppose $\phi \in \Lambda^p V$ is a calibration. Define $W \subset V$ by*

$$W^{\perp} \ \equiv \ \bigcap_{\xi \in G(\phi)} (\mathrm{span}\,\xi)^{\perp}$$

and set $\psi = \phi\big|_W$. Then $\psi \in \Lambda^p W$ is a calibration and $G(\psi)$ involves all the variables in W. Moreover, $G(\phi) = G(\psi)$ and the reduced operators $\overline{\mathcal{H}}^{\phi}$ and $\overline{\mathcal{H}}^{\psi}$ agree.

Proof. Obviously ψ is a calibration and $G(\psi) \subset G(\phi)$. If $\xi \in G(\phi)$, then $\operatorname{span} \xi \subset W$ and hence $\phi(\xi) = \psi(\xi)$. Thus $G(\phi) = G(\psi)$. By construction $G(\psi)$ involves all the variables in W. Finally, for all $\xi \in G(\phi)$, we have $\overline{\mathcal{H}}^{\phi}(f)(\xi) = \mathrm{tr}_{\xi}\mathrm{Hess} f = \overline{\mathcal{H}}^{\psi}(f)(\xi)$. ∎

EXAMPLE . Let $\Psi \in \Lambda^4_{\mathbf{R}} \mathbf{H}^n$ be the quaternionic calibration (1.5) on $\mathbf{H}^n$. One can show that $dd^{\Psi} f = 0$ if and only if $\mathrm{Hess} f = 0$. However,

$$\overline{d} d^{\Psi} f \ = \ \overline{\mathcal{H}}^{\Psi}(f) \ = \ \lambda_{\Psi} \left(\left(\frac{\partial^2 f}{\partial q_\alpha \partial \bar{q}_\beta} \right) \right),$$

that is, the reduced hessian is isomorphic to the quaternionic hessian $\left(\frac{\partial^2 f}{\partial q_\alpha \partial \bar{q}_\beta} \right)$

6. Duval-Sibony Duality.

In this section we extend the fundamental duality results established in [DS] in the complex case to calibrated manifolds (X, ϕ). The Duval-Sibony duality results involve plurisubharmonic functions, pseudoconvex hulls, positive currents and Poisson-Jensen formulas.

DEFINITION 6.1. Suppose R is a $(p-1)$-dimensional current on X. The operator ∂_ϕ is defined by

$$(\partial_\phi R)(f) \;=\; R(d^\phi f)$$

for all $f \in C^\infty_{\text{cpt}}(X)$.

In other words, $\partial_\phi : \mathcal{D}'_p(X) \longrightarrow \mathcal{D}'_0(X)$ is the formal adjoint of $d^\phi : \mathcal{E}^0(X) \longrightarrow \mathcal{E}^p(X)$. Let $\partial : \mathcal{D}'_p(X) \longrightarrow \mathcal{D}'_{p-1}(X)$ denote the **boundary operator** on currents. This is the formal adjoint of $d : \mathcal{E}^{p-1} \longrightarrow \mathcal{E}^p(X)$ and is related to the deRham differential on currents by $\partial = (-1)^{n-p}d$. The formal adjoint of $dd^\phi : \mathcal{E}^0(X) \longrightarrow \mathcal{E}^p(X)$ is the operator

$$\partial_\phi\partial : \mathcal{D}'_p(X) \;\longrightarrow\; \mathcal{D}'_0(X) \tag{6.1}$$

REMARK . Throughout the remainder of this section we assume that (X, ϕ) is a non-compact connected calibrated manifold. We also assume that ϕ is parallel. This assumption enables us to use the operator $\partial_\phi\partial$, but it is not necessary. We leave it to the reader to verify that all of the results of this section extend to the case where ϕ is not parallel by replacing the operator $\partial_\phi\partial$ with $\mathcal{H}_\phi : \mathcal{D}'_p(X) \to \mathcal{D}'_0(X)$, the formal adjoint of $\mathcal{H}^\phi : \mathcal{E}^0(X) \to \mathcal{E}^p(X)$. Of course, $\mathcal{H}_\phi$ is defined by $(\mathcal{H}_\phi(T))(f) = T(\mathcal{H}^\phi(f))$ for all $f \in C^\infty_{\text{cpt}}(X)$.

LEMMA 6.2. (The Support Lemma). *Suppose K is a compact subset of X. Suppose T is a $\Lambda_+(\phi)$-positive current with compact support in X. If $\partial_\phi\partial T$ is ≤ 0 (a non-positive measure) on $X - \widehat{K}$, then* $\operatorname{supp} T \subset \widehat{K} \cup \operatorname{Core}(X)$.

Proof. Lemma 3.2 states that for each $x \notin \widehat{K} \cup \operatorname{Core}(X)$ there exists a non-negative ϕ-plurisubharmonic function f on X which is identically zero on a neighborhood of K and strict at x. Since f is strict at x, there exists a small ball B about x and $\epsilon > 0$ so that $dd^\phi f - \epsilon\phi$ is $\Lambda^+(\phi)$-positive at each point in B. By equation (5.3), $M(\chi_B T) = (\chi_B T, \phi)$. Therefore, $\epsilon M(\chi_B T) = (\chi_B T, \epsilon\phi) \leq (\chi_B T, dd^\phi f) \leq (T, dd^\phi f) = (\partial_\phi\partial T, f) \leq 0$. This proves that $M(\chi_B T) = 0$ and hence $\operatorname{supp} T \subset \widehat{K} \cup \operatorname{Core}(X)$. ∎

The case where $K = \emptyset$ is a generalization of Proposition 3.12.

COROLLARY 6.3. *If T is a $\Lambda_+(\phi)$-positive current with compact support and $\partial_\phi\partial T \leq 0$, then*

$$\operatorname{supp} T \;\subset\; \operatorname{Core}(X).$$

When $\operatorname{Core}(X) = \emptyset$ we have

COROLLARY 6.4. *Suppose (X, ϕ) is strictly convex and K is ϕ-convex. Suppose T is $\Lambda_+(\phi)$-positive with compact support. If* $\operatorname{supp}\{\partial_\phi\partial T\} \subset K$, *then* $\operatorname{supp} T \subset K$. *In particular, there are no $\Lambda_+(\phi)$-positive currents which are compactly supported without boundary on X.*

.

Suppose $\overline{M} = M \cup \partial M$ is a compact oriented submanifold with boundary in X, and that M has no compact components. Let G_x denote the Green's function for $(M, \partial M)$ with singularity at $x \in M$. Let μ_x denote harmonic measure (i.e., the Poisson kernel) on ∂M and let $[x]$ denote the point-mass measure at $x \in M$. Then

$$*_M \Delta_M G_x \;=\; \mu_x - [x] \qquad \text{on } \;\overline{M}. \tag{6.2}$$

If $\overline{M}$ is a ϕ-submanifold of a calibrated manifold (X, ϕ), this equation can be reformulated as a current $\partial_\phi \partial$-equation on X.

Lemma 6.5. *Suppose M is a ϕ-submanifold of X, and that $u \in \mathcal{D}'^0(M)$ is a generalized function on M. Then*

$$\partial_\phi \partial (u[M]) \;=\; (*_M \Delta_M u)[M].$$

Proof. Consider the inclusion map $i : M \hookrightarrow X$. Then, by definition, $u[M] = i_* u$ and $(*_M \Delta_M u)[M] = i_*(*_M \Delta_M u)$. For any test function f on X we have $((\partial_\phi \partial)(i_* u), f) = (i_* u, dd^\phi f) = (u, i^*(dd^\phi f))_M$ where $(\cdot,\cdot)_M$ denotes the pairing of functions with currents on M. Proposition 1.13 states that $i^*(dd^\phi f) = *_M \Delta_M (i^* f)$ if M is a ϕ-submanifold. Finally, $(u, *_M \Delta_M (i^* f))_M = (*_M \Delta_M u, i^* f)_M = (i_*(*_M \Delta_M u), f)_X$. ■

Corollary 6.6. *Suppose $\overline{M} = M \cup \partial M$, as above, is a ϕ-submanifold with boundary. Then*

$$\partial_\phi \partial (G_x[M]) \;=\; \mu_x - [x]$$

as a current equation on X.

Assume K is a compact subset of X, and let $\mathcal{M}_K$ denote the set of probability measures with support in K.

Definition 6.7. If T_x is a $\Lambda_+(\phi)$-positive current with compact support and T_x satisfies:

$$\partial_\phi \partial\, T_x \;=\; \mu_x - [x] \tag{6.3}$$

with $\mu_x \in \mathcal{M}_K$, then: T_x is a **Green's current for** (K, x), μ_x is a **Poisson-Jensen measure** for (K, x), and the equation (6.3) is the **Poisson-Jensen equation**.

Theorem 6.8. *Suppose X is strictly ϕ-convex, K is a compact subset of X, and $x \in X - K$. Then there exists a Green's current T_x for (K, x) if and only if $x \in \widehat{K}$.*

To prove this we begin with the following.

Proposition 6.9. *Suppose (X, ϕ) is non-compact calibrated manifold. If there exists a Green's current for (K, x), then $x \in \widehat{K}$.*

Proof. This follows immediately from Lemma 6.2 since $x \in \operatorname{supp} T_x$. ■

Second Proof. Since $\partial_\phi \partial\, T_x = \mu_x - [x]$, we have $\int f \mu_x - f(x) = (T_x, dd^\phi f)$ for all $f \in C^\infty(X)$. If f is ϕ-plurisubharmonic , this implies that $f(x) \le \int f \mu_x \le \sup_K f$, since $\mu_x \in \mathcal{M}_K$. Thus $x \in \widehat{K}$. ■

The set $\mathcal{P}_X \equiv \mathcal{PSH}(X,\phi) \subset C^\infty(X)$ of all ϕ-plurisubharmonic -functions on X is clearly a closed convex cone in $C^\infty(X)$. Let

$$C_X \;\equiv\; \{u \in \mathcal{D}'_{0,\mathrm{cpt}}(X) : u = \partial_\phi \partial T \text{ for some } \Lambda_+(\phi)\text{−positive } T \in \mathcal{D}'_{p,\mathrm{cpt}}(X)\}. \tag{6.4}$$

This is a convex cone in $\mathcal{D}'_{0,\mathrm{cpt}}(X)$.

LEMMA 6.10. *Suppose X is non-compact with calibration ϕ. Then $\mathcal{P}$ is the polar of C, that is,*

$$\mathcal{P} \;=\; C^0 \;\equiv\; \{f \in C^\infty(X) \;:\; (u,f) \geq 0 \;\;\forall u \in C\}.$$

Proof. Consider $u = \partial_\phi \partial(\delta_x \xi)$, with $\xi \in G_x(\phi)$. Clearly $u \in C$. If $f \in C^\infty(X)$ belongs to C^0, then $0 \leq (u,f) = (\partial_\phi \partial(\delta_x \xi), f) \;=\; (\delta_x \xi, dd^\phi f) \;=\; (dd^\phi f)_x(\xi)$. Hence $C^0 \subseteq \mathcal{P}$.

Conversely, if $f \in \mathcal{P}$, then for all $u \in C$, $(u,f) \;=\; (\partial_\phi \partial T, f) \;=\; (T, dd^\phi f) \geq 0$, since T is $\Lambda_+(\phi)$-positive. This proves that $\mathcal{P} \subseteq C^0$. ∎

LEMMA 6.11. *If X is strictly ϕ-convex, then the convex cone $C \subset \mathcal{D}'_{0,\mathrm{cpt}}(X)$ is closed.*

Proof. It suffices to show that $C \cap \mathcal{D}'_{0,K}(X)$ is closed for an exhaustive family of compact subsets $K \subset X$. We may assume K is ϕ-convex. Suppose u_j converges to u in $C \cap \mathcal{D}'_{0,K}(X)$ with each $u_j \in C$, i.e., $u_j = \partial_\phi \partial T_j$ where T_j is a $\Lambda_+(\phi)$-positive current with compact support. By Corollary 6.4 the support of each T_j is contained in K. Consider a strictly ϕ-plurisubharmonic function f on X. Pick $\epsilon > 0$ so that $dd^\phi f - \epsilon\phi$ is $\Lambda^+(\phi)$-positive at each point of K. Then $\epsilon M(T_j) \;=\; (T_j, \epsilon\phi) \;\leq\; (T_j, dd^\phi f) \;=\; (\partial_\phi \partial T_j, f) \;=\; (u_j, f)$ which converges to (u,f). Therefore the masses $M(T_j)$ are bounded. By compactness there exists a weakly convergent subsequence $T_j \to T$. Now $\operatorname{supp} T \subset K$ and T must be $\Lambda_+(\phi)$-positive. Hence $u = \partial_\phi \partial T \in C \cap \mathcal{D}'_{0,K}(X)$. This proves that $C \cap \mathcal{D}'_{0,K}(X)$ is closed for each compact set K which is ϕ-convex. ∎

COROLLARY 6.12. *Suppose X is strictly ϕ-convex. Then*

$$C \;=\; \mathcal{P}^0.$$

Equivalently, the equation

$$\partial_\phi \partial T \;=\; u$$

has a solution T which is a $\Lambda_+(\phi)$-positive current with compact support if and only if

$$0 \;\leq\; u(f) \qquad \textit{for all} \quad f \in \mathcal{PSH}(X,\phi)$$

Proof. Apply the Bipolar Theorem. ∎

Proof of Theorem 6.8. Suppose there does not exist a Green's current for (K,x), that is, suppose $\mathcal{M}_K - [x]$ is disjoint from the cone C. By the Hahn-Banach Theorem (note that $\mathcal{M}_K - [x]$ is a compact convex set) there exist $f \in C^0 = \mathcal{P}$ with f, considered as a linear functional on $\mathcal{D}'_{0,\mathrm{cpt}}(X)$, satisfying $u(f) \leq -\epsilon < 0$ for all $u \in (\mathcal{M}_K - [x])$. That is, $\int f d\mu - f(x) \leq -\epsilon < 0$ for all $\mu \in \mathcal{M}_K$. Consequently,

$$\sup_K f \ = \ \sup_{\mu\in\mathcal{M}_K} \int f d\mu \ \leq \ f(x) - \epsilon$$

or $\sup_K f + \epsilon \leq f(x)$ so that $x \notin \widehat{K}$. ∎

One could define the "Poisson-Jensen hull" of a compact set K to be the set of points x for which there exists a Poisson-Jensen measure μ_x and a Green's current T_x satisfying (6.3). Then Proposition 6.9 states that on any (non-compact) calibrated manifold (X, ϕ), the Poisson-Jensen hull of a compact set is contained in the ϕ-plurisubharmonic hull, while Theorem 6.8 states that the two hulls are equal if (X, ϕ) is strictly convex.

The next "hull" obviously contains the Poisson-Jensen hull.

Definition 6.13. The **current hull** of a compact subset $K \subset X$ is the union

$$\widetilde{K} \ \equiv \ \bigcup_{T\in\mathcal{P}(K)} \operatorname{supp} T$$

where $\mathcal{P}(K)$ consists of all $\Lambda_+(\phi)$-positive currents with compact support on X satisfying $\partial_\phi \partial T \leq 0$ on $X - K$.

Lemma 6.14. *If (X, ϕ) is strictly ϕ-convex, then $\widetilde{K} = \widehat{K}$.*

Proof. The support Lemma 6.2 states that

$$\widetilde{K} \ \subset \ \widehat{K} \cup \mathrm{Core}(X).$$

for any calibrated manifold. Now $\widetilde{K}$ contains the Poisson-Jensen hull which equals $\widehat{K}$ if (X, ϕ) is strictly ϕ-convex by Theorem 5.8. ∎

Suppose (X, ϕ) is non-compact and connected.

Definition 6.15. An open subset $\Omega \subset X$ is **ϕ-convex relative to** X if $K \subset\subset \Omega$ implies $\widehat{K}_X \subset\subset \Omega$.

Note that if X is ϕ-convex this condition implies that Ω is ϕ-convex since $\widehat{K}_\Omega \subseteq \widehat{K}_X$. Moreover, if X is strictly ϕ-convex , then $\mathrm{Core}(\Omega) \subseteq \mathrm{Core}(X)$ is empty so that Ω is strictly ϕ-convex (by Proposition 2.13).

Proposition 6.16. *Suppose (X, ϕ) is strictly ϕ-convex . An open subset $\Omega^{\mathrm{open}} \subset X$ is ϕ-convex relative to X if and only if $\mathcal{PSH}(X, \phi)$ is dense in $\mathcal{PSH}(\Omega, \phi)$.*

Proof. Let $L : C^\infty(X) \to C^\infty(\Omega)$ denote restriction. The adjoint $L^* : \mathcal{D}'_{0,\mathrm{cpt}}(\Omega) \to \mathcal{D}'_{0,\mathrm{cpt}}(\Omega)$ is inclusion. Suppose $v \in (L^*)^{-1}(C_X)$. i.e., $v \in \mathcal{D}'_{0,\mathrm{cpt}}(\Omega)$ with $v = \partial_\phi \partial T$ for some $\Lambda_+(\phi)$-positive current T compactly supported in X. Then $K \equiv \operatorname{supp} v$ satisfies $\widetilde{K}_X = \widehat{K}_X$ by Lemma 6.14. Hence, $\widehat{K}_X \subset \Omega$ implies $\operatorname{supp} v \subset \Omega$ or that $v \in C_\Omega$. This proves that Ω is ϕ-convex relative to X if and only if

$$(L^*)^{-1}(C_X) \ = \ C_\Omega. \tag{6.5}$$

By Corollary 6.12 we may replace C_X by $\mathcal{P}_X^0 = C_X$. In general, $[L(\mathcal{P}_X)]^0 = (L^*)^{-1}(\mathcal{P}_X^0)$, so that (6.5) is equivalent to

$$[L(\mathcal{P}_X)]^0 \;=\; C_\Omega. \tag{6.6}$$

By Lemma 6.10, $\mathcal{P}_\Omega = C_\Omega^0$. Hence (6.6) is equivalent to

$$\overline{L(\mathcal{P}_X)} \;=\; \mathcal{P}_\Omega \tag{6.7}$$

■

7. ϕ-Free Submanifolds

Suppose (X, ϕ) is a calibrated manifold. A plane ξ is said to be *tangential* to a submanifold M if $\operatorname{span} \xi \subset T_x M$.

DEFINITION 7.1. A closed submanifold $M \subset X$ is *ϕ-free* if there are no ϕ-planes $\xi \in G(\phi)$ which are tangential to M. If the restriction of the calibration ϕ to M vanishes, M is called *ϕ-isotropic*.

Note that ϕ-isotropic submanifolds are ϕ-free . Each submanifold of dimension strictly less than the degree of ϕ is ϕ-isotropic and hence automatically ϕ-free . Furthermore, in dimension p the generic local submanifold is ϕ-free . Depending on the geometry, this may continue through a range of dimensions greater than p.

THEOREM 7.2. *Suppose M is a closed submanifold of (X, ϕ) and let $f_M(x) \equiv \frac{1}{2}\mathrm{dist}(x, M)^2$ denote half the square of the distance to M. Then M is ϕ-free if and only if the function f_M is strictly ϕ-plurisubharmonic at each point in M (and hence in a neighborhood of M).*

Proof. We begin with the following.

LEMMA 7.3. *Fix $x \in M$ and let $P_N : T_x X \to N$ denote orthogonal projection onto the normal plane of M at x. Then for each $\xi \in G(\phi)$ one has*

$$\{\lambda_\phi(\mathrm{Hess}_x f_M)\}(\xi) \;=\; \langle P_N, P_\xi \rangle \tag{7.1}$$

Proof. By Lemma 1.15b)

$$\{\lambda_\phi(\mathrm{Hess}_x f)\}(\xi) \;=\; \langle \mathrm{Hess}_x f, P_\xi \rangle \tag{7.2}$$

for any function f. The lemma then follows from the assertion that

$$\mathrm{Hess}_x f_M \;=\; P_N. \tag{7.3}$$

To see this we first note that the Hessian of any function f can be written

$$\mathrm{Hess} f(V, W) \;=\; \langle V, \nabla_W(\nabla f) \rangle \tag{7.4}$$

for all $V, W \in T_xX$. It follows that if $\nabla f = 0$ on the submanifold M, then $T_xM \subset \mathrm{Null}(\mathrm{Hess}_x f)$. Thus, with respect to the decomposition $T_xX = T_xM \oplus N$ we have

$$\mathrm{Hess}_x f_M \;=\; \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

and it remains to show that A is the identity. To see this, set $\delta(x) = \mathrm{dist}(x, M)$ and note that $\nabla\delta = n$ is a smooth unit-length vector field near (but not on) M whose integral curves are geodesics emanating from M. Hence,

$$\nabla_n(\nabla f_M) = \nabla_n(\nabla \frac{1}{2}\delta^2) = \nabla_n(\delta n) = n + \delta \nabla_n n = n.$$

Taking limits along normal geodesics down to M gives the result. ∎

Theorem 7.2 now follows from the fact that

$$\langle P_N, P_\xi \rangle \;\geq\; 0 \text{ with equality iff } \mathrm{span}\,\xi \subset N^\perp = T_xM.$$

∎

The existence of ϕ-free submanifolds insures the existence of lots of strictly ϕ-convex domains in (X, ϕ).

THEOREM 7.4. *Suppose M is a ϕ-free submanifold of (X, ϕ). Then there exists a fundamental neighborhood system $\mathcal{F}(M)$ of M such that:*

(a) M is a deformation retract of each $U \in \mathcal{F}(M)$.

(b) Each neighborhood $U \in \mathcal{F}(M)$ is strictly ϕ-convex .

(c) $\mathcal{PSH}(V, \phi)$ is dense in $\mathcal{PSH}(V, \phi)$ if $U \subset V$ and $V, U \in \mathcal{F}(M)$.

(d) Each compact set $K \subset M$ is $\mathcal{PSH}(U, \phi)$-convex for each $U \in \mathcal{F}(M)$.

Proof. We construct tubular neighborhoods of M as follows. Let $\epsilon \in C^\infty(M)$ be a smooth function which vanishes at infinity and has the property that for each $x \in M$ the ball $\{y \in X : \frac{1}{2}\mathrm{dist}(y, x)^2 \leq \epsilon(x)\}$ is compact and geodesically convex. Assume also that ϵ is sufficiently small so that the exponential map gives a diffeomorphism

$$\exp : N_\epsilon \;\longrightarrow\; U_\epsilon$$

from the open set N_ϵ in the normal bundle N defined by $\frac{1}{2}\|n_x\|^2 < \epsilon(x)$ to the neighborhood

$$U_\epsilon \;=\; \{x \in X : f_M(x) < \epsilon(x)\}. \tag{7.5}$$

of M in X. Each U_ϵ admits a deformation retraction onto M.

By Theorem 7.2 the function $f_M = \frac{1}{2}\mathrm{dist}(\cdot, M)^2$ is strictly ϕ-plurisubharmonic on a neighborhood of M, which we can assume to be W. We impose the following additional condition on the function $\epsilon \in C^\infty(W)$.

$$f_M - t\epsilon \quad \text{is strictly } \phi-\text{plurisubharmonic on } \; W \; \text{ for } \; 0 \leq t \leq 1. \tag{7.6}$$

Since (7.6) is valid as long as ϵ and its first and second derivatives vanish sufficiently fast at infinity, it is easy to see that the family $\mathcal{F}(M)$ of neighborhoods U_ϵ constructed above with ϵ satisfying (7.6) is a fundamental neighborhood system for M.

Obviously, the function $\psi \equiv (\epsilon - f_M)^{-1}$ is a proper exhaustion for U_ϵ. To prove (b), recall that if g is a positive concave function, then $1/g$ is convex, or more directly, calculate that

$$\text{Hess}\psi \;=\; \psi^2\text{Hess}(f_M - \epsilon) + \psi^3\nabla(\epsilon - f_M)\circ\nabla(\epsilon - f_M). \tag{7.7}$$

Applying λ_ϕ to (7.7) proves that ψ is strictly ϕ-plurisubharmonic on $\{\psi > 0\} = U_\epsilon$.

To prove parts (c) and (d) one uses Proposition 6.16 and argues exactly as on page 302 of [HW_1]. ∎

EXAMPLE 7.5. As mentioned above, Theorem 7.4 exhibits a rich family of ϕ-convex domains in (X,ϕ). For example, let $M \subset X$ be *any submanifold of dimension* $< p = \deg\phi$. Then by 7.4, M has a fundamental system of neighborhoods each of which is a strictly ϕ-convex domain homotopy equivalent to M.

EXAMPLE 7.6. Interesting examples occur in all the calibrated geometries examined in depth in [HL_3]. Suppose for instance that X is a Calabi-Yau manifold with special Lagrangian calibration ϕ. Then *any complex submanifold* $Y \subset X$ *is ϕ-free* . It follows that any smooth submanifold of Y is also ϕ-free .

We now consider the following two classes of subsets of (X,ϕ).

(1) Closed subsets A of ϕ-free submanifolds.

(2) Zero sets of non-negative strictly ϕ-plurisubharmonic functions f.

These two classes are basically the same, as described in the following two propositions.

PROPOSITION 7.7. *Suppose A is a closed subset of a ϕ-free submanifold M of X. Then there exists a non-negative function $f \in C^\infty(X)$ with*

(a) $A = \{x \in X : f(x) = 0\}$

(b) f is strictly ϕ-plurisubharmonic at each point in M (and hence in a neighborhood of M in X).

Proof. Since M is a closed submanifold, the function f_M in Theorem 7.2 can be extended to $h \in C^\infty(X)$ which agrees with f_M in a neighborhood of M and satisfies

$$h \geq 0 \qquad \text{and} \qquad \{h = 0\} = M.$$

Choose $\psi \in C^\infty(X)$ with $\psi \geq 0$ and $A = \{x \in X : \psi = 0\}$. Now choose $\epsilon \in C^\infty(X)$ with $\epsilon(x) > 0$ for all $x \in X$, and with ϵ and its derivatives sufficiently small so that $f \equiv h + \epsilon\psi$ is strictly ϕ-plurisubharmonic on M. ∎

PROPOSITION 7.8. *Suppose $f \in C^\infty(X)$ is a non-negative function which is strictly ϕ-plurisubharmonic at each point in $A \equiv \{x \in X : f(x) = 0\}$. Given a point $x \in A$ there exists a neighborhood U of x and a proper ϕ-free submanifold M of U such that $A \cap U \subset M$.*

Proof. Given $x \in A$ we may choose geodesic normal coordinates (z, y) in a neighborhood U at x so that

$$\mathrm{Hess}_x f \;=\; \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} \tag{7.8}$$

where Λ is the diagonal matrix $\mathrm{diag}\{\lambda_1, ..., \lambda_r\}$, r is the rank of $\mathrm{Hess}_x f$, and $\lambda_j \neq 0$ for $j = 1, ..., r$. Set

$$M \;=\; \left\{ w \in U : \frac{\partial f}{\partial y_1} = \cdots = \frac{\partial f}{\partial y_r} = 0 \right\}.$$

Since $\nabla \frac{\partial f}{\partial y_1}, ..., \nabla \frac{\partial f}{\partial y_r}$ are linearly independent at x, M is a codimension r submanifold locally near x.

Note that $\ker(\mathrm{Hess}_x f) = T_x M$. It remains to show that $\ker(\mathrm{Hess}_x f)$ is totally real, since if M is ϕ-free at x, then M is ϕ-free in a neighborhood of x. This is proved in Lemma 7.9 below. ■

Lemma 7.9. *Suppose f is strictly ϕ-plurisubharmonic at $x \in X$. Then* $\ker(\mathrm{Hess}_x f) \subset T_x X$ *is ϕ-free .*

Proof. If $\ker(\mathrm{Hess}_x f) \subset T_x X$ is not ϕ-free , there exists $\xi \in G(\phi)$ with $(\mathrm{Hess}_x f)\big|_{\mathrm{span}\,\xi} = 0$. Consequently, $dd^\phi f(\xi) = \lambda_\phi(\mathrm{Hess}_x f)(\xi) = \mathrm{tr}_\xi(\mathrm{Hess}_x f) = 0$, and f is not strict at x. ■

Remark 7.10. Parts (b), (c) and (d) of Theorem 7.4 can be generalized as follows. Suppose $M = \{f = 0\}$ is the zero set of a non-negative strictly ϕ-plurisubharmonic function f on (X, ϕ). Then there exists a fundamental neighborhood system $\mathcal{F}(M)$ of M satisfying (b), (c) and (d) of Theorem 7.4. The neighborhoods $U_\epsilon \in \mathcal{F}(M)$ are defined by $U_\epsilon = \{x \in X : f(x) < \epsilon(x)\}$ where $\epsilon > 0$ is a C^∞ function on X vanishing at infinity along with its first and second derivatives so that $f - \epsilon$ remains strictly ϕ-plurisubharmonic . The proofs of (b), (c) and (d) are essentially the same as in Theorem 7.4.

We conclude with a the following general observation.

Proposition 7.11. *Let M be a submanifold of (X, ϕ) and f a smooth function defined on a neighborhood of M such that:*

(1) $\nabla f \equiv 0$ on M, and

(2) f is strictly ϕ-plurisubharmonic at all points of M.

Then M is ϕ-free .

Proof. By (7.4) we see that $TM \subseteq \ker(\mathrm{Hess} f)$ at all points of M. We then apply Lemma 7.9. ■

Corollary 7.12. *Let f be a non-negative, real analytic function on (X, ϕ) and consider the real analytic subvariety $Z \equiv \{f = 0\}$. If f is strictly ϕ-plurisubharmonic at points of Z, then each stratum of Z is ϕ-free .*

8. Hodge Manifolds

In this section we pose some highly speculative questions for calibrated manifolds in the spirit of those posed in the complex case (cf. [HK, p.58], [$L_{4,5}$]). Assume that (X, ϕ) is a compact calibrated n-manifold with a parallel calibration ϕ of degree p. Let

$$\psi \;=\; *\phi \tag{8.1}$$

denote the dual calibration. Note that a ϕ-submanifold or, more generally, any ϕ-cycle on X is a current of dimension p and degree $n-p$. By contrast a ψ-submanifold or ψ-cycle is a current of dimension $n-p$ and degree p. Denote by $\widetilde{H}^p(X, \mathbf{Z})$ the image of the map $H^p(X, \mathbf{Z}) \to H^p(X, \mathbf{R})$.

Definition 8.1. If the de Rham class of the calibration ϕ lies in $\widetilde{H}^p(X, \mathbf{Z})$, i.e., if ϕ has integral periods, then (X, ϕ) will be referred to as a **Hodge manifold**.

Remark . If (X, ω) is a Kähler manifold, then this coincides with standard terminology. The Kodaira Embedding Theorem states that in this case each Hodge manifold is projective algebraic with $N\omega - [H] = d\alpha$, where H is a hyperplane section, N a positive integer, and α a current of degree 1.

The Hodge Question (for the class of ϕ). Suppose (X, ϕ) is a Hodge manifold. When does there exist a $*\phi$-cycle T cohomologous to $N\phi$ for some positive integer N, i.e.,

$$N\phi - T \;=\; d\alpha \tag{8.2}$$

for some current α of degree $p-1$?

Recall that a $*\phi$-cycle is automatically $*\phi$-positive, so this is, more precisely, the "Hodge Question with Positivity" for ϕ.

Remark . If equation (8.2) (called the spark equation) has a solution, then α determines a differential character on X. (See [HLZ] for more details.)

Example 8.3. In [L_3] an example is constructed of a Hodge manifold (X, ϕ) for which no such cycle exists. More specifically, a parallel self-dual 4-form ϕ of comass 1 is constructed on a flat torus X of dimension 8 with the property that $[\phi] \in H^4(X, \mathbf{R})$ is an integral class, but there exist no ϕ-cycles whatsoever on X.

Example 8.4. Consider the fundamental bi-invariant 3-form Ω on a compact simple Lie group G, normalized to be the generator of $H^3(G, \mathbf{Z}) \cong \mathbf{Z}$. Then (G, Ω) is a Hodge manfiold, and R. Bryant [B] has shown that, indeed, Ω is always cohomologous to a $*\Omega$-cycle. These $*\Omega$-cycles are always sums of singular semi-analytic subvarieties congruent to irreducible components of the cut-locus of the exponential map.

For a general class in $\widetilde{H}^p(X, \mathbf{Z})$ to be represented by a $*\phi$-cycle (or for a class in $\widetilde{H}_p(X, \mathbf{Z}) \cong \widetilde{H}^{n-p}(X, \mathbf{Z})$ to be represented by a ϕ-cycle), there is a natural necessary condition coming from the Hodge decomposition. Note that since ϕ and $*\phi$ are parallel, the subspaces $\Lambda_x(\phi) \equiv \operatorname{span} G(\phi) \subset \Lambda_p T_x X$ and $\Lambda_x(*\phi) \equiv \operatorname{span} G(*\phi) \subset \Lambda_{n-p} T_x X$, under metric equivalence $TX \cong T^*X$, define *parallel subbundles*

$$\Lambda(\phi) \ \subset \Lambda^p T^* X \qquad \text{and} \qquad \Lambda(*\phi) \ \subset \Lambda^{n-p} T^* X$$

The orthogonal projections $P_{\Lambda(\phi)}$ and $P_{\Lambda(*\phi)}$ onto these subbundles are *parallel operators on forms*. It was proved by Chern [Ch] that any such operator commutes with harmonic projection. Recall that the Hodge decomposition: $\mathcal{E}^p(X) = \mathbf{H}^p(X) \oplus \mathrm{Image}(d) \oplus \mathrm{Image}(d^*)$, is a C^∞-decomposition, and therefore induces a corresponding decomposition of currents: $\mathcal{E}'_p(X) = \mathbf{H}_p(X) \oplus \mathrm{Image}(\partial) \oplus \mathrm{Image}(\partial^*)$.

Definition 8.5. A p-dimensional current T, representable by integration, is said to be of **type** $\Lambda(\phi)$ if $\overrightarrow{T}_x \in \Lambda(\phi) \subset \Lambda^p T_x X$ for $\|T\|$-a.a. x.

This definition extends to arbitrary currents $T$of dimension p by requiring that $T(\psi) = 0$ for all smooth p-forms ψ such that $\psi\big|_{\Lambda(\phi)} = 0$.

Proposition 8.6. *If a class $c \in \widetilde{H}_p(X, \mathbf{Z})$ is represented by a current of type $\Lambda(\phi)$, then the harmonic representative of c must be of type $\Lambda(\phi)$.*

Proof. Any $\Lambda(\phi)$-current is fixed by the parallel bundle projection, and that projection commutes with the harmonic projector. ∎

Definition 8.7. A $\Lambda(\phi)$**-cycle** is a d-closed, p-dimensional locally rectifiable current of type $\Lambda(\phi)$.

I. The Hodge Question . Suppose $u \in \widetilde{H}_p(X, \mathbf{Z})$ is a class whose harmonic representative is of type $\Lambda(\phi)$. When does there exist an integer N and a $\Lambda(\phi)$-cycle T with $T \in Nc$?

Remark 8.8. Example 8.3 above gives a parallel calibration ϕ on a flat 8-dimensional torus $X$and an integral class $c \in \widetilde{H}_4(X, \mathbf{Z})$ of type $\Lambda(\phi)$ for which no such current exists.

Remark 8.9. The Hodge Question is a direct generalization of the standard Hodge Conjecture for algebraic cycles on a complex projective manifold, since we know from [HS], [Sh] and [Alex] that for $\phi = \omega^p/p!$ (ω = the Kähler form), any $\Lambda(\phi)$-cycle is an algebraic p-cycle.

Any locally finite integer sum of ϕ-cycles is a $\Lambda(\phi)$-cycle. However, the converse is completely open outside of the Kähler case. Moreover, even though it holds in the Kähler case (cf. Remark 8.9), there is no proof of this fact by the standard methods of regularity in Geometric Measure Theory.

Before trying to prove that a general $\Lambda(\phi)$-cycle is a sum of ϕ-cycles, one would like the calibration ϕ to have the algebraic property displayed in the next remark.

Remark 8.10. Equation (5.4) says that $G(p, T_x X) \cap \Lambda_+(\phi) \ = \ G(\phi)$ so that ϕ-cycles and $\Lambda_+(\phi)$-cycles are the same thing. Most parallel calibrations(see [HL$_3$, p. 68]) are known to satisfy

$$G(p, T_x X) \cap \Lambda(\phi) \ = \ G(\phi) \cup (-G(\phi)).$$

In this case T is a $\Lambda_+(\phi)$-cycle if and only if $\pm\overrightarrow{T_x} \in G_x(\phi)$ for $\|T\|$-a. a. x. Consequently, T decomposes inito $T^+ - T^-$ with both $\overrightarrow{T_x}^{\pm} \in G_x(\phi)$, but, even in the Kähler case, one can not show directly that T^+ and T^- are d-closed.

There are versions of the Hodge Question involving "positivity" which may have more hope. For example:

II. The Hodge Question (with positivity). Suppose $c \in \widetilde{H}_p(X, \mathbf{Z})$ is a class whose harmonic representative is strictly $\Lambda_+(\phi)$-positive. When does there exist an integer N and a ϕ-cycle T with $T \in Nc$?

Remark 8.11. If the current $*\phi$ (of dimension p) is strictly $\Lambda_+(\phi)$-positive, then for any form ψ of type $\Lambda(\phi)$, there exists an integer ℓ such that $\psi + \ell(*\phi)$ is strictly $\Lambda_+(\phi)$-positive. This applies for example to the harmonic representative of c in Hodge Question II. Consequently, one can see that *if $(X, *\phi)$ is a Hodge manifold with a solution to (8.2), then the Hodge Questions I and II are equivalent.*

Remark 8.12. The point of Hodge Question II is that one is asking for a ϕ-cycle which is automatically $\Lambda_+(\phi)$-positive and therefore satisfies the strong regularity theorem 5.9.

9. Boundaries of ϕ-submanifolds.

In this section we take up the following general question. Suppose (X, ϕ) is a non-compact strictly ϕ-convex manifold. Given a compact oriented submanifold $\Gamma \subset X$ of dimension $p-1$, when does there exist a ϕ-submanifold M with boundary Γ? More generally, when does there exist a $\Lambda_+(\phi)$-positive current T with $\partial T = \Gamma$?

Theorem 9.1. *Suppose ϕ is exact. Given $S \in \mathcal{E}'_{p-1}(X)$, there exists a $\Lambda_+(\phi)$-positive current $T \in \mathcal{E}'_p(X)$ with $S = \partial T$ if and only if*

$$\int_S \alpha \;\geq\; 0 \qquad \text{for all } \alpha \in \mathcal{E}^{p-1}(X) \text{ such that } d\alpha \text{ is } \;\Lambda^+(\phi)\text{−positive}$$

Proof. Consider the following convex cones.

$$A \;=\; \{\alpha \in \mathcal{E}^{p-1}(X) : \; d\alpha \text{ is } \;\Lambda^+(\phi)\text{−positive}\}$$

$$B \;=\; \{S \in \mathcal{E}'_{p-1}(X) : \; S = \partial T \text{ for some } \Lambda_+(\phi)\text{−positive } T \in \mathcal{E}'_p(X)\}$$

If $\alpha \in A$ and $S \in B$, then

$$S(\alpha) = \partial T(\alpha) = T(d\alpha) \geq 0,$$

that is,

$$A \;\subset\; B^0 \qquad \text{and} \qquad B \;\subset\; A^0$$

where B^0 denotes the polar of B. If $\xi \in G_x(\phi)$, then $T = \delta_x\xi$ is $\Lambda_+(\phi)$-positive, so that $\partial(\delta_x\xi) \in B$. Therefore, if $\alpha \in B^0$, then $0 \leq \partial(\delta_x\xi)(\alpha) = (\delta_x\xi)(d\alpha) = (d\alpha)_x(\xi)$. This proves that $B^0 \subset A$, and hence $A = B^0$. (In particular, note that A is closed.) Theorem 9.1 is just the statement that $B = A^0$. Now since $A = B^0$, the Bipolar Theorem states that $\overline{B} = A^0$. Thus it remains to show that B is closed.

Suppose $S_j \in B$ and $S_j \to S$ in $\mathcal{E}'_{p-1}(X)$. Then $S_j = \partial T_j$ for some T_j which is $\Lambda_+(\phi)$-positive. The calibration ϕ is exact, i.e., $\phi = d\eta$ for some $\eta \in \mathcal{E}^{p-1}(X)$. Therefore

$$M(T_j) \;=\; T_j(\phi) \;=\; T_j(d\eta) \;=\; (\partial T_j)(\eta) \;=\; S_j(\eta) \;\longrightarrow\; S(\eta).$$

In particular, there exists a constant C such that $M(T_j) \leq C$ for all j. By Lemma 6.2

$$\mathrm{supp} T_j \;\subset\; \widehat{\mathrm{supp} S_j}$$

for each j. Pick a compact subset K with $\mathrm{supp} S_j \subset K$ for all j. Then

$$\mathrm{supp} T_j \;\subset\; \widehat{K} \qquad \text{for all } j.$$

This proves that $\{T_j\}$ is a precompact set in $\mathcal{E}'_p(X)$. Therefore, there exists a convergent subsequence $T_j \to T$ in $\mathcal{E}'_p(X)$. Obviously, $\partial T = S$ and T is $\Lambda_+(\phi)$-positive. Hence, $S \in B$. ∎

Remark 9.2. The same proof combined with the Federer-Fleming compactness theorem for integral currents proves the following. Let $\mathcal{R}_p(X)$ denote the compactly supported rectifiable currents of dimension p on X. Then, if ϕ is exact, the set

$$B_{\mathrm{rect}} \;\equiv\; \{\Gamma \in \mathcal{R}_{p-1}(X) : \; S = \partial T \text{ for some } \Lambda_+(\phi)\text{−positive } T \in \mathcal{R}_p(X)\}\}$$

is weakly closed in $\mathcal{R}_{p-1}(X)$.

10. ϕ-Flat Hypersurfaces and Functions which are ϕ-Pluriharmonic mod d.

The ϕ-pluriharmonic functions are the closest thing to holomorphic functions on a calibrated manifold (X, ϕ). Usually there are very few ϕ-pluriharmonic functions. An attempt has been made in this paper to remedy this situation by emphasizing the ϕ-plurisubharmonic functions. By comparison these functions exist in abundance. For some purposes another extension of the concept of ϕ-pluriharmonic functions is more useful — namely the *ϕ-pluriharmonic functions mod d.*

This section is, for the most part, a straightforward extension of the results of Lei Fu [Fu] from the special Lagrangian case to the general calibrated manifold (X, ϕ).

Definition 10.1. A function $f \in C^\infty(X)$ is **ϕ-pluriharmonic mod d** if

$$dd^\phi f \;=\; df \wedge \alpha_f + \sigma_f \tag{10.1}$$

for some 1-form α_f and some p-form σ_f of type $\Lambda(\phi)^\perp$, i.e., $\sigma_f(\xi) = 0$ for all $\xi \in G(\phi)$.

If f is ϕ-pluriharmonic mod d, then λf, $\lambda \in \mathbf{R}$, is also ϕ-pluriharmonic mod d. However, the sum of two such functions need not be ϕ-pluriharmonic mod d.

PROPOSITION 10.2. *Suppose that df never vanishes so that $\mathcal{H} \equiv \ker df$ defines a hypersurface foliation. The condition that f be ϕ-pluriharmonic mod d is independent of the function defining the foliation $\mathcal{H}$.*

Proof. Recall that locally f and g define the same foliation $\mathcal{H}$ if and only if $g = \chi(f)$ for some function $\chi : \mathbf{R} \to \mathbf{R}$ for which χ' is never zero. To prove this fact assume that $f = x_1$ is a local coordinate. Since g is constant on the leaves $\{x_1 = \text{constant}\}$, g must be independent of $x_2, ..., x_n$, i.e., $g = \chi(x_1)$. Since dg is never zero, χ' is never zero. Finally,

$$dd^\phi g \;=\; \chi'(f)dd^\phi f + \chi''(f)df \wedge d^\phi f \;=\; dg \wedge \left(\alpha_f + \frac{\chi''(f)}{\chi'(f)} d^\phi f\right) + \chi'(f)\sigma_f$$

which proves that if f is ϕ-pluriharmonic mod d, then $g = \chi(f)$ is also. ∎

PROPOSITION 10.3. *If f is ϕ-pluriharmonic mod d, then each (non-critical) hypersurface $\{f = C\}$ is ϕ-flat.*

Proof. Suppose $\xi \in G(\phi)$ is tangent to $\{f = C\}$, i.e., $\nabla f \lrcorner \xi = 0$. Then $(dd^\phi f)(\xi) = (df \wedge \alpha_f)(\xi) + \sigma_f(\xi)$. Since σ_f is of type $\Lambda(\phi)^\perp$, we have $\sigma_f(\xi) = 0$, and $(df \wedge \alpha_f)(\xi) = \alpha_f(\nabla f \lrcorner \xi) = 0$. ∎

Recall from Proposition 1.13 that for any $f \in C^\infty$ and any ϕ-submanifold M, we have

$$(dd^\phi f - df \wedge \alpha_f)\big|_M \;=\; *_M(\Delta_M f) - d(f\big|_M) \wedge \alpha_f\big|_M. \tag{10.2}$$

This proves

PROPOSITION 10.4. *If f is ϕ-pluriharmonic mod d and M is a ϕ-submanifold, then $u \equiv f\big|_M$ satisfies the partial differential equation*

$$\Delta_M u \;=\; *(du \wedge \beta) \qquad \text{on} \quad M \tag{10.3}$$

where $\beta = \alpha_f\big|_M$.

The maximum principle is applicable to solutions to (10.3). See for instance [BJS].

COROLLARY 10.5. *Suppose (M, Γ) is a compact ϕ-submanifold with boundary. Then for each function f which is ϕ-pluriharmonic mod d and each point $x \in M$, one has*

$$\inf_\Gamma \;\le f(x) \;\le\; \sup_\Gamma f \tag{10.4}$$

COROLLARY 10.6. *Suppose (M, Γ) is as above. If $\Gamma \subset \{f = C\}$, then $M \subset \{f = C\}$.*

PROPOSITION 10.7. *Suppose (M, Γ) is a compact ϕ-submanifold with boundary, and suppose f is a function on X which is ϕ-pluriharmonic mod d. If f is constant on Γ, then*

$$d^\phi f\big|_\Gamma \;\equiv\; 0 \qquad \text{(pointwise)}. \tag{10.5}$$

Proof. By Corollary 10.6, f is constant on M. We then apply the following.

LEMMA 10.8. *For any function f constant on M, $d^\phi f\big|_\Gamma \equiv 0$.*

Proof. At $x \in \Gamma$, we have $\overrightarrow{M} = e \wedge \overrightarrow{\Gamma}$ for some e tangent to M. Since f is constant on M, $\nabla f \perp \operatorname{span} \overrightarrow{M}$. Now $(d^\phi f)(\overrightarrow{\Gamma}) = (\nabla f \lrcorner \phi)(e \lrcorner \overrightarrow{M}) = \phi((\nabla f) \wedge (e \lrcorner \overrightarrow{M})) = 0$ since $\nabla f \wedge (e \lrcorner \overrightarrow{M})$ is a first cousin of $\overrightarrow{M} \in G(\phi)$. ■ ■

Our next objective is to show that, for the large class of *normal* calibrations, a function f is ϕ-pluriharmonic mod d if and only if its level sets are ϕ-flat.

Suppose $\phi \in \Lambda^p V$ is a calibration on a euclidean vector space V. For each hyperplane $W \subset V$, $\phi\big|_W \in \Lambda^p W$ has comass ≤ 1 and, in fact, < 1 if and only if $G(\phi\big|_W)$ is empty.

DEFINITION 10.9. The calibration $\phi \in \Lambda^p V$ is **normal** if, for every hyperplane $W \subset V$

$$\Lambda(\phi\big|_W)^\perp = \Lambda(\phi)^\perp\big|_W$$

as subspaces of $\Lambda^p W$. A calibration ϕ on a manifold X is **normal** if $\phi_x \in \Lambda^p T_x X$ is normal for each $x \in X$.

PROPOSITION 10.10. *Suppose ϕ is a normal calibration on X, and $f \in C^\infty(X)$ has a never-vanishing gradient. Then*

$$f \text{ is } \phi\text{-pluriharmonic mod } d$$

if and only if

$$\text{each level set } \{f = C\} \text{ is } \phi\text{-flat.}$$

Proof. Suppose each level set $\{f = C\}$ is ϕ-flat. That is

$$(dd^\phi f)(\xi) = 0 \qquad \text{for all } \xi \in G(\phi) \text{ which are tangential to } \{f = C\}. \tag{10.6}$$

Note that at a point $x \in X$, $G(\phi\big|_W) = \{\xi \in G(\phi) : \xi \text{ is tangential to } W\}$. Let $W = \ker df$. Then (10.6) is equivalent to

$$dd^\phi f\big|_W \in \Lambda\left(\phi\big|_W\right)^\perp \tag{10.7}$$

Now f is ϕ-pluriharmonic mod d if

$$dd^\phi f = df \wedge \alpha_f + \sigma_f \qquad \sigma_f \in \Lambda(\phi)^\perp \tag{10.8}$$

or equivalently

$$dd^\phi f\big|_W \in \Lambda(\phi)^\perp\big|_W \tag{10.9}$$

If ϕ is normal, then

$$\Lambda\left(\phi\big|_W\right)^\perp \subset \Lambda(\phi)^\perp\big|_W$$

and (10.7) implies (10.9). ■

PROPOSITION 10.11. *The following calibrations are normal.*

1. *A Kähler or pth power Kähler calibration.*
2. *A Special Lagrangian calibration.*
3. *An associative, coassociative or Cayley calibration.*
4. *A quaternionic calibration.*

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