THE PROJECTIVE HULL OF CERTAIN CURVES IN C^2.

Reese Harvey, Blaine Lawson* and John Wermer

Abstract

The projective hull $\widehat{X}$ of a compact set $X \subset \mathbf{P}^n$ is an analogue of the classical polynomial hull of a set in $\mathbf{C}^n$. In the special case that $X \subset \mathbf{C}^n \subset \mathbf{P}^n$, the affine part $\widehat{X} \cap \mathbf{C}^n$ can be defined as the set of points $x \in \mathbf{C}^n$ for which there exists a constant M_x so that

$$|p(x)| \ \leq \ M_x^d \sup_X |p|$$

for all polynomials p of degree $\leq d$, and any $d \geq 1$. Let $\widehat{X}(M)$ be the set of points x where M_x can be chosen $\leq M$. Using an argument of E. Bishop, we show that if $\gamma \subset \mathbf{C}^2$ is a compact real analytic curve (not necessarily connected), then for any linear projection $\pi : \mathbf{C}^2 \to \mathbf{C}^1$, the set $\widehat{\gamma}(M) \cap \pi^{-1}(z)$ *is finite for almost all* $z \in \mathbf{C}$. It is then shown that for any compact stable real-analytic curve $\gamma \subset \mathbf{P}^n$, the set $\widehat{\gamma} - \gamma$ *is a 1-dimensional complex analytic subvariety of* $\mathbf{P}^n - \gamma$.

*Partially supported by the N.S.F.

§**1. Introduction.** The classical **polynomial hull** of a compact subset $X \subset \mathbf{C}^n$ is the set of points $x \in \mathbf{C}^n$ such that

$$|p(x)| \ \leq \ \sup_X |p| \qquad \text{for all polynomials} \ \ p. \tag{1.1}$$

In [4] the first two authors introduced an analogue for compact subsets of projective space. Given $X \subset \mathbf{P}^n$, the **projective hull of** X is the set $\widehat{X}$ of points $x \in \mathbf{P}^n$ for which there exists a constant $C = C_x$ such that

$$\|P(x)\| \ \leq \ C_x^d \sup_X \|P\| \qquad \text{for all sections} \ \ P \in H^0(\mathbf{P}^n, \mathcal{O}(d)) \tag{1.2}$$

and all $d \geq 1$. Here $\mathcal{O}(d)$ is the dth power of the hyperplane bundle with its standard metric. Recall that $H^0(\mathbf{P}^n, \mathcal{O}(d))$ is given naturally as the set of homogeneous polynomials of degree d in homogeneous coordinates. If X is contained in an affine chart $X \subset \mathbf{C}^n \subset \mathbf{P}^n$ and $x \in \mathbf{C}^n$, then condition (1.2) is equivalent to

$$|p(x)| \ \leq \ M_x^d \sup_X |p| \qquad \text{for all polynomials} \ \ p \ \text{of degree} \ d \tag{1.3}$$

and all $d \geq 1$ where $M_x = \rho\sqrt{1 + \|x\|^2}C_x$ and ρ depends only on X. Therefore the set $\widehat{X} \cap \mathbf{C}^n$ consists exactly of those points $x \in \mathbf{C}^n$ for which there exists an M_x satisfying condition (1.3).

This paper is concerned with the case where $X = \gamma$ is a real analytic curve. In [4] evidence was given for the following conjecture.