AN INTRODUCTION TO POTENTIAL THEORY IN CALIBRATED GEOMETRY

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ABSTRACT

In this paper we introduce and study the notion of plurisubharmonic functions in calibrated geometry. These functions generalize the classical plurisubharmonic functions from complex geometry and enjoy their important properties. Moreover, they exist in abundance whereas the corresponding pluriharmonics are generally quite scarce. A number of the results established in complex analysis via plurisubharmonic functions are extended to calibrated manifolds. This paper introduces and investigates questions of pseudoconvexity in the context of a general calibrated manifold (X, ϕ). Analogues of totally real submanifolds are introduced and used to construct enormous families of strictly ϕ-convex spaces with every topological type allowed by Morse Theory. Specific calibrations are used as examples throughout.

In a sequel, the duality between ϕ-plurisubharmonic functions and ϕ-positive currents is investigated. This study involves boundaries, generalized Jensen measures, and other geometric objects on a calibrated manifold.

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0. Introduction.

Calibrated geometries, as introduced in [HL_1], are geometries of distinguished submanifolds determined by a fixed, closed differential form ϕ on a riemannian manifold X. The basic example is that of a Kähler manifold (or more generally a symplectic manifold, with compatible almost complex structure) where the distinguished submanifolds are the holomorphic curves. However, there exist many other interesting geometries, each carrying a wealth of ϕ-submanifolds, particularly on spaces with special holonomy. The relationship between spinors and calibrations established in [DH] provides additional interest. Calibrated manifolds have attracted particular attention in recent years due to their appearance in generalized Donaldson theories ([DT], [Ti]) and in modern versions of string theory in Physics ([GLW], [GP], [AFS], [Her], [MS], [G], [EM], [GW], [MC] for example).

Unfortunately, analysis on these spaces (X, ϕ) has been difficult, in part because there is generally no reasonable analogue of the holomorphic functions and transformations which exist in the Kähler case. However, in complex analysis there are many important results which can be established using only the plurisubharmonic functions (cf. [Ho], [D]). It turns out that analogues of these functions exist in abundance on any calibrated manifold, and they enjoy almost all the pleasant properties of their cousins from complex analysis. The point of this paper is to introduce and study these functions and related notions of convexity.

In a sequel [HL_2] these notions will be related to ϕ-positive currents and their boundaries, generalized Jensen measures, and other geometric objects on a calibrated manifold.

We begin by defining our notion of ϕ-plurisubharmonicity for smooth functions on any calibrated manifold (X, ϕ). In the Kähler case they are exactly the classical plurisubharmonic functions. We then study the basic properties of these functions, and subsequently use them to establish a series of results in geometry and analysis on (X, ϕ).

A fundamental result is that:

The restriction of a ϕ-plurisubharmonic function to a ϕ-submanifold M is subharmonic in the induced metric on M.

Any convex function on the riemannnian manifold X is ϕ-plurisubharmonic. Moreover, at least locally, there exists an abundance of ϕ-plurisubharmonic functions which are not convex.

The definition of ϕ-plurisubharmonicity extends from smooth functions to arbitrary distributions on X. Such distributions enjoy all the nice properties of generalized subharmonic functions. In this paper, however, we shall focus mainly on the smooth case, except for Section 3.

To define ϕ-plurisubharmonic functions on a calibrated manifold (X, ϕ) where $\deg(\phi) = p$, we introduce a second order differential operator $\mathcal{H}^\phi : C^\infty(X) \to \mathcal{E}^p(X)$, the *$\phi$-Hessian*, given by

$$\mathcal{H}^\phi(f) \;=\; \lambda_\phi(\mathrm{Hess} f)$$

where $\mathrm{Hess} f$ is the riemannian hessian of f and $\lambda_\phi : \mathrm{End}(TX) \to \Lambda^p T^*X$ is the bundle map given by $\lambda_\phi(A) = D_{A^*}(\phi)$ where $D_{A^*} : \Lambda^p T^*X \longrightarrow \Lambda^p T^*X$ is the natural extension of $A^* : T^*X \to T^*X$ as a derivation.

When the calibration ϕ is parallel there is a natural factorization

$$\mathcal{H}^\phi \;=\; dd^\phi$$

where d is the de Rham differential and $d^\phi : C^\infty(X) \;\to\; \mathcal{E}^{p-1}(X)$ is given by

$$d^\phi f \;\equiv\; \nabla f \lrcorner \phi.$$

In general these operators are related by the equation: $\mathcal{H}^\phi f = dd^\phi f - \nabla_{\nabla f}(\phi)$.

Recall that a calibration ϕ of degree p is a closed p-form with the property that $\phi(\xi) \leq 1$ for all unit simple tangent p-vectors ξ on X. Those ξ for which $\phi(\xi) = 1$ are called ϕ-planes, and the set of ϕ-planes is denoted by $G(\phi)$. With this understood, a function $f \in C^\infty(X)$ is defined to be **ϕ-plurisubharmonic** if $\mathcal{H}^\phi(f)(\xi) \geq 0$ for all $\xi \in G(\phi)$. It is **strictly ϕ-plurisubharmonic** at a point $x \in X$ if $\mathcal{H}^\phi(f)(\xi) > 0$ for all ϕ-planes ξ at x. In a similar fashion, f is called **ϕ-pluriharmonic** if $\mathcal{H}^\phi(f)(\xi) = 0$ for all $\xi \in G(\phi)$. Denote by $\mathrm{PSH(X},\phi)$ the convex cone of ϕ-plurisubharmonic functions on X.

When X is a complex manifold with a Kähler form ω, one easily computes that $d^\omega = d^c$, the conjugate differential. In this case, $\mathcal{H}^\omega = dd^\omega = dd^c$ and the ω-planes correspond to the complex lines in TX. Hence, the definitions above coincide with the classical notions of plurisubharmonic and pluriharmonic functions on X.

With this said, we must remark that in many calibrated manifolds the ϕ-pluriharmonic functions are scarce. For the calibrations on manifolds with strict G_2 or Spin_7 holonomy, for example, every pluriharmonic function is constant. For the Special Lagrangian calibration $\phi = \mathrm{Re}\{dz\}$, every ϕ-pluriharmonic function f defined locally in $\mathbf{C}^n$ is of the form $f = a + q$ where a is affine and q is a traceless Hermitian quadratic form (cf. [Fu] and Proposition 1.10.) Nevertheless, as we stated above, the ϕ-plurisubharmonic functions in any calibrated geometry are locally abundant.

The fundamental property of the ϕ-Hessian:

$$\left(\mathcal{H}^\phi f\right)(\xi) \;=\; \mathrm{trace}\left\{\mathrm{Hess} f\big|_\xi\right\} \qquad \text{for all } \phi - \text{planes } \xi$$

is established in Section 2 (Corollary 2.5). This gives the useful fact that

$$f \text{ is } \phi\text{-plurisubharmonic} \quad\Longleftrightarrow\quad \mathrm{tr}_\xi\left\{\mathrm{Hess} f\big|_\xi\right\} \;\geq\; 0 \quad \forall\, \xi \in G(\phi).$$

Elliptic Calibrations.

This brief section is an introduction to the theory of ϕ-plurisubharmonic distributions. A very mild condition on the calibration is needed to ensure "ellipticity", namely, $G(\phi)$ should involve all the variables. (This is stronger than requiring that the calibration ϕ involve all the variables. See Example 2 in Section 3.) Under this assumption, each ϕ-plurisubharmonic distribution is, in fact, L^1_{loc} (locally Lebesgue integrable) and has a canonical point-wise representative which is $[-\infty, \infty)$-valued and upper semi-continuous, given by the limit of the means over balls. The usual properties of plurisubharmonic function in complex analysis are valid for these ϕ-plurisubharmonic functions. See [HL_4] for a more comprehensive development, which is also calibration independent.

Beginning with Section 4 the ϕ-plurisubharmonic functions are used to study geometry and analysis on calibrated manifolds. The first concept to be addressed is the analogue of pseudoconvexity in complex geometry.

Convexity.

Let (X, ϕ) be a calibrated manifold and $K \subset X$ a closed subset. By the **ϕ-convex hull of K** we mean the subset

$$\widehat{K} \;=\; \{x \in X : f(x) \leq \sup_K f \;\text{ for all } f \in \mathrm{PSH(X}, \phi)\}$$

The manifold (X, ϕ) is said to be **ϕ-convex** if $K \subset\subset X \;\Rightarrow\; \widehat{K} \subset\subset X$ for all K.

Theorem 4.3. *A calibrated manifold (X, ϕ) is ϕ-convex if and only if it admits a ϕ-plurisubharmonic proper exhaustion function $f : X \to \mathbf{R}$.*

The manifold (X, ϕ) will be called **strictly ϕ-convex** if it admits an exhaustion function f which is strictly ϕ-plurisubharmonic, and it will be called **strictly ϕ-convex at infinity** if f is strictly ϕ-plurisubharmonic outside of a compact subset. It is shown that in the second case, f can be assumed to be ϕ-plurisubharmonic everywhere. Analogues of Theorem 4.3 are established in each of these cases.

Note that in complex geometry, strictly ϕ-convex manifolds are Stein and manifolds which are strictly ϕ-convex at infinity are called strongly pseudoconvex.

We next consider the **core** of X which is defined to be the set of points $x \in X$ with the property that no $f \in \mathrm{PSH(X}, \phi)$ is strictly ϕ-plurisubharmonic at x. The following results are established:

1) The manifold X is strictly ϕ-convex at infinity if and only if $\mathrm{Core}(X)$ is compact.

2) The manifold X is strictly ϕ-convex if and only if $\mathrm{Core}(X) = \emptyset$.

Examples of complete calibrated manifolds with compact cores are given in the final subsection of §4. A very general construction of strictly ϕ-convex manifolds is presented in §6. We next examine the analogues of pseudoconvex boundaries in calibrated geometry.

Boundary Convexity.

Let $\Omega \subset X$ be a domain with smooth boundary $\partial\Omega$, and let $\rho : X \to \mathbf{R}$ be a **defining function for** $\partial\Omega$, that is, a smooth function defined on a neighborhood of $\overline{\Omega}$ with $\Omega = \{x : \rho(x) < 0\}$, and $\nabla\rho \neq 0$ on $\partial\Omega$. Then $\partial\Omega$ is said to be **ϕ-convex** if

$$\mathcal{H}^\phi(\rho)(\xi) \;\geq\; 0 \qquad \text{for all } \phi - \text{planes } \xi \text{ tangential to } \partial\Omega, \tag{0.1}$$

i.e., for all $\xi \in G(\phi)$ with $\mathrm{span}(\xi) \subset T(\partial\Omega)$. The boundary ϕ is **strictly ϕ-convex** if the inequality in (0.1) is strict everywhere on $\partial\Omega$. These conditions are independent of the choice of defining function ρ.

Theorem 5.6. *Let $\Omega \subset\subset X$ be a compact domain with strictly ϕ-convex boundary, and let $\delta = -\rho$ where ρ is an arbitrary defining function for $\partial\Omega$. Then $-\log\delta : \Omega \to \mathbf{R}$ is strictly ϕ-plurisubharmonic outside a compact subset. In particular, the domain Ω is strictly ϕ-convex at infinity.*

Elementary examples show that the converse of this theorem does not hold in general. However, there is a weak partial converse.

PROPOSITION 5.9. *Let $\Omega \subset\subset X$ be a compact domain with smooth boundary. Suppose ϕ is parallel and consider the function $\delta = \text{dist}(\bullet, \partial\Omega)$. If $-\log\delta$ is ϕ-plurisubharmonic near $\partial\Omega$, then $\partial\Omega$ is ϕ-convex.*

We note that boundary convexity can be interpreted geometrically as follows. Let II denote the second fundamental form of the hypersurface $\partial\Omega$ oriented by the outward-pointing normal. Then $\partial\Omega$ is ϕ-convex if and only if $\text{trace}(II\big|_\xi) \leq 0$ for all ϕ-planes ξ which are tangent to $\partial\Omega$. In the strict case one also has the following.

THEOREM 5.14. *Let (X, ϕ) be a strictly ϕ-convex manifold and $\Omega \subset\subset X$ a domain with smooth boundary. Then the following are equivalent.*

(i) *$\partial\Omega$ is strictly ϕ-convex .*

(ii) *$\text{tr}_\xi \{II_{\partial\Omega}\} < 0$ for all tangential ϕ-planes ξ.*

(iii) *There exists a smooth defining function ρ for $\partial\Omega$ which is strictly ϕ-plurisubharmonic on a neighborhood of $\overline{\Omega}$.*

ϕ-FREE SUBMANIFOLDS AND STRICTLY ϕ-CONVEX SUBDOMAINS.

We next examine the analogues in calibrated geometry of the totally real submanifolds in complex analysis. Using the methods of [HW$_{1,2}$] we then show how to construct strictly ϕ-convex manifolds in enormous families with every topological type allowed by Morse theory.

Let (X, ϕ) be any fixed calibrated manifold. A closed submanifold $M \subset X$ is called **ϕ-free** if there are no ϕ-planes tangential to M, i.e., no $\xi \in G(\phi)$ with $\text{span}\,\xi \subset TM$.

Note that M is automatically ϕ-free if it is **ϕ-isotropic**, that is, if $\phi\big|_M \equiv 0$. .

Any submanifold of dimension $< p$ is ϕ-free, and generic local submanifolds of dimension p are ϕ-free. Furthermore, any submanifold of a ϕ-free submanifold is again ϕ-free.

The **ϕ-free dimension** of (X, ϕ), denoted $\text{fd}(\phi)$, is defined to be the largest dimension of a ϕ-free vector subspace of T_xX for $x \in X$. The first result is the following generalization of the Andreotti-Frenkel Theorem [AF] for Stein manifolds.

THEOREM 6.2. *Suppose (X, ϕ) is a strictly ϕ-convex manifold. Then X has the homotopy type of a CW complex of dimension $\leq \text{fd}(\phi)$.*

For a Kähler manifold (X, ω) of complex dimension n, the ω-free dimension is n and the ω-free submanifolds are those which are totally real (e.g., the Lagrangian submanifolds). Furthermore the $\omega^p/p!$-free dimension is $n + p - 1$ and a submanifold M is $\omega^p/p!$-free if there are no complex p-planes tangent to M at any point.

In Special Lagrangian geometry on an n-dimensional Calabi-Yau manifold (X, ω, ϕ), the ϕ-free dimension is $2n - 2$ and the ϕ-free submanifolds are exactly the symplectic submanifolds (e.g., the complex hypersurfaces).

For a quaternionic Kähler manifold (X, Ψ) of dimension $4n$, where $\Psi = \frac{1}{6}(\omega_I^2 + \omega_J^2 + \omega_K^2)$ is the fundamental 4-form, the Ψ-free dimension is $3n$. For the higher degree calibrations $\Psi_p \equiv \frac{1}{(2p+1)!}(\omega_I^2 + \omega_J^2 + \omega_K^2)^p$ the free dimension is $3(n - p + 1)$.

If (X, Φ) is an 8-dimensional Spin$_7$-manifold with Cayley calibration Φ, $\text{fd}(\Phi) = 4$.

If (X, ϕ) is a 7-dimensional G_2-manifold with associative calibration ϕ, then $\mathrm{fd}(\phi) = 4$. So if X is ϕ-convex , it has homotopy dimension ≤ 4. Recently, I.Unal [U] has shown that *for every connected manifold M of dimension < 4 (compact or non-compact) there exists a strictly ϕ-convex G_2-manifold which is homotopy-equivalent to M.*

The relationship between ϕ-free submanifolds and convexity is expressed in the next two results.

THEOREM 6.4. *Suppose M is a closed submanifold of (X, ϕ) and let $\mathrm{dist}^2_M(x)$ denote the square of the distance to M. Then M is ϕ-free if and only if the function $\mathrm{dist}^2_M(x)$ is strictly ϕ-plurisubharmonic at each point in M (and hence in a neighborhood of M).*

The existence of ϕ-free submanifolds ensures the existence of many strictly ϕ-convex domains in (X, ϕ).

THEOREM 6.6. *Suppose M is a ϕ-free submanifold of (X, ϕ). Then there exists a fundamental system $\mathcal{F}(M)$ of strictly ϕ-convex neighborhoods of M, each of which admits a deformation retraction onto M.*

This result provides rich families of strictly convex domains. The neighborhoods in $\mathcal{F}(M)$ include the sets $\{x : \mathrm{dist}_M(x) < \epsilon(x)\}$ for positive functions $\epsilon \in C^\infty(M)$ which die arbitrarily rapidly at infinity. As noted, any submanifold of dimension $< p$ is ϕ-free. Furthermore, any submanifold of a ϕ-free submanifold is again ϕ-free.

For example if X is a Calabi-Yau manifold with Special Lagrangian calibration ϕ, then any symplectic submanifold $Y \subset X$ is ϕ-free, as is any smooth submanifold $A \subset Y$. The topological type of such manifolds A can be quite complicated.

This construction can be refined even further by replacing the submanifold $A \subset Y$ with an arbitrary closed subset. It turns out that the following two classes of subsets:

(1) Closed subsets A of ϕ-free submanifolds

(2) Zero sets of non-negative strictly ϕ-plurisubharmonic functions f

are essentially the same.

We mention that the operator d^ϕ has been independently found by M. Verbitsky [V] who studied the generalized Kähler theory (in the sense of Chern) on G_2-manifolds. The authors would like to thank Robert Bryant for useful comments and conversations related to this paper.

1. Plurisubharmonic Functions

Suppose ϕ is a calibration on a riemannian manifold X. The ϕ-Grassmannian, denoted $G(\phi)$, consists of the unit simple p-vectors ξ with $\phi(\xi) = 1$, i.e., the ϕ-planes. An oriented submanifold M is a *ϕ-submanifold*, or is *calibrated by* ϕ, if the oriented unit tangent space $\vec{T}_x M$ lies in $G_x(\phi)$ for each $x \in M$, or equivalently, if ϕ restricts to M to be the volume form on M. Let $n = \dim X$ and $p = \text{degree}(\phi)$.

DEFINITION 1.1. The **d^ϕ-operator** is defined by

$$d^\phi f \;=\; \nabla f \lrcorner \phi$$

for all smooth functions f on X.

Hence

$$d^\phi : \mathcal{E}^0(X) \;\longrightarrow\; \mathcal{E}^{p-1}(X) \qquad \text{and} \qquad dd^\phi : \mathcal{E}^0(X) \;\longrightarrow\; \mathcal{E}^p(X)$$

where $\mathcal{E}^p(X)$ denotes the space of C^∞ p-forms on X. This dd^ϕ operator provides a way of defining plurisubharmonic functions in calibrated geometry when the calibration ϕ is parallel.

If ω is a Kähler form on a complex manifold, then $d^\omega \;=\; d^c \;=\; -J \circ d$ is the conjugate differential. Thus, the dd^ϕ-operator generalizes the dd^c-operator in complex geometry. Although no analogue of a holomorphic function exists on a calibrated manifold, there is an analogue of the real part of a holomorphic function.

DEFINITION 1.2. Suppose $\nabla\phi = 0$. A function $f \in C^\infty(X)$ is **ϕ-plurisubharmonic** if

$$(dd^\phi f)(\xi) \;\geq 0 \qquad \text{for all} \;\; \xi \in G(\phi).$$

The set of such functions will be denoted PSH(X, ϕ). If $(dd^\phi f)(\xi) > 0$ for all $\xi \in G(\phi)$, then f is **strictly ϕ-plurisubharmonic**. If $(dd^\phi f)(\xi) = 0$ for all $\xi \in G(\phi)$, then f is **ϕ-pluriharmonic**. Finally, f is **partially ϕ-pluriharmonic** if f is ϕ-plurisubharmonic and, at each point, there exists a ϕ-plane ξ with $(dd^\phi f)(\xi) = 0$.

Note that f is ϕ-pluriharmonic if and only if both f and $-f$ are ϕ-plurisubharmonic, and that f is partially ϕ-pluriharmonic if and only if f is ϕ-plurisubharmonic but not strict at any point.

REMARK 1.3. If ϕ is not parallel, we define ϕ-plurisubharmonic functions by replacing $dd^\phi f$, in Definition 1.2, with

$$\mathcal{H}^\phi(f) = dd^\phi f - \nabla_{\nabla f}\phi.$$

This modified dd^ϕ-operator is discussed in detail in Section 2. Note that the difference $\nabla_{\nabla f}\phi$ is a first order operator.

EXAMPLE. Consider the Special Lagrangian calibration $\phi = \text{Re}(dz)$ on $\mathbf{C}^n$. Let Z_{ij} denote the bidegree $(n-1, 1)$ form obtained from $dz = dz_1 \wedge \cdots \wedge dz_n$ by replacing dz_i with $d\bar{z}_j$ (in the ith position). A short calculation shows that

$$dd^\phi f \;=\; 2\text{Re}\Big\{ \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} Z_{ij} \Big\} + (\Delta f)\text{Re}(dz) \tag{1.1}$$

NOTE THAT: 1) The constant functions are ϕ-pluriharmonic .

2) If $a, b > 0$ and $f, g \in \mathrm{PSH}(X, \phi)$, then $af + bg \in \mathrm{PSH}(X, \phi)$.

The next result justifies the use of the word plurisubharmonic in the context of a ϕ-geometry. A calibration ϕ is **integrable** if for each point $x \in X$ and each $\xi \in G_x(\phi)$ there exists a ϕ-submanifold M through x with $\overrightarrow{T}_x M = \xi$.

THEOREM 1.4. *Let (X, ϕ) be any calibrated manifold. If a function $f \in C^\infty(X)$ is ϕ-plurisubharmonic, then the restriction of f to any ϕ-submanifold $M \subset X$ is subharmonic in the induced metric. If ϕ is integrable, then the converse holds.*

Theorem 1.4 is an immediate consequence of the formula

$$\mathcal{H}^\phi f\big|_M \;=\; \left(dd^\phi f - \nabla_{\nabla f}\phi\right)\big|_M \;=\; (\Delta_M f)\,\mathrm{vol}_M \tag{1.2}$$

This formula follows from the three equations (2.7), (2.12), and (2.15), proved below, and the fact that ϕ-submanifolds are minimal submanifolds. We continue for the moment to present results whose proofs will be given in Section 2.

The ϕ-plurisubharmonic functions enjoy many of the useful properties of their classical cousins in complex analysis. The next result is useful, in particular if one wishes to only consider smooth ϕ-plurisubharmonic functions

LEMMA 1.5. *Let $f, g \in C^\infty(X)$ be ϕ-plurisubharmonic.*

a) If $\psi \in C^\infty(\mathbf{R})$ is convex and increasing, then $\psi \circ f$ is ϕ-plurisubharmonic.

b) If $\psi \in C^\infty(\mathbf{R}^2)$ is convex, and is increasing in each variable, then $\psi(f, g)$ is ϕ-plurisubharmonic .

Proof. See Appendix B.

REMARK 1.6. Part b) can be used to construct a ϕ-plurisubharmonic smoothing h_ϵ of the maximum $h = \max\{f, g\}$ of two ϕ-plurisubharmonic functions f, g with:

1) h_ϵ decreasing as $\epsilon \to 0$,

2) $h_\epsilon - \epsilon \leq \max\{f, g\} \leq h_\epsilon$ for all $\epsilon > 0$,

3) $h_\epsilon = \max\{f, g\}$ on the set where $|f - g| \geq \epsilon$.

To see this, note first that $\max\{t_1, t_2\} = \frac{1}{2}(t_1 + t_2) + \frac{1}{2}|t_1 - t_2|$. Now choose a convex function $\varphi \in C^\infty(\mathbf{R})$ with $\varphi(0) = \frac{1}{2}$, $|\varphi'| \leq 1$, and $\varphi(s) \geq |s|$ with equality when $|s| \geq 1$. Then $\varphi_\epsilon(s) = \epsilon\varphi(\frac{s}{\epsilon})$ provides a smooth approximation to the function $|s|$, namely $\varphi_\epsilon(s) - \epsilon \leq |s| \leq \varphi_\epsilon(s)$. The function $\psi_\epsilon(t_1, t_2) = \frac{1}{2}(t_1 + t_2) + \frac{1}{2}\varphi_\epsilon(t_1 - t_2)$ approximates $\max\{t_1, t_2\}$, and the function $h_\epsilon = \psi_\epsilon(f, g)$ approximates $h = \max\{f, g\}$. To complete the proof, note that

$$\frac{\partial \psi_\epsilon}{\partial t_1} \;=\; \tfrac{1}{2}\left(1 + \varphi'\left(\tfrac{t_1 - t_2}{\epsilon}\right)\right), \qquad \frac{\partial \psi_\epsilon}{\partial t_2} \;=\; \tfrac{1}{2}\left(1 - \varphi'\left(\tfrac{t_1 - t_2}{\epsilon}\right)\right)$$

and

$$2\epsilon \mathrm{Hess}\psi_\epsilon \;=\; \varphi''\left(\tfrac{t_1 - t_2}{\epsilon}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

While pluriharmonic functions are often scarce, the partially pluriharmonic functions represent the calibrated analogue of solutions to the homogeneous Monge-Ampére equation, and they are sufficiently abundant to solve the Dirichlet Problem [$HL_{3,6}$].

For now we mention a "fundamental" example.

PROPOSITION 1.7. *Suppose $\phi \in \Lambda^p \mathbf{R}^n$ is a parallel calibration. Set*

$$E(x) \;=\; \log|x| \;\text{ if } p=2 \qquad \text{and} \qquad E(x) \;=\; -\frac{1}{(p-2)}\,\frac{1}{|x|^{p-2}} \;\text{ if } p \geq 3 \tag{1.3}$$

Then E is ϕ-plurisubharmonic on $\mathbf{R}^n - \{0\}$. Moreover, E is partially ϕ-pluriharmonic on $\mathbf{R}^n - \{0\}$ if and only if each unit vector $e \in \mathbf{R}^n$ is contained in a ϕ-plane $\xi \in G(\phi)$.

REMARK **(The Abundance of ϕ-plurisubharmonic Functions)**. We shall see in the next section that any convex function on the riemannian manifold X is automatically ϕ-plurisubharmonic. However, there always exist huge families of locally defined ϕ-plurisubharmonic functions which are not convex. This follows, for example, from duality considerations as in Remark 2.9 below. However, in section 6 we give a general construction of ϕ-plurisubharmonic functions from any ϕ-free submanifold, which shows that such functions exist in abundance.

Pluriharmonic Functions

The ϕ-pluriharmonic functions are a natural replacement for the holomorphic functions in complex geometry. However, while ϕ-plurisubharmonic functions are abundant, the ϕ-pluriharmonic functions are often quite scarce. To illustrate this phenomenon we shall sketch some of the basic facts in the "classical" cases.

To begin we note that for some calibrations ϕ, one has that:

$$dd^\phi f \;=\; 0 \qquad \text{if and only if} \qquad (dd^\phi f)(\xi) \;=\; 0 \quad \text{for all } \xi \in G(\phi) \tag{1.4}$$

while for others this is not true. It is the right hand side that defines pluriharmonicity. If (1.4) holds and the basic map λ_ϕ, defined in section 2, is everywhere injective (as in Example 1.14), then the only pluriharmonic functions are the affine functions, i.e., the functions with parallel gradient. Note that if f is affine, then ∇f splits the manifold locally as a riemannian product $X = \mathbf{R} \times X_0$.

EXAMPLE 1.8. **(Complex geometry)**. Let ω be a Kähler form on a complex manifold X. Then $d^\omega = d^c$ is the conjugate differential, $dd^c f$ is the complex hermitian Hessian of f, $G(\omega)$ is the grassmannian of complex lines, and the statement (1.4) is valid. In particular, the ω-pluriharmonic functions are just the classical pluriharmonic functions on X.

For the higher divided powers $\Omega_p = \frac{1}{p!}\omega^p$ one computes that $dd^{\Omega_p} f = \Omega_{p-1} dd^c f$. Furthermore, it can be deduced from the discussion in Remark 2.13 that (1.4) holds in this case. Therefore, the Ω_p-pluriharmonic functions are also just the classical pluriharmonic functions.

EXAMPLE 1.9. **(Quaternionic-Kähler geometry).** Let $\mathbf{H}$ denote the quaternions and consider $\mathbf{H}^n$ as a right-$\mathbf{H}$ vector space. Each of the complex structures I, J, K (right multiplication by i, j, k) determines a Kähler form $\omega_I, \omega_J, \omega_K$ respectively. The 4-form

$$\Psi \equiv \frac{1}{6}(\omega_I^2 + \omega_J^2 + \omega_K^2) \tag{1.5}$$

on $\mathbf{H}^n \equiv \mathbf{R}^{4n}$ is a calibration with $G(\Psi)$ consisting of the oriented quaternion lines in $\mathbf{H}^n$. In this case, $dd^\Psi f \equiv 0$ if and only if $\mathrm{Hess} f \equiv 0$. However, the assertion (1.4) is not valid in this case, and in fact there is a rich family of Ψ-pluriharmonic functions. For example, if f is ω_I-pluriharmonic, then f is Ψ-pluriharmonic. Hence, so is any ω-pluriharmonic f where $\omega = a\omega_I + b\omega_J + c\omega_K$ with $a^2 + b^2 + c^2 = 1$.

It is well known that the only Ψ-submanifolds in $\mathbf{H}^n$ are the affine quaternion lines.

Of course the calibration (1.5) exists on any quaternionic Kähler manifold, i.e., one with $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$-holonomy. (See [GL] for examples.) With this full holonomy group it seems unlikely that there are many Ψ-pluriharmonic functions. However, if the holonomy is contained in Sp_n, they exist in abundance as seen in the next example.

EXAMPLE 1.10. **(Hyper-Kähler manifolds).** Let $(X, \omega_I, \omega_J, \omega_K)$ be a hyper-Kähler manifold. Then X carries several parallel calibrations. There are, of course, the Kähler forms $\omega = a\omega_I + b\omega_J + c\omega_K$ with $a^2 + b^2 + c^2 = 1$, and two others of particular interest.

(1) Let $\Psi = \frac{1}{6}(\omega_I^2 + \omega_J^2 + \omega_K^2)$. Then as in Example 1.9, *any ω-pluriharmonic function is Ψ-pluriharmonic*. Hence, the sheaf of Ψ-pluriharmonic functions is quite rich on any manifold with SP_n-holonomy. On the other hand there are precious few Ψ-submanifolds.

(2) Consider the generalized Cayley form $\Xi \equiv \frac{1}{2}(\omega_I^2 - \omega_J^2 - \omega_K^2)$. For this calibration there exist no interesting pluriharmonic functions, at least in dimension 8, but there are many Ξ-submanifolds (cf. [BH]).

EXAMPLE 1.11. **(Double point geometry).** Let $\phi = dx_1 \wedge \cdots \wedge dx_n + dy_1 \wedge \cdots \wedge dy_n$ in $\mathbf{R}^{2n}$ for $n \geq 3$. The only ϕ-planes are those parallel to the x or y axes. An easy calculation shows that $dd^\phi f = 0$ if and only if $f(x, y) = g(x) + h(y)$ for harmonic functions g and h. However, a function $f(x, y)$ is ϕ-pluriharmonic if and only if it is harmonic in x and y separately. This is a simple example where (1.4) fails.

In all the following examples ϕ-pluriharmonic functions are quite scarce.

EXAMPLE 1.12. **(Special Lagrangian geometry).** Consider the Special Lagrangian calibration $\phi = \mathrm{Re}(dz)$ on $\mathbf{C}^n$. For this calibration one can show that (1.4) is valid. Consequently, Lei Fu [Fu] has described all the ϕ-pluriharmonic functions.

PROPOSITION 1.13. *Let f be a Special Lagrangian pluriharmonic function defined locally on $\mathbf{C}^n$, $n \geq 3$. Then $f = A + Q$ where A is affine and Q is a traceless hermitian quadratic function.*

Proof. If $dd^\phi f = 0$ and $n \geq 3$ (so that Z_{ij} and $\overline{Z}_{ij}$ are of different bi-degrees), then (1.1) implies that $\frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} = 0$ for all i, j. Therefore, all third partial derivatives of f are zero. For polynomials of degree ≤ 2 the result is transparent from (1.1). ∎

EXAMPLE 1.14. **(Associative, Coassociative and Cayley geometry).** Consider one of the calibrations:

1. (Associative) $\phi(x \wedge y \wedge z) = \langle x, yz \rangle$ for $x, y, z \in \mathrm{Im}\mathbf{O}$
2. (Coassociative) $\psi = *\phi$ on $\mathrm{Im}\mathbf{O}$
3. (Cayley) $\Phi(x \wedge y \wedge z \wedge w) = \langle x, y \times z \times w \rangle$ for $x, y, z, w \in \mathbf{O}$

where $\mathbf{O}$ denotes the octonions. As in the Special Lagrangian case one can show that (1.4) is valid for each of these calibrations. Furthermore, an application of representation theory shows the maps $\lambda_\phi, \lambda_\psi$ and λ_Φ are injective. These calculations carry over to manifolds with G_2 or Spin_7-holonomy to establish the following.

PROPOSITION 1.15. *Let X be a manifold with holonomy contained in G_2 or $Spin_7$ and having dimension 7 or 8 respectively. Suppose ϕ is a parallel calibration on X of one of the three types above. Then every ϕ-pluriharmonic function on X is affine. Moreover, if the holonomy is exactly G_2 or $Spin_7$, every ϕ-pluriharmonic function is constant.*

Proof. The first assertion follows because (1.4) is valid and the λ-maps are injective. The second follows because any non-constant affine function on X would reduce its holonomy to a subgroup of $\{1\} \times SO_{n-1}$. ∎

EXAMPLE 1.16. **(Lie group geometry).** Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$, defined as the set of left-invariant vector fields on G.

1) Consider the fundamental 3-form ϕ on G defined by $\langle x, [y, z] \rangle$ and normalized to have comass one. Calculations indicate that in all but a finite number of cases non-constant pluriharmonic functions do not exist, however there are ϕ-submanifolds, namely the "minimal" SU_2-subgroups (cf. [B], [T], [Th]).

2) Consider $*\phi$. The $*\phi$-submanifolds are given by certain components of the cut locus C. Is $G - C$ strictly ϕ-convex?

EXAMPLE 1.17. **(Gromov manifolds).** By a *Gromov manifold* we mean an ensemble $(X, \omega, J, \langle \cdot, \cdot \rangle)$ where (X, ω) is a symplectic manifold, J is an almost complex structure on X and $\langle \cdot, \cdot \rangle$ is a riemannian metric with the property that

$$\omega(v, w) = \langle Jv, w \rangle$$

for all $v, w \in T_x X$ at all $x \in X$. Every symplectic manifold has many Gromov structures. Generically the almost complex structure J is not integrable, and the only ω-pluriharmonic functions are the constants. However, there are generally many ω-submanifolds (the pseudo-holomorphic curves) and there are many ω-plurisubharmonic functions as we shall see below. It is important to note here that the operator dd^c is not appropriate for this context since $\nabla\omega \neq 0$. However, our notion of plurisubharmonicity works well and has the property that ω-plurisubharmonic functions are subharmonic on all pseudo-holomorphic curves.

We note that on a Gromov manifold there exists a class of *Lagrangian plurisubharmonic functions* with many good properties. For example, they are subharmonic when restricted to any Lagrangian submanifold which is minimal. This is explored in a separate paper [HL_5].

2. The ϕ-Hessian.

In this section we prove Theorem 1.4 and Proposition 1.7. The arguments will involve ideas and notation important for the rest of the paper. A generalization of Theorem 1.4 to submanifolds which are *ϕ-critical* can be found in Appendix A.

Recall (cf. [ON], p. 86) that the **Hessian**, or second covariant derivative, of a smooth function f on a riemannian manifold X is defined on tangent vector fields V, W by

$$\text{Hess}(f)(V,W) \;\equiv\; V(Wf) - (\nabla_V W)f \tag{2.1}$$

where ∇ denotes the riemannian connection. Note that $V(Wf) - (\nabla_V W)f = V(\langle W, \nabla f\rangle) - \langle \nabla_V W, \nabla f\rangle = \langle W, \nabla_V(\nabla f)\rangle$ so that at a point $x \in X$, the Hessian is the symmetric 2-tensor, or the symmetric linear map of $T_x X$ given by

$$\text{Hess}(f)(V) \;=\; \nabla_V(\nabla f). \tag{2.2}$$

In terms of local coordinate vector fields

$$\text{Hess}(f)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) \;=\; \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x_k}$$

where Γ_{ij}^k are the standard Christoffel symbols of the riemannian connection.

Let V be a real inner product space. Given an element $\phi \in \Lambda^p V^*$, we define a linear map, central to this paper,

$$\lambda_\phi : \text{End}(V) \;\longrightarrow\; \Lambda^p V^* \tag{2.3}$$

by

$$\lambda_\phi(A) \;\equiv\; D_{A^t}(\phi)$$

where D_{A^t} denotes the extension of the transpose $A^t : V^* \to V^*$ to $D_{A^t} : \Lambda^p V^* \to \Lambda^p V^*$ as a derivation. That is, on simple vectors, one has

$$D_{A^t}(v_1 \wedge \ldots \wedge v_p) \;=\; \sum_{k=1}^{p} v_1 \wedge \ldots \wedge A^t(v_k) \wedge \ldots \wedge v_n$$

NOTE . Recall that the natural inner product on $\text{End}(V)$ is given by:

$$\langle A, B\rangle \;=\; \text{tr} AB^* \qquad \text{for } A, B \in \text{End}(V)$$

Using this inner product we have the adjoint map

$$\lambda_\phi^* : \Lambda^p V^* \;\longrightarrow\; \text{End}(V) \tag{2.4}$$

which will also be important.

NOTE . If we identify $\text{End}(V)$ with the Lie algebra $\mathfrak{gl}(V)$ of $GL(V)$, then λ_ϕ is the differential of the standard representation of $GL(V)$ on $\Lambda^p V^*$ at ϕ. Therefore, $\ker(\lambda_\phi)$ is the

Lie algebra of the subgroup $H_\phi \equiv \{g \in GL(V) : g(\phi) = \phi\}$ and $\ker(\lambda_\phi) \cap \mathrm{SkewEnd}(V)$ is the Lie algebra of the compact subgroup $K_\phi = H_\phi \cap \mathrm{O}(V)$.

DEFINITION 2.1. The **ϕ-Hessian** of a function $f \in C^\infty(X)$ is the p-form $\mathcal{H}^\phi(f)$ defined by letting the symmetric endomorphism $\mathrm{Hess} f$ act on ϕ as a derivation, i.e.,

$$\mathcal{H}^\phi(f) \;\equiv\; D_{\mathrm{Hess}^t f}(\phi). \tag{2.5}$$

In terms of the bundle map $\lambda_\phi : \mathrm{End}(TX) \to \Lambda^p T^*X$,

$$\mathcal{H}^\phi(f) \;\equiv\; \lambda_\phi(\mathrm{Hess} f) \tag{2.6}$$

is the image of the Hessian of f.

The second order differential operators dd^ϕ and $\mathcal{H}^\phi$ differ by a pure first order operator. This is the first of the three equations needed to prove Theorem 1.4.

THEOREM 2.2. *If ϕ is a closed form on X, then*

$$\mathcal{H}^\phi(f) \;=\; dd^\phi f - \nabla_{\nabla f}\phi \tag{2.7}$$

Proof. By (2.2) we have $(\mathrm{Hess} f)(V) = \nabla_V \nabla f = [V, \nabla f] + \nabla_{\nabla f} V$, i.e.,

$$\mathrm{Hess} f \;=\; -\mathcal{L}_{\nabla f} + \nabla_{\nabla f}$$

as operators on vector fields ($\mathcal{L}$ = the Lie derivative). The right hand side of this formula has a standard extension to all tensor fields as a derivation that commutes with contractions. It is zero on functions, that is, it is a bundle endomorphism whose value on T^*X is minus the transpose of its value on TX. In particular, we find that $D_{\mathrm{Hess}^t f} = \mathcal{L}_{\nabla f} - \nabla_{\nabla f}$ on p-forms, i.e.,

$$\mathcal{H}^\phi(f) \;=\; \mathcal{L}_{\nabla f}(\phi) - \nabla_{\nabla f}\phi \tag{2.8}$$

Finally, since $d\phi = 0$, the classical formula $d \circ \lrcorner + \lrcorner \circ d = \mathcal{L}$ gives

$$dd^\phi f \;=\; d(\nabla f \lrcorner \phi) \;=\; \mathcal{L}_{\nabla f}(\phi) \qquad \blacksquare$$

Many of the nice results for the dd^ϕ-operator continue to hold in the non-parallel case after replacing it with the ϕ-Hessian. Perhaps even more importantly, many properties of the dd^ϕ-operator in the parallel case can best be understood by considering the ϕ-Hessian.

The second formula needed for the proof of Theorem 1.4 is algebraic in nature, involving the bundle map $\lambda_\phi : \mathrm{End}(TX) \to \Lambda^p T^*X$. Consequently, as before, we replace T_xX by a general inner product space V. If ξ is a p-plane in V (not necessarily oriented), let $P_\xi : V \to \xi$ denote orthogonal projection. The following, along with its reinterpretations $(2.9)'$ and (2.12), is a central result of this paper.

THEOREM 2.3. *Suppose ϕ has comass one. For each $A \in \mathrm{End}(V)$,*

$$(\lambda_\phi A)(\xi) \;=\; \langle A, P_\xi \rangle \qquad \text{if } \xi \in G(\phi). \tag{2.9}$$

Equivalently,

$$(\lambda_\phi^*)(\xi) \;=\; P_\xi \qquad \text{if } \xi \in G(\phi). \tag{2.10}$$

Note that if $e_1, ..., e_p$ is an orthonormal basis for the p-plane ξ, then

$$\langle A, P_\xi \rangle \;=\; \sum_{j=1}^{p} \langle e_j, Ae_j \rangle.$$

Consequently, it is natural to refer to $\langle A, P_\xi \rangle$ as the ξ**-trace of** A and to use the notation

$$\mathrm{tr}_\xi A \;\equiv\; \langle A, P_\xi \rangle.$$

In particular, for each $A \in \mathrm{End}(V)$,

$$(\lambda_\phi A)(\xi) \;=\; \mathrm{tr}_\xi A \qquad \text{if } \xi \in G(\phi). \tag{2.9$'$}$$

Suppose $\xi \in G(p, V) \subset \Lambda_p V$ is a unit simple p-vector. If a, b are unit vectors in V with $a \in \mathrm{span}\, \xi$ and $b \perp \mathrm{span}\, \xi$, then

$$b \wedge (a \lrcorner \xi)$$

is called a **first cousin** of ξ. The first cousins of ξ span the tangent space to the Grassmannian $G(p, V) \subset \Lambda_p V$ at the point ξ. Since ϕ restricted to $G(p, V)$ is a maximum on $G(\phi)$, this fact implies the following result, which we shall use frequently.

LEMMA 2.4. **(The First Cousin Principle).** *If $\phi \in \Lambda^p V^*$ has comass one and $\xi \in G(\phi)$, then*

$$\phi(\eta) \;=\; 0$$

for all first cousins $\eta = b \wedge (a \lrcorner \xi)$ of ξ.

Note that $D_{(b \otimes a)^t} \phi = D_{a \otimes b} \phi = a \wedge (b \lrcorner \phi)$ and $D_{b \otimes a} \xi = b \wedge (a \lrcorner \xi)$ so that if $A = b \otimes a$ is rank one, then

$$\lambda_\phi(b \otimes a)(\xi) \;=\; (D_{a \otimes b} \phi)(\xi) \;=\; \phi(D_{b \otimes a} \xi) \;=\; \phi(b \wedge (a \lrcorner \xi)) \tag{2.11}$$

Proof of Theorem 2.3. Pick an orthonormal basis for ξ and extend to an orthonormal basis of V. It suffices to prove (2.9) when $A = b \otimes a$ with a and b elements of this basis. It is easy to see that $\langle b \otimes a, P_\xi \rangle = 0$ unless $a = b \in \xi$, in which case $\langle a \otimes a, P_\xi \rangle = 1$. By equation (2.11) we have $\lambda_\phi(b \otimes a)(\xi) = \phi(b \wedge (a \lrcorner \xi))$ and $b \wedge (a \lrcorner \xi) = 0$ unless $a \in \xi$ and either $b \in \xi^\perp$ or $b = a$. If $b \in \xi^\perp$, then $(b \wedge (a \lrcorner \xi)$ is a first cousin of ξ and $\phi((b \wedge (a \lrcorner \xi)) = 0$ by the First Cousin Principle. If $a = b \in \xi$, then $b \wedge (a \lrcorner \xi) = \xi$ and therefore $\phi((b \wedge (a \lrcorner \xi)) = \phi(\xi) = 1$. ∎

Theorem 2.3 has many consequences. We mention several. From (2.9)$'$ we have:

COROLLARY 2.5. *Suppose (X, ϕ) is a calibrated manifold. For each function $f \in C^\infty(X)$,*

$$\mathcal{H}^\phi(f)(\xi) \;=\; \mathrm{tr}_\xi(\mathrm{Hess} f) \qquad \text{if } \xi \in G(\phi). \tag{2.12}$$

This equation (2.12) is the second equation needed in the proof of Theorem 1.4.

REMARK . Equation (2.12) provides an alternative definition of ϕ-plurisubharmonic (as well as strictly ϕ-plurisubharmonic and ϕ-pluriharmonic) functions, which bypasses the bundle map λ_ϕ and uses only the trace of the Hessian of f on ϕ-planes ξ.

Another application of Theorem 2.3 is given by:

COROLLARY 2.6. *If $A \in \mathrm{End}(V)$ is skew, then the p-form $\lambda_\phi A$ vanishes on $G(\phi)$.*

See Remarks A.5 and A.7 for an extension of this to a recent result in [R].

Theorem 2.3 has another useful consequence used to prove Lemma 1.5. Note that for $A, B \in \mathrm{Sym}^2(\mathrm{V}) \subset \mathrm{End(V)}$, if $A \geq 0$, $B \geq 0$, then $\langle A, B\rangle = \mathrm{tr}AB \geq 0$. Hence for all $\xi \in G_p(V)$ one has $\langle e \otimes e, P_\xi\rangle \geq 0$, and more generally $\langle A, P_\xi\rangle \geq 0$ whenever $A \geq 0$. Since df and ∇f are metrically equivalent,

$$\lambda_\phi(\nabla f \otimes \nabla f) = df \wedge (\nabla f \lrcorner \phi) = df \wedge d^\phi f. \tag{2.13}$$

Therefore, Theorem 2.3 has the following consequence.

COROLLARY 2.7. *For any $f \in C^\infty(X)$,*

$$(df \wedge d^\phi f)(\xi) \;=\; |\nabla f \lrcorner \xi|^2 \;\geq\; 0 \qquad \textit{for all } \; \xi \in G(\phi). \tag{2.14}$$

Theorem 2.3 can also be used to understand the relationship between convex functions and ϕ-plurisubharmonic functions. A function $f \in C^\infty(X)$ is called **convex** if $\mathrm{Hess} f \geq 0$ at each point, and it is called **affine** if $\mathrm{Hess} f \equiv 0$ on X. (If f is affine, ∇f splits X locally as a riemannian product $\mathbf{R} \times X_0$.)

COROLLARY 2.8. *Every convex function is ϕ-plurisubharmonic, and every strictly convex function is strictly ϕ-plurisubharmonic (and every affine function is ϕ-pluriharmonic).*

REMARK 2.9. The converse always fails; there are always ϕ-plurisubharmonic functions which are not convex. To see this, consider first the euclidean case with $X = V$ and ϕ parallel. Recall that the orthogonal projections P_e onto lines in V generate the extreme rays of the convex cone of convex functions (positive semi-definite quadratic forms) in $\mathrm{Sym}^2\mathrm{V} \subset \mathrm{End(V)}$. This cone is self-dual. The projections $P_\xi = \lambda_\phi^*(\xi)$ for $\xi \in G(\phi)$ generate a proper convex subcone (in fact a proper convex subcone of the cone generated by orthogonal projections onto p-planes). Hence, by the Bipolar Theorem there must exist a non-convex quadratic function $Q \in \mathrm{Sym}^2\mathrm{V}$ with $\langle Q, P_\xi\rangle \geq 0$ for all $\xi \in G(\phi)$. By (2.9), Q is ϕ-plurisubharmonic. (Recall that for a convex cone $C \subset \mathbf{R}^n$ with vertex at the origin, the Bipolar Theorem states that $(C^0)^0 = \overline{C}$ where $C^0 \equiv \{w \in \mathbf{R}^n : \langle w, v\rangle \geq 0 \text{ for all } v \in C\}$ is the dual cone.)

We now recall some elementary facts about submanifolds. Given a submanifold $\overline{X} \subset X$, let $(\bullet)^T$ and $(\bullet)^N$ denote orthogonal projection of T_xX onto the tangent and normal spaces of $\overline{X}$ respectively. Then the canonical riemannian connection $\overline{\nabla}$ of the induced metric on $\overline{X}$ is given by $\overline{\nabla}_V W = (\nabla_V W)^T$ for tangent vector fields V, W on $\overline{X}$. The **second fundamental form** is defined by

$$B_{V,W} \equiv (\nabla_V W)^N = \nabla_V W - \overline{\nabla}_V W.$$

This is a symmetric bilinear form on $T\overline{X}$ with values in the normal space. Its trace $H = \mathrm{trace}\, B$ is the **mean curvature vector field** of $\overline{X}$, and $\overline{X}$ is called a **minimal submanifold** if $H \equiv 0$. Finally, let $\overline{\Delta}$ denote the Laplace-Beltrami operator on $\overline{X}$ and $\overline{\mathrm{Hess}}$ denote the Hessian operator on $\overline{X}$. The proof of the following is straightforward.

$$\overline{\mathrm{Hess}}(f)(V,W) = \mathrm{Hess}(f)(V,W) - B_{V,W} \cdot f$$

for tangent vectors $V, W \in T\overline{X}$. Taking the $T\overline{X}$-trace yields:

$$\overline{\Delta} f = \mathrm{tr}_{T\overline{X}} \mathrm{Hess} f - H(f).$$

With a change of notation, this is the final formula needed to prove Theorem 1.4.

PROPOSITION 2.10. *Suppose M is a p-dimensional submanifold of X with mean curvature vector field H. Then for each $f \in C^\infty(X)$,*

$$\Delta_M f = \mathrm{tr}_{TM} \mathrm{Hess} f - H(f) \qquad \text{on } M. \tag{2.15}$$

COROLLARY 2.11. *Suppose M is a ϕ-submanifold of X. Then*

$$\mathcal{H}^\phi(f)\big|_M = (\Delta_M f)\mathrm{vol}_M. \tag{2.16}$$

Proof. Combine (2.12) and (2.15) with the fact that $H = 0$. ∎

Combining this with (2.7) gives equation (1.2) and proves Theorem 1.4.

Given vectors $u, v \in V$, define $u \circ v \in \mathrm{Sym}^2(\mathrm{V})$ by $u \circ v(w) = \frac{1}{2}(\langle v, w\rangle u + \langle u, w\rangle v)$.

Proof of Proposition 1.7. In all cases $2 \le p \le n$

$$\mathrm{Hess}_x E = \frac{p}{|x|^p}\left(\frac{1}{p} \cdot I - e \circ e\right) \qquad \text{with } e = \frac{x}{|x|}.$$

Set $H = \frac{1}{p} I - e \circ e$. Then

$$\lambda_\phi(H) = \phi - e \wedge (e \lrcorner \phi) = e \lrcorner (e \wedge \phi).$$

Since $(e \wedge (e \lrcorner \phi))(\xi) = \phi(e \wedge (e \lrcorner \xi))$ and $e \wedge (e \lrcorner \xi)$ is a simple p-vector of norm ≤ 1, for each unit simple p-vector $\xi \in G(\phi)$ we have

$$\lambda_\phi(H)(\xi) \ge 0 \qquad \text{for all } \xi \in G(p, \mathbf{R}^n).$$

This proves that E is ϕ-plurisubharmonic for all calibrations ϕ.

Finally, suppose $\xi \in G(\phi)$, i.e., $\phi(\xi) = 1$. Then we have $\lambda_\phi(H) = 0$ if and only if $(e \wedge (e \lrcorner \phi))(\xi) = 1$, which is equivalent to $(e \wedge (e \lrcorner \xi)) = \xi$ or $e \in \operatorname{span} \xi$. This proves that E is partially ϕ-pluriharmonic on $\mathbf{R}^n - \{0\}$ if and only if every vector e is contained in a ϕ-plane. ∎

For future reference we add a remark.

REMARK 2.12. When ϕ is harmonic, the operator d^ϕ can be expressed in terms of the Hodge d^*-operator as

$$d^\phi f \;=\; -d^*(f\phi)$$

and therefore

$$dd^\phi f \;=\; -dd^*(f\phi).$$

To prove this, first note that if $v \in T_xX$ and $\alpha \in T_x^*X$ are metrically equivalent, then $v \lrcorner \phi = (-1)^{(n-p)(p-1)} * (\alpha \wedge *\phi)$. Hence, $d^\phi f = \nabla f \lrcorner \phi = (-1)^{(n-p)(p-1)} * (df \wedge *\phi) = (-1)^{(n-p)(p-1)}\{*(d(f * \phi) - f(d * \phi))\} = (-1)^{(n-p)(p-1)}\{*d * (f\phi) - f * d * \phi\}$, and since $d^* = (-1)^{np+n+1} * d*$, we conclude that

$$d^\phi f \;=\; f d^*\phi - d^*(f\phi)$$

so that the first equation holds if ϕ is a harmonic form, and in particular if ϕ is parallel. Note also that for $\psi = *\phi$

$$d^\psi f \;=\; \pm * d(f\phi) \qquad \text{and} \qquad dd^\psi f \;=\; \pm * d^* d(f\phi).$$

REMARK 2.13. (Examples). Corollary 2.5 gives us the following basic fact.

$$f \textit{ is } \phi\textit{-plurisubharmonic} \quad \Longleftrightarrow \quad \operatorname{tr}_\xi \operatorname{Hess} f \;\geq\; 0 \quad \forall\, \xi \in G(\phi).$$

This infinitesimal version of Theorem 1.4 gives insight into the condition of ϕ-plurisubharmonicity. Consider for example the calibration

$$\Omega_p = \tfrac{1}{p!}\omega^p$$

where (X, ω) is a Kähler, or more generally a Gromov, manifold. In this case

$$G(\Omega_p) \;=\; G_{\mathbf{C}}(p, TX) \;\subset\; G_{\mathbf{R}}(2p, TX)$$

is exactly the set of **complex p-planes** in TX. Thus a function f is Ω_p-plurisubharmonic if and only if $\operatorname{tr}_\xi \operatorname{Hess} f \geq 0$ for all complex p-planes ξ. In the Kähler case this means that *f is Ω_p-plurisubharmonic if and only if it is subharmonic on all p-dimensional complex submanifolds*. This condition can be expressed somewhat differently as follows. Any symmetric endomorphism $A : TX \to TX$ can be decomposed as $A = A_{\text{sym}} + A_{\text{sk}}$ where $A_{\text{sym}} = \frac{1}{2}(A - JAJ)$ is hermitian symmetric with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If ξ is a complex p-plane, then $\operatorname{tr}_\xi A = \operatorname{tr}_\xi A_{\text{sym}}$. Moreover, the infimum of $\operatorname{tr}_\xi A_{\text{sym}}$ over such planes is $2(\lambda_1 + \cdots + \lambda_p)$. From this it follows that a function *f is Ω_p-plurisubharmonic if*

and only if the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ *of its hermitian symmetric hessian* $\{\mathrm{Hess}\, f\}_{\mathrm{sym}}$ *satisfy*

$$\lambda_1 + \cdots + \lambda_p \ \geq \ 0 \qquad \textit{at all points of } X.$$

There is a parallel story on a quaternionic Kähler manifold (X, Ψ) where Ψ is locally of the form $\Psi = \frac{1}{6}\{\omega_I^2 + \omega_J^2 + \omega_K^2\}$ for orthogonal almost complex structures I, J, K satisfying the standard relations. For the calibration

$$\Psi_p \ \equiv \ \tfrac{1}{(2p+1)!}\{\omega_I^2 + \omega_J^2 + \omega_K^2\}^p, \quad \text{one has that}$$

$$G(\Psi_p) \ = \ G_{\mathbf{H}}(p, TX) \ \subset \ G_{\mathbf{R}}(4p, TX)$$

is the set of **quaternionic** p**-planes** in TX. Each symmetric endomorphism A has a quaternionic hermitian symmetric part $A_{\mathrm{qsym}} = \frac{1}{4}(A - IAI - JAJ - KAK)$ with real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, and a function f is Ψ_p-plurisubharmonic if and only if the eigenvalues of $\{\mathrm{Hess}\, f\}_{\mathrm{qsym}}$ satisfy $\lambda_1 + \cdots + \lambda_p \ \geq \ 0$ at each point.

In Section 6 we show that on a general (X, ϕ) the squared distance to any ϕ-free submanifold M is strictly ϕ-plurisubharmonic on a neighborhood of M. This constructs huge families of ϕ-plurisubharmonic functions with topologically interesting level sets.

3. Elliptic Calibrations

In this paper we primarily restrict attention to C^∞-functions. However, in this section we give the foundations for a theory of more general ϕ-plurisubharmonic functions developed in [HL$_4$]. Suppose (X, ϕ) is a calibrated manifold and assume that throughout this section that $G(\phi)$ is a fibre bundle over X.

Definition 3.1. Given an everywhere positive definite section A of $\mathrm{Sym}^2(\mathrm{TX})$, the associated differential operator

$$\Delta_A f \ \equiv \ \langle \mathrm{Hess} f, A \rangle \tag{3.1}$$

will be called a **Laplacian** on X.

The standard riemannian Laplacian is associated with the identity section of $\mathrm{Sym}^2(\mathrm{TX})$. For each Laplacian Δ on X, classical potential theory is applicable.

Definition 3.2. A calibration ϕ is said to be **mollified by a Laplacian** Δ_A (or **subordinate to a Laplacian** Δ_A) if every $f \in \mathrm{PSH}(X, \phi)$ is Δ_A-subharmonic, i.e., $\Delta_A f \geq 0$.

Example 1. Suppose that ϕ is a parallel calibration on $\mathbf{R}^n$ and that $\phi = \sum_{j=1}^N \alpha_j \xi_j$ is a positive linear combination of $\xi_j \in G(\phi)$. Assume that $\mathrm{Sym}^2(\mathbf{R}^n)$ has only one irreducible component of dimension 1 (the span of the identity I) under the subgroup of $\mathrm{O}(n)$ that fixes ϕ. Then ϕ is mollified by Δ_A with $A = \lambda_\phi^* \phi$.

Proof. If f is ϕ-plurisubharmonic then

$$\Delta_A f \ = \ \langle \mathrm{Hess} f, \lambda_\phi^* \phi \rangle \ = \ \left\langle \mathrm{Hess} f, \sum_{j=1}^N \alpha_j \lambda_\phi^* \xi_j \right\rangle \ = \ \sum_{j=1}^N \alpha_j \mathcal{H}^\phi(f)(\xi_j) \ \geq \ 0.$$

By the hypothesis $\lambda_\phi^*\lambda_\phi(I) = cI$ for some constant c. For any calibration ϕ, one has $\lambda_\phi(I) = p\phi$. Hence, $A = \lambda_\phi^*\phi = \frac{c}{p}I$. Finally, $p^2\langle\phi,\phi\rangle = \langle\lambda_\phi(I),\lambda_\phi(I)\rangle = \langle\lambda_\phi^*\lambda_\phi(I), I\rangle = cn$, proving that $A = \lambda_\phi^*\phi = \frac{p}{n}|\phi|^2 \cdot I$ is positive definite.

Moreover, $\Delta_A = \frac{p}{n}|\phi|^2\Delta$ where Δ is the standard Laplacian.

REMARK . In general, the operator $\langle \mathrm{Hess} f, \lambda_\phi^*\phi\rangle$ is not useful, as ϕ may not be a positive combination of elements $\xi_j \in G(\phi)$.

DEFINITION 3.3. A calibration ϕ is said to be **elliptic** and $G(\phi)$ is said to **involve all the variables** if for every tangent vector $v \neq 0$, there exists a ϕ-plane ξ with $v \lrcorner \xi \neq 0$.

EXAMPLE 2. Let $\phi = dx_1 \wedge dy_1 + \lambda dx_2 \wedge dy_2$, with $0 < \lambda < 1$ on $\mathbf{R}^4$. Then $G(\phi) = \{\xi_0\}$, a single point in $G(2, \mathbf{R}^4)$, so that ϕ is *not* elliptic. Note that the 2-form ϕ *does* involve all the variables.

EXAMPLE 3. Suppose $\phi = dx_1 \wedge \cdots \wedge dx_n + dy_1 \wedge \cdots \wedge dy_n \in \Lambda^n\mathbf{R}^{2n}$ with $n \geq 3$. Then $G(\phi)$ consists of only two points in $G(n, \mathbf{R}^{2n})$, but ϕ is elliptic. A function $f(x, y)$ is ϕ-plurisubharmonic if and only if it is separately subharmonic in x and y. An example is $f(x, y) = u(x)v(y)$ with $u, v \geq 0$ and subharmonic.

EXAMPLE 4. Suppose $\Psi = \frac{1}{3}(\frac{1}{2}\omega_I^2 + \frac{1}{2}\omega_J^2 + \frac{1}{2}\omega_K^2)$ on $\mathbf{H}^n$ is the quaternion calibration (cf. Example 1.9). Then the standard Laplacian Δ is a mollifying Laplacian since

$$\Delta f \;=\; \sum_{j=1}^{n} \mathrm{tr}_{\xi_j}\mathrm{Hess} f \;=\; \sum_{j=1}^{n} (\mathcal{H}^\Psi f)(\xi_j)$$

where $\xi_1, ..., \xi_n$ are the axis **H**-lines. However, Ψ is not a positive linear combination of elements in $G(\phi)$.

THEOREM 3.4. *There exists a mollifying Laplacian for ϕ if and only if ϕ is elliptic.*

THEOREM 3.5. *Suppose ϕ is elliptic. A function $f \in C^\infty(X)$ is ϕ-plurisubharmonic if and only if f is Δ-subharmonic for every mollifying Laplacian Δ.*

The proofs of these two results will be given in the case $\phi \in \Lambda^p V$ is a parallel calibration on euclidean n-space. Arguments for the more general case are essentially the same.

Let $\mathcal{P}_+$ denote the convex cone in $\mathrm{Sym}^2(\mathrm{V})$ on $\{P_\xi : \xi \in G(\phi)\}$. That is, $A \in \mathcal{P}_+$ if and only if

$$A \;=\; \sum_{j=1}^{N} \lambda_j P_{\xi_j} \qquad \text{with} \;\; \lambda_j > 0 \text{ and } \xi_j \in G(\phi). \tag{3.2}$$

Let $\mathcal{P}^+$ denote the polar cone. That is, $H \in \mathcal{P}^+$ if and only if

$$\langle H, P_\xi\rangle \;=\; \mathrm{tr}_\xi H \;\geq\; 0 \;\; \text{for all } \xi \in G(\phi) \tag{3.3}$$

The Bipolar Theorem (cf. [S]) states that $A \in \mathrm{Sym}^2(\mathrm{V})$ can be expressed as in (3.2) if and only if

$$\langle H, A\rangle \;\geq\; 0 \;\; \text{for all } H \in \mathcal{P}^+ \tag{3.4}$$

LEMMA 3.6. *Given $A \in \mathrm{Sym}^2(\mathrm{V})$, the associated operator Δ_A is a mollifying Laplacian for ϕ if and only if*

1) $A = \sum_{j=1}^N \lambda_j P_{\xi_j}$ with $\lambda_j > 0$ and $\xi_j \in G(\phi)$, *and*

2) $\xi_1, ..., \xi_N$ *involve all the variables, i.e.,* $v \lrcorner \xi_j = 0, j = 1, ..., N$ *implies* $v = 0$.

Proof. Suppose 1) and 2) are valid. Then $\langle Av, v\rangle = \sum \lambda_j |v \lrcorner \xi_j|^2$ and 2) implies that A is positive definite. Moreover,

$$\Delta_A f = \langle \mathrm{Hess} f, A\rangle = \sum \lambda_j \mathrm{tr}_{\xi_j} \mathrm{Hess} f \tag{3.5}$$

so that if f is ϕ-plurisubharmonic , then f is Δ_A-subharmonic.

Conversely, suppose Δ_A is a mollifying Laplacian for ϕ. Take f to be a quadratic function with $H = \mathrm{Hess} f \in \mathcal{P}^+$, so that f is ϕ-plurisubharmonic. Then $\Delta_A f = \langle H, A\rangle \geq 0$ for all such H. As noted above (3.4) implies (3.2). Finally, note that $\langle Av, v\rangle = \sum_{j=1}^N \lambda_j |v \lrcorner \xi_j|^2$ and therefore, since A is positive definite 2) is verified. ∎

Proof of Theorem 3.4 Suppose there exists a mollifying Laplacian Δ_A for ϕ. Then by Lemma 3.6 1), we have $A = \sum_{j=1}^N \lambda_j P_{\xi_j}$, and by 2) we have that given $v \neq 0$, there exists $\xi_j \in G(\phi)$ with $v \lrcorner \xi_j \neq 0$. Thus ϕ is elliptic.

Conversely, if ϕ is elliptic, then by compactness, there exists a finite number of $\xi_j \in G(\phi)$ such that

$$A = \sum_{j=1}^N P_{\xi_j} \in \mathcal{P}_+$$

is positive definite, thereby verifying 1) and 2). ∎

Proof of Theorem 3.5 Suppose $f \in C^\infty(X)$ is subharmonic for every mollifying Laplacian Δ_A. Suppose $\xi \in G(\phi)$ and $A = \sum_{j=1}^N \lambda_j P_{\xi_j}$ is a mollifying Laplacian. Then $A(t) = tA + (1-t)P_\xi$, $0 < t \leq 1$ also determines a mollifying Laplacian by Lemma 3.6. Hence,

$$\Delta_{A(t)} f = t \sum_{j=1}^N \lambda_j \mathrm{tr}_{\xi_j} \mathrm{Hess} f + (1-t)\mathrm{tr}_\xi \mathrm{Hess} f \geq 0.$$

Taking the limit as $t \to 0$, we obtain $\mathrm{tr}_\xi \mathrm{Hess} f \geq 0$. ∎

Generalized ϕ-plurisubharmonic Functions

Throughout this subsection we assume that ϕ is an elliptic calibration. The differential operator $\mathrm{Hess} f$ extends to distributions f on X via duality producing a well-defined distributional section $\mathrm{Hess} f$ of the bundle $\mathrm{Sym}^2(\mathrm{TX})$. By definition, a distributional section of a vector bundle $E \to X$ is a continuous linear functional on the space of smooth compactly supported sections of $E^* \otimes \Lambda^n T^*X$, or equivalently, on the space of $\widetilde{s} \equiv s \otimes *1$ for $s \in \Gamma_{\mathrm{cpt}} E^*$.

DEFINITION 3.7. A distribution f on X is ϕ-plurisubharmonic if

$$(\mathrm{tr}_\xi \mathrm{Hess} f)(\lambda) \geq 0$$

for every smooth section ξ of $G(\phi)$ and every smooth compactly supported non-negative multiple λ of the volume form on X.

EXAMPLE. The fundamental function $E(x)$ in Proposition 1.7 defines an $L^1_{\rm loc}(\mathbf{R}^n)$ function and, hence, is a distribution on $\mathbf{R}^n$. It is ϕ-plurisubharmonic for any calibration ϕ on $\mathbf{R}^n$ of degree p. This is easy to prove since (for $p < n$) the distributional hessian

$$\mathrm{Hess}E \;=\; \frac{p}{|x|^p}\left(\frac{1}{p}I - \frac{x}{|x|}\circ\frac{x}{|x|}\right) \qquad \text{on } \mathbf{R}^n$$

has ξ-trace

$$\mathrm{tr}_\xi \mathrm{Hess}E \;=\; \frac{p}{|x|^p}\left(1 - \frac{|x\lrcorner\,\xi|^2}{|x|^2}\right).$$

Theorem 3.5 extends from functions $f \in C^\infty(X)$ to distributions $f \in \mathcal{D}'(X)$.

THEOREM 3.8. *Suppose Δ_A is a mollifying Laplacian for ϕ. If f is a ϕ-plurisubharmonic distribution, then $\Delta_A f \geq 0$ is a non-negative measure, i.e., f is Δ_A-subharmonic. Conversely, if $\Delta_A f \geq 0$ for each mollifying Laplacian Δ_A, then f is a ϕ-plurisubharmonic distribution.*

Proof. Assume that A is of the form $A = \sum_{j=1}^N \lambda_j P_{\xi_j}$ with $\lambda_j > 0$ smooth and each ξ_j a smooth section of $G(\phi)$. Then

$$\Delta_A f \;=\; \langle A, \mathrm{Hess}f\rangle \;=\; \sum_{j=1}^N \lambda_j \mathrm{tr}_{\xi_j}\mathrm{Hess}f$$

is a well defined distribution on X, and, by hypothesis, it pairs with every smooth, compactly supported non-negative multiple λ of the volume form, to give $(\Delta_A f)(\lambda) \geq 0$. Hence $\Delta_A f$ is a non-negative regular Borel measure on X. The proof of the converse is similar to that of Theorem 3.5. ∎

This theorem has a multitude of corollaries, deducible from the classical potential theory for Δ_A. We list just two of the facts.

1) Each Δ_A-subharmonic distribution (and therefore each ϕ-plurisubharmonic distribution) belongs to $L^1_{\rm loc}(X)$, the space of locally Lebesgue integrable functions on X.

2) Each Δ_A-subharmonic distribution (and therefore each ϕ-plurisubharmonic distribution) has a canonical **classical** representative defined by

$$f(x) \;=\; \lim_{r\to 0}\frac{1}{|B_r(x)|}\int_{B_r(x)} f\, d\mathrm{vol}$$

which is $[-\infty,\infty)$-valued and upper semi-continuous on X. Here $B_r(x)$ denotes the ball of radius r about x and $|B_r(x)|$ denotes its volume.

See [HL$_4$] for a development of upper semi-continuous ϕ-plurisubharmonic functions using these results.

4. Convexity in Calibrated Geometries

We suppose throughout this section that (X, ϕ) is a non-compact, connected calibrated manifold and all ϕ-plurisubharmonic functions are of class C^∞.

DEFINITION 4.1. If K is a compact subset of X, we define the (X, ϕ)**-convex hull of** K by

$$\widehat{K} \equiv \{x \in X : f(x) \leq \sup_K f \text{ for all } f \in \mathrm{PSH(X}, \phi)\}.$$

If $\widehat{K} = K$, then K is called (X, ϕ)**-convex.**

LEMMA 4.2. *Suppose K is a compact subset of X. Then $x \notin \widehat{K}$ if and only if there exists a smooth non-negative ϕ-plurisubharmonic function f on X which is identically zero on a neighborhood of K and has $f(x) > 0$. Furthermore, if there exists a ϕ-plurisubharmonic function on X which is strict at x, then f can be chosen to be strict at x.*

Proof. Suppose $x \notin \widehat{K}$. Then there exists $g \in \mathrm{PSH(X}, \phi)$ with $\sup_K g < 0 < g(x)$. Pick $\varphi \in C^\infty(\mathbf{R})$ with $\varphi \equiv 0$ on $(-\infty, 0]$ and with $\varphi > 0$ and convex increasing on $(0, \infty)$. Then $f = \varphi \circ g$ satisfies the required conditions (See Lemma 1.5a). Furthermore, assume $h \in \mathrm{PSH(X}, \phi)$ is strict at x. Then take $\overline{g} = g + \epsilon h$. For small enough ϵ, $\sup_K \overline{g} < 0 < \overline{g}(x)$. If φ is also strictly increasing on $(0, \infty)$, then $f = \varphi \circ \overline{g}$ is strict at x. ∎

Note: One sees easily that $\widehat{\widehat{K}} = \widehat{K}$. Therefore, if $\widehat{K}$ is compact, the function f in Lemma 4.2 can be taken to be zero on a neighborhood of $\widehat{K}$ (since one can replace K with $\widehat{K}$).

THEOREM 4.3. *The following two conditions are equivalent.*

1) If $K \subset\subset X$, then $\widehat{K} \subset\subset X$.

2) There exists a ϕ-plurisubharmonic proper exhaustion function f on X.

DEFINITION 4.4. If the equivalent conditions of Theorem 4.3 are satisfied, then (X, ϕ) is a **convex calibrated manifold** and X **is ϕ-convex .**

Proof that 2) ⇒ 1)**.** If K is compact, then $c \equiv \sup_K f$ is finite and $\widehat{K}$ is contained in the compact pre-level set $\{x \in X : f(x) \leq c\}$.

Proof that 1) ⇒ 2)**.** A ϕ-plurisubharmonic proper exhaustion function on X is constructed as follows. Choose an exhaustion of X by compact (X, ϕ)-convex subsets $K_1 \subset K_2 \subset K_3 \subset \cdots$ with $K_m \subset K^0_{m+1}$ for all m. By Lemma 4.2 and the compactness of $K_{m+2} - K^0_{m+1}$, there exists a ϕ-plurisubharmonic function $f_m \geq 0$ on X with f_m identically zero on a neighborhood of K_m and $f_m > 0$ on $K_{m+2} - K^0_{m+1}$. By re-scaling we may assume $f_m > m$ on $K_{m+2} - K^0_{m+1}$. The locally finite sum $f = \sum_{m=1}^\infty f_m$ satisfies 2). ∎

LEMMA 4.5. *Condition 2) in Theorem 4.3 is equivalent to the* a priori *weaker condition:*

2)′ There exists a continuous proper exhaustion function f on X which is smooth and ϕ-plurisubharmonic outside a compact subset of X.

In fact if f satisfies 2)′, then f can be modified on a compact subset to be ϕ-plurisubharmonic on all of X. Consequently, if f satisfies 2)′ and is strict outside a compact set, then its modification is also strict outside a compact set.

Proof. For large enough c, f is smooth and ϕ-plurisubharmonic outside the compact set $\{x \in X : f(x) \leq c-1\}$. Pick a convex increasing function $\varphi \in C^\infty(\mathbf{R})$ with $\varphi \equiv c$ on a neighborhood of $(-\infty, c-1]$ and $\varphi(t) = t$ on $(c+1, \infty)$. Then by Lemma 1.5 the composition $\varphi \circ f$ is ϕ-plurisubharmonic on all of X (in particular smooth) and equal to f outside of the compact set $\{x \in X : f(x) \leq c+1\}$. ∎

THEOREM 4.6. *The following two conditions are equivalent:*

1) $K \subset\subset X \Rightarrow \widehat{K} \subset\subset X$, and X carries a strictly ϕ-plurisubharmonic function.

2) There exists a strictly ϕ-plurisubharmonic proper exhaustion function for X.

DEFINITION 4.7. If the equivalent conditions of Theorem 4.6 are satisfied, then (X, ϕ) is a **strictly convex calibrated manifold** or X is **strictly ϕ-convex.**

Proof of Theorem 4.6. Suppose that X is equipped with both a ϕ-plurisubharmonic proper exhaustion function f and a strictly ϕ-plurisubharmonic function g. Then the sum $f + e^g$ is a strictly ϕ-plurisubharmonic exhaustion function. Now Theorem 4.6 follows immediately from Theorem 4.3. ∎

We shall construct many ϕ-convex manifolds in the course of our discussion (See, in particular, §6). However, we present some elementary examples here.

EXAMPLE 1. Suppose $\phi \in \Lambda^p \mathbf{R}^n$ is a parallel calibration on $\mathbf{R}^n$. Let $f(x) = \frac{1}{2}\|x\|^2$. Then $dd^\phi f = p\phi$ and hence f is a strictly ϕ-plurisubharmonic exhaustion. That is, $(\mathbf{R}^n, \phi)$ is a strictly convex calibrated manifold.

EXAMPLE 2. Suppose $\phi = dx_1 \wedge \cdots \wedge dx_n$ on a domain X in $\mathbf{R}^n$. Then $dd^\phi f = (\Delta f)\phi$ and f is ϕ-plurisubharmonic if and only if f is subharmonic. Recall that if $K \subset\subset X$, then $\widehat{K} = K \cup \{\text{all the "holes" in } K \text{ relative to } X\}$, (connected components of $X - K$ which are relatively compact in X). Thus (X, ϕ) is strictly convex for any open set $X \subset \mathbf{R}^n$.

It is instructive to extend this elementary example.

EXAMPLE 3. Suppose $\phi = dx_1 \wedge \cdots \wedge dx_p$ on a domain X in $\mathbf{R}^n$ with coordinates $(x_1, ..., x_p, y_1, ..., y_{n-p})$. A function $f \in C^\infty(X)$ is ϕ-plurisubharmonic if and only if $\Delta_x f \geq 0$ on X. For a set $K \subset \mathbf{R}^n$, let K_y denote the horizontal slice $\{x \in \mathbf{R}^p : (x, y) \in K\}$ of K. Suppose that for each $y \in \mathbf{R}^{n-p}$, the horizontal slice X_y has no holes in $\mathbf{R}^p$. Then (X, ϕ) is strictly convex. To prove this fact, it suffices to exhaust X by compact sets K with the same property and show that each such K is equal to its (X, ϕ)-hull. Suppose $z_0 = (x_0, y_0) \in X - K$. Since x_0 is not in a hole of K_{y_0} in $\mathbf{R}^p$, we may choose (by Example 2) an entire subharmonic function $g(x)$ with $g(x_0) >> 0$ and $\sup_{K_{y_0}} g << 0$. Now pick $\psi \in C^\infty_{\text{cpt}}(\{y : |y - y_0| < \epsilon\})$ with $0 \leq \psi \leq 1$ and $\psi(y_0) = 1$. Then $f(x, y) = g(x)\psi(y)$ is ϕ-plurisubharmonic and $f(z_0) = g(x_0) >> 0$. For ϵ sufficiently small, $\sup_K f \leq 0$. This proves z_0 does not belong to the (X, ϕ)-hull of K.

EXAMPLE 4. Let $\phi = dx$ in $\mathbf{R}^2$ and set $X = \{(x, y) : x^2 - c < y < x^2,\ |x| < 1\}$. Then X is not ϕ-convex. The closure of the hull of the compact subset $K = ([-\epsilon, \epsilon] \times \{-\epsilon\}) \cup (\{\pm\epsilon\} \times [-\epsilon, 0])$ of X is easily seen to contain the origin. Similarly, a domain of "U"-shape, whose upper boundary along the bottom has a flat segment, is not ϕ-convex even though it is locally ϕ-convex (by Example 3).

It is important to "weaken" this notion of strict convexity.

THEOREM 4.8. *The following two conditions are equivalent:*

1) $K \subset\subset X \Rightarrow \widehat{K} \subset\subset X$, *and there exists a strictly* ϕ*-plurisubharmonic function defined outside a compact subset of* X

2) There exists a ϕ*-plurisubharmonic proper exhaustion function on* X *which is strict outside a compact subset of* X.

DEFINITION 4.9. If the equivalent conditions of Theorem 4.8 are satisfied, then the calibrated manifold (X, ϕ) is **strictly convex at** ∞ or X is **strictly** ϕ**-convex at** ∞**.**

REMARK . This is not the standard terminology used in complex geometry where such spaces are called "strongly (pseudo) convex".

Proof of Theorem 4.8. Obviously 2) implies 1). We will prove that 1) implies the following weakening of 2).

2)$'$ There exists a continuous proper exhaustion function f on X which is smooth and strictly ϕ-plurisubharmonic outside a compact subset of X.

By Lemma 4.5, Condition 2)$'$ implies Condition 2).

Now assume 1). Since $K \subset\subset X$ implies $\widehat{K} \subset\subset X$, we know from Theorem 4.3 that there exists a ϕ-plurisubharmonic exhaustion function f for X. Let g denote the strictly ϕ-plurisubharmonic function which is only defined outside of a compact set. We can assume this compact set is $\{x \in X : f(x) \leq c\}$ for some large c. Then $h \equiv \max\{f + e^g, c\}$ is a continuous proper exhaustion function which, outside the compact set $\{x \in X : f(x) \leq c\}$, is strictly ϕ-plurisubharmonic (in fact, equal to$f + e^g$). This proves 2)$'$ and completes the proof of the theorem. ∎

COROLLARY 4.10. (X, ϕ) *is strictly convex at* ∞ *if and only if Condition* 2)$'$ *holds.*

Cores.

In each non-compact calibrated manifold (X, ϕ) there are certain distinguished subsets which play an important role in the ϕ-geometry of the space. (In complex manifolds which are strongly pseudoconvex, these sets correspond to the compact exceptional subvarieties.) The remainder of this section is devoted to a discussion of these subsets.

Given a function $f \in \mathrm{PSH}(X, \phi)$, consider the closed set

$$W(f) \equiv \{x \in X : f \text{ is partially } \phi - \text{pluriharmonic at } x\}$$

That is, $W(f)$ is the complement of the set

$$S(f) \equiv \{x \in X : f \text{ is strictly } \phi - \text{plurisubharmonic at } x\}$$

Note that

$$W(\lambda f + \mu g) \subseteq W(f) \cap W(g)$$

for $f, g \in \mathrm{PSH}(X, \phi)$ and $\lambda, \mu > 0$.

DEFINITION 4.11. The **core** of X is defined to be the intersection

$$\mathrm{Core}(X) \equiv \bigcap W(f)$$

over all $f \in \mathrm{PSH}(X,\phi)$. The **inner core** of X is defined to be the set $\mathrm{InnerCore}(X)$ of points x for which there exists $y \neq x$ with the property that $f(x) = f(y)$ for all $f \in \mathrm{PSH}(X,\phi)$.

PROPOSITION 4.12. $\mathrm{InnerCore}(C) \subset \mathrm{Core}(X)$.

Proof. If $x \notin \mathrm{Core}(X)$, then there exists $g \in \mathrm{PSH(X},\phi)$ with g strict at x. Suppose $y \neq x$. Then if ψ is compactly supported in a small neighborhood of x missing y, and ψ has sufficiently small second derivatives, one has $f = g + \psi \in \mathrm{PSH(X},\phi)$. Obviously for such f, the values $f(x)$ and $f(y)$ can be made to differ, so therefore $x \notin \mathrm{InnerCore}(X)$. ∎

PROPOSITION 4.13. *Every compact ϕ-submanifold M without boundary in X is contained in the inner core.*

Proof. Each $f \in \mathrm{PSH(X},\phi)$ is subharmonic on M by Theorem 1.4. Hence, f is constant on M. ∎

PROPOSITION 4.14. *Suppose X is ϕ-convex. Then* $\mathrm{Core}(X)$ *is compact if and only if X is strictly ϕ-convex at ∞, and* $\mathrm{Core}(X) = \emptyset$ *if and only if X is strictly ϕ-convex .*

Proof. If X is strictly ϕ-convex at ∞, then choosing f to satisfy 2) in Theorem 4.8, we see that the $\mathrm{Core}(X) \subset W(f)$ is compact. Obviously, strict ϕ-convexity implies that $\mathrm{Core}(X) = \emptyset$.

Conversely, if $\mathrm{Core}(X)$ is compact, then in the construction of the ϕ-plurisubharmonic exhaustion function in the proof of Theorem 4.3 we may choose K_1 to be the ϕ-convex hull of $\mathrm{Core}(X)$. Then by the definition of $\mathrm{Core}(X)$ and Lemma 4.2, each of the functions f_m in that proof can be chosen to be strictly ϕ-plurisubharmonic on $K_{m+2} - K^0_{m+1}$. Hence the exhaustion $f = \sum_m f_m$ is strictly ϕ-plurisubharmonic outside a compact set containing the core. ∎

A slight modification of this construction gives the following general result.

PROPOSITION 4.15. *Suppose X is strictly ϕ-convex at ∞, and $K \subset X$ is a compact, ϕ-convex subset containing the core of X. Let U be any neighborhood of K. Then there exists a proper ϕ-plurisubharmonic exhaustion function $f : X \to \mathbf{R}^+$ which is strictly ϕ-plurisubharmonic on $X - U$, and identically zero on a neighborhood of K.*

Proof. Choose K_1 as in the construction of the ϕ-plurisubharmonic exhaustion function given in the proof of Theorem 4.3. Let $K(\epsilon)$ denote the compact ϵ-neighborhood of K_1. Then

$$K_1 = \bigcap_{\epsilon>0} \widehat{K(\epsilon)}. \tag{4.1}$$

If $x \in \bigcap_{\epsilon>0} \widehat{K(\epsilon)}$, then for each $f \in \mathrm{PSH(X},\phi)$, we have $f(x) \leq \sup_{K(\epsilon)} f$. However, $\inf_\epsilon \sup_{K(\epsilon)} f = \sup_K f$, and we conclude that $x \in \widehat{K}$. Thus we can choose $K_2 \equiv \widehat{K}(\epsilon)$ in

our construction of f, and for small enough ϵ we have $K_2 \subset U$ as well as $K_1 \subset K_2^0$. The proof is now completed as in the proof of Proposition 4.14. ∎

Obviously, many questions concerning

$$\text{InnerCore}(X) \ \subseteq \ \text{Core}(X) \ \subseteq \ \widehat{\text{Core}(X)}$$

remain to be answered.

Examples of Complete Convex Manifolds and Cores

In §6 (Theorem 6.4) we shall show that there are many strictly ϕ-convex domains in any calibrated manifold (X, ϕ). They can have quite arbitrary topological type within the strictures imposed by Morse Theory and ϕ-positivity of the Hessian. However, it is also interesting geometrically to ask for convex manifolds which are complete.

In fact, there exist enormous families of complete calibrated manifolds (X, ϕ) with $\nabla\phi = 0$ which are strictly ϕ-convex at infinity. For example, any (X, ϕ) which is asymptotically locally euclidean (ALE) is such a creature. In this case the radial function on the asymptotic cone at infinity is strictly convex. It appears likely the corresponding assertion also holds for quasi ALE spaces. For the general construction of such spaces with SU(n), Sp(n), G_2, or Spin_7 holonomy, the reader is referred to the book of Joyce [J].

Certain manifolds of this type have been quite explicitly constructed, and in these cases one can explicitly construct ϕ-plurisubharmonic exhaustion functions and identify the cores. We indicate how to do this below.

We begin however with an observation in dimension 4. Every crepant resolution of singularities of $\mathbf{C}^2/\Gamma$, for a finite subgroup $\Gamma \subset \text{SU}(2)$, admits a Ricci-flat ALE Kähler metric. On each such manifold there exists an S^2-family of parallel calibrations

$$\mathcal{C} \ = \ \{u\omega + v\varphi + \ w\psi : u^2 + v^2 + w^2 = 1\}$$

where ω is the given Kähler form, $\varphi = \text{Re}\{\Phi\}$ and $\psi = \text{Im}\{\Phi\}$ and Φ is a parallel section of the canonical bundle κ_X. Let $E = \pi^{-1}(0)$ be the exceptional locus of the resolution. Then for any $\phi \in \mathcal{C}$ we have

$$\text{Core}(X, \phi) \ = \ \begin{cases} E & \text{if } \phi = \omega \\ \emptyset & \text{otherwise.} \end{cases}$$

This follows from the fact that each $\phi \in \mathcal{C}$ is in fact the Kähler form for a complex structure on X compatible with the given metric. With this complex structure X is pseudo-convex, and by the Stein Reduction Theorem (cf. [GR, p. 221]) we know its core is the union of its compact complex subvarieties. For $\phi \neq \omega$ there are no such subvarieties since by the Wirtinger inequality (cf. [$L_{1,2}$]), applied to ϕ, they would necessarily be homologically mass-minimizing, and by the same result applied to ω any such subvariety is ω-complex (and therefore a component of E).

EXAMPLE 1. (Calabi Spaces). Let $X \to \mathbf{C}^n/\mathbf{Z}_n$ be a crepant resolution of $\mathbf{C}^n/\mathbf{Z}_n$ where the action on $\mathbf{C}^n$ is generated by scalar multiplication by $\tau = e^{2\pi i/n}$. Following Calabi [C] we define the function $F : \mathbf{C}^n/\mathbf{Z}_n \to \mathbf{R}$ by

$$F(\rho) \;=\; \sqrt[n]{\rho^n + 1} + \frac{1}{n}\sum_{k=0}^{n-1} \tau^k \log\Big(\sqrt[n]{\rho^n+1} - \tau^k\Big)$$

where $\rho \equiv \|z\|^2$ (pushed down to $\mathbf{C}^n/\mathbf{Z}_n$), and the log is defined by choosing $\arg\zeta \in (-\pi, \pi)$. We then define a Kahler metric on $\mathbf{C}^n/\mathbf{Z}_n - \{0\}$ by setting

$$\omega \;=\; \frac{1}{4} dd^c F.$$

Calabi shows that *this metric is Ricci-flat and (when pulled back) extends to a Ricci flat metric on X. The parallel form $\Phi = dz_1 \wedge \cdots \wedge dz_n$ extends to a parallel section of κ_X.* This metric is given explicitly on $\mathbf{R}^{2n}/\mathbf{Z}_n$ by

$$ds^2 \;=\; F'(\rho)|dx|^2 + \rho F''(\rho) dr \circ d^c r$$

where $r = \|x\|$. Define $G(\rho)$ by setting $G'(\rho) = F'(\rho) + \rho F''(\rho)$ and $G(0) = 0$. Then direct calculation shows that

$$dd^\phi G \;=\; 2n\phi$$

where $\phi = \mathrm{Re}\{\Phi\}$. Hence, *$X$ is a complete, strictly ϕ-convex manifold.*

EXAMPLE 2. (Bryant-Salamon Spaces). Let P denote the principal Spin_3-bundle of S^3 and

$$S \;\equiv\; P \times_{\mathrm{Sp}_1} \mathbf{H}$$

the associated spinor bundle, where $\mathbf{H}$ denotes the quaternions. Bryant and Salamon have explicitly constructed a complete riemannian metric with G_2-holonomy on the total space of S. (See [BS, page 838, Case ii].) Let $\rho = |a|$ for $a \in \mathbf{H}$ (pushed-down to S) and let $Z \subset S$ denote the zero section. Then a direct calculation shows that the function

$$F(\rho) \;=\; (1+\rho)^{\frac{5}{6}} \;\; \textit{is strictly } \varphi - \textit{plurisubharmonic on } S - Z$$

where φ denotes the associative calibration on S. Since Z is an associative submanifold we conclude that

$$\mathrm{Core}(S) \;=\; Z.$$

In an analogous fashion the authors construct a complete riemannian metric with Spin_7-holonomy on the total space $\widetilde{S}$ of a spinor bundle over S^4. (See [BS, page 847, Case ii].) A similar calculation shows that there exists an exhaustion function which is strictly Φ-plurisubharmonic on $\widetilde{S} - \widetilde{Z}$ where Φ denotes the Cayley calibration $\widetilde{Z}$ the zero-section of $\widetilde{S}$. Since $\widetilde{Z}$ is a Cayley submanifold, we conclude that

$$\mathrm{Core}(\widetilde{S}) \;=\; \widetilde{Z}.$$

5. Boundary Convexity.

Suppose $\Omega \subset\subset X$ is an open set with smooth boundary $\partial\Omega$, where (X, ϕ) is a non-compact calibrated manifold. A p-plane $\xi \in G(\phi)$ at a point $x \in \partial\Omega$ will be called **tangential** if $\operatorname{span}\xi \subset T_x\partial\Omega$.

DEFINITION 5.1. Suppose that ρ is a **defining function** for $\partial\Omega$, that is, ρ is a smooth function defined on a neighborhood of $\overline{\Omega}$ with $\Omega = \{x : \rho(x) < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$. If

$$\mathcal{H}^\phi\rho(\xi) \;\geq\; 0 \quad \text{for all tangential } \xi \in G_x(\phi),\ x \in \partial\Omega, \tag{5.1}$$

then $\partial\Omega$ is called **ϕ-convex**. If the inequality in (5.1) is strict for all ξ, then $\partial\Omega$ is called **strictly ϕ-convex**. If $\mathcal{H}^\phi(\xi) = 0$ for all ξ as in (5.1), then $\partial\Omega$ is **ϕ-flat**.

Each of these conditions is a local condition on $\partial\Omega$. In fact:

LEMMA 5.2. *Each of the three conditions in Definition 5.1 is independent of the choice of defining function ρ. In fact, if $\overline{\rho} = u\rho$ is another choice with $u > 0$ on $\partial\Omega$, then on $\partial\Omega$*

$$\mathcal{H}^\phi(\overline{\rho})(\xi) \;=\; u\mathcal{H}^\phi(\rho)(\xi) \quad \text{for all tangential} \quad \xi \in G(\phi) \tag{5.2}$$

Proof. Since $\rho = 0$ on $\partial\Omega$ and $\nabla\rho \perp \partial\Omega$, we have by (2.12) that

$$\begin{aligned}\mathcal{H}^\phi(\overline{\rho})(\xi) \;&=\; \operatorname{tr}_\xi\left[\operatorname{Hess}(u\rho)\right] \;=\; \operatorname{tr}_\xi\left[u\operatorname{Hess}(\rho) + 2\nabla u \circ \nabla\rho + \rho\operatorname{Hess}u\right] \\ &=\; u\operatorname{tr}_\xi(\operatorname{Hess}\rho) + 2\operatorname{tr}_\xi(\nabla u \circ \nabla\rho) + \rho\operatorname{tr}_\xi(\operatorname{Hess}u) \;=\; u\operatorname{tr}_\xi(\operatorname{Hess}\rho) = u\,\mathcal{H}^\phi(\rho). \;\blacksquare\end{aligned}$$

COROLLARY 5.3. *Assume $\phi \in \Lambda^p\mathbf{R}^n$ is a calibration. Suppose $\partial\Omega$ is (strictly) ϕ-convex in $\mathbf{R}^n$, and locally near a point $p \in \partial\Omega$, let $\partial\Omega$ be graphed over its tangent space by a function $x_n = u(x')$ for linear coordinates (x', x_n) on $\mathbf{R}^n$. Then each nearby hypersurface: $x_n = u(x') + c$ is also (strictly) ϕ-convex.*

The next lemma and its corollary will be used to establish the main results of this section.

LEMMA 5.4. *Suppose ρ is a smooth real-valued function on a riemannian manifold, and $\psi : \mathbf{R} \to \mathbf{R}$ is smooth on the image of ρ. Then*

$$\operatorname{tr}_\xi\operatorname{Hess}\psi(\rho) \;=\; \psi'(\rho)\operatorname{tr}_\xi\operatorname{Hess}\rho + \psi''(\rho)|\nabla\rho \lrcorner \xi|^2 \tag{5.3}$$

for all oriented tangent p-planes ξ.

Proof. We first calculate that $\operatorname{Hess}\psi(\rho) = \psi'(\rho)\operatorname{Hess}\rho + \psi''(\rho)\nabla\rho \circ \nabla\rho$ and then note that $\operatorname{tr}_\xi(\nabla\rho \circ \nabla\rho) = |\nabla\rho \lrcorner \xi|^2$. $\blacksquare$

COROLLARY 5.5. *With $\delta = -\rho$ and $\rho < 0$, one has*

$$\operatorname{tr}_\xi\operatorname{Hess}(-\log\delta) \;=\; \frac{1}{\delta}\operatorname{tr}_\xi\operatorname{Hess}\rho + \frac{1}{\delta^2}|\nabla\rho \lrcorner \xi|^2 \tag{5.4}$$

Proof. Take $\psi(t) = -\log(-t)$ for $t < 0$, and note that $\psi'(t) = -1/t$ and $\psi''(t) = 1/t^2$, so that $\psi'(\rho) = 1/\delta$ and $\psi''(\rho) = 1/\delta^2$. ∎

We now come to the main result of this section.

THEOREM 5.6. *Let $\Omega \subset\subset X$ be a compact domain with strictly ϕ-convex boundary. Suppose $\delta = -\rho$ is an arbitrary "distance function" for $\partial\Omega$, i.e., ρ is an arbitrary defining function for $\partial\Omega$. Then $-\log\delta$ is strictly ϕ-plurisubharmonic outside a compact subset of Ω. Thus, in particular, the domain Ω is strictly ϕ-convex at ∞.*

Proof. Applying (2.12) to Corollary 5.5 shows that at each point $x \in \Omega$ near $\partial\Omega$, we have

$$\mathcal{H}^\phi(-\log\delta)(\xi) \;=\; \frac{1}{\delta}\mathcal{H}^\phi(\rho)(\xi) + \frac{1}{\delta^2}|\nabla\rho \lrcorner \xi|^2 \tag{5.5}$$

for all $\xi \in G(\phi)$. Note that at $x \in \partial\Omega$, $|\nabla\rho \lrcorner \xi|^2$ vanishes if and only if ξ is tangential to $\partial\Omega$. For notational convenience we set

$$\cos^2\theta(\xi) \;=\; \frac{|\nabla\rho \lrcorner \xi|^2}{|\nabla\rho|^2} \;=\; \langle P_{\operatorname{span}\nabla\rho}, P_{\operatorname{span}\xi}\rangle.$$

Then the inequality $|\cos\theta| < \epsilon$ defines a fundamental neighborhood system for $G(p, T\partial\Omega) \subset G(p, TX)$. By restriction $|\cos\theta| < \epsilon$ defines a fundamental neighborhood system for $G(\phi) \cap G(p, T\partial\Omega) \subset G(\phi)$. The hypothesis of strict ϕ-convexity for $\partial\Omega$ implies that there exists $\overline{\epsilon} > 0$ so that $(\mathcal{H}^\phi\rho)(\xi) \geq \overline{\epsilon}$ for all ϕ-planes ξ at points of $\partial\Omega$ with $|\cos\theta| < \epsilon$ for some $\epsilon > 0$. (Note that if there are no ϕ-planes tangent to $\partial\Omega$ at a point x, then there are no ϕ-planes with $|\cos\theta| < \epsilon$ for sufficiently small ϵ in a neighborhood of x.) Consequently, we have by equation (5.5) that

$$\mathcal{H}^\phi(-\log\delta)(\xi) \;\geq\; \frac{\overline{\epsilon}}{2\delta}$$

near $\partial\Omega$ for all ϕ-planes ξ with $|\cos\theta| < \epsilon$.

Now choose $M >> 0$ so that $\mathcal{H}^\phi(\rho)(\xi) \geq -M$ in a neighborhood of $\partial\Omega$ for all $\xi \in G(\phi)$. Then, by (5.5)

$$\mathcal{H}^\phi(-\log\delta)(\xi) \;\geq\; -\frac{M}{\delta} + \frac{1}{\delta^2}|\nabla\rho \lrcorner \xi|^2.$$

If $|\cos\theta| \geq \epsilon$, this is positive in a neighborhood of $\partial\Omega$ in Ω. This proves that $-\log\delta$ is strictly ϕ-plurisubharmonic near $\partial\Omega$. By Corollary 4.10 the domain Ω is strictly ϕ-convex at infinity. ∎

Although a general defining function for a strictly ϕ-convex boundary may not be ϕ-plurisubharmonic, for some applications the following is useful.

PROPOSITION 5.7. *Suppose $\Omega \subset\subset X$ has strictly ϕ-convex boundary $\partial\Omega$ with defining function ρ. Then, for A sufficiently large, the function $\overline{\rho} \equiv \rho + A\rho^2$ is strictly ϕ-convex in a neighborhood of $\partial\Omega$ and also a defining function for $\partial\Omega$.*

Proof. By Lemma 5.4 and (2.12)

$$\mathcal{H}^\phi(\rho)(\xi) \;=\; (1 + 2A\rho)\mathcal{H}^\phi(\rho)(\xi) + 2A|\nabla\rho \lrcorner \xi|^2 \qquad \text{for all } \xi \in G(\phi). \tag{5.6}$$

As noted in the proof of Theorem 5.6, strict boundary convexity implies the existence of $\epsilon, \overline{\epsilon} > 0$ so that, along $\partial\Omega$, $\mathcal{H}^\phi(\rho)(\xi) \geq \overline{\epsilon}$ if $\xi \in G(\phi)$ with $|\cos\theta(\xi)| < \epsilon$. Therefore $\mathcal{H}^\phi(\overline{\rho})(\xi) \geq (1 + 2A\rho)\overline{\epsilon}$ for all such ξ. Choose a lower bound $-M$ for $\mathcal{H}^\phi(\rho)(\xi)$ over all $\xi \in G(\phi)$ in a neighborhood of $\partial\Omega$. Then by (5.6), $\mathcal{H}^\phi(\rho)(\xi) \geq -(1 + 2A\rho)M + 2|\nabla\rho|^2 A\epsilon^2$ for $\xi \in G(\phi)$ with $|\cos\theta(\xi)| \geq \epsilon$ at points of $\partial\Omega$. For A sufficiently large, the right hand side is > 0 in some neighborhood of $\partial\Omega$. ∎

One might hope for a converse to Theorem 5.6, e.g., if the domain Ω is ϕ-convex then the boundary is ϕ-convex. However, the following elementary example shows that this is false.

EXAMPLE . Let $\phi \equiv dx \wedge dy$ in $\mathbf{R}^3$ as in Example 3 of section 4. Let X denote the solid torus obtained by rotating the disk $\{(y, z) : y^2 + (z - R)^2 < r^2\}$ about the y-axis. Since each slice X_z has no holes in $\mathbf{R}^3$, the domain X is ϕ-convex (cf. Example 3 of §4). However, the boundary torus ∂X is ϕ-convex if and only if $2r \leq R$. This follows from an elementary calculation which uses the obvious defining function and Definition 5.1 (or by using Proposition 5.13 below)

QUESTION 5.8. For which strictly convex calibrated manifolds is it true that ϕ-convex subdomains have ϕ-convex boundaries? More generally, when is the ϕ-convexity of a domain a local condition at the boundary?

A weak partial converse to Theorem 5.6 is given by the following.

PROPOSITION 5.9. *Suppose the calibration is parallel, and set* $\delta = \mathrm{dist}(\bullet, \partial\Omega)$ *in* Ω. *If* $-\log\delta$ *is* ϕ*-plurisubharmonic near* $\partial\Omega$, *then* $\partial\Omega$ *is* ϕ*-convex .*

NOTE 5.10. Examples show that the strict convexity of $-\log\delta$ near $\partial\Omega$ is stronger than ϕ-convexity for $\partial\Omega$.

Proof. Set $\rho = -\delta$ on $\overline{\Omega}$ near $\partial\Omega$. Suppose that $\partial\Omega$ is not ϕ-convex. Then there exist $x \in \partial\Omega$ and $\xi_x \in G_x(\phi)$ with $\mathrm{span}\,(\xi) \subset T_x(\partial\Omega)$ and $(\mathcal{H}^\phi\rho)(\xi_x) < 0$. Let γ denote the geodesic segment in Ω which emanates orthogonally from $\partial\Omega$ at x. Since δ is the distance function, γ is an integral curve of $\nabla\delta$. Let ξ_y, $y \in \gamma$ denote the parallel translation of ξ_x along γ. Then ξ_y is a ϕ-plane with $\mathrm{span}\,(\xi_y) \perp \nabla\rho$ for all y. By formula (5.5), since $\nabla\rho \lrcorner \xi_y = 0$, we have

$$\mathcal{H}^\phi(-\log\delta)(\xi_y) \;=\; \frac{1}{\delta}\mathcal{H}^\phi(\rho)(\xi_y) \;<\; 0$$

for all y sufficiently close to x. Hence, $-\log\delta$ is not ϕ-plurisubharmonic near $\partial\Omega$. ∎

The Second Fundamental Form

The ϕ-convexity of a boundary can be equivalently defined in terms of its second fundamental form. Note that if $M \subset X$ is a smooth hypersurface with a chosen unit normal field n we have a quadratic form II defined on TM by

$$II(V, W) \;=\; \langle B_{V,W}, n\rangle$$

where B denotes the second fundamental form of M discussed in §2. For example, when $H = S^{n-1}(r) \subset \mathbf{R}^n$ is the euclidean sphere of radius r, oriented by the outward-pointing unit normal, we find that $II(V,W) = -\frac{1}{r}\langle V, W\rangle$.

For the sake of completeness we include a proof of the following standard fact.

LEMMA 5.11. *Suppose ρ is a defining function for Ω and let II denote the second fundamental form of the hypersurface $\partial\Omega$ oriented by the outward-pointing normal. Then*

$$\text{Hess}\,\rho\,\big|_{T\partial\Omega} = -|\nabla\rho|\,II$$

and therefore

$$\text{tr}_\xi \text{Hess}\rho \;=\; -|\nabla\rho|\,\text{tr}_\xi II$$

for all $\xi \in G(p, T\partial\Omega)$.

Proof. Suppose e is a tangent field on $\partial\Omega$. Extend e to a vector field tangent to the level sets of ρ. By definition $II(e,e) = \langle \nabla_e e, n\rangle$ where $n = \nabla\rho/|\nabla\rho|$ is the outward normal. Then $(\text{Hess}\,\rho)(e,e) = e(e\rho) - (\nabla_e e)\rho = -(\nabla_e e)\rho = -\langle \nabla_e e, \nabla\rho\rangle = -|\nabla\rho|\langle \nabla_e e, n\rangle$. ∎

REMARK . Recall that a defining function ρ for Ω satisfies $|\nabla\rho| \equiv 1$ in a neighborhood of $\partial\Omega$ if and only if ρ is the signed distance to $\partial\Omega$ (< 0 in Ω and > 0 outside of Ω). In fact any function ρ with $|\nabla\rho| \equiv 1$ in a riemannian manifold is, up to an additive constant, the distance function to (any) one of its level sets. In this case it is easy to see that

$$\text{Hess}\rho \;=\; \begin{pmatrix} 0 & 0 \\ 0 & -II \end{pmatrix} \tag{5.7}$$

where II denotes the second fundamental form of the hypersurface $H = \{\rho = \rho(x)\}$ with respect to the normal $n = \nabla\rho$ and the blocking in (5.7) is with respect to the splitting $T_xX = \text{span}\,(n_x) \oplus T_xH$. For example let $\rho(x) = \|x\| \equiv r$ in $\mathbf{R}^n$. Then direct calculation shows that $\text{Hess}\,\rho = \frac{1}{r}(I - \hat{x}\circ\hat{x})$ where $\hat{x} = x/r$.

COROLLARY 5.12. *For all tangential $\xi \in G(\phi)$*

$$(\mathcal{H}^\phi\rho)(\xi) \;=\; -|\nabla\rho|\text{tr}_\xi II.$$

Proof. Apply Theorem 2.3. ∎

As an immediate consequence we have

PROPOSITION 5.13. *Let $\Omega \subset X$ be a domain with smooth boundary $\partial\Omega$ oriented by the outward-pointing normal. Then $\partial\Omega$ is ϕ-convex if and only if its second fundamental form satisfies*

$$\text{tr}_\xi II \;\le\; 0$$

for all ϕ-planes ξ which are tangent to $\partial\Omega$. This can be expressed more geometrically by saying that

$$\text{tr}\left\{B\big|_\xi\right\} \quad \textit{must be inward} - \textit{pointing}$$

for all tangential ϕ-planes ξ.

REMARK . If ρ is the signed distance to $\partial\Omega$, then equation (5.7) together with Lemma 5.11 can be used to simplify (5.4). An arbitrary p-plane ξ at a point can be put in the canonical form $\xi = (\cos\theta n + \sin\theta e_1) \wedge e_2 \wedge \cdots \wedge e_p$ with $n = \nabla\rho$ and $n, e_1, ..., e_p$ orthonormal. Then $\eta = e_1 \wedge \cdots \wedge e_p$ is the tangential projection of ξ. Note that $\mathrm{tr}_\xi \mathrm{Hess}\rho = -\sin^2\theta \mathrm{tr}_\eta II$ and that $|\nabla\rho \lrcorner \xi|^2 = \cos^2\theta$, so that (5.4) becomes

$$\mathrm{tr}_\xi \mathrm{Hess}(-\log\delta) \;=\; -\frac{1}{\delta}\sin^2\theta\,\mathrm{tr}_\eta II + \frac{1}{\delta^2}\cos^2\theta.$$

We finish this section with a useful characterization of strictly convex domains.

THEOREM 5.14. *Let (X, ϕ) be a strictly ϕ-convex manifold and $\Omega \subset\subset X$ a domain with smooth boundary. Then the following are equivalent.*

(i) $\partial\Omega$ is strictly ϕ-convex .

(ii) $\mathrm{tr}_\xi \{II_{\partial\Omega}\} < 0$ for all tangential ϕ-planes ξ.

(iii) There exists a defining function $\rho \in C^\infty(\overline{\Omega})$ for $\partial\Omega$ which is strictly ϕ-plurisubharmonic on a neighborhood of $\overline{\Omega}$.

Proof. It is clear that (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii), so we need only prove that (i) $\Rightarrow$ (iii). Suppose $\partial\Omega$ is strictly ϕ-convex. By Proposition 5.7 we may assume that $\partial\Omega$ has a defining function ρ_0 which is strictly ϕ-plurisubharmonic in a neighborhood of $\partial\Omega$. By the Inverse Function Theorem there is a neighborhood U of $\partial\Omega$ and a diffeomorphism:

$$\partial\Omega \times [-2\epsilon, 2\epsilon] \xrightarrow{\cong} U \quad \text{ such that } \; \rho_0(x,t) = t.$$

Let $\rho_1 : X \to \mathbf{R}^+$ be a strictly ϕ-plurisubharmonic proper exhaustion function. Choose δ with $0 < \delta << \epsilon$. By replacing ρ_1 with $a\rho_1 - b$ for suitable $a, b > 0$ we may assume that

$$-\epsilon - \delta \;<\; \rho_1 \;<\; -\epsilon + \delta < 0 \qquad \text{on the neighborhood } U$$

Note that $\max\{\rho_0, \rho_1\} = \rho_1$ in the region where $t \equiv \rho_0 < -\epsilon - \delta$, and that $\max\{\rho_0, \rho_1\} = \rho_0$ where $t \equiv \rho_0 > -\epsilon + \delta$ (in particular, in a neighborhood of $\partial\Omega$).

By Remark 1.6 the function $\max\{\rho_0, \rho_1\}$ can be approximated by a smooth strictly ϕ-plurisubharmonic function ρ on U which agrees with $\max\{\rho_0, \rho_1\}$ outside the compact subset of U where $|\rho_0 - \rho_1| \leq \delta$. We see that $\rho = \rho_1$ when $t < -\epsilon - 2\delta$ and $\rho = \rho_0$ $(= t)$ when $t > -\epsilon + 2\delta$. Therefore, $\rho = \rho_0$ in a neighborhood of $\partial\Omega$, and ρ extends smoothly to Ω by setting $\rho = \rho_1$ in $\Omega - U$. ∎

6. ϕ-Free Submanifolds and Topology

Somewhat surprisingly, for any calibration ϕ there is a precise integer bound on the homotopy dimension of any strictly ϕ-convex domain. This is the first result below. After establishing it, we show that on the other hand, subject to this bound, there exist strictly convex domains of almost arbitrary topological type.

DEFINITION 6.1. The *free dimension*, denoted $\mathrm{fd}(\phi)$, of a calibrated manifold (X, ϕ) is the maximum dimension of a linear subspace in TX which contains no ϕ-planes. Such subspaces will be called *ϕ-free*.

THEOREM 6.2. *Suppose (X, ϕ) is a strictly ϕ-convex manifold. Then X has the homotopy type of a CW complex of dimension $\leq \mathrm{fd}(\phi)$.*

Proof. Let $f : X \to \mathbf{R}^+$ be a strictly ϕ-plurisubharmonic proper exhaustion function. By perturbing we may assume that f has non-degenerate critical points. The theorem follows if we show that each critical point has Morse index $\leq \mathrm{fd}(\phi)$ (cf. [M]). If this fails, then there is a critical point x at which $\mathrm{Hess}_x f$ has at least $\mathrm{fd}(\phi)+1$ negative eigenvalues. In particular, there exists a subspace $W \subset T_x X$ of dimension $= \mathrm{fd}(\phi)+1$ with $\mathrm{Hess}_x f\big|_W < 0$. However, by definition of $\mathrm{fd}(\phi)$, W must contain a ϕ-plane $\xi \in G(\phi)$, and since f is strictly ϕ-convex , we must have $\mathrm{tr}_\xi \mathrm{Hess}_x f > 0$, a contradiction. ∎

EXAMPLES:

(a) If (X, ω) is a Kähler manifold of real dimension $2n$, then $\mathrm{fd}(\omega) = n$.
More generally one has $\mathrm{fd}(\frac{1}{p!}\omega^p) = n - p + 1$.

(b) If (X, φ) is a Ricci-flat Kähler manifold (Calabi-Yau manifold) of real dimension $2n$ with Special Lagrangian calibration φ, then $\mathrm{fd}(\varphi) = 2n - 2$.

(c) If (X, Ψ) is a quaternionic Kähler manifold or hyperKähler manifold of real dimension $4n$ with the quaternionic calibration, then $\mathrm{fd}(\Psi) = 3n$. More generally for the calibration $\Psi_p \equiv \frac{1}{(2p+1)!}(\omega_I^2 + \omega_J^2 + \omega_K^2)^p$ one has $\mathrm{fd}(\Psi_p) = 3(n - p + 1)$.

(d) If (X, ϕ) is a 7-manifold with an associative calibration ϕ, then $\mathrm{fd}(\phi) = 4$.

(e) If (X, ψ) is a 7-manifold with a coassociative calibration ψ, then $\mathrm{fd}(\psi) = 4$.

(f) If (X, Φ) is an 8-manifold with a Cayley calibration Φ, then $\mathrm{fd}(\Phi) = 4$.

COMMENTS:

For (a), note that every real subspace of dimension $n+1$ in $\mathbf{C}^n$ contains a complex line and is therefore not free. The free subspaces of dimension n are exactly the **totally real** n-planes – those for which $JW \cap W = \{0\}$.

For the second statement, recall that the ϕ-planes are exactly the J-invariant subspaces of dimension $2p$. Now if $W \subset \mathbf{C}^n$ has codimension $\leq n-p$, then $\dim_{\mathbf{R}}\{W \cap JW\} \geq 2p$, and so W is not ϕ-free. However for a generic W of codimension $n-p+1$, the maximal complex subspace of W satisfies $\dim_{\mathbf{R}}\{W \cap JW\} = 2p - 2$.

For (b), we first show that every real hyperplane $H \subset \mathbf{C}^n$ contains a Special Lagrangian n-plane and is therefore not free. Choose a unit vector $n \perp H$ and consider the orthogonal decomposition $\mathbf{C}^n = (\mathbf{R}n) \oplus (\mathbf{R}Jn) \oplus H_0$ where $H_0 = H \cap J(H)$ is the maximal complex subspace of H. If $L_0 \subset H_0$ is a Lagrangian subspace of H_0, then $L = (\mathbf{R}Jn) \oplus L_0$

is a Lagrangian subspace of H. Rotating L_0 in H_0 makes L Special Lagrangian as claimed. We now observe that for a real subspace $W \subset \mathbf{C}^n$ of dimension $2n-2$,

$$W \text{ is } \varphi \text{ free} \iff J(W^\perp) \not\subset W \iff W \text{ is symplectic, i.e., } \omega^{n-1}\big|W \neq 0.$$

For the first equivalence note that if $J(W^\perp) \subset W$, then the construction above gives a Special Lagrangian $L \subset W$. Conversely, given $L \subset W$, $J(L) = L^\perp = (L^\perp \cap W) \oplus W^\perp$ and so $J(W^\perp) \subset W$. For the second equivalence, note that $J(W^\perp) \subset W$ implies that $J(W^\perp)$ lies in the null space of $\omega\big|_W$. Conversely, if $v \in W$ lies in the null space of $\omega\big|_W$, then $J(\operatorname{span}\{v, Jv\}) \subset W^\perp$

For (c) suppose $V \subset \mathbf{H}^n$ has codimension $< n$. Then $V \cap I(V) \cap J(V) \cap K(V)$ is a non-trivial quaternionic subspace of V and so V cannot be free. A subspace W of real codimension n is free if and only if $W^\perp \cap IW^\perp = W^\perp \cap JW^\perp = W^\perp \cap KW^\perp = \{0\}$.

For the second statement we use that fact that $G(\Psi_p)$ is exactly the set of quaternionic linear subspaces of quaternionic dimension p in $\mathbf{H}^n$. The argument then proceeds as in part two of (a).

For (d), suppose $V \subset \mathrm{Im}\mathbf{O}$ has dimension 5. Let x, y be an orthonormal basis of $V^\perp$. Choose any unit vector $\epsilon \in V$ perpendicular to $z \equiv xy$. Then $\operatorname{span}\{z, \epsilon, z\epsilon\} \subset V$ is associative, and so V is not free. (To see that $z\epsilon \in V$, note that left multiplication by z is an isometry which preserves the quaternion subalgebra $\operatorname{span}\{x, y, z\}$ and therefore also preserves its orthogonal complement.) We now claim that

$$\text{a 4-plane } W \subset \mathrm{Im}\mathbf{O} \text{ is free if and only if } W^\perp \text{ is not } \phi\text{-isotropic, i.e., } \phi\big|_{W^\perp} \neq 0.$$

To see this, suppose there exists an associative 3-plane $V \subset W$. Then $V^\perp = \mathbf{R}\epsilon \oplus W^\perp$ is coassociative (where $\epsilon \in W$ is a unit vector perpendicular to V). Choose an orthonormal basis x', y', z' of $W^\perp$. Coassociativity is equivalent to the fact that the 4-form $(*\phi)(\epsilon, x', y', z') = \langle \epsilon, [x', y', z'] \rangle = \pm 1$ where $[\cdot, \cdot, \cdot]$ is the associator. We now recall the general equality $\phi(x', y', z')^2 + \|[x', y', z']\|^2 = 1$ (cf. [HL$_1$]) from which it follows that $\phi(x', y', z') = 0$, that is, $W^\perp$ is ϕ-isotropic. Conversely, supposing $W^\perp$ is ϕ-isotropic, the equality shows that $[x', y', z']$ is a unit vector and therefore $U \equiv \operatorname{span}\{x', y', z', [x', y', z']\}$ is coassociative. Hence, $U^\perp \subset W$ is associative.

For (e), suppose $V \subset \mathrm{Im}\mathbf{O}$ has dimension 5. Let x, y be an orthonormal basis of $V^\perp$. Then $U = \operatorname{span}\{x, y, xy\}$ is associative, and so $U^\perp \subset V$ is coassociative. Hence, V is not free. Of course the free 4-planes are exactly those which are not coassociative.

For (f), suppose $V \subset \mathbf{O}$ has dimension 5. Let x, y, z be an orthonormal basis of $V^\perp$. Then $W = \operatorname{span}\{x, y, z, x \times y \times z\}$ is a Cayley plane, and so $W^\perp \subset V$ is also Cayley. Hence, V is not free. The free 4-planes are exactly those which are not Cayley.

We now show that within the homotopy restrictions imposed by Theorem 6.2, the possible topologies for strictly ϕ-convex manifolds are vast.

Let (X, ϕ) be a calibrated manifold. A p-plane ξ is said to be *tangential* to a submanifold $M \subset X$ if $\operatorname{span}\xi \subset T_xM$.

DEFINITION 6.3. A closed submanifold $M \subset X$ is **ϕ-free** if there are no ϕ-planes $\xi \in G(\phi)$ which are tangential to M. If the restriction of the calibration ϕ to M vanishes, M is called **ϕ-isotropic**.

Note that ϕ-isotropic submanifolds are ϕ-free. Each submanifold of dimension strictly less than the degree of ϕ is ϕ-isotropic and hence automatically ϕ-free. Furthermore, in dimension p the generic local submanifold is ϕ-free. Depending on the geometry, this may continue through a range of dimensions greater than p.

THEOREM 6.4. *Suppose M is a closed submanifold of (X, ϕ) and let $f_M(x) \equiv \frac{1}{2}\text{dist}_M^2(x)$ denote half the square of the distance to M. Then M is ϕ-free if and only if the function f_M is strictly ϕ-plurisubharmonic at each point in M (and hence in a neighborhood of M).*

Proof. We begin with the following.

LEMMA 6.5. *Fix $x \in M$ and let $P_N : T_xX \to N$ denote orthogonal projection onto the normal plane of M at x. Then for each $\xi \in G(\phi)$ one has*

$$\{\lambda_\phi(\text{Hess}_x f_M)\}(\xi) \;=\; \langle P_N, P_\xi \rangle \tag{6.1}$$

Proof. By Theorem 2.3

$$\{\lambda_\phi(\text{Hess}_x f)\}(\xi) \;=\; \langle \text{Hess}_x f, P_\xi \rangle \tag{6.2}$$

for any function f. The lemma then follows from the assertion that

$$\text{Hess}_x f_M \;=\; P_N. \tag{6.3}$$

To see this we first note that the Hessian of any function f can be written

$$\text{Hess} f(V, W) \;=\; \langle V, \nabla_W(\nabla f) \rangle \tag{6.4}$$

for all $V, W \in T_xX$. It follows that if $\nabla f = 0$ on the submanifold M, then $T_xM \subset \text{Null}(\text{Hess}_x f)$. Thus, with respect to the decomposition $T_xX = T_xM \oplus N$ we have

$$\text{Hess}_x f_M \;=\; \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

and it remains to show that A is the identity. To see this, set $\delta(x) = \text{dist}_M(x)$ and note that $\nabla\delta = n$ is a smooth unit-length vector field near (but not on) M whose integral curves are geodesics emanating from M. Hence,

$$\nabla_n(\nabla f_M) = \nabla_n(\nabla \frac{1}{2}\delta^2) = \nabla_n(\delta n) = n + \delta \nabla_n n = n.$$

Taking limits along normal geodesics down to M gives the result. ∎

Theorem 6.4 now follows from the fact that

$$\langle P_N, P_\xi \rangle \;\geq\; 0 \text{ with equality iff } \text{span}\,\xi \subset N^\perp = T_xM.$$

■

The following result gives us a powerful, very general method for constructing strictly ϕ-convex domains in (X, ϕ).

THEOREM 6.6. *Suppose M is a ϕ-free submanifold of (X, ϕ). Then there exists a fundamental neighborhood system $\mathcal{F}(M)$ of M consisting of strictly ϕ-convex domains. Moreover,*

(a) M is a deformation retract of each $U \in \mathcal{F}(M)$.

(b) $\mathrm{PSH}(V, \phi)$ is dense in $\mathrm{PSH}(U, \phi)$ if $U \subset V$ and $V, U \in \mathcal{F}(M)$.

(c) Each compact set $K \subset M$ is $\mathrm{PSH}(U, \phi)$-convex for each $U \in \mathcal{F}(M)$.

REMARK . The existence of ϕ-free submanifolds gives the existence of strictly ϕ-convex domains with essentially every topological type permitted by Morse Theory (Theorem 6.2). Note in particular that if $M \subset X$ is ϕ-free, then **every submanifold of M is also ϕ-free.**

Proof. We construct tubular neighborhoods of M as follows. Let $\epsilon \in C^\infty(M)$ be a smooth function which vanishes at infinity and has the property that for each $x \in M$ the ball $\{y \in X : \frac{1}{2}\mathrm{dist}(y, x)^2 \leq \epsilon(x)\}$ is compact and geodesically convex. Assume also that ϵ is sufficiently small so that the exponential map gives a diffeomorphism

$$\exp : N_\epsilon \longrightarrow U_\epsilon$$

from the open set N_ϵ in the normal bundle N defined by $\frac{1}{2}\|n_x\|^2 < \epsilon(x)$ to the neighborhood

$$U_\epsilon = \{x \in X : f_M(x) < \epsilon(x)\}. \tag{6.5}$$

of M in X. Each U_ϵ admits a deformation retraction onto M.

By Theorem 6.4 the function $f_M = \frac{1}{2}\mathrm{dist}^2_M(\cdot)$ is strictly ϕ-plurisubharmonic on a neighborhood of M, which we can assume to be W. We impose the following additional condition on the function $\epsilon \in C^\infty(W)$.

$$f_M - t\epsilon \quad \text{is strictly } \phi - \text{plurisubharmonic on } \ W \ \text{ for } \ 0 \leq t \leq 1. \tag{6.6}$$

Since (6.6) is valid as long as ϵ and its first and second derivatives vanish sufficiently fast at infinity, it is easy to see that the family $\mathcal{F}(M)$ of neighborhoods U_ϵ constructed above with ϵ satisfying (6.6) is a fundamental neighborhood system for M.

Obviously, the function $\psi \equiv (\epsilon - f_M)^{-1}$ is a proper exhaustion for U_ϵ. Recall that if g is a positive concave function, then $1/g$ is convex, or more directly, calculate that

$$\mathrm{Hess}\psi = \psi^2 \mathrm{Hess}(f_M - \epsilon) + \psi^3 \nabla(\epsilon - f_M) \circ \nabla(\epsilon - f_M). \tag{6.7}$$

Applying λ_ϕ to (6.7) proves that $(\epsilon - f_M)^{-1}$ is strictly ϕ-plurisubharmonic on $\{f_M < \epsilon\} = U_\epsilon$. Hence, U_ϵ is strictly ϕ-convex.

To prove parts (b) and (c) one uses Proposition 3.16 in [HL_2], characterizing denseness of PSH(V, ϕ) in PSH(U, ϕ) in terms of relative convexity, and argues exactly as on page 302 of [HW_1]. ■

EXAMPLE 6.7. As mentioned above, Theorem 6.6 exhibits a rich family of ϕ-convex domains in (X, ϕ). For example, let $M \subset X$ be *any submanifold of dimension* $< p = \deg\phi$. Then by 6.6, M has a fundamental system of neighborhoods each of which is a strictly ϕ-convex domain homotopy equivalent to M.

EXAMPLE 6.8. Interesting examples occur in all the calibrated geometries examined in depth in [HL$_1$]. Suppose for instance that X is a Calabi-Yau manifold with Special Lagrangian calibration ϕ. Then *any complex submanifold* $Y \subset X$ *(of positive codimension) is* ϕ*-free.* It follows that any smooth submanifold of Y is also ϕ-free.

Zero Sets of Non-negative Strictly ϕ-plurisubharmonic Functions

We now consider the following two classes of subsets of (X, ϕ).

(1) Closed subsets A of ϕ-free submanifolds.

(2) Zero sets of non-negative strictly ϕ-plurisubharmonic functions f.

These two classes are basically the same, as described in the following two propositions.

PROPOSITION 6.9. *Suppose A is a closed subset of a ϕ-free submanifold M of X. Then there exists a non-negative function $f \in C^\infty(X)$ with*

(a) $A = \{x \in X : f(x) = 0\}$

(b) f is strictly ϕ-plurisubharmonic at each point in M (and hence in a neighborhood of M in X).

Proof. Since M is a closed submanifold, the function f_M in Theorem 6.4 can be extended to $h \in C^\infty(X)$ which agrees with f_M in a neighborhood of M and satisfies

$$h \geq 0 \qquad \text{and} \qquad \{h = 0\} = M.$$

Choose $\psi \in C^\infty(X)$ with $\psi \geq 0$ and $A = \{x \in X : \psi = 0\}$. Now choose $\epsilon \in C^\infty(X)$ with $\epsilon(x) > 0$ for all $x \in X$, and with ϵ and its derivatives sufficiently small so that $f \equiv h + \epsilon\psi$ is strictly ϕ-plurisubharmonic on M. ∎

PROPOSITION 6.10. *Suppose $f \in C^\infty(X)$ is a non-negative function which is strictly ϕ-plurisubharmonic at each point in $A \equiv \{x \in X : f(x) = 0\}$. Given a point $x \in A$ there exists a neighborhood U of x and a proper ϕ-free submanifold M of U such that $A \cap U \subset M$.*

Proof. Given $x \in A$ we may choose geodesic normal coordinates (z, y) in a neighborhood U at x so that

$$\text{Hess}_x f \;=\; \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} \tag{6.8}$$

where Λ is the diagonal matrix $\text{diag}\{\lambda_1, ..., \lambda_r\}$, r is the rank of $\text{Hess}_x f$, and $\lambda_j \neq 0$ for $j = 1, ..., r$. Set

$$M \;=\; \left\{ w \in U : \frac{\partial f}{\partial y_1} = \cdots = \frac{\partial f}{\partial y_r} = 0 \right\}.$$

Since $\nabla\frac{\partial f}{\partial y_1}, ..., \nabla\frac{\partial f}{\partial y_r}$ are linearly independent at x, M is a codimension r submanifold locally near x.

Note that $\ker(\mathrm{Hess}_x f) = T_x M$. It remains to show that $\ker(\mathrm{Hess}_x f)$ is ϕ-free (since if M is ϕ-free at x, then M is ϕ-free in a neighborhood of x). This is proved in Lemma 6.11 below. ∎

LEMMA 6.11. *Suppose f is strictly ϕ-plurisubharmonic at $x \in X$. Then* $\ker(\mathrm{Hess}_x f) \subset T_x X$ *is ϕ-free.*

Proof. If $\ker(\mathrm{Hess}_x f) \subset T_x X$ is not ϕ-free, there exists $\xi \in G(\phi)$ with $(\mathrm{Hess}_x f)\big|_{\mathrm{span}\,\xi} = 0$. Consequently, $(\mathcal{H}^\phi f)(\xi) = \lambda_\phi(\mathrm{Hess}_x f)(\xi) = \mathrm{tr}_\xi(\mathrm{Hess}_x f) = 0$, and f is not strict at x. ∎

REMARK 6.12. Theorem 6.6 can be generalized as follows. Suppose $M = \{f = 0\}$ is the zero set of a non-negative strictly ϕ-plurisubharmonic function f on (X, ϕ). Then there exists a fundamental neighborhood system $\mathcal{F}(M)$ of M consisting of strictly ϕ-convex domains which satisfy (c) of Theorem 6.6. The neighborhoods $U_\epsilon \in \mathcal{F}(M)$ are defined by $U_\epsilon = \{x \in X : f(x) < \epsilon(x)\}$ where $\epsilon > 0$ is a C^∞ function on X vanishing at infinity along with its first and second derivatives so that $f - \epsilon$ remains strictly ϕ-plurisubharmonic . The proofs of (b), (c) and (d) are essentially the same as in Theorem 6.6.

We conclude with the following useful observation.

PROPOSITION 6.13. *Let M be a submanifold of (X, ϕ) and f a smooth function defined on a neighborhood of M such that:*

(1) $\nabla f \equiv 0$ on M, and

(2) f is strictly ϕ-plurisubharmonic at all points of M.

Then M is ϕ-free.

Proof. By (6.4) we see that $TM \subseteq \ker(\mathrm{Hess} f)$ at all points of M. We then apply Lemma 6.11. ∎

COROLLARY 6.14. *Let f be a non-negative, real analytic function on (X, ϕ) and consider the real analytic subvariety $Z \equiv \{f = 0\}$. If f is strictly ϕ-plurisubharmonic at points of Z, then each stratum of Z is ϕ-free.*

Appendix A. Submanifolds which are ϕ-critical.

Here we establish a useful extension of Theorem 2.3 to certain ξ which are not ϕ-planes. Let $G \equiv G(p, V)$ denote the Grassmannian of oriented p-planes in the inner product space V, considered as the subset $G \subset \Lambda_p V$ of unit simple vectors.

DEFINITION A.1. Given $\phi \in \Lambda^p V^*$ an element $\xi \in G$ is said to be a **ϕ-critical point** if $\xi \in G$ is a critical point of the function $\phi\big|_G$. Equivalently, ϕ must vanish on $T_\xi G \subset \Lambda_p V$. Let $G^{\mathrm{cr}}(\phi)$ denote the set of ϕ-critical points.

Note that if ϕ is a calibration on G, i.e., $\sup \phi\big|_G = 1$, then

$$G(\phi) \;\subset\; G^{\mathrm{cr}}(\phi)$$

since for each $\xi \in G(\phi)$ the form ϕ attains its maximum value 1 at ξ. Equation (2.9)$'$ extends from $G(\phi)$ to $G^{\rm cr}(\phi)$ as follows

PROPOSITION A.2. *Suppose $\phi \in \Lambda^p V^*$ and $A \in \mathrm{End}(V)$. Then for all $\xi \in G^{\rm cr}(\phi)$*

$$\lambda_\phi(A)(\xi) \;=\; (\mathrm{tr}_\xi A)\phi(\xi)$$

Proof. This is an immediate consequence of the more general Proposition A.4 below. ∎

Recall that at a point $\xi \in G$ there is a canonical isomorphism:

$$T_\xi G \;\cong\; \mathrm{Hom}\,(\mathrm{span}\,\xi, (\mathrm{span}\,\xi)^\perp). \tag{A.1}$$

On the other hand, $T_\xi G$ is canonically a subspace of $\Lambda_p V$. It is exactly the subspace spanned by the first cousins of ξ. More specifically, the isomorphism (A.1) associates to $L : \mathrm{span}\,\xi \to (\mathrm{span}\,\xi)^\perp$ the p-vector $D_L\xi$.

DEFINITION A.3. Let $A \in \mathrm{End}(V)$ be a linear map. At each point $\xi \in G$ we define a tangent vector

$$D_{\widetilde{A}}\xi \;\in\; T_\xi G$$

where $\widetilde{A} = P_{\xi^\perp} \circ A \circ P_\xi$. This vector field $\xi \to D_{\widetilde{A}}\xi$ on G is called the **A-vector field.**

REMARK . A straightforward calculation shows that if A is symmetric, this A-vector field on G is the gradient of the height function $F_A : G \to \mathbf{R}$ given by $F_A(\xi) = \langle A, P_\xi \rangle$.

PROPOSITION A.4. *Suppose $\phi \in \Lambda^p V^*$ and $A \in \mathrm{End}(V)$. Then for all p-planes $\xi \in G(p,V)$,*

$$\lambda_\phi(A)(\xi) \;=\; (\mathrm{tr}_\xi A)\phi(\xi) + \phi(D_{\widetilde{A}}\xi) \tag{A.2}$$

Proof. Pick an orthonormal basis for ξ and extend to an orthonormal basis of V. It suffices to prove (A.2) when $A = b \otimes a$ with a and b elements of this basis. Using formula (2.11) we see the following.

(1) If $a \in \xi^\perp$, then all terms in (A.2) are zero.

(2) If $a \in \xi$ and $b \in \xi^\perp$, then $\widetilde{A} = A = b \otimes a$, $\mathrm{tr}_\xi A = 0$, and $\lambda_\phi(b \otimes a)(\xi) = (a \wedge (b \lrcorner \phi))(\xi) = \phi(b \wedge (a \lrcorner \xi)) = \phi(D_A\xi)$

(3) If $a = b \in \xi$, then $\lambda_\phi(A)(\xi) = \phi(a \wedge (a \lrcorner \xi) = \phi(\xi)$ and $\mathrm{tr}_\xi(A) = 1$. Since $\widetilde{A} = 0$, equation (A.2) holds in this case.

(4) If $a, b \in \xi$ and $a \perp b$, then $b \wedge (a \lrcorner \xi) = 0$, and one sees easily that all three terms in (A.2) are zero. ∎

REMARK A.5. Proposition A.2 can be restated as

$$\lambda^*_\phi(\xi) \;=\; \phi(\xi)P_\xi \qquad \text{for all} \quad \xi \in G^{\rm cr}(\phi). \tag{A.3}$$

Conversely, if $\lambda^*_\phi(\xi) \;=\; cP_\xi$ for some $\xi \in G(p,V)$, then $c = \phi(\xi)$ and ξ is ϕ-critical.

Proof. For all $\xi \in G(p,V)$ we have $\langle P_\xi, \lambda_\phi^*(\xi)\rangle = (\lambda_\phi P_\xi)(\xi) = (D_{P_\xi^t}\phi)(\xi) = \phi(D_{P_\xi}\xi) = p\phi(\xi)$ since $D_{P_\xi}\xi = p\xi$. Therefore, $\lambda_\phi^*(\xi) = cP_\xi$ implies that $pc = p\phi(\xi)$ and equation (A.3) holds. Equation (A.2) now implies that $\phi(D_{\widetilde{A}}\xi) = 0$ for all $A \in \mathrm{End}(V)$ and, in particular, $\phi(D_L\xi) = 0$ for all $L : \xi \to \xi^\perp$. That is, ϕ vanishes on $T_\xi G \subset \Lambda_p V$, i.e., $\xi \in G^{\mathrm{cr}}(\phi)$. ∎

We now define an oriented submanifold M of X to be ϕ-**critical** if $\overrightarrow{T}_x M \in G^{\mathrm{cr}}(\phi)$ for all $x \in M$. We leave it to the reader to use Proposition A.2 to establish the following extension of the previous results.

THEOREM A.6. *Suppose ϕ is a p-form on a riemannian manifold X and $M \subset X$ is a ϕ-critical submanifold with mean curvature vector field H. Then for all $f \in C^\infty(X)$,*

$$\lambda_\phi(\mathrm{Hess} f) \;=\; [\Delta_M(f) + H(f)]\phi$$

when restricted to M. In particular, if M is minimal, then on M

$$\lambda_\phi(\mathrm{Hess} f) \;=\; (\Delta_M f)\phi$$

EXAMPLE. Let $\phi = \frac{1}{6}\{\omega_I^2 + \omega_J^2 + \omega_K^2\}$ be the quaternion calibration on $\mathbf{H}^n$. Then $\pm\frac{1}{3}$ are critical values and the ϕ-critical submanifolds with critical value $\pm\frac{1}{3}$ include all complex Lagrangian submanifolds for any complex structure defined by right multiplication by a unit imaginary quaternion (cf. [U]).

REMARK A.7. In a very interesting recent paper Colleen Robles [R] has shown that for any given parallel calibration ϕ, $\lambda_\phi(\mathrm{SkewEnd}(TX))$ generates an exterior differential system whose integral submanifolds are exactly the ϕ-critical submanifolds. At a point x this can be stated equivalently as follows. Given $\xi \in G_p(T_xX)$

$$\lambda_\phi(A)(\xi) \;=\; 0 \;\; \forall\, A \in \mathrm{SkewEnd}(T_xX) \qquad \Longleftrightarrow \qquad \xi \in G^{\mathrm{cr}}(\phi). \tag{A.4}$$

This can be derived from (A.2) and (A.3).

Appendix B. Constructing ϕ-plurisubharmonic functions.

Straightforward calculation shows that if $F(x) = g(u_1(x), ..., u_m(x))$, then

$$\text{Hess}F \;=\; \sum_{j=1}^{m} \frac{\partial g}{\partial t_j} \text{Hess} u_j + \sum_{i,j=1}^{m} \frac{\partial^2 g}{\partial t_i \partial t_j} (\nabla u_i \circ \nabla u_j) \tag{$B.1$}$$

and hence

$$\mathcal{H}^{\phi}(F) \;=\; \sum_{j=1}^{m} \frac{\partial g}{\partial t_j} \mathcal{H}^{\phi}(u_j) + \sum_{i,j=1}^{m} \frac{\partial^2 g}{\partial t_i \partial t_j} \lambda_{\phi}(\nabla u_i \circ \nabla u_j) \tag{$B.1'$}$$

PROPOSITION B.1. *If $u_1, ..., u_m$ are ϕ-pluriharmonic and $g(t_1, ..., t_m)$ is convex, then $F = g(u_1, ..., u_m)$ is ϕ-plurisubharmonic . More generally, if $\frac{\partial g}{\partial t_j} \geq 0$ for $j = 1, ..., m$ and g is convex, then $F = g(u_1, ..., u_m)$ is ϕ-plurisubharmonic whenever each u_j is ϕ-plurisubharmonic.*

Proof. Under our assumptions the first term in equation (B.1)$'$ is ≥ 0 on any $\xi \in G(\phi)$. To show that the second term is ≥ 0 is suffices to consider the case where the matrix $((\frac{\partial^2 g}{\partial t_i \partial t_j}))$ is rank one, i.e., equal to $((x_i x_j))$ for some vector $x \in \mathbf{R}^n$. Then the second term equals $\lambda_{\phi}\{(\sum_i x_i \nabla u_i) \circ (\sum_j x_j \nabla u_j)\}$ which is ≥ 0 on $\xi \in G(\phi)$ by (2.13) and Corollary 2.7. ∎

We now analyze the case where $m = 2$ and determine necessary and sufficient conditions for $F = g(u_1, u_2)$ to be ϕ-plurisubharmonic.

LEMMA B.2. *Fix $v, w \in \mathbf{R}^n$ and $\xi \in G(\phi)$. Let v_0 and w_0 denote the orthogonal projections of v and w respectively onto ξ (considered as a p-plane in $\mathbf{R}^n$). Then*

$$\lambda_{\phi}(v \circ w)(\xi) \;=\; \langle v_0, w_0 \rangle.$$

Proof. Write $v = v_0 + v_1$ and $w = w_0 + w_1$ with respect to the decomposition $\mathbf{R}^n = \text{span}\,\xi \oplus (\text{span}\,\xi)^{\perp}$. Then for $\xi \in G(\phi)$ we have

$$\lambda_{\phi}(v \circ w)(\xi) \;=\; \phi\{(v_0 + v_1) \wedge ((w_0 + w_1) \lrcorner \xi))\} \;=\; \phi\{(v_0 + v_1) \wedge (w_0 \lrcorner \xi)\}$$

$$= \; \phi(v_0 \wedge (w_0 \lrcorner \xi)) \;=\; \langle v_0, w_0 \rangle \phi(\xi) \;=\; \langle v_0, w_0 \rangle.$$

where the third equality follows from the First Cousin Principle. ∎

By Lemma B.2 we have that for $\xi \in G(\phi)$,

$$\begin{aligned} &\lambda_{\phi}\{a\, v \circ v + 2b\, v \circ w + c\, w \circ w\}(\xi) \\ &\qquad = \; a\|v_0\|^2 + 2b\langle v_0, w_0 \rangle + c\|w_0\|^2 \\ &\qquad = \; \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} \|v_0\|^2 & \langle v_0, w_0 \rangle \\ \langle v_0, w_0 \rangle & \|w_0\|^2 \end{pmatrix} \right\rangle \end{aligned} \tag{$B.2$}$$

REMARK B.3. A symmetric $n \times n$-matrix A is ≥ 0 iff $\langle A, P \rangle \geq 0$ for all rank-one symmetric $n \times n$-matrices P.

REMARK B.4. The matrix $\begin{pmatrix} \|v_0\|^2 & \langle v_0, w_0 \rangle \\ \langle v_0, w_0 \rangle & \|w_0\|^2 \end{pmatrix}$ is rank-one iff v_0 and w_0 are linearly dependent.

LEMMA B.5. *Let $v, w \in \mathbf{R}^n$ be linearly independent. Suppose that for every line*

$$\ell \ \subset \ \text{span}\,\{v, w\}$$

there exists a $(p-1)$-plane $\xi_0 \subset \text{span}\,\{v, w\}^\perp$ such that $\ell \oplus \xi_0$ (when properly oriented) is a ϕ-plane. Then $\lambda_\phi\{av \circ v + 2bv \circ w + cw \circ w\}$ is ϕ-positive if and only if $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \geq 0$.

Proof. Necessity is already done. For sufficiency fix a, b, c. For each $\ell \subset \text{span}\,\{v, w\}$ let $\xi \in G(\phi)$ be the oriented p-plane $\ell \oplus \xi_0$ given in the hypothesis, and note that by equation (B.2)

$$\lambda_\phi\{av \circ v + 2bv \circ w + cw \circ w\}(\xi) \ = \ \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} v_\ell^2 & v_\ell w_\ell \\ v_\ell w_\ell & w_\ell^2 \end{pmatrix} \right\rangle \ \geq \ 0$$

where $v_\ell = \langle v, e \rangle e$, $w_\ell = \langle w, e \rangle e$, and $\ell = \text{span}\,\{e\}$. Now the matrix $\begin{pmatrix} v_\ell^2 & v_\ell w_\ell \\ v_\ell w_\ell & w_\ell^2 \end{pmatrix}$ is rank-one, and every rank-one 2×2 matrix, up to positive scalars, occurs in this family. The result follows from Remark B.3. ∎

DEFINITION B.6. A calibration ϕ on a manifold X is called **rich** (or **2-rich**) if for any 2-plane $P \subset T_x X$ at any point x, and for any line $\ell \subset P$, there exists a $(p-1)$-plane $\xi_0 \subset P^\perp$ so that $\pm\ell \oplus \xi_0$ is a ϕ-plane.

PROPOSITION B.7. *Let (X, ϕ) be a rich calibrated manifold. Suppose u_1, u_2 are ϕ-pluriharmonic functions on X with $\nabla u_1 \wedge \nabla u_2 \neq 0$ on a dense set. Then for any C^2-function $g(t_1, t_2)$*

$$F \ = \ g(u_1, u_2) \in \text{PSH}(X, \phi) \quad \Leftrightarrow \quad g \ \text{ is convex}$$

Proof. Apply Proposition B.1, equation (B.2) and Lemma B.5. ∎

PROPOSITION B.8. *The Special Lagrangian calibration on a Calabi-Yau n-fold, $n \geq 3$, and the associative and coassociative calibrations on a G_2-manifold are rich calibrations.*

Proof. For the Special Lagrangian case it suffices to consider $\phi = \text{Re}(dz)$ on $\mathbf{C}^n$, $n \geq 3$. Let $e_1, Je_1, ..., e_n, Je_n$ be the standard hermitian basis of $\mathbf{C}^n$. By unitary invariance we may assume that $\ell = \text{span}\,\{e_1\}$ and $P = \text{span}\,\{e_1, \alpha Je_1 + \beta e_2\}$ (see [HL$_1$, Lemma 6.13] for example). Then the $(p-1)$-plane $\xi_0 = -Je_2 \wedge Je_3 \wedge e_4 \wedge \cdots \wedge e_n$ does the job.

Consider now the associative calibration $\phi(x, y, z) = \langle x \cdot y, z \rangle$ on the imaginary octonians $\text{Im}(\mathbf{O}) = \text{Im}(\mathbf{H}) \oplus \mathbf{H} \cdot \epsilon$ where $\mathbf{H}$ denotes the quaternions and ϵ is any unit vector in $\text{Im}(\mathbf{O})^\perp$. By the transitivity of the group G_2 on $S^6 = G_2/\text{SU}(3)$ and the transitivity of

SU(3) on the tangent space, we may assume $\ell = \text{span}\{i\}$ and $P = \text{span}\{i, j\}$ in $\text{Im}(\mathbf{H})$. We now choose $\xi_0 = \epsilon \wedge (i \cdot \epsilon)$. For the coassociative calibration we choose $\xi_0 = k \wedge (i\epsilon) \wedge (k\epsilon)$ and note that $i \wedge \xi_0 = i \wedge k \wedge (i\epsilon) \wedge (k\epsilon)$ is coassociative because its orthogonal complement is $j \wedge \epsilon \wedge (j\epsilon)$ which is associative.

Alternatively, as noted by the referee, these latter cases follow easily from the fact that G_2 acts transitively on the Stiefel manifolds $V_{2,7}$ of ordered pairs of orthonormal 2-vectors in $\mathbf{R}^7$ (see for example [HL$_1$, Prop. IV.1.10]). ∎

We now give some examples and applications of the material above. We start with Special Lagrangian geometry where the ϕ-pluriharmonic functions are given by Proposition 1.13. Hence, we may apply Proposition B.7 to conclude the following. Let $u_1(z)$ and $u_2(z)$ be two traceless hermitian quadratic forms on $\mathbf{C}^n$. (For example, take $u_1(z) = |z_1|^2 - |z_2|^2$ and $u_2(z) = (n-2)|z_1|^2 - |z_3|^2 - \cdots - |z_n|^2$.) Then $g(u_1(z), u_2(z))$ is ϕ-plurisubharmonic if and only if g is convex.

Formula (B.1)′ can be usefully applied to more general functions u_j. For example, in the Special Lagrangian case on $\mathbf{C}^n$ with $\phi = \text{Re}(dz)$, one has that $dd^\phi(\frac{1}{2}|z_k|^2) = \phi$, for any complex coordinate z_k in any unitary coordinate system on $\mathbf{C}^n$. Hence a linear combination of these functions has the property that $dd^\phi u = c\phi$ for some constant c.

PROPOSITION B.9. *Let (X, ϕ) be a rich calibrated manifold. Suppose $u_1, ..., u_n \in C^\infty(X)$ satisfy the equations $dd^\phi u_i = c_i\phi$ for constants $c_1, ..., c_n$. Then for any C^2-function $g(t_1, ..., t_n)$*

$$F \;=\; g(u_1, ..., u_n) \in \text{PSH}(X, \phi) \;\;\Leftrightarrow\;\; \left\{\sum_{i=1}^n c_i \frac{\partial g}{\partial t_i}\right\} \mathbf{Id} + \left\langle \text{Hess}_g, ((\langle (\nabla u_i)^\xi, (\nabla u_j)^\xi \rangle)) \right\rangle \;\geq\; 0$$

for all ϕ-planes ξ at all points of X.

Appendix C. Structure of the Core.

Let (X, ϕ) be a calibrated manifold and consider the set

$$\mathcal{N} \equiv \{\xi \in G(\phi) : (\mathcal{H}^\phi f)(\xi) = 0 \text{ for all } f \in \mathrm{PSH}(X, \phi)\}.$$

PROPOSITION C.1. *Let $\pi : G(\phi) \to X$ denote the projection. Then*

$$\pi(\mathcal{N}) = \mathrm{Core}(X).$$

Proof. Suppose $x \notin \mathrm{Core}(X)$. Then by definition there exists $f \in \mathrm{PSH}(X, \phi)$ with $(\mathcal{H}^\phi f)(\xi) > 0$ for all $\xi \in \pi^{-1}(x)$. Hence, $x \notin \pi(\mathcal{N})$.

Conversely, suppose $x \notin \pi(\mathcal{N})$. Then for each $\xi \in \pi^{-1}(x)$ there exists $f_\xi \in \mathrm{PSH}(X, \phi)$ with $(\mathcal{H}^\phi f_\xi)(\xi) > 0$. Let $W_\xi = \{\eta \in \pi^{-1}(x) : (\mathcal{H}^\phi f_\xi)(\eta) > 0\}$ and choose a finite cover $W_{\xi_1}, ..., W_{\xi_\ell}$ of $\pi^{-1}(x)$. Then $f \equiv f_{\xi_1} + \cdots + f_{\xi_\ell}$ is strictly ϕ-plurisubharmonic at x, and so $x \notin \mathrm{Core}(X)$. ∎

PROPOSITION C.2. *If $\xi \in \mathcal{N}$, then for each vector $v \in \operatorname{span} \xi$,*

$$df(v) = 0 \qquad \textit{for all} \quad f \in \mathrm{PSH}(X, \phi) \tag{C.1}$$

Proof. Suppose $f \in \mathrm{PSH}(X, \phi)$ and set $F = e^f$. Then $F \in \mathrm{PSH}(X, \phi)$, and by equation (B.1)' and Corollary 2.7 we see that $0 = (\mathcal{H}^\phi F)(\xi) = e^f \{df \wedge d^\phi f + \mathcal{H}^\phi f\}(\xi) = e^f \{df \wedge d^\phi f\}(\xi) = e^f |\nabla f \lrcorner \xi|^2$. ∎

DEFINITION C.3. The **tangential core** of X is the set

$$T\mathrm{Core}(X) \equiv \{v \in TX : v \neq 0 \text{ and satisfies condition (C.1)}\}.$$

Thus $T\mathrm{Core}(X) \subset TX$ is a subset defined by the vanishing of the family of smooth functions $df : TX \to \mathbf{R}$ for $f \in \mathrm{PSH}(X, \phi)$. Propositions C.1 and C.2 show that the restriction of the bundle map $p : TX \to X$ gives a surjective mapping

$$p : T\mathrm{Core}(X) \to \mathrm{Core}(X)$$

and for each $x \in X$, the vector space $T_x\mathrm{Core}(C) \equiv p^{-1}(x)$ contains the non-empty space generated by all $v \in \operatorname{span} \xi$ for $\xi \in \mathcal{N}_x$.

Consider a point $v \in T\mathrm{Core}(X)$ and suppose we have functions $f_1, ..., f_\ell \in \mathrm{PSH}(X, \phi)$ such that $\nabla df_1, ..., \nabla df_\ell$ are linearly independent at v. Then $T\mathrm{Core}(C)$ is locally contained in the codimension-ℓ submanifold $\{df_1 = \cdots = df_\ell = 0\}$.

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