DUALITY OF POSITIVE CURRENTS AND PLURISUBHARMONIC FUNCTIONS IN CALIBRATED GEOMETRY

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ABSTRACT

Recently the authors showed that there is a robust potential theory attached to any calibrated manifold (X, ϕ). In particular, on X there exist ϕ-plurisubharmonic functions, ϕ-convex domains, ϕ-convex boundaries, etc., all inter-related and having a number of good properties. In this paper we show that, in a strong sense, the plurisubharmonic functions are the polar duals of the ϕ-submanifolds, or more generally, the ϕ-currents studied in the original paper on calibrations. In particular, we establish an analogue of Duval-Sibony Duality which characterizes points in the ϕ-convex hull of a compact set $K \subset X$ in terms of ϕ-positive Green's currents on X and Jensen measures on K. We also characterize boundaries of ϕ-currents entirely in terms of ϕ-plurisubharmonic functions. Specific calibrations are used as examples throughout. Analogues of the Hodge Conjecture in calibrated geometry are considered.

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1. Introduction.

Calibrated geometries are geometries of distinguished minimal varieties determined by a fixed, closed differential form ϕ on a riemannian manifold X. A basic example is that of a Kähler manifold (or more generally a symplectic manifold, with compatible complex structure) where the distinguished submanifolds are the holomorphic curves. However, there exist many other interesting geometries, each carrying a wealth of ϕ-submanifolds, particularly on spaces with special holonomy. These have attracted particular attention in recent years due to their appearance in generalized Donaldson theories and in modern versions of string theory in Physics.

Recently it was shown [HL_5] that a surprisingly large part of classical pluripotential theory can be carried over to any calibrated manifold (X, ϕ). In particular, there is a natural family PSH(X, ϕ) of ϕ-plurisubharmonic functions possessing essentially all of the important properties of the plurisubharmonic functions in complex analysis. For example, the family is closed under composition with convex increasing functions, under taking the maximum of elements in the family, and under taking upper envelopes. Furthermore, there is a notion of ϕ-plurisubharmonic distribution, and if ϕ is *elliptic* (i.e., ϕ involves all the variables at each point – see below), then the cone of these is weakly closed, and every such distribution has a unique upper semi-continuous, $L^1_{\rm loc}$-representative.

There is a notion of ϕ-convexity for (X, ϕ), defined in terms of the ϕ-plurisubharmonic hulls of compact subsets, and characterized by the existence of a proper ϕ-plurisubharmonic exhaustion function. There is also a notion of boundary convexity for domains $\Omega \subset\subset X$, and various aspects of the Levi problem hold. For example, if $\partial\Omega$ is strictly ϕ-convex, then so is Ω (provided X admits some strictly psh-function). Furthermore, on strictly convex domains, the Perron-Bremermann method can be applied to solve the Dirichlet problem for ϕ-partially pluriharmonic functions — the analogue of the homogeneous Monge-Ampère equation on X, [HL_6]. These methods also lead to various notions of capacity in calibrated geometry.

Now it is known that ϕ-plurisubharmonic functions and ϕ-submanifolds are intimately related. For example:

THEOREM 1.1. ([HL_5]). *The restriction of a ϕ-plurisubharmonic function to a ϕ-submanifold M is subharmonic in the induced riemannian metric on M.*

The main thrust of this paper could be summarized in the following

General Principle: *The ϕ-plurisubharmonic functions and ϕ-submanifolds (more specifically, the positive ϕ-currents) are polar duals of one-another.*

We shall see this in §3 where points x in the ϕ-convex hull of a compact set $K \subset X$ are characterized by the existence of a ϕ-positive *Green's current* T and a Jensen measure μ on K satisfying the generalized *Poisson-Jensen equation*

$$\partial_\phi \partial T \;=\; \mu - [x].$$

This is an extension of Duval-Sibony duality [DS] to general calibrated geometries.

We shall also see this in §4 where the boundaries of ϕ-positive currents are characterized.

For the reader's sake, here in the introduction, we now briefly review the basic definitions and results from [HL$_5$] to which this paper is a sequel. A *calibrated manifold* is a pair (X, ϕ) where X is a riemannian manifold and ϕ is an exterior p-form on X which is d-closed and has comass one, i.e.,

$$\phi(\xi) \leq \ 1$$

for every unit simple p-vector ξ on X. The *ϕ-Grassmannian* is the set of *ϕ-planes:*

$$G(\phi) \ \equiv \ \{\xi : \phi(\xi) = 1\}.$$

The ϕ-plurisubharmonic functions on (X, ϕ) are defined by a second order differential operator $\mathcal{H}^\phi : C^\infty(X) \ \rightarrow \ \mathcal{E}^p(X)$, called the *$\phi$-Hessian*, defined by

$$\mathcal{H}^\phi(f) \ = \ \lambda_\phi(\text{Hess} f)$$

where Hess$f$ is the riemannian hessian of f and $\lambda_\phi : \text{End}(TX) \rightarrow \Lambda^p T^*X$ is the bundle map given by $\lambda_\phi(A) = D_{A^*}(\phi)$ where $D_{A^*} : \Lambda^p T^*X \longrightarrow \Lambda^p T^*X$ is the natural extension of $A^* : T^*X \rightarrow T^*X$ as a derivation. When $\nabla\phi = 0$, there is a natural factorization

$$\mathcal{H}^\phi \ = \ dd^\phi$$

where $d^\phi : C^\infty(X) \ \rightarrow \ \mathcal{E}^{p-1}(X)$ is given by

$$d^\phi f \ \equiv \ \nabla f \lrcorner \phi.$$

In general these operators are related by the equation: $\mathcal{H}^\phi f = dd^\phi f - \nabla_{\nabla f}(\phi)$.

A function $f \in C^\infty(X)$ is defined to be **ϕ-plurisubharmonic** if

$$\mathcal{H}^\phi(f)(\xi) \geq 0 \qquad \text{for all } \xi \in G(\phi)$$

The function f is called *strictly ϕ-plurisubharmonic* at a point $x \in X$ if $\mathcal{H}^\phi(f)(\xi) > 0$ for all ϕ-planes ξ at x. It is *ϕ-pluriharmonic* if $\mathcal{H}^\phi(f)(\xi) = 0$ for all ϕ-planes ξ. We denote by PSH(X, ϕ) the convex cone of smooth ϕ-plurisubharmonic functions on X.

When X is a complex manifold with a Kähler form ω, one easily computes that $d^\omega = d^c$, the conjugate differential. In this case, $\mathcal{H}^\omega = dd^\omega = dd^c$ and the ω-planes correspond to the complex lines in TX. Hence, the definitions above coincide with the classical notions of plurisubharmonic and pluriharmonic functions on X.

A fundamental property of the ϕ-Hessian is that for any ϕ-plane ξ, one has

$$\left(\mathcal{H}^\phi f\right)(\xi) \ = \ \text{trace}\left\{\text{Hess} f\big|_\xi\right\}.$$

The first concept addressed in [HL$_5$] is the analogue of pseudoconvexity in complex geometry. Let (X, ϕ) be a calibrated manifold and $K \subset X$ a closed subset. By the **ϕ-convex hull of K** we mean the subset

$$\widehat{K} \ = \ \{x \in X : f(x) \leq \sup_K f \ \text{ for all } f \in \text{PSH(X}, \phi)\}$$

The manifold (X, ϕ) is said to be **ϕ-convex** if $K \subset\subset X \;\Rightarrow\; \widehat{K} \subset\subset X$ for all K.

THEOREM 1.2. ([HL_5]). *A calibrated manifold (X, ϕ) is ϕ-convex if and only if it admits a ϕ-plurisubharmonic proper exhaustion function $f : X \to \mathbf{R}$.*

The manifold (X, ϕ) will be called *strictly ϕ-convex* if it admits an exhaustion function f which is strictly ϕ-plurisubharmonic, and it will be called *strictly ϕ-convex at infinity* if f is strictly ϕ-plurisubharmonic outside of a compact subset.

Note that in complex geometry, ϕ-convex manifolds are Stein, and manifolds which are ϕ-convex at infinity are strongly pseudoconvex. In the latter case there is a distinguished compact subset consisting of exceptional subvarieties. This set has the following general analogue.

The **core** of X is the set of points $x \in X$ with the property that no $f \in \mathrm{PSH(X}, \phi)$ is strictly ϕ-plurisubharmonic at x.

THEOREM 1.3. ([HL_5]). *Suppose X is ϕ-convex. Then X is strictly ϕ-convex at infinity if and only if* $\mathrm{Core}(X)$ *is compact, and X is strictly ϕ-convex if and only if* $\mathrm{Core}(X) = \emptyset$.

A very general construction of strictly ϕ-convex manifolds is given in [HL_5, §6]. The construction is based on the notion of *ϕ-free submanifolds*, which directly generalize the totally real submanifolds in complex geometry.

We now sketch the contents of this paper.

ϕ-POSITIVE CURRENTS.

In §2 we review the theory of ϕ-positive currents originally introduced in [HL_3]. We recall that a p-dimensional current T is called **ϕ-positive** if it is representable by integration and its generalized tangent p-vector

$$\overrightarrow{T} \in \mathrm{ConvexHull}(G(\phi)) \qquad \|T\| - a.e.$$

where $\|T\|$ denotes the total variation measure of T. Examples include ϕ-submanifolds and, more generally, rectifiable ϕ-currents. By Almgren's Theorem [A] we know that rectifiable ϕ-currents T with $dT = 0$ are regular, that is, given by integration over ϕ-submanifolds with positive integer multiplicities, outside a closed subset of Hausdorff dimension $p - 2$.

ϕ-Positive currents generalize the positive currents in complex geometry, and d-closed rectifiable ϕ-currents generalize positive holomorphic chains.

If T is a ϕ-positive current (with compact support), then

$$\mathrm{supp}\, T \;\subset\; \widehat{\mathrm{supp}(dT)} \cup \mathrm{Core}(X). \tag{1.1}$$

In particular, if $dT = 0$, then $\mathrm{supp}T \subset \mathrm{Core}(X)$, and if X is strictly ϕ-convex (i.e., $\mathrm{Core}(X) = \emptyset$), then there exist no d-closed ϕ-positive currents with compact support on X.

Section 2 summarizes the known facts concerning ϕ-positive currents. These include compactness theorems, regularity theorems, mass-minimizing properties, monotonicity properties, and dual characterizations.

THE SUPPORT LEMMA.

Assume for now that the calibration ϕ is parallel, and consider the adjoint of the operator $dd^\phi : \mathcal{E}^0(X) \to \mathcal{E}^p(X)$ which can be written as

$$\partial_\phi \partial : \mathcal{E}'_p(X) \longrightarrow \mathcal{E}'_0(X)$$

where $\partial : \mathcal{E}'_p(X) \to \mathcal{E}'_{p-1}(X)$ denotes the usual adjoint of $d : \mathcal{E}^{p-1}(X) \to \mathcal{E}^p(X)$ and $\partial_\phi : \mathcal{E}'_{p-1}(X) \longrightarrow \mathcal{E}'_0(X)$ is the adjoint of d^ϕ, defined by $(\partial_\phi R)(f) \equiv R(d^\phi f)$.

Positive currents T with the property: $\partial_\phi \partial T \leq 0$ (i.e., $\partial_\phi \partial T$ is a non-positive measure), satisfy a version of (1.1) above.

LEMMA 3.2. *Suppose T is a ϕ-positive current with compact support on X which satisfies $\partial_\phi \partial T \leq 0$ outside a compact subset $K \subset X$. Then*

$$\operatorname{supp} T \subset \widehat{K} \cup \operatorname{Core}(X).$$

In particular, if $\partial_\phi \partial T \leq 0$ on X, then $\operatorname{supp} T \subset \operatorname{Core}(X)$.

Another consequence is the following. Suppose X is strictly ϕ-convex. If T is a ϕ-positive current with $\partial_\phi \partial T \leq 0$ on $X - K$, then $\operatorname{supp} T \subset \widehat{K}$. In fact, it turns out that the points $x \in \widehat{K}$ can be characterized in terms of certain ϕ-positive currents T which satisfy $\partial_\phi \partial T = -[x]$ in $X - K$. We discuss this next in greater detail.

DUVAL - SIBONY DUALITY.

Points in the ϕ-convex hull of a compact set $K \subset X$ have a useful characterization in terms of ϕ-positive currents and certain Poisson-Jensen measures. The following results generalize work of Duval and Sibony [DS] in the complex case. They remain valid (as does Lemma 3.2 above) when ϕ is not parallel if the operator $\partial_\phi \partial$ is replaced with $\mathcal{H}_\phi$.

Let $K \subset X$ be a compact subset and x a point in $X - K$. By a **Green's current for** (K, x) we mean a ϕ-positive current T which satisfies

$$\partial_\phi \partial T = \mu - [x] \tag{1.2}$$

where μ is a probability measure with support on K and $[x]$ denotes the Dirac measure at x. In this case μ is called a **Poisson-Jensen** measure for (K, x). By the remarks above we see that $x \in \widehat{K}$. In fact we have the following.

THEOREM 3.8. *Suppose ϕ is parallel and X is strictly ϕ-convex. Let $K \subset X$ be a compact subset and $x \in X - K$. Then there exists a Green's current for (K, x) if and only if $x \in \widehat{K}$.*

We note that if $M \subset X$ is a compact ϕ-submanifold with boundary, and if G_x is the Green's function for the riemannian laplacian on M with singularity at $x \in M - \partial M$, then $\partial_\phi \partial (G_x[M]) = \mu - [x]$ for a probability measure μ on ∂M.

As a application we obtain the following approximation result. A domain $\Omega \subset X$ is said to be **ϕ-convex relative to** X if $K \subset\subset \Omega \Rightarrow \widehat{K}_X \subset\subset \Omega$.

PROPOSITION 3.16. *Suppose ϕ is parallel and X is strictly ϕ-convex. An open subset $\Omega \subset X$ is ϕ-convex relative to X if and only if* $\mathrm{PSH}(\mathrm{X}, \phi)$ *is dense in* $\mathrm{PSH}(\Omega, \phi)$.

Boundary Duality.

A very natural question in calibrated geometry is the following: Given a compact oriented submanifold $\Gamma \subset X$ of dimension $p-1$, when does there exist a compact ϕ-submanifold M with $\partial M = \Gamma$? A companion question is: Given a compactly supported current S of dimension $p-1$ in X, when does there exist a compactly supported ϕ-positive current T with $\partial T = \Gamma$? For this second question there is a complete answer when X is strictly ϕ-convex and ϕ is exact.

Theorem 4.1. *Fix (X, ϕ) as above, and consider a current $S \in \mathcal{E}'_{p-1}(X)$. Then $S = \partial T$ for some ϕ-positive current $T \in \mathcal{E}'_p(X)$ if and only if*

$$\int_S \alpha \;\geq\; 0 \qquad \text{for all } \alpha \in \mathcal{E}^{p-1}(X) \quad \text{such that } d\alpha \text{ is } \Lambda^+(\phi)-\text{positive}$$

Note . $\Lambda^+(\phi)$-positive means that $d\alpha(\xi) \geq 0$ for all $\xi \in G(\phi)$.

There is a similar result for compact manifolds (X, ϕ) with no condition on ϕ.

Theorem 4.3. *Suppose (X, ϕ) is a compact calibrated manifold. Fix $S \in \mathcal{E}'_{p-1}(X)$ and $\lambda > 0$. Then the following are equivalent.*

(i) There exists a $\Lambda_+(\phi)$-positive current $T \in \mathcal{E}'_p(X)$ with $S = \partial T$ and $M(T) \leq \lambda$.

(ii) $\int_S \alpha \;\geq\; -\lambda$ for all $\alpha \in \mathcal{E}^{p-1}(X)$ such that $d\alpha + \phi$ is $\Lambda^+(\phi)$-positive.

There is also a result for non-compact boundaries on general manifolds (X, ϕ) (Theorem 4.4 below). Versions of these theorems in Kähler geometry appear in [HL$_4$].

ϕ-Flat Hypersurfaces.

In Section 5 we expand the notion of ϕ-pluriharmonic functions to include functions f which are ϕ-pluriharmonic modulo the ideal generated by df. In most interesting geometries these functions are characterized by the fact that their level sets are **ϕ-flat**, i.e., the trace of the second fundamental form on all tangential ϕ-planes is zero. These functions are important for the boundary problem. If f is such a function defined in a neighborhood of a compact ϕ-submanifold with boundary $M \subset X$, then

$$\inf_{\partial M} f \;\leq\; f(x) \;\leq\; \sup_{\partial M} f \qquad \text{for } \; x \in M.$$

Generalized Hodge Manifolds.

In Section 6 we discuss analogues of Hodge manifolds in the general calibrated setting. We also examine various analogues of the Hodge Conjecture in these spaces.

Note. For simplicity we always assume the manifold X to be oriented. Following de Rham we denote by $\mathcal{E}^p(X)$ the space of smooth differential forms of degree p on X, and by $\mathcal{E}'_p(X)$ its topological dual space of currents of dimension p with compact support on X

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2. Positive Currents in Calibrated Geometries.

The important classical notion of a positive current on a complex manifold has an analogue on any calibrated manifold. This concept was introduced in section II of [HL_3]. In this Section we review that material with some of the terminology and notation updated.

Suppose ϕ is a calibration on a riemannian manifold X. Let $p = \deg\phi$ and $n = \dim X$. The ϕ-Grassmannian, denoted $G(\phi)$, consists of the unit simple vectors $\xi \in G(p, TX) \subset \Lambda_p TX$ with $\phi(\xi) = 1$, i.e., the ϕ-planes.

On a calibrated manifold (X, ϕ) the three concepts we wish to discuss are:

a) ϕ-submanifolds,

b) rectifiable ϕ-currents, and ϕ-cycles

c) ϕ-positive currents.

A ϕ-submanifold is, of course, a smooth oriented submanifold M whose oriented tangent space is a ϕ-plane at every point, i.e., $\overrightarrow{M}_x \equiv \overrightarrow{T}_x M \in G(\phi)$ for all $x \in M$.

Suppose T is a locally rectifiable p-dimensional current (cf. [F_1]) on X. Then its generalized tangent space is a unit simple vector $\overrightarrow{T} \in G(p, TX)$ at $\|T\|$ almost every point, where $\|T\|$ denotes the generalized volume measure associated with T.

DEFINITION 2.1. A **rectifiable ϕ-current** is a locally rectifiable current T with $\overrightarrow{T} \in G(\phi)$ for $\|T\|$- a.a. points in X. A **ϕ-cycle** is a rectifiable ϕ-current which is d-closed.

REMARK 2.2. We shall see below (Theorem 2.10) that ϕ-cycles always have a particularly nice local structure. The strongest result of this kind occurs in the Kähler case, with $\phi = \omega^p/p!$, where a theorem of J. King [K] states that each ϕ-cycle is a positive holomorphic cycle, i.e., a locally finite sum of p-dimensional complex analytic subvarieties with positive integer coefficients. Results about the singular structure in the special Lagrangian case have been obtained by Joyce, Haskins, Kapouleas, Pacini, and others (cf. [J_*], [Ha_*], [HaK], [HaP]).

REMARK 2.3. On a general calibrated manifold (X, ϕ) one can also consider d-closed rectifiable currents T with $\pm\overrightarrow{T} \in G(\phi)$ for $\|T\|$-a.a. points. In the Kähler case T must be a holomorphic chain by results in [HS], [S] and [Alex]. However, nothing is known about the structure of such currents for any of the other standard calibrations.

An understanding of the definition of a ϕ-positive current is a little more complicated.

Recall (cf. [F_1]) that a current T is **representable by integration** if T has measure coefficients when expressed as a generalized differential form. Equivalently, the mass norm $M_K(T)$ of T on each compact set K, is finite. Associated with such a current T is a Radon measure $\|T\|$ and a generalized tangent space $\overrightarrow{T}_x \in \Lambda_p T_x X$ defined for $\|T\|$-a.a. points x. Recall that each $\overrightarrow{T}_x$ has mass norm one. For any p-form α with compact support

$$T(\alpha) \;=\; \int \alpha(\overrightarrow{T})\, d\|T\| \tag{2.1}$$

DEFINITION 2.4. At each point $x \in X$ let $\Lambda(\phi)$ denote the span of $G(\phi) \subset \Lambda_p TX$, and let

$$\Lambda_+(\phi) \;\subset\; \Lambda(\phi)$$

denote the convex cone on $G(\phi)$ with vertex the origin. The p-vectors $\xi \in \Lambda_+(\phi)$ will be called $\Lambda_+(\phi)$**-positive.**

Note that $\Lambda_+(\phi)$ is just the cone on $\operatorname{ch} G(\phi)$, the convex hull of the ϕ-Grassmannian.

The following Lemma is needed for a robust understanding of the definition of a ϕ-positive current.

LEMMA 2.5. *The following conditions are equivalent:*

1) $\overrightarrow{T} \in \Lambda_+(\phi)$ $\quad$ $\|T\|$-*a.e.*

2) $\overrightarrow{T} \in \operatorname{ch} G(\phi)$ $\quad$ $\|T\|$-*a.e.*

3) $\phi(\overrightarrow{T}) = 1$ $\quad$ $\|T\|$-*a.e.*

The proof is provided later.

DEFINITION 2.6. A ϕ**-positive current** is a p-dimensional current T which is representable by integration and for which the equivalent conditions of Lemma 2.5 are satisfied.

By Condition 3) of 2.5 and (2.1)

$$T(\phi) \;=\; \int \phi(\overrightarrow{T})\, d\|T\| \;= M(T) \tag{2.2}$$

for all ϕ-positive currents T. This fact has important implications.

PROPOSITION 2.7. *Suppose T is a compactly supported p-dimensional current which is representable by integration. Then*

$$T(\phi) \;\leq\; M(T)$$

with equality if and only if T is a ϕ-positive current.

Consequently, any ϕ-positive current T_0 with compact support is homologically mass-minimizing, i.e.,

$$M(T_0) \;\leq\; M(T) \tag{2.3}$$

for any $T = T_0 + dS$ where S is a $(p+1)$-dimensional current with compact support. Furthermore, equality holds in (2.3) if and only if T is also ϕ-positive.

Proof. Note that $T(\phi) = \int \phi(\overrightarrow{T}) d\|T\| \leq \int d\|T\| = M(T)$ since $\phi(\overrightarrow{T}) \leq \|\overrightarrow{T}\| = 1$. Equality occurs if and only if $\phi(\overrightarrow{T}) = 1$ ($\|T\|$-a.e.). This is Condition 3) in Lemma 2.5. The second assertion follows from the fact that $T_0(\phi) = T(\phi)$. ∎

The reader may note that only Condition 3) of 2.5 was used in this proof. However, it is Conditions 1) and 2) which give a genuine understanding of ϕ-positive currents.

The closed currents which are ϕ-positive have a *monotonicity property* which says that the function $\|T\|(B_x(r))/r^p$ is monotone increasing in r. This implies that the density of $\|T\|$ is well-defined everywhere and upper semi-continuous. This is disussed in detail in [HL_3].

Deep results in geometric measure theory have important applications here.

THEOREM 2.8. *Fix a compact set $K \subset X$ and a constant $c > 0$. Then the set $\mathcal{P}(\phi, K, c)$ of ϕ-positive currents T with $M(T) \leq c$ and $\mathrm{supp}(T) \subseteq K$ is compact in the weak topology.*

Proof. Proposition 2.7 easily implies that a weak limit of ϕ-positive currents is ϕ-positive. The result then follows from standard compactness theorems for measures. ∎

THEOREM 2.9. *Fix a compact set $K \subset X$ and a constant $c > 0$. Then the set $\mathcal{R}(\phi, K, c)$ of rectifiable ϕ-currents T with rectifiable boundaries, such that $M(T) + M(\partial T) \leq c$ and $\mathrm{supp}(T) \subseteq K$, is compact in the weak topology.*

Proof. This follows from Proposition 2.7 and the Federer-Fleming weak compactness theorem for rectifiable currents [FF], [F_1]. ∎

THEOREM 2.10. *Let T be a ϕ-cycle on X. Then there is a closed subset $\Sigma \subset \mathrm{supp}(T)$ of Hausdorff dimension $p-2$ such that $M \equiv \mathrm{supp}(T) - \Sigma$ is a proper ϕ-submanifold with finite volume in $X - \Sigma$ and*

$$T = \sum_k n_k [M_k]$$

where the n_k's are positive integers and the M_k's are the connected components of M.

Proof. This is a direct consequence of Almgren's big regularity theorem [A]. ∎

We now present the dual characterization of ϕ-positive currents.

Suppose $\phi \in \Lambda^p V$ is a calibration on an inner product space V. Let $\Lambda^+(\phi) \subset \Lambda^p V$ denote the polar cone of $\Lambda_+(\phi) \subset \Lambda_p V$. By definition this is the set of $\alpha \in \Lambda^p V$ such that $\alpha(\xi) \geq 0$ for all $\xi \in \Lambda_+(\phi)$, or equivalently,

$$\Lambda^+(\phi) = \{\alpha \in \Lambda^p V : \alpha(\xi) \geq 0 \ \text{ for all } \xi \in G(\phi)\}.$$

A p-form $\alpha \in \Lambda^p V$ is said to be $\Lambda^+(\phi)$**-positive** if $\alpha \in \Lambda^+(\phi)$, and **strictly** $\Lambda^+(\phi)$**-positive** if $\alpha(\xi) > 0$ for all $\xi \in G(\phi)$ (or equivalently, α belongs to the interior of $\Lambda^+(\phi)$).

REMARK 2.11. Note that ϕ itself is strictly $\Lambda^+(\phi)$-positive, i.e., an interior point of the cone $\Lambda^+(\phi) \subset \Lambda^p V$. If a closed convex cone has one interior point, then there exists a basis for the vector space consisting of interior points. Consequently, $\Lambda^p V$ has a basis of strictly $\Lambda^+(\phi)$-positive p-forms.

If (X, ϕ) is a calibrated manifold, the considerations and definitions above apply to the tangent space $V = T_x X$ at each point $x \in X$.

DEFINITION 2.12. A smooth p-form α on X is $\Lambda^+(\phi)$**-positive (strictly** $\Lambda^+(\phi)$**-positive)** if α is $\Lambda^+(\phi)$-positive (strictly $\Lambda^+(\phi)$-positive) at each point $x \in X$.

DEFINITION 2.13. A (twisted) current T of dimension p is said to be $\Lambda_+(\phi)$**-positive** if

$$T(\alpha) \geq 0$$

for all $\Lambda^+(\phi)$-positive p-forms α with compact support.

THEOREM 2.14. *A current T is $\Lambda_+(\phi)$-positive if and only if it is ϕ-positive.*

This result is proven in [HL$_3$, Prop. A.2 and Remark on page 83]. However, for the sake of completeness we include a proof.

Proof. First assume that T is representable by integration. Then from (2.1) we see that T is $\Lambda_+(\phi)$-positive if and only if

$$T(g\alpha) \;=\; \int g\alpha(\overrightarrow{T})d\|T\| \;\geq\; 0$$

for all functions $g \geq 0$ and all compactly supported $\Lambda^+(\phi)$-positive p-forms α. Equivalently, each measure $\alpha(\overrightarrow{T})\|T\|$ is ≥ 0 for the same set of p-forms. In turn, this is equivalent to

$$\alpha(\overrightarrow{T}) \;\geq\; 0 \qquad \|T\|-\text{a.e..}$$

for all compactly supported $\Lambda^+(\phi)$-positive p-forms. Finally, by the Bipolar Theorem [S] this last condition is equivalent to the Condition 1) in the Lemma 2.5.

It remains to prove that if T is $\Lambda_+(\phi)$-positive, then T is representable by integration. For this we may assume that T has compact support in a small neighborhood U of X, and by Remark 2.11, we may choose a frame $\alpha_1, ..., \alpha_N$ of smooth p-forms which are strictly $\Lambda^+(\phi)$-positive on U. Let $\xi_1, ..., \xi_N$ denote the dual frame of p-vector fields, i.e., $(\alpha_i, \xi_j) \equiv \delta_{ij}$ on U. Every such current T has a unique representation as $T = \sum_{j=1}^N u_j\xi_j$ with $u_j \in \mathcal{D}'(U)$ a distribution defined by $u_j(f) \equiv T(f\alpha_j)$ for all test functions f. (Note that $\alpha = \sum_j f_j\alpha_j$ implies that $T(\alpha) = \sum_j T(f_j\alpha_j) = \sum_j u_j(f_j) = (\sum_j u_j\xi_j)(\sum_i f_i\alpha_i) = (\sum_j u_j\xi_j)(\alpha)$.) Since T is $\Lambda_+(\phi)$-positive, each u_j satisfies

$$u_j(f) \;\geq\; 0 \qquad \text{for all} \;\; f \geq 0.$$

By the Riesz Representation Theorem this proves that each u_j is a measure. Therefore $T = \sum_j u_j\xi_j$ is representable by integration. ∎

Now we give the proof of Lemma 2.5. As before $\phi \in \Lambda^pV$ is a calibration. Let K denote the unit mass ball in Λ_pV, that is, the convex hull of the Grassmannian $G(p, V) \subset \Lambda_pV$.

LEMMA 2.15.

$$\text{ch}\, G(\phi) \;=\; \{\phi = 1\} \cap \partial K \;=\; \Lambda_+(\phi) \cap \partial K$$

Proof. Note that:

a) $\text{ch}\, G(\phi) \subset \{\phi = 1\}$ since $G(\phi) \subset \{\phi = 1\}$.

b) $\text{ch}\, G(\phi) \subset K$ since $G(\phi) \subset G(p, V)$.

c) $K \cap \{\phi = 1\} \;=\; \partial K \cap \{\phi = 1\}$ since $K \subset \{\phi \leq 1\}$.

Hence, $\text{ch}\, G(\phi) \subset \{\phi = 1\} \cap \partial K$.

Conversely, suppose $\phi(\xi) = 1$ and $\|\xi\| = 1$. Since $\xi \in K$,

$$\xi \;=\; \sum_j \lambda_j\xi_j \qquad \text{with each} \;\; \xi_j \in G(p, V), \;\text{each}\; \lambda_j > 0, \;\text{and}\; \sum_j \lambda_j = 1.$$

Hence, $1 = \phi(\xi) = \sum \lambda_j \phi(\xi_j) \le \sum \lambda_j = 1$ forcing each $\phi(\xi_j) = 1$ and therefore each $\xi_j \in G(\phi)$. This proves the first equality.

We have shown $\operatorname{ch} G(\phi) \subset \partial K$, and by definition, $\operatorname{ch} G(\phi) \subset \Lambda_+(\phi)$. Hence, $\operatorname{ch} G(\phi) \subset \Lambda^+(\phi) \cap \partial K$. Finally, suppose $\xi \in \partial K \cap \Lambda_+(\phi)$, i.e., $\|\xi\| = 1$ and there exists some $\lambda > 0$ such that $\lambda\xi \in \operatorname{ch} G(\phi)$. We have already shown that $\operatorname{ch} G(\phi) \subset \partial K$, therefore $\|\lambda\xi\| = 1$, and hence $\lambda = 1$ proving that $\xi \in \operatorname{ch} G(\phi)$. ∎

COROLLARY 2.16. *Suppose $\xi \in \Lambda_p V$ has mass norm $\|\xi\| = 1$. Then*

$$\xi \in \Lambda_+(\phi) \quad \Longleftrightarrow \quad \phi(\xi) = 1 \quad \Longleftrightarrow \quad \xi \in \operatorname{ch} G(\phi).$$

This is the required restatement of Lemma 2.5

REMARK . Note also that the equation

$$G(\phi) \;=\; G(p, V) \cap \Lambda_+(\phi) \tag{2.4}$$

follows easily from Lemma 2.15. This clarifies the notion of a rectifiable ϕ-current. Namely, this proves that a rectifiable current is $\Lambda_+(\phi)$-positive if and only if it is a rectifiable ϕ-current, and eliminates a potential conflict in terminology.

We finish this section with a lemma and corollary that are often useful.

Note that a form $\alpha \in \Lambda^+(\phi)$ lies on the topological boundary $\partial\Lambda^+(\phi)$ of $\Lambda^+(\phi)$ if and only if there exists some $\xi \in G(\phi)$ with $\alpha(\xi) = 0$.

LEMMA 2.17. *For any $\psi \in \Lambda^p V$*

$$\phi - \psi \in \partial\Lambda^+(\phi) \quad \Longleftrightarrow \quad \psi \le 1 \text{ on } G(\phi) \;\text{ and } \psi(\xi) = 1 \text{ for some } \xi \in G(\phi)$$

Proof. By definition $\phi - \psi \in \Lambda^+(\phi)$ if and only if $\phi(\xi) - \psi(\xi) = 1 - \psi(\xi) \ge 0$ for all $\xi \in G(\phi)$. As remarked above $\phi - \psi$ lies in the boundary of $\Lambda^+(\phi)$ iff $\phi(\xi) - \psi(\xi) = 1 - \psi(\xi) = 0$ at some point $\xi \in G(\phi)$. ∎

COROLLARY 2.18. *For each unit vector $e \in V$, let $\phi_e = e \lrcorner (e \wedge \phi) \cong \phi\big|_W$ where $W \equiv (\operatorname{span} e)^\perp$. Then $\phi \in \Lambda^+(\phi)$ and*

$$\phi_e \in \partial\Lambda^+(\phi) \qquad \Longleftrightarrow \qquad e \in \operatorname{span} \xi \;\text{ for some }\; \xi \in G(\phi).$$

Proof. Note first that $\phi_e = e \lrcorner (e \wedge \phi) = \phi - e \wedge (e \lrcorner \phi)$. Now write $e = a + b$ with $a \in \operatorname{span} \xi$ and $b \perp \operatorname{span} \xi$. Then

$$e \wedge (e \lrcorner \phi)(\xi) = |a|^2. \tag{2.5}$$

because $e \wedge (e \lrcorner \phi)(\xi) = \phi(e \wedge (e \lrcorner \xi)) = \phi((a+b) \wedge (a \lrcorner \xi)) = \phi(|a|^2 \xi) + \phi(b \wedge (a \lrcorner \xi)) = |a|^2$ since $\phi(b \wedge (a \lrcorner \xi)) = 0$ by the First Cousin Principle [HL$_5$, Lemma 2.4] and $\phi(\xi) = 1$.

Now by Lemma 2.17, $\phi_e \in \partial\Lambda^+(\phi)$ if and only if there exists $\xi \in G(\phi)$ with $|a| = 1$, that is, with $e = a \in \operatorname{span} \xi$. ∎

REMARK . Both $df \wedge (\nabla f \lrcorner \phi) = df \wedge d^\phi f$ and $\nabla f \lrcorner (df \wedge \phi) = \|\nabla f\|^2 \phi - df \wedge d^\phi f$ take values in $\Lambda^+(\phi) \subset \Lambda^p T^*X$. Furthermore,

1) $df \wedge d^\phi f \in \partial\Lambda^+(\phi) \iff \exists\, \xi \in G(\phi)$ tangential to the level set of f.

2) $\nabla f \lrcorner (df \wedge \phi) \in \partial\Lambda^+(\phi) \iff \exists\, \xi \in G(\phi)$ with $\nabla f \in \operatorname{span} \xi$.

Note that for some calibrations, condition 2) is true for all f, i.e., given a vector $n \in V$, there always exists a p-vector $\xi \in G(\phi)$ with $n \in \operatorname{span} \xi$.

3. Duality with Plurisubharmonic Functions.

As noted in the introduction, the ϕ-plurisubharmonic functions represent, in a general sense, the polar dual of the ϕ-positive currents. More specifically, there are many situations where the polar dual of an interesting family of ϕ-positive currents is some explicit and useful family of ϕ-plurisubharmonic functions. In this section we examine an example of this phenomenon. We extend the fundamental duality results of Duval-Sibony [DS] in complex geometry, to a general calibrated manifold (X, ϕ). The Duval-Sibony Duality Theorem involves plurisubharmonic functions, pseudoconvex hulls, positive currents and Poisson-Jensen formulas.

DEFINITION 3.1. Suppose R is a $(p-1)$-dimensional current on X. The operator ∂_ϕ is defined by

$$(\partial_\phi R)(f) \;=\; R(d^\phi f)$$

for all $f \in C^\infty_{\text{cpt}}(X)$.

In other words, $\partial_\phi : \mathcal{E}'_{p-1}(X) \longrightarrow \mathcal{E}'_0(X)$ is the adjoint of $d^\phi : \mathcal{E}^0(X) \longrightarrow \mathcal{E}^{p-1}(X)$. Let $\partial : \mathcal{E}'_p(X) \longrightarrow \mathcal{E}'_{p-1}(X)$ denote the **boundary operator** on currents. This is the adjoint of $d : \mathcal{E}^{p-1}(X) \longrightarrow \mathcal{E}^p(X)$ and is related to the deRham differential on currents by $\partial = (-1)^{n-p} d$. The adjoint of $dd^\phi : \mathcal{E}^0(X) \longrightarrow \mathcal{E}^p(X)$ is the operator

$$\partial_\phi \partial : \mathcal{E}'_p(X) \;\longrightarrow\; \mathcal{E}'_0(X) \tag{3.1}$$

REMARK . Throughout the remainder of this section we assume that (X, ϕ) is a non-compact connected calibrated manifold. We also assume that ϕ is parallel. This assumption enables us to use the operator $\partial_\phi \partial$, but it is not necessary. We leave it to the reader to verify that all of the results of this section extend to the case where ϕ is not parallel by replacing the operator $\partial_\phi \partial$ with $\mathcal{H}_\phi : \mathcal{E}'_p(X) \to \mathcal{E}'_0(X)$, the adjoint of $\mathcal{H}^\phi : \mathcal{E}^0(X) \to \mathcal{E}^p(X)$. Of course, $\mathcal{H}_\phi$ is defined by $(\mathcal{H}_\phi(T))(f) = T(\mathcal{H}^\phi(f))$ for all $f \in C^\infty_{\text{cpt}}(X)$.

LEMMA 3.2. (The Support Lemma). *Suppose K is a compact subset of X. Suppose T is a $\Lambda_+(\phi)$-positive current with compact support in X. If $\partial_\phi \partial T$ is ≤ 0 (a non-positive measure) on $X - \widehat{K}$, then* $\operatorname{supp} T \subset \widehat{K} \cup \operatorname{Core}(X)$.

Proof. The Note following Lemma 4.2 in [HL$_5$] states that for each $x \notin \widehat{K} \cup \operatorname{Core}(X)$ there exists a non-negative ϕ-plurisubharmonic function f on X which is identically zero on a neighborhood of $\widehat{K}$ and strict at x. Since f is strict at x, there exists a small ball B about

x and $\epsilon > 0$ so that $dd^\phi f - \epsilon\phi$ is $\Lambda^+(\phi)$-positive at each point in B. By equation (2.2), $M(\chi_B T) = (\chi_B T, \phi)$. Therefore, $\epsilon M(\chi_B T) = (\chi_B T, \epsilon\phi) \leq (\chi_B T, dd^\phi f) \leq (T, dd^\phi f) = (\partial_\phi \partial T, f) \leq 0$. This proves that $M(\chi_B T) = 0$ and hence $\operatorname{supp} T \subset \widehat{K} \cup \operatorname{Core}(X)$. ∎

The case where $K = \emptyset$ is a generalization of Proposition 4.13 in [HL$_5$] .

COROLLARY 3.3. *If T is a $\Lambda_+(\phi)$-positive current with compact support and $\partial_\phi \partial T \leq 0$, then*

$$\operatorname{supp} T \ \subset \ \operatorname{Core}(X).$$

When $\operatorname{Core}(X) = \emptyset$ we have

COROLLARY 3.4. *Suppose (X, ϕ) is strictly convex and K is ϕ-convex. Suppose T is $\Lambda_+(\phi)$-positive with compact support. If $\operatorname{supp}\{\partial_\phi \partial T\} \subset K$, then $\operatorname{supp} T \subset K$. In particular, there are no $\Lambda_+(\phi)$-positive currents which are compactly supported without boundary on X.*

.

We now introduce the notion of a Green's current on (X, ϕ). We shall begin with a description of the "classical" case. Let $M \subset X$ be an p-dimensional oriented submanifold having no compact components, and fix a compact domain $D \subset\subset M$ with smooth boundary ∂D. (One can assume that M is just D with an external collar added to it.) Let G_x denote the Green's function for $(D, \partial D)$ with singularity at $x \in \operatorname{Int} D$. Let μ_x denote harmonic measure (i.e., the Poisson kernel) on ∂D and let $[x]$ denote the point-mass measure at x. Extend G_x to a continuous function on M (also denoted by G_x) by defining it to be zero on $M - D$. Then

$$*_M \Delta_M G_x \ = \ \mu_x - [x] \qquad \text{on} \ \ M. \tag{3.2}$$

If M is a ϕ-submanifold of the calibrated manifold (X, ϕ), this equation can be reformulated as a current $\partial_\phi \partial$-equation on X.

LEMMA 3.5. *Suppose M is a ϕ-submanifold of X, and that $u \in \mathcal{D}'^0_{\text{cpt}}(M)$ is a generalized function with compact support on M. Then*

$$\partial_\phi \partial(u[M]) \ = \ (*_M \Delta_M u)[M].$$

Proof. Consider the inclusion map $i : M \hookrightarrow X$. Then, by definition, $u[M] = i_* u$ and $(*_M \Delta_M u)[M] = i_*(*_M \Delta_M u)$. For any test function f on X we have $((\partial_\phi \partial)(i_* u), f) \ = \ (i_* u, dd^\phi f) \ = \ (u, i^*(dd^\phi f))_M$ where $(\cdot, \cdot)_M$ denotes the pairing of functions with currents on M. Equation (1.2) of [HL$_5$] states that $i^*(dd^\phi f) \ = \ *_M \Delta_M (i^* f)$ if M is a ϕ-submanifold. Finally, $(u, *_M \Delta_M (i^* f))_M \ = \ (*_M \Delta_M u, i^* f)_M \ = \ (i_*(*_M \Delta_M u), f)_X$. ∎

COROLLARY 3.6. *Suppose M is a ϕ-submanifold and G_x is defined as above. Then*

$$\partial_\phi \partial(G_x[M]) \ = \ \mu_x - [x]$$

as a current equation on X.

We now generalize these currents $G_x[M]$. Assume K is a compact subset of X, and let $\mathcal{M}_K$ denote the set of probability measures with support in K.

DEFINITION 3.7. If T_x is a $\Lambda_+(\phi)$-positive current with compact support and T_x satisfies:

$$\partial_\phi \partial T_x \;=\; \mu_x - [x] \tag{3.3}$$

with $\mu_x \in \mathcal{M}_K$, then: T_x is a **Green's current for** (K,x), μ_x is a **Poisson-Jensen measure** for (K,x), and the equation (3.3) is the **Poisson-Jensen equation.**

THEOREM 3.8. *Suppose X is strictly ϕ-convex, K is a compact subset of X, and $x \in X - K$. Then there exists a Green's current T_x for (K,x) if and only if $x \in \widehat{K}$.*

To prove this we begin with the following.

PROPOSITION 3.9. *Suppose (X,ϕ) is noncompact calibrated manifold. If there exists a Green's current for (K,x), then $x \in \widehat{K}$.*

Proof. Since $\partial_\phi \partial T_x \;=\; \mu_x - [x]$, we have $\int f\mu_x - f(x) \;=\; (T_x, dd^\phi f)$ for all $f \in C^\infty(X)$. If f is ϕ-plurisubharmonic , this implies that $f(x) \;\le\; \int f\mu_x \;\le\; \sup_K f$, since $\mu_x \in \mathcal{M}_K$. Thus $x \in \widehat{K}$. ∎

The set $\mathcal{P}_X \equiv \mathrm{PSH}(X,\phi) \subset C^\infty(X)$ of all ϕ-plurisubharmonic functions on X is clearly a closed convex cone in $C^\infty(X)$. Let

$$C_X \;\equiv\; \{u \in \mathcal{E}'_0(X) : u = \partial_\phi \partial T \text{ for some } \Lambda_+(\phi)\text{−positive } T \in \mathcal{E}'_p(X)\}. \tag{3.4}$$

This is a convex cone in $\mathcal{D}'_{0,\mathrm{cpt}}(X)$. Let's abbreviate $\mathcal{P} = \mathcal{P}_X$ and $C = C_X$.

LEMMA 3.10. *Suppose X is non-compact with calibration ϕ. Then $\mathcal{P}$ is the polar of C, that is,*

$$\mathcal{P} \;=\; C^0 \;\equiv\; \{f \in C^\infty(X) \;:\; (u,f) \ge 0 \;\; \forall u \in C\}.$$

Proof. Consider $u = \partial_\phi \partial(\delta_x \xi)$, with $\xi \in G_x(\phi)$. Clearly $u \in C$. If $f \in C^\infty(X)$ belongs to C^0, then $0 \le (u,f) = (\partial_\phi \partial(\delta_x \xi), f) \;=\; (\delta_x \xi, dd^\phi f) \;=\; (dd^\phi f)_x(\xi)$. Hence $C^0 \subseteq \mathcal{P}$.

Conversely, if $f \in \mathcal{P}$, then for all $u \in C$, $(u,f) \;=\; (\partial_\phi \partial T, f) \;=\; (T, dd^\phi f) \ge 0$, since T is $\Lambda_+(\phi)$-positive. This proves that $\mathcal{P} \subseteq C^0$. ∎

LEMMA 3.11. *If X is strictly ϕ-convex, then the convex cone $C \subset \mathcal{D}'_{0,\mathrm{cpt}}(X)$ is closed.*

Proof. Let $\mathcal{E}'_{0,K}(X) = \{T \in \mathcal{E}'_0(X) : \mathrm{supp}\, T \subset K\}$. It suffices to show that $C \cap \mathcal{E}'_{0,K}(X)$ is closed for an exhaustive family of compact subsets $K \subset X$. We may assume K is ϕ-convex. Suppose u_j converges to u in $C \cap \mathcal{E}'_{0,K}(X)$ with each $u_j \in C$, i.e., $u_j = \partial_\phi \partial T_j$ where T_j is a $\Lambda_+(\phi)$-positive current with compact support. By Corollary 3.4 the support of each T_j is contained in K. Consider a strictly ϕ-plurisubharmonic function f on X. Pick $\epsilon > 0$ so that $dd^\phi f - \epsilon\phi$ is $\Lambda^+(\phi)$-positive at each point of K. Then $\epsilon M(T_j) \;=\; (T_j, \epsilon\phi) \;\le\; (T_j, dd^\phi f) \;=\; (\partial_\phi \partial T_j, f) \;=\; (u_j, f)$ which converges to (u,f). Therefore the masses $M(T_j)$ are bounded. By compactness there exists a weakly convergent subsequence $T_j \to T$. Now $\mathrm{supp}\, T \subset K$ and T must be $\Lambda_+(\phi)$-positive. Hence $u = \partial_\phi \partial T \in C \cap \mathcal{E}'_{0,K}(X)$. This proves that $C \cap \mathcal{E}'_{0,K}(X)$ is closed for each compact set K which is ϕ-convex. ∎

COROLLARY 3.12. *Suppose X is strictly ϕ-convex. Then*

$$C \;=\; \mathcal{P}^0.$$

Equivalently, the equation

$$\partial_\phi \partial T \;=\; u$$

has a solution T which is a $\Lambda_+(\phi)$-positive current with compact support if and only if

$$0 \;\le\; u(f) \qquad \text{for all} \quad f \in \mathrm{PSH}(X,\phi)$$

Proof. Apply the Bipolar Theorem (cf. [S]). ∎

Proof of Theorem 3.8. Suppose there does not exist a Green's current for (K,x), that is, suppose $\mathcal{M}_K - [x]$ is disjoint from the cone C. By the Hahn-Banach Theorem (note that $\mathcal{M}_K - [x]$ is a compact convex set) there exists $f \in C^0 = \mathcal{P}$ with f, considered as a linear functional on $\mathcal{D}'_{0,\mathrm{cpt}}(X)$, satisfying $(u,f) \le -\epsilon < 0$ for all $u \in (\mathcal{M}_K - [x])$. That is, $\int f d\mu - f(x) \le -\epsilon < 0$ for all $\mu \in \mathcal{M}_K$. Consequently,

$$\sup_K f \;=\; \sup_{\mu \in \mathcal{M}_K} \int f d\mu \;\le\; f(x) - \epsilon$$

or $\sup_K f + \epsilon \le f(x)$ so that $x \notin \widehat{K}$. In light of Proposition 3.9 we are done. ∎

One could define the "Poisson-Jensen hull" of a compact set K to be the set of points x for which there exists a Poisson-Jensen measure μ_x and a Green's current T_x satisfying (3.3). Then Proposition 3.9 states that on any (non-compact) calibrated manifold (X,ϕ), the Poisson-Jensen hull of a compact set is contained in the ϕ-plurisubharmonic hull, while Theorem 3.8 states that the two hulls are equal if (X,ϕ) is strictly convex.

The next "hull" obviously contains the Poisson-Jensen hull.

DEFINITION 3.13. The **current hull** of a compact subset $K \subset X$ is the union

$$\widetilde{K} \;\equiv \bigcup_{T \in \mathcal{P}(K)} \mathrm{supp}\, T$$

where $\mathcal{P}(K)$ consists of all $\Lambda_+(\phi)$-positive currents with compact support on X satisfying $\partial_\phi \partial T \le 0$ on $X - K$.

LEMMA 3.14. *If (X,ϕ) is strictly ϕ-convex, then $\widetilde{K} = \widehat{K}$.*

Proof. The Support Lemma 3.2 states that

$$\widetilde{K} \;\subset\; \widehat{K} \cup \mathrm{Core}(X).$$

for any calibrated manifold. Now $\widetilde{K}$ contains the Poisson-Jensen hull which equals $\widehat{K}$ if (X,ϕ) is strictly ϕ-convex by Theorem 3.8. Strict ϕ-convexity also implies $\mathrm{Core}(X) = \emptyset$ by Theorem 1.3. ∎

Suppose (X, ϕ) is non-compact and connected.

DEFINITION 3.15. An open subset $\Omega \subset X$ is **ϕ-convex relative to X** if $K \subset\subset \Omega$ implies $\widehat{K}_X \subset\subset \Omega$.

Note that if X is ϕ-convex this condition implies that Ω is ϕ-convex since $\widehat{K}_\Omega \subseteq \widehat{K}_X$. Moreover, if X is strictly ϕ-convex , then $\mathrm{Core}(\Omega) \subseteq \mathrm{Core}(X)$ is empty so that Ω is strictly ϕ-convex (by Theorem 1.3).

PROPOSITION 3.16. *Suppose (X, ϕ) is strictly ϕ-convex . An open subset $\Omega \subset X$ is ϕ-convex relative to X if and only if* $\mathrm{PSH}(X, \phi)$ *is dense in* $\mathrm{PSH}(\Omega, \phi)$.

Proof. Let $L : \mathcal{E}^0(X) \to \mathcal{E}^0(\Omega)$ denote restriction. The adjoint $L^* : \mathcal{E}'_0(\Omega) \to \mathcal{E}'_0(X)$ is inclusion. Suppose $v \in (L^*)^{-1}(C_X)$, i.e., $v \in \mathcal{E}'_0(\Omega)$ with $v = \partial_\phi \partial T$ for some $\Lambda_+(\phi)$-positive current T compactly supported in X. Then $K \equiv \mathrm{supp}\, v \subset \Omega$ satisfies $\widetilde{K}_X = \widehat{K}_X$ by Lemma 3.14. Hence, $\widehat{K}_X \subset \Omega$ implies $\mathrm{supp}\, T \subset \Omega$, i.e., that $v \in C_\Omega$. This proves that Ω is ϕ-convex relative to X if and only if

$$(L^*)^{-1}(C_X) \;=\; C_\Omega. \tag{3.5}$$

By Corollary 3.12 we may replace C_X by $\mathcal{P}^0_X = C_X$. In general, $[L(\mathcal{P}_X)]^0 = (L^*)^{-1}(\mathcal{P}^0_X)$, so that (3.5) is equivalent to

$$[L(\mathcal{P}_X)]^0 \;=\; C_\Omega. \tag{3.6}$$

By Lemma 3.10, $\mathcal{P}_\Omega = C^0_\Omega$. Hence (3.6) is equivalent to

$$\overline{L(\mathcal{P}_X)} \;=\; \mathcal{P}_\Omega \tag{3.7}$$

■

4. Boundary Duality.

In this section we take up the following general question. Suppose (X, ϕ) is a non-compact strictly ϕ-convex manifold. Given a compact oriented submanifold $\Gamma \subset X$ of dimension $p-1$, when does there exist a ϕ-submanifold M with boundary Γ? More generally, when does there exist a $\Lambda_+(\phi)$-positive current T with $\partial T = \Gamma$?

THEOREM 4.1. *Suppose ϕ is exact. Given $S \in \mathcal{E}'_{p-1}(X)$, there exists a $\Lambda_+(\phi)$-positive current $T \in \mathcal{E}'_p(X)$ with $S = \partial T$ if and only if*

$$\int_S \alpha \;\geq\; 0 \qquad \textit{for all } \alpha \in \mathcal{E}^{p-1}(X) \textit{ such that } d\alpha \text{ is } \Lambda^+(\phi)\textit{−positive}$$

Proof. Consider the following convex cones.

$$A \;=\; \{\alpha \in \mathcal{E}^{p-1}(X) :\; d\alpha \text{ is } \Lambda^+(\phi)\text{−positive}\}$$

$$B \;=\; \{S \in \mathcal{E}'_{p-1}(X) :\; S = \partial T \text{ for some } \Lambda_+(\phi)\text{−positive } T \in \mathcal{E}'_p(X)\}$$

If $\alpha \in A$ and $S \in B$, then

$$S(\alpha) = \partial T(\alpha) = T(d\alpha) \geq 0,$$

that is,

$$A \;\subseteq\; B^0 \qquad \text{and} \qquad B \;\subseteq\; A^0$$

where B^0 denotes the polar of B. If $\xi \in G_x(\phi)$, then $T = \delta_x\xi$ is $\Lambda_+(\phi)$-positive, so that $\partial(\delta_x\xi) \in B$. Therefore, if $\alpha \in B^0$, then $0 \leq \partial(\delta_x\xi)(\alpha) = (\delta_x\xi)(d\alpha) = (d\alpha)_x(\xi)$. This proves that $B^0 \subseteq A$, and hence $A = B^0$. (In particular, note that A is closed.) Theorem 4.1 is just the statement that $B = A^0$. Now since $A = B^0$, the Bipolar Theorem states that $\overline{B} = A^0$. Thus it remains to show that B is closed.

Suppose $S_j \in B$ and $S_j \to S$ in $\mathcal{E}'_{p-1}(X)$. Then $S_j = \partial T_j$ for some T_j which is $\Lambda_+(\phi)$-positive. The calibration ϕ is exact, i.e., $\phi = d\eta$ for some $\eta \in \mathcal{E}^{p-1}(X)$. Therefore

$$M(T_j) \;=\; T_j(\phi) \;=\; T_j(d\eta) \;=\; (\partial T_j)(\eta) \;=\; S_j(\eta) \;\longrightarrow\; S(\eta).$$

In particular, there exists a constant C such that $M(T_j) \leq C$ for all j. By the Support Lemma 3.2 and Theorem 1.3 we have

$$\text{supp}T_j \;\subseteq\; \widehat{\text{supp}S_j}$$

for each j. Pick a compact subset K with $\text{supp}S_j \subseteq K$ for all j. Then

$$\text{supp}T_j \;\subseteq\; \widehat{K} \qquad \text{for all } j.$$

This proves that $\{T_j\}$ is a precompact set in $\mathcal{E}'_p(X)$. Therefore, there exists a convergent subsequence $T_j \to T$ in $\mathcal{E}'_p(X)$. Obviously, $\partial T = S$ and T is $\Lambda_+(\phi)$-positive. Hence, $S \in B$. ∎

REMARK 4.2. The same proof combined with the compactness Theorem 2.9 proves the following. Let $\mathcal{R}_p(X)$ denote the compactly supported rectifiable currents of dimension p on X. Then, if ϕ is exact, the set

$$B_{\text{rect}} \;\equiv\; \{\Gamma \in \mathcal{R}_{p-1}(X) :\; S = \partial T \text{ for some } \Lambda_+(\phi)\text{−positive } T \in \mathcal{R}_p(X)\}\}$$

is weakly closed in $\mathcal{R}_{p-1}(X)$.

We now turn attention to the case where X is compact.

THEOREM 4.3. *Suppose (X, ϕ) is a compact calibrated manifold. Fix $S \in \mathcal{E}'_{p-1}(X)$ and $\lambda > 0$. Then the following are equivalent.*

(i) There exists a $\Lambda_+(\phi)$-positive current $T \in \mathcal{E}'_p(X)$ with $S = \partial T$ and $M(T) \leq \lambda$.

(ii) $\int_S \alpha \;\geq\; -\lambda$ for all $\alpha \in \mathcal{E}^{p-1}(X)$ such that $d\alpha + \phi$ is $\Lambda^+(\phi)$-positive.

Proof. Consider the convex sets:

$$A_\lambda \;=\; \{\alpha \in \mathcal{E}^{p-1}(X):\; d\alpha + \tfrac{1}{\lambda}\phi \quad \text{is } \Lambda^+(\phi)\text{-positive}\},$$

$$B_\lambda \;=\; \{S \in \mathcal{E}'_{p-1}(X):\; S = \partial T \quad \text{for some } \Lambda_+(\phi)\text{-positive } T \in \mathcal{E}'_p(X) \text{ with } M(T) \le \lambda\}.$$

If $\alpha \in A_\lambda$ and $S \in B_\lambda$, then

$$S(\alpha) = \partial T(\alpha) = T(d\alpha) = T(d\alpha + \tfrac{1}{\lambda}\phi - \tfrac{1}{\lambda}\phi) \ge -\tfrac{1}{\lambda}T(\phi) = -\tfrac{1}{\lambda}M(T) \ge -1,$$

that is,

$$A_\lambda \;\subseteq\; B^0_\lambda \qquad \text{and} \qquad B_\lambda \;\subseteq\; A^0_\lambda$$

where $B^0_\lambda = \{\alpha : (S,\alpha) \ge -1 \text{ for all } S \in B_\lambda\}$ denotes the polar of the convex set B_λ. If $\xi \in G_x(\phi)$, then $T = \lambda\delta_x\xi$ is $\Lambda_+(\phi)$-positive, so that $\partial(\lambda\delta_x\xi) \in B_\lambda$. Therefore, if $\alpha \in B^0_\lambda$, then $-1 \le \partial(\lambda\delta_x\xi)(\alpha) = \lambda(\delta_x\xi)(d\alpha) = \lambda(d\alpha)_x(\xi)$. Since $\phi(\xi) = 1$ we conclude that $(d\alpha + \frac{1}{\lambda}\phi)(\xi) \ge 0$. This proves that $B^0_\lambda \subseteq A_\lambda$, and hence $A_\lambda = B^0_\lambda$. (Note, in particular, that A_λ is closed.) Theorem 4.3 is just the statement that $B_\lambda = A^0_\lambda$. Now the Bipolar Theorem states that $\overline{B}_\lambda = A^0_\lambda$. However it follows from the Compactness Theorem 2.8 that $\overline{B}_\lambda = B_\lambda$. ∎

Similar arguments can be applied to prove other versions of boundary duality. For example we have the following result concerning boundaries of ϕ-positive currents whose support is not necessarily compact.

THEOREM 4.4. *Suppose (X,ϕ) is an arbitrary calibrated manifold. Fix $S \in \mathcal{D}'_{p-1}(X)$ and $\lambda > 0$. Then the following are equivalent.*

(i) There exists a $\Lambda_+(\phi)$-positive current $T \in \mathcal{D}'_p(X)$ with $S = \partial T$ and $M(T) \le \lambda$.

(ii) $\int_S \alpha \;\ge\; -\lambda$ for all compactly supported forms $\alpha \in \mathcal{D}^{p-1}(X)$ such that $d\alpha + \phi$ is $\Lambda^+(\phi)$-positive.

Note that here the currents S and T do not necessarily have compact support. The proof of Theorem 4.4 is left to the reader. Arguments for the special case of complex geometry appear in [HL$_4$].

5. ϕ-Flat Hypersurfaces and Functions which are ϕ-Pluriharmonic mod d.

The ϕ-pluriharmonic functions are the closest thing to holomorphic functions on a calibrated manifold (X, ϕ). Usually there are very few ϕ-pluriharmonic functions. An attempt has been made in this paper to remedy this situation by emphasizing the ϕ-plurisubharmonic functions. By comparison these functions exist in abundance. For some purposes another extension of the concept of ϕ-pluriharmonic functions is more useful — namely the *ϕ-pluriharmonic functions mod d.*

This section is, for the most part, a straightforward extension of the results of Lei Fu [Fu] from the special Lagrangian case to the general calibrated manifold (X, ϕ).

DEFINITION 5.1. A function $f \in C^\infty(X)$ is **ϕ-pluriharmonic mod d** if

$$dd^\phi f \;=\; df \wedge \alpha_f + \sigma_f \tag{5.1}$$

for some $(p-1)$-form α_f and some p-form σ_f of type $\Lambda(\phi)^\perp$, i.e., $\sigma_f(\xi) = 0$ for all $\xi \in G(\phi)$.

If f is ϕ-pluriharmonic mod d, then λf, $\lambda \in \mathbf{R}$, is also ϕ-pluriharmonic mod d. However, the sum of two such functions need not be ϕ-pluriharmonic mod d.

PROPOSITION 5.2. *Suppose that df never vanishes so that $\mathcal{H} \equiv \ker df$ defines a hypersurface foliation. The condition that f be ϕ-pluriharmonic mod d is independent of the function defining the foliation $\mathcal{H}$.*

Proof. Recall that locally f and g define the same foliation $\mathcal{H}$ if and only if $g = \chi(f)$ for some function $\chi : \mathbf{R} \to \mathbf{R}$ for which χ' is never zero. To prove this fact assume that $f = x_1$ is a local coordinate. Since g is constant on the leaves $\{x_1 = \text{constant}\}$, g must be independent of $x_2, ..., x_n$, i.e., $g = \chi(x_1)$. Since dg is never zero, χ' is never zero. Finally,

$$dd^\phi g \;=\; \chi'(f) dd^\phi f + \chi''(f) df \wedge d^\phi f \;=\; dg \wedge \left(\alpha_f + \frac{\chi''(f)}{\chi'(f)} d^\phi f \right) + \chi'(f)\sigma_f$$

which proves that if f is ϕ-pluriharmonic mod d, then $g = \chi(f)$ is also. ∎

Recall that a real hypersurface $Y \subset X$ is *ϕ-flat* if its second fundamental form II_Y has that property that $\text{tr}_\xi II_Y = 0$ for all tangential ϕ-planes $\xi \subset TY$. This is equivalent to the condition that $dd^\phi f(\xi) = 0$ for all such ξ, where f is any defining function for Y. (See Definition 5.1 and Lemmas 5.2 and 5.11 in [HL$_5$].)

PROPOSITION 5.3. *If f is ϕ-pluriharmonic mod d, then each (non-critical) hypersurface $\{f = C\}$ is ϕ-flat.*

Proof. Suppose $\xi \in G(\phi)$ is tangent to $\{f = C\}$, i.e., $\nabla f \lrcorner \xi = 0$. Then $(dd^\phi f)(\xi) = (df \wedge \alpha_f)(\xi) + \sigma_f(\xi)$. Since σ_f is of type $\Lambda(\phi)^\perp$, we have $\sigma_f(\xi) = 0$, and $(df \wedge \alpha_f)(\xi) = \alpha_f(\nabla f \lrcorner \xi) = 0$. ∎

By Corollary 2.11 in [HL$_5$] we have that for any $f \in C^\infty$ and any ϕ-submanifold M,

$$(dd^\phi f - df \wedge \alpha_f)\big|_M \;=\; *_M(\Delta_M f) - d(f\big|_M) \wedge \alpha_f\big|_M. \tag{5.2}$$

This proves the following.

PROPOSITION 5.4. *If f is ϕ-pluriharmonic mod d and M is a ϕ-submanifold, then $u \equiv f\big|_M$ satisfies the partial differential equation*

$$\Delta_M u \;=\; *(du \wedge \beta) \qquad \text{on} \;\; M \tag{5.3}$$

where $\beta = \alpha_f\big|_M$.

The maximum principle is applicable to solutions to (5.3). See [BJS].

COROLLARY 5.5. *Suppose (M, Γ) is a compact ϕ-submanifold with boundary. Then for each function f which is ϕ-pluriharmonic mod d and each point $x \in M$, one has*

$$\inf_\Gamma f \;\le\; f(x) \;\le\; \sup_\Gamma f \tag{5.4}$$

COROLLARY 5.6. *Suppose (M, Γ) is as above. If $\Gamma \subset \{f = C\}$, then $M \subset \{f = C\}$.*

PROPOSITION 5.7. *Suppose (M, Γ) is a compact ϕ-submanifold with boundary, and suppose f is a function on X which is ϕ-pluriharmonic mod d. If f is constant on Γ, then*

$$d^\phi f\big|_\Gamma \;\equiv\; 0 \qquad (\text{point} - \text{wise}). \tag{5.5}$$

Proof. By Corollary 5.6, f is constant on M. We then apply the following.

LEMMA 5.8. *For any function f constant on M, $d^\phi f\big|_\Gamma \;\equiv\; 0$.*

Proof. At $x \in \Gamma$, we have $\overrightarrow{M} = e \wedge \overrightarrow{\Gamma}$ for some e tangent to M. Since f is constant on M, $\nabla f \perp \operatorname{span} \overrightarrow{M}$. Now $(d^\phi f)(\overrightarrow{\Gamma}) = (\nabla f \lrcorner \phi)(e \lrcorner \overrightarrow{M}) = \phi((\nabla f) \wedge (e \lrcorner \overrightarrow{M})) = 0$ since $\nabla f \wedge (e \lrcorner \overrightarrow{M})$ is a first cousin of $\overrightarrow{M} \in G(\phi)$ (cf. [HL$_5$, 2.4]). ■■

Our next objective is to show that, for the large class of *normal* calibrations, a function f is ϕ-pluriharmonic mod d if and only if its level sets are ϕ-flat.

Suppose $\phi \in \Lambda^p V$ is a calibration on a euclidean vector space V. For each hyperplane $W \subset V$, $\phi\big|_W \in \Lambda^p W$ has comass ≤ 1 and, in fact, < 1 if and only if $G(\phi\big|_W)$ is empty.

DEFINITION 5.9. The calibration $\phi \in \Lambda^p V$ is **normal** if, for every hyperplane $W \subset V$

$$\Lambda(\phi\big|_W)^\perp \;= \Lambda(\phi)^\perp\big|_W$$

as subspaces of $\Lambda^p W$. A calibration ϕ on a manifold X is **normal** if $\phi_x \in \Lambda^p T_x X$ is normal for each $x \in X$.

PROPOSITION 5.10. *Suppose ϕ is a normal calibration on X, and $f \in C^\infty(X)$ has a never-vanishing gradient. Then*

f is ϕ-pluriharmonic mod d

if and only if

each level set $\{f = C\}$ is ϕ-flat.

Proof. Suppose each level set $\{f = C\}$ is ϕ-flat. That is

$$(dd^\phi f)(\xi) \;=\; 0 \qquad \text{for all } \xi \in G(\phi) \;\text{ which are tangential to } \{f = C\}. \tag{5.6}$$

Let $W = \ker df_x$ at some $x \in X$. Note that $G(\phi|_W) = \{\xi \in G(\phi) : \xi \text{ is tangential to } W\}$. Then (5.6) is equivalent to

$$dd^\phi f|_W \in \Lambda\left(\phi|_W\right)^\perp \tag{5.7}$$

Now f is ϕ-pluriharmonic mod d if

$$dd^\phi f \;=\; df \wedge \alpha_f + \sigma_f \qquad \sigma_f \in \Lambda(\phi)^\perp \tag{5.8}$$

or equivalently

$$dd^\phi f|_W \in \Lambda(\phi)^\perp|_W \tag{5.9}$$

If ϕ is normal, then

$$\Lambda\left(\phi|_W\right)^\perp \subset \Lambda(\phi)^\perp|_W$$

and (5.7) implies (5.9). ∎

PROPOSITION 5.11. *The following calibrations are normal.*

1. *A Kähler or pth power Kähler calibration.*
2. *A Special Lagrangian calibration.*
3. *An associative, coassociative or Cayley calibration.*
4. *A quaternionic calibration.*

The proof is left to the reader.

6. Hodge Manifolds

In this section we pose some highly speculative questions for calibrated manifolds in the spirit of those posed in the complex case (cf. [HK, p.58], [$L_{4,5}$]). Assume that (X, ϕ) is a compact calibrated n-manifold with a parallel calibration ϕ of degree p. Let $*\phi$ denote the dual calibration. Note that a ϕ-submanifold or, more generally, any ϕ-cycle (see Definition 2.1) is a current of dimension p and degree $n-p$. By contrast a $*\phi$-submanifold or $*\phi$-cycle is a current of dimension $n-p$ and degree p.

Denote by $\widetilde{H}^p(X, \mathbf{Z})$ the image of the map $H^p(X, \mathbf{Z}) \to H^p(X, \mathbf{R})$, with analogous notation for homology.

DEFINITION 6.1. If the de Rham class of the calibration ϕ lies in $\widetilde{H}^p(X, \mathbf{Z})$, i.e., if ϕ has integral periods, then (X, ϕ) will be referred to as a **Hodge manifold**.

REMARK 6.2. If (X, ω) is a Kähler manifold, then this coincides with standard terminology. The Kodaira Embedding Theorem states that in this case each Hodge manifold is projective algebraic with $N\omega - [H] = d\alpha$, where H is a hyperplane section, N a positive integer, and α a current of degree 1. Note that $[H]$ is a $*\omega$-submanifold not an ω-submanifold.

The Hodge Question for the Class of ϕ. Suppose (X, ϕ) is a Hodge manifold. When does there exist a $*\phi$-cycle T cohomologous to $N\phi$ for some positive integer N, i.e.,

$$N\phi - T \;=\; d\alpha \tag{6.1}$$

for some current α of degree $p-1$?

Recall that by definition a $*\phi$-cycle is automatically $*\phi$-positive, so this is, more precisely, the "Hodge Question with Positivity" for ϕ.

REMARK . If equation (6.1) (called the *spark equation*) has a solution, then α determines a differential character on X. (See [HLZ] for more details.)

EXAMPLE 6.3. In [L_3] an example is constructed of a Hodge manifold (X, ϕ) for which no such cycle exists. More specifically, a parallel self-dual 4-form ϕ of comass 1 is constructed on a flat torus X of dimension 8 with the property that $[\phi] \in \widetilde{H}^4(X, \mathbf{Z})$, but there exist no ϕ-cycles whatsoever on X.

EXAMPLE 6.4. Consider the fundamental bi-invariant 3-form Ω on a compact simple Lie group G, normalized to be the generator of $H^3(G, \mathbf{Z}) \cong \mathbf{Z}$. By rescaling the bi-invariant metric on G we may assume that Ω has comass 1, i.e., that (G, Ω) is a Hodge manifold. H. Tasaki [T, Thm. 7], building on work of Dao Čong Thi [Thi], showed that $*\Omega$-cycles exist and so there is a positive answer to the Hodge Question for Ω. Later R. Bryant [B] proved that Ω is itself cohomologous to a $*\Omega$-cycle, i.e., we can take $N = 1$ in this case. He also showed that all $*\Omega$-cycles are sums of singular semi-analytic subvarieties congruent to irreducible semi-analytic components of the cut-locus of the exponential map.

Recall from Definition 2.1 that

$$\Lambda(\phi) \;\equiv\; \text{span}\{G(\phi)\} \;\subset\; \Lambda_p TX.$$

DEFINITION 6.5. A p-dimensional current T, representable by integration, is said to be of **type** $\Lambda(\phi)$ if $\overrightarrow{T}_x \in \Lambda(\phi) \subset \Lambda_p T_x X$ for $\|T\|$-a.a. x. This definition extends to arbitrary currents T of dimension p. If $T(\psi) = 0$ for all smooth p-forms ψ such that $\psi\big|_{\Lambda(\phi)} = 0$, then T is said to be of **type** $\Lambda(\phi)$.

Each ϕ-cycle T is of type $\Lambda(\phi)$ since $\overrightarrow{T}_x \in G(\phi)$ $\|T\|$-almost everywhere.

DEFINITION 6.6. A **$\Lambda(\phi)$-cycle** is a d-closed, p-dimensional locally rectifiable current of type $\Lambda(\phi)$.

There is a natural necessary condition for a class $c \in \widetilde{H}_p$ to be represented by a $\Lambda(\phi)$-cycle. In fact we have the following more general statement.

PROPOSITION 6.7. *If a class $c \in \widetilde{H}_p(X, \mathbf{Z})$ is represented by a current of type $\Lambda(\phi)$, then the harmonic representative of c must be of type $\Lambda(\phi)$.*

Proof. Recall that the Hodge decomposition: $\mathcal{E}^p(X) = \mathbf{H}^p(X) \oplus \text{Image}(d) \oplus \text{Image}(d^*)$, is a C^∞-decomposition, and therefore induces a corresponding decomposition of currents: $\mathcal{E}'_p(X) = \mathbf{H}_p(X) \oplus \text{Image}(\partial) \oplus \text{Image}(\partial^*)$. Note that the orthogonal bundle projection $P_{\Lambda(\phi)} : \Lambda_p TX \to \Lambda(\phi)$ is a parallel operator. It was proved by Chern [Ch] that any such operator commutes with harmonic projection $\mathbf{H}$. Suppose now that c is represented by a current T of type $\Lambda(\phi)$. Then $P_{\Lambda(\phi)}(T) = T$ and therefore $P_{\Lambda(\phi)}\mathbf{H}(T) = \mathbf{H}(T)$. ∎

I. The Hodge Question. Suppose $c \in \widetilde{H}_p(X, \mathbf{Z})$ is a class whose harmonic representative is of type $\Lambda(\phi)$. When does there exist an integer N and a $\Lambda(\phi)$-cycle T with $T \in Nc$?

REMARK 6.8. Example 6.3 above gives a parallel calibration ϕ on a flat 8-dimensional torus X and an integral class $c \in \widetilde{H}_4(X, \mathbf{Z})$ of type $\Lambda(\phi)$ for which no such current exists.

REMARK 6.9. The Hodge Question is a direct generalization of the standard Hodge Conjecture for algebraic cycles on a complex projective manifold, since we know from [HS], [Sh] and [Alex] that for $\phi = \omega^p/p!$ (ω = the Kähler form), any $\Lambda(\phi)$-cycle is an algebraic p-cycle.

REMARK 6.10. Any locally finite integer sum of ϕ-cycles is a $\Lambda(\phi)$-cycle. However, the converse is completely open outside of the Kähler case. Moreover, even though it holds in the Kähler case (cf. Remark 6.9), there is no proof of this fact by the standard methods of regularity in Geometric Measure Theory.

Before trying to prove that a general $\Lambda(\phi)$-cycle is a sum of ϕ-cycles, one would like the calibration ϕ to have the following algebraic property (6.2). Equation (2.4) says that $G(p, T_x X) \cap \Lambda_+(\phi) = G(\phi)$ so that ϕ-cycles and $\Lambda_+(\phi)$-cycles are the same thing. Most parallel calibrations (see [HL$_3$, p. 68]) are known to satisfy

$$G(p, T_x X) \cap \Lambda(\phi) = G(\phi) \cup (-G(\phi)). \tag{6.2}$$

In this case T is a $\Lambda_+(\phi)$-cycle if and only if $\pm\overrightarrow{T_x} \in G_x(\phi)$ for $\|T\|$-a.a. x. Consequently, T decomposes into $T^+ - T^-$ with both $\overrightarrow{T_x}^{\pm} \in G_x(\phi)$, but, even in the Kähler case, one can not show directly that T^+ and T^- are d-closed.

There are versions of the Hodge Question involving "positivity" which may have more hope. For example:

II. The Hodge Question (with positivity). Suppose $c \in \widetilde{H}_p(X, \mathbf{Z})$ is a class whose harmonic representative is strictly $\Lambda_+(\phi)$-positive. When does there exist an integer N and a ϕ-cycle T with $T \in Nc$?

REMARK 6.11. If the current $*\phi$ (of dimension p) is strictly $\Lambda_+(\phi)$-positive, then for any form ψ of type $\Lambda(\phi)$, there exists an integer ℓ such that $\psi + \ell(*\phi)$ is strictly $\Lambda_+(\phi)$-positive. This applies for example to the harmonic representative of c in Hodge Question II. Consequently, one can see that *if $(X, *\phi)$ is a Hodge manifold with a solution to (6.2), then the Hodge Question I follows from II.*

REMARK 6.12. The point of Hodge Question II is that one is asking for a ϕ-cycle T. These are automatically $\Lambda_+(\phi)$-positive and therefore satisfy the strong regularity Theorem 2.10.

Federer and Fleming [FF] showed that each class $c \in H_p(X, \mathbf{Z})$ contains a rectifiable cycle T with $M(T) \leq M(S)$ for all other rectifiable cycles $S \in c$. Let $\|c\|_{\mathbf{Z}} = M(T)$ denote this minimum. Let $\|\widetilde{c}\|_{\mathbf{R}}$ denote the infimum of the masses $M(S)$ taken over *all* closed currents homologous to T, i.e., over all currents in the real homology class $\widetilde{c} \in \widetilde{H}_p(X, \mathbf{Z})$ corresponding to c.

PROPOSITION 6.13. *Fix $c \in H_p(X, \mathbf{Z})$ and suppose the corresponding class $\widetilde{c} \in \widetilde{H}_p(X, \mathbf{Z})$ has a smooth representative ψ which is $\Lambda_+(\phi)$-positive. Then there exists a ϕ-cycle $T \in c$ if and only if $\|c\|_{\mathbf{Z}} = \|\widetilde{c}\|_{\mathbf{R}}$.*

Proof. If c contains a ϕ-cycle T, then $\|c\|_{\mathbf{Z}} \leq M(T) = T(\phi) = S(\phi) \leq M(S)$ for all currents $S \in \widetilde{c}$. Hence, $\|c\|_{\mathbf{Z}} \leq \|\widetilde{c}\|_{\mathbf{R}}$, and the inequality $\|\widetilde{c}\|_{\mathbf{R}} \leq \|c\|_{\mathbf{Z}}$ is clear from the definitions.

Conversely, if $\|c\|_{\mathbf{Z}} = \|\widetilde{c}\|_{\mathbf{R}}$, then the Federer-Fleming solution T (a rectifiable current) satisfies $M(T) = \|c\|_{\mathbf{Z}} = \|\widetilde{c}\|_{\mathbf{R}} \leq M(\psi) = \int \psi \wedge \phi = T(\phi)$ since T and ψ are homologous. However, $T(\phi) \leq M(T)$ with equality iff T if a ϕ-cycle. ∎

Thus the positive Hodge question can be rephrased as follows: When does there exist an integer k with $\|kc\|_{\mathbf{Z}} = \|k\widetilde{c}\|_{\mathbf{R}}$?

Final Note: Since $\|k\widetilde{c}\|_{\mathbf{R}} = k\|\widetilde{c}\|_{\mathbf{R}}$, we have $\|kc\|_{\mathbf{Z}} = \|k\widetilde{c}\|_{\mathbf{R}}$ iff $\|\widetilde{c}\|_{\mathbf{R}} = \frac{1}{k}\|kc\|_{\mathbf{Z}}$. Federer has shown in [$F_1$] that

$$\|\widetilde{c}\|_{\mathbf{R}} \;=\; \lim_{k\to\infty} \tfrac{1}{k}\|kc\|_{\mathbf{Z}}$$

Appendix: The Reduced ϕ-Hessian.

We assume throughout this section that $\Lambda(\phi)$ is a vector subbundle of $\Lambda_p TX$, and we let $\Lambda(\phi) \subset \Lambda^p T^*X$ denote the corresponding bundle under the metric equivalence $\Lambda_p TX \cong \Lambda^p T^*X$.

DEFINITION A.1. The **reduced ϕ-hessian** $\overline{\mathcal{H}}^\phi : C^\infty(X) \to \Gamma(X, \Lambda(\phi))$ is defined to be $\mathcal{H}^\phi$ followed by orthogonal projection onto the subbundle $\Lambda(\phi) \subset \Lambda^p T^*X$.

Note that a function f is ϕ-pluriharmonic if and only if $\overline{\mathcal{H}}^\phi(f) = 0$.

Note also that if ϕ is parallel, then $\overline{\mathcal{H}}^\phi = \overline{d} d^\phi$ where $\overline{d}$ denotes the exterior derivative followed by orthogonal projection onto $\Lambda(\phi)$.

For most of the calibrations considered as examples in this paper, the image of the map $\lambda_\phi : \mathrm{Sym}^2(\mathrm{TX}) \to \Lambda^{\mathrm{p}}\mathrm{T}^*\mathrm{X}$ is contained in $\Lambda(\phi)$, or equivalently, $\overline{\mathcal{H}}^\phi = \mathcal{H}^\phi$. For reference, $\overline{\mathcal{H}}^\phi = \mathcal{H}^\phi$ in the following cases.

(1) $\phi = \frac{1}{p!}\omega^p$, the pth power of the Kähler form,
(2) ϕ Special Lagrangian
(3) ϕ Associative, Coassociative or Cayley
(4) ϕ the fundamental 3-form on a simple Lie group.

Exceptions will be discussed at the end of this appendix.

Even when $\overline{\mathcal{H}}^\phi = \mathcal{H}^\phi$ the following proposition is important.

PROPOSITION A.2. *Suppose f is a distribution on X. Then f is ϕ-plurisubharmonic if and only if $\overline{\mathcal{H}}^\phi(f) \equiv R$ is representable by integration and $\overrightarrow{R} \in \Lambda^+(\phi)$ $\|R\|$-a.e., that is, if and only if $\overline{\mathcal{H}}^\phi(f)$ is a $\Lambda^+(\phi)$-positive current.*

The proof is similar to the proof of Theorem 2.14 and is omitted.

DEFINITION A.3. The ϕ-Grassmannian $G(\phi)$ **involves all the variables** if, for $u \in TX$, the condition $u \lrcorner \xi = 0$ for all $\xi \in G(\phi)$ implies $u = 0$

EXAMPLE . The 2-form $\phi \equiv dx_1 \wedge dx_2 + \lambda dx_3 \wedge dx_4$ with $|\lambda| < 1$ is a calibration on $\mathbf{R}^4$ which involves all the variables (See Section 1), but the only $\xi \in G(\phi)$ is the x_1, x_2 plane so that $G(\phi)$ does not involve all the variables.

PROPOSITION A.4. *The operator $\overline{\mathcal{H}}^\phi$ is over-determined elliptic if and only if $G(\phi)$ involves all the variables.*

Proof. We need only consider the case $\phi \in \Lambda^p V$, where V is an inner product space. The symbol of $\mathcal{H}^\phi$ at $u \in V$ is $u \wedge (u \lrcorner \phi)$. Hence, the reduced operator $\overline{\mathcal{H}}^\phi$ is elliptic if and only if

$$(u \wedge (u \lrcorner \phi))(\xi) = 0 \ \ \forall \ \ \xi \in G(\phi) \qquad \Rightarrow \qquad u = 0.$$

For $\xi \in G(p, V)$ and $u \in V$, let $u = a + b$ with $a \in \operatorname{span} \xi$ and $b \perp \operatorname{span} \xi$. Then $(u \wedge (u \lrcorner \phi))(\xi) = \phi(u \wedge (u \lrcorner \xi)) = \phi((a+b) \wedge (a \lrcorner \xi)) = |a|^2 \phi(\xi) + \phi(b \wedge (a \lrcorner \xi))$. If $\xi \in G(\phi)$, then $\phi(b \wedge (a \lrcorner \xi)) = 0$ by the First Cousin Principle [HL$_5$, Lemma 2.4] , and $\phi(\xi) = 1$. Hence, $(u \wedge (u \lrcorner \phi))(\xi) = |a|^2 = 0$ if and only if $u \lrcorner \xi = 0$. ∎

One can easily reduce a calibration to the elliptic case.

PROPOSITION A.5. *Suppose $\phi \in \Lambda^p V$ is a calibration. Define $W \subset V$ by*

$$W^\perp \equiv \bigcap_{\xi \in G(\phi)} (\operatorname{span} \xi)^\perp$$

and set $\psi = \phi\big|_W$. Then $\psi \in \Lambda^p W$ is a calibration and $G(\psi)$ involves all the variables in W. Moreover, $G(\phi) = G(\psi)$ and the reduced operators $\overline{\mathcal{H}}^\phi$ and $\overline{\mathcal{H}}^\psi$ agree.

Proof. Obviously ψ is a calibration and $G(\psi) \subset G(\phi)$. If $\xi \in G(\phi)$, then $\operatorname{span} \xi \subset W$ and hence $\phi(\xi) = \psi(\xi)$. Thus $G(\phi) = G(\psi)$. By construction $G(\psi)$ involves all the variables in W. Finally, for all $\xi \in G(\phi)$, we have $\overline{\mathcal{H}}^\phi(f)(\xi) = \operatorname{tr}_\xi \mathrm{Hess} f = \overline{\mathcal{H}}^\psi(f)(\xi)$. ∎

EXAMPLE . Let $\Psi \in \Lambda^4_{\mathbf{R}} \mathbf{H}^n$ be the quaternionic calibration on $\mathbf{H}^n$. One can show that $dd^\Psi f = 0$ if and only if $\mathrm{Hess} f = 0$. However,

$$\overline{d} d^\Psi f \;=\; \overline{\mathcal{H}}^\Psi(f) \;=\; \lambda_\Psi \left(\left(\frac{\partial^2 f}{\partial q_\alpha \partial \bar{q}_\beta} \right) \right),$$

that is, the reduced hessian is isomorphic to the quaternionic hessian $\left(\frac{\partial^2 f}{\partial q_\alpha \partial \bar{q}_\beta} \right)$

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