ANALYTIC DISKS AND THE PROJECTIVE HULL

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ABSTRACT

Let X be a complex manifold and γ a simple closed curve in X. We address the question: What conditions on γ insure the existence of a 1-dimension complex variety Σ with boundary γ in X. When $X = \mathbf{C}^n$, an answer to this question involves the polynomial hull of gamma. When $X = \mathbf{P}^n$, complex projective space, the projective hull $\widehat{\gamma}$ of γ, a generalization of polynomial hull, comes into play. One always has $\Sigma \subset \widehat{\gamma}$, and for analytic γ they conjecturally coincide.

In this paper we establish an approximate analogue of this idea which holds without analyticity. We characterize points in $\widehat{\gamma}$ as those which lie on a sequence of analytic disks whose boundaries converge down to γ. This is in the spirit of work of Poletsky and of Larusson-Sigurdsson, whose results are essential here.

The results are applied to construct a remarkable example of a closed curve $\gamma \subset \mathbf{P}^2$, which is real analytic at all but one point, and for which the closure of $\widehat{\gamma}$ is $W \cup L$ where L is a projective line and W is an analytic (non-algebraic) subvariety of $\mathbf{P}^2 - L$. Furthermore, $\widehat{\gamma}$ itself is the union of W with two points on L.

Introduction.

Let γ be a simple closed real curve in a complex manifold X. Consider the problem of finding conditions which guarantee that γ forms the boundary of a complex analytic subvariety in X. When X is $\mathbf{C}^n$ (or, more generally, Stein), there is a solution [W] which involves the polynomial hull of γ. When X is $\mathbf{P}^n$ (or, more generally, projective), there is a notion of the projective hull of γ, denoted $\widehat{\gamma}$, which is related to the polynomial hull and has the following property. If

$$f : \Sigma \ \rightarrow \ X$$

is a map of a compact Riemann surface with boundary, which is holomorphic on IntΣ and continuous up to the boundary with

$$f(\partial\Sigma) = \gamma,$$

then

$$f(\Sigma) \ \subseteq \ \widehat{\gamma}.$$

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(See [HL$_2$].) It is conjectured that if $X = \mathbf{P}^n$ and γ is real analytic, then either $\widehat{\gamma} = \gamma$ or $\widehat{\gamma}$ is a 1-dimensional analytic subvariety with boundary γ or $\widehat{\gamma}$ is an algebraic curve containing γ. Real analyticity is fundamental for this conjecture. It fails for any C^∞-curve γ which is not pluripolar [HL$_2$, Cor. 4.4]. However, the conjecture has been established for real analytic curves which are "stable" by using arguments of E. Bishop (see [HLW]).

In this paper we show that for any closed curve $\gamma \subset \mathbf{P}^n$, all points z_0 in the projective hull $\widehat{\gamma}$ are essentially characterized by the fundamental property discussed above. We show that if $z_0 \in \gamma$, then for any sequence $\epsilon_j \searrow 0$, there exists a sequence of holomorphic maps of the unit disk

$$f_j : \Delta \ \rightarrow \ \mathbf{P}^n$$

with

$$f_j(0) \ = \ z_0$$

and

$$\mathrm{dist}\,(f_j(\partial\Delta), \gamma) \ \leq \ \epsilon_j.$$

In fact, from any such sequence we extract limiting holomorphic maps $f : \Delta \rightarrow \mathbf{P}^n$, possibly constant $\equiv z_0$, and show in Theorem 2 that

$$f(\Delta) \subset \widehat{\gamma}.$$

If we choose an affine coordinate chart $\mathbf{C}^n \subset \mathbf{P}^n$ containing γ, then the sequence of maps f_j can be chosen so that the poles $\zeta_1^j, ..., \zeta_{N_j}^j$ of f_j, viewed as a $\mathbf{C}^n$-valued function, are simple and satisfy $\sum_k \log|\zeta_k^j| \geq -M$ for some constant M independent of j. In this form **the existence of the sequence is necessary and sufficient for z_0 to lie in $\widehat{\gamma}$.** This is Theorem 1 below.

Theorem 2 is applied to produce interesting examples related to the conjecture above. We construct a simple closed curve $\gamma \subset \mathbf{P}^2$, which is real analytic at all but one point p, and has the following properties. There exists a projective line L through p, a point $q \in L$, and a proper complex analytic subvariety $W \subset \mathbf{P}^2 - L$ such that

$$\widehat{\gamma} \ = \ W \cup \{p, q\} \qquad \text{and} \qquad \overline{\{\widehat{\gamma}\}} \ = \ W \cup L.$$

Thus, if real analyticity breaks down at a single point, the conjecture fails. Moreover, if this happens, then $\widehat{\gamma}$ may not be closed.

Related examples with other interesting properties are also given

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The Main Theorems.

Consider a connected closed curve γ lying in complex projective n-space $\mathbf{P}^n$. Write $\mathbf{P}^n = \mathbf{C}^n \cup H$, where H denotes the hyperplane at infinity, and assume that $\gamma \subset \mathbf{C}^n$. We wish to characterize those points $z_0 \in \mathbf{C}^n$ which lie in the projective hull $\widehat{\gamma}$ of γ in terms of analytic disks. We shall make use of the work of Lárusson and Sigurdsson [LS$_1$], who give a formula for the Siciak-Zahariuta extremal function V_X of a connected open subset X of $\mathbf{C}^n$.

For each $r > 0$ let K_r denote the open tube of radius r around γ, i.e.,

$$K_r \;=\; \{z \in \mathbf{C}^n : \mathrm{dist}(z, \gamma) < r\}$$

We fix a point $z_0 \in \mathbf{C}^n - \gamma$. Thus $z_0 \notin K_r$ for r sufficiently small.

Let $\{f_r\}_r$ be a family of analytic maps of the unit disk Δ into $\mathbf{P}^n$, indexed by numbers $r > 0$ converging to zero. We consider the following four conditions on the family $\{f_r\}_r$:

For all r

(i) $f_r(\partial\Delta) \;\subset\; K_r$,

(ii) $f_r(0) \;=\; z_0$,

(iii) There exists a number $M > 0$ such that if $\zeta_1^{(r)}, ..., \zeta_{N_r}^{(r)}$ are the poles of f_r in Δ (i.e., $f_r(\zeta_j^{(r)}) \in H$ for $j = 1, ..., N_r$), then we have

$$\sum_{j=1}^{N_r} \log|\zeta_j^{(r)}| \;\geq\; -M.$$

(iv) The poles are simple, that is, the $\mathbf{C}^n$-valued function $(\zeta - \zeta_j^{(r)}) f_r(\zeta)$ is holomorphic in a neighborhood of $\zeta_j^{(r)}$ for each j.