SINGULARITIES AND CHERN-WEIL THEORY, II

Geometric Atomicity

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Abstract.

This paper introduces a general method for relating characteristic classes to singularities of a bundle map. The method is based on the notion of *geometric atomicity*. This is a property of bundle maps $\alpha : E \to F$ which universally guarantees the existence of certain limits arising in the theory of singular connections. Under this hypothesis each characteristic form Φ of E or F satisfies an equation of the form

$$\Phi \;=\; L + dT$$

where L is an explicit localization of Φ along the singularities of α and T is a canonical form with locally integrable coefficients. The method is constructive and leads to explicit calculations. For normal maps (those transversal to the universal singularity sets) it retrieves classical formulas of MacPherson at the level of forms and currents (cf. Part I). It also produces such formulas for direct sum and tensor product mappings. These are new even at the topological level. The condition of geometric atomicity is quite broad and holds in essentially every case of interest, including all real analytic bundle maps. An important aspect of the theory is that it applies even in cases of "excess dimension", i.e., where the the singularity sets of α have dimensions greater than those of the generic map. The method yields explicit calculations in this general context. A number of examples are worked out in detail.

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§0. Introduction. Among the most useful and interesting results in geometry are those which relate the singularities of bundle maps to topological invariants. Such theorems cover a broad range of topics including:

1. The relationship between the Euler class of a bundle and the zeros of a cross-section (e.g., Hopf's Theorem on vector fields) or more generally,
2. The relationship between the Chern or Pontrjagin classes of a bundle and the linear dependency locus of a family of cross-sections.
3. Formulas relating the characteristic classes of a manifold X to the higher complex tangencies to a smooth immersion $X \to \mathbb{C}^n$.
4. Formulas relating topological invariants to the high-order tangency sets of a pair of foliations.
5. Thom-Porteous invariants associated to the singularities of a smooth mapping between manifolds.
6. The differentiable Riemann-Roch-Grothendieck Theorem for embeddings.

We present here a method for deriving such results in quite general circumstances. It retrieves all known formulas of the type above (as shown in [$\mathrm{HL}_{2,3}$]) and generates many new ones, several of which are derived in this paper. More importantly the method applies in highly non-generic cases where, say, the singularity sets are of greater than "expected" dimension. In particular it applies to *any real analytic bundle map* α, and even when the singularity sets of α have excess dimension, it yields straightforward calculations and explicit formulas.

The method is also interesting in that it delivers formulas which are *local* and *canonical* on the underlying manifold. Given a smooth map $\alpha : E \to F$ between bundles with connection over a manifold X and given a Chern-Weil characteristic form $\Phi(\Omega)$ in the curvature for either E or F, we obtain a formula of the type

$$\Phi(\Omega) \;=\; \sum_k \mathrm{Res}_k\, S_k(\alpha) + dT_\Phi \tag{0.1}$$

where $S_k(\alpha)$ is a current associated to the locus where α drops rank by k, Res_k is an explicit residue form defined along $S_k(\alpha)$, and T_Φ is an L^1_{loc}-form on X. In most cases of interest one has that

$$dS_k(\alpha) \;=\; 0 \qquad \text{and} \qquad d(\mathrm{Res}_k\, S_k(\alpha)) \;=\; 0$$

for all k. When the residues are constants, the form T_Φ represents a Cheeger-Simons differential character associated to α and the connections on E and F (See [CS], [HLZ]).

The key to our method is the concept of *geometric atomicity*, a property of bundle maps which holds in surprizing generality and is often easily verified. The main fact is that whenever $\alpha : E \to F$ has this property, canonical formulas of type (0.1) exist for every Φ.

Actually more is true. To study bundle maps α, the authors have developed a theory of singular connections and characteristic currents [HL$_2$]. For given α we introduced a smooth family of connections D_t, $0 < t \le \infty$, on E (or F), which starts with the given connection at $t = \infty$ and tends to a singular "pull-back" (or "push-forward") connection as $t \to 0$. For each Ad-invariant polynomial Φ, classical Chern-Weil theory gives formulas

$$\Phi(\Omega) \;=\; \Phi(\Omega_t) + dT_t \tag{0.2}$$

where T_t is a canonical, smooth transgression form on X. The natural questions, posed in [HL$_2$] and answered in special cases, are the following.

Question 0.1. When does $\lim_{t\to 0} T_t$ exist as a current on X?

If the **transgression current** $\lim_{t\to 0} T_t$ exists, so does the **characteristic current** $\lim_{t\to 0} \Phi(\Omega_t)$.

Question 0.2. If $\lim_{t\to 0} T_t$ exists, how does one compute $\lim_{t\to 0} \Phi(\Omega_t)$?

This paper provides a very general answer.

Theorem. *Suppose $\alpha : E \to F$ is geometrically atomic. Then for every Φ, the limit $\lim_{t\to 0} T_t$ exists. Furthermore, there is a canonical method for computing $\lim_{t\to 0} \Phi(\Omega_t)$.*

The limit as $t \to 0$ of formula (0.2) is our formula (0.1).

The concept of geometric atomicity is elementary. Think of α as a cross-section of the bundle $H \equiv \mathrm{Hom}(E, F)$, and define its **radial span** to be the subset

$$T_\alpha \;\equiv\; \{\tfrac{1}{t}\alpha_x \in H : 0 < t < \infty \;\text{ and }\; x \in X\} \;\subset\; H.$$

Consider the compactification $H \subset G \equiv G_m(E \oplus F)$ by the Grassmann bundle of m-planes in $E \oplus F$, where $m = \mathrm{rank}(E)$. Note that $T^0_\alpha \equiv T_\alpha - \mathrm{Zero}(\alpha)$ is a submanifold of dimension $= \dim(X) + 1$. Then α is defined to be **geometrically atomic (GA)** if T^0_α has locally finite volume in G.

This property of geometric atomicity holds in nearly every situation of interest.

To begin, any α which is **normal**, i.e., transversal to the submanifolds $\Sigma_k \subset H$ where the rank drops by k, is geometrically atomic. The resulting formulas consititute a local version of the classical MacPherson Formulas [Mac_*]. This was established in Part I [HL_5] and is reviewed here for the sake of completeness in §3.

A fundamental and surprizing fact is that **any real analytic bundle map α is GA**. Thus, no matter how badly the singularities of α behave – no matter how bizarre the degeneracy sets of α are – the limiting currents exist for all characteristic forms. Furthermore, these limits are very often *explicitly computable*. For a simple illustration consider a real analytic section α of a rank-n complex bundle whose zero set $Z(\alpha)$ is a submanifold of codimension $2k$, and take $\Phi = c_n$ the n^{th} Chern form. When $k = n$ and α vanishes to first order, formula (0.1) has the form $c_n(\Omega) = [Z(\alpha)] + dT$. This is the normal case. However, when $k < n$ one easily computes the limit and generically finds $c_n(\Omega) = c_{n-k}(\Omega)[Z(\alpha)] + dT$. (See Example 9.1 below.)

Interestingly *any* real-valued function $f : X \to \mathbb{R}$, considered as an endomorphism of any vector bundle E is GA.

An important area to which this theory applies is the one where bundle maps are constrained in a specific manner, such as requiring that α be a section of a given subbundle S of $\text{Hom}(E, F)$. This precludes α being normal, but the GA condition is still generic within the constrained class. Perhaps the simplest example is the global direct sum of bundle maps. These are essentially never normal; their singularity sets have the wrong dimension. However, generic direct sums (and tensor products) **are GA**. Also, Clifford multiplication on spinor bundles by a GA-section of a vector bundle is again GA.

As just noted, generic direct sums are GA. In fact we show that for a generic pair of bundle maps $\alpha : E \to F$ and $\alpha' : E' \to F'$ both α and α' are individually normal and their singularity sets are mutually transversal. In sections 4 and 5 we apply our theory to derive detailed formulas of type (0.1) for both direct sums and tensor products

$$\alpha \oplus \alpha' : E \oplus E' \longrightarrow F \oplus F' \qquad \text{and} \qquad \alpha \otimes \alpha' : E \otimes E' \longrightarrow F \otimes F'$$

where in each case the right hand side is a sum of explicitly calculated residues times the currents $[\Sigma_k(\alpha) \cap \Sigma_{k'}(\alpha')]$ where $\Sigma_k(\alpha) = \{x \in X : \text{rank } \alpha_x = \min(\dim(E), \dim(F)) - k\}$. These formulas are new even at the topological level.

In fact the direct sum and tensor product formulas are new and interesting even for the simplest case of sections of vector bundles. These cases are examined in §§6 and 7 where detailed calculations are made for Chern classes, the Chern character and for general multiplicative sequences.

We point out that there is a certain geometric simplicity in our approach which can mislead the reader into thinking that some of these results are easy. Establishing general formulas even in the "universal" and "normal" cases is a formidible task by more conventional means. C^∞-stratified sets are complicated objects. Just to produce the topological formulas as in [$\text{Mac}_{1,2}$] requires a careful resolution of the singularities of the stratification and computation of the residue classes. With our method a harder problem is solved. We show in all basic cases that each characteristic form is explicitly cohomologous to a finite sum of locally closed submanifolds (defined by the singularities of the bundles mappings) multiplied by intrinsically defined C^∞-forms. We then prove that each of these summands has an extension as a current to the manifold X. In fact we show that each summand extends as a current of *finite mass* and furthermore is d-*closed* on X. These facts are difficult to establish by conventional analytic means.

An important consequence of our approach is that it leads to a **general ansatz** which applies directly and enables explicit computations in very general circumstances – where bundle maps can have badly behaved singularities. The principles of this general method are presented in §9. It essentially boils down to computing the boundary of the current on G given by integration over the submanifold T^0_α. This is often a straightforward matter. Once $\partial[T^0_\alpha]$ is computed, formulas are derived by the operator calculus introduced in §2. Many examples of this method are worked out here in detail.

Formally this general method is like the algebro-geometric procedure of "pulling to the normal cone", which gives an intrinsic computation of intersection classes even when intersections are not of proper dimension (cf. [Fu]).

It is tempting to think that the currents T_Φ from formula (0.1) fit into a theory of secondary characteristic classes. If all the residues Res_k are integer constants, this is indeed the case. Then for such pairs (Φ, α), the current T_Φ represents a canonical differential character via the deRham-Federer Theory presented in [HLZ]. To date there is no analogous theory which takes into account transgressions between smooth forms and "partially recitifiable" currents which appear in (0.1). However, if one restricts attention to forms with constant integer residues, the situation is already quite rich. For example one can give an new proof of the Chern product formula for differential characters [CS].

Standing Assumptions. Throughout the paper $E \to X$ and $F \to X$ will denote smooth bundles which are either both real or both complex. It will be assumed that each bundle is furnished with a metric and a connection. However, in general the connection need not respect the metric. The manifold X is not assumed to be compact.

For simplicity X is assumed to be oriented. This hypothesis is generally unnecessary. Any submanifold with oriented normal bundle defines a current (cf. [deR]). So if the bundle $\text{Hom}(E, F)$ is oriented (which is automatic when E and F are oriented or of even rank), the radial span of any section will have oriented normal bundle, and the operator calculus of §2 can be carried through.

A Remark on Terminology. On the total space of the bundle $\pi : \text{Hom}(E, F) \to X$ there is a tautologically defined bundle map $\boldsymbol{\alpha} : \pi^*E \to \pi^*F$ given by $\boldsymbol{\alpha}(a) = a$. Under local trivializations of E and F, $\boldsymbol{\alpha}$ reduces to the tautological map over $\text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$, i.e., it is independent of base variables. Furthermore, every bundle mapping $\alpha : E \to F$ over X is the pull-back $\alpha = \alpha^*(\boldsymbol{\alpha})$ of $\boldsymbol{\alpha}$. For these reasons $\boldsymbol{\alpha}$ is called the *universal* mapping. There are analogous universal mappings of direct sum and tensor product type.

Historical Comment: Notions of Atomicity. The first notion of atomicity, introduced in [HS], applied only to sections of a vector bundle. It consisted of analytic conditions on the section under which its vanishing determines a well-defined, integrally flat current dual to the Euler class. In [HL$_2$], the notion of k-atomicity was introduced for bundle maps $\alpha : E \to F$ (for $0 \leq k \leq \text{rank}\, \alpha$). The condition of k-atomicity guaranteed the existence of a k^{th} degeneracy current for α which was integrally flat and of the expected dimension.

These currents appear in a variety of important geometric situations and play a role in residue theorems relating singularities to characteristic forms (cf. [HL$_3$]).

The notion of geometric atomicity for a bundle map α is substantially more general than these. It is a minimal condition under which the limits discussed in the introduction exist and lead to formulas of type (0.1). Geometrically atomic bundle maps need not be k-atomic for any k.

§1. **Geometric Atomicity.** In this section we re-examine the concept of geometric atomicity introduced in [HL$_5$]. The definition given here is different from but equivalent to the original one. In practice the new definition is easier to apply

Let $E \to X$ and $F \to X$ be smooth vector bundles (both real or both complex) of rank $m \leq n$ respectively over a manifold X of dimension ν, and let

$$\alpha : E \longrightarrow F$$

be a smooth bundle mapping. This mapping can be considered as a section of the bundle

$$H \equiv \mathrm{Hom}(E, F)$$

for which we have the **Grassmann compactification**

$$G \equiv G_m(E \oplus F)$$

by the bundle of m-dimensional linear subspaces of $E \oplus F$. The embedding $H \subset G$ is defined by taking the graph.

We now consider the **radial span** of α in H defined by

$$T_\alpha \equiv \left\{ \tfrac{1}{t}\alpha(x) : 0 < t < \infty \text{ and } x \in X \right\}.$$

Note that $T_\alpha^0 \equiv T_\alpha - \mathrm{Zero}(\alpha)$ is an oriented submanifold of dimension $\nu + 1$ in H. Of course $\mathrm{Zero}(\alpha)$ is contained in the 0-section of H, which is a submanifold of dimension ν.

Definition 1.1. The bundle mapping α is called **geometrically atomic** if T_α has locally-finite $(\nu + 1)$-measure in G, or equivalently, if T_α^0 is a submanifold of locally finite volume in G. When this holds, integration over T_α^0 defines a current of dimension $\nu + 1$ on G which will also be denoted by T_α.

Example 1.2. *When $\dim_{\mathbb{R}}(E) = \dim_{\mathbb{R}}(F) = 1$, every bundle mapping $a : E \to F$ is geometrically atomic.* The proof is obvious since T_α^0 is an open subset of the S^1-bundle $G = \mathbb{P}_{\mathbb{R}}(\mathbb{R} \oplus \mathrm{Hom}(E, F))$. Note in particular that every real-valued continuous function is geometrically atomic, when considered an endomorphism of the trivial line bundle – or any other bundle for that matter.

Example 1.3. *Any real analytic bundle map $\alpha : E \to F$ is geometrically atomic.*

This is proved below.

Note that sections of a vector bundle F are in natural one-to-one correspondence with bundle mappings $\alpha : \underline{K} \to F$ where $\underline{K}$ is the trivial (real or complex) line bundle.

Example 1.4. *Any section of a vector bundle which is atomic in the sense of [HS], is geometrically atomic.* See [HL_6].

Example 1.5. *Any normal bundle map is geometrically atomic. Such maps are open and dense in the C^1-topology. (See §3 for details.)*

Example 1.6. *Suppose that $\alpha : E \to F$ is a geometrically atomic bundle map, and suppose there exists a map*

$$\rho : \text{Hom}(E, F) \longrightarrow \text{Hom}(E', F')$$

with the properties that for each $x \in X$, $\rho_x : \text{Hom}(E_x, F_x) \longrightarrow \text{Hom}(E'_x, F'_x)$ is a homogeneous polynomial map, and ρ extends smoothly to the Grassmann compactifications

$$\rho : G_m(E \oplus F) \longrightarrow G_m(E' \oplus F')$$

Then $\rho \circ \alpha$ is geometrically atomic.

Proof. If T_α has locally-finite volume, then so does $\rho(T_\alpha) = T_{\rho\circ T}$. ▮

Example 1.7. *Let $\sigma \in \Gamma(V)$ be a geometrically atomic section of a vector bundle V which acts by Clifford multiplication on associated spinor bundles $S^\pm$. Then the bundle mapping $\sigma\bullet : S^+ \to S^-$ given by Clifford multiplication is also geometrically atomic.*

Proof. Take the embedding $V \subset \text{Cliff}(V) \subset \text{Hom}(S^+, S^-)$ and apply 1.6. ▮

Example 1.8. *Direct sums and tensor products of bundle mappings are generically geometrically atomic.* See Appendix A.

The following sufficient criterion is often easy to verify.

Lemma 1.9. *Given a bundle map $\alpha : E \to F$ as above, consider the submanifold*

$$\mathcal{T} \overset{\text{def}}{=} \{(t, \tfrac{1}{t}\alpha(x)) : 0 < t < \infty \text{ and } x \in X\} \subset \mathbb{P}^1_{\mathbb{R}} \times G.$$

If $\mathcal{T}$ has locally finite volume in $\mathbb{P}^1_{\mathbb{R}} \times G$, then α is geometrically atomic.

Proof. Note that $T = p_*\mathcal{T}$ where $p : \mathbb{P}^1_{\mathbb{R}} \times G \to G$ is projection, and recall that the property of having locally finite mass is preserved under smooth proper maps. ▮

Proof of 1.3. Choose local C^ω-trivializations of E and F above a local C^ω-coordinate system on $U \subset X$. Consider the closure

$$\overline{T} \equiv \text{Cl}\{(x, \tfrac{1}{t}\alpha(x)) : x \in U, \alpha(x) \neq 0 \text{ and } 0 < t < \infty\} \subset U \times \mathbb{P}_{\mathbb{R}}(\mathbb{R} \oplus H_0)$$

where H_0 is the space of $m \times n$-matrices. Let $(y_0, y_1, ..., y_{mn})$ be the homogeneous coordinates for $\mathbb{P}_{\mathbb{R}}(\mathbb{R} \oplus H_0)$ where $(y_1, ..., y_{mn})$ are the standard linear coordinates on H_0. Our claim is that $\overline{T}$ is an irreducible semi-analytic subset of dimension $\nu + 1$ and therefore has locally finite $(\nu+1)$-measure in $U \times \mathbb{P}_{\mathbb{R}}(\mathbb{R} \oplus H_0)$. It then follows that the image of $\overline{T}$ under the map $U \times \mathbb{P}_{\mathbb{R}}(\mathbb{R} \oplus H_0) \to U \times G_m(H_0)$ has locally finite $(\nu+1)$-measure as desired.

To prove the claim note that in the affine coordinate system where $y_0 = 1$, T^0_α is defined by the conditions: $y_i\alpha_j(x) = y_j\alpha_i(x)$, and $y_i\alpha_i(x) \geq 0$ for all i, j whenever $\alpha(x) \neq 0$. Thus, after excluding the trivial case where $\alpha \equiv 0$, we see that $\overline{T}$ is a real semi-analytic subset of this chart as claimed. Now consider the affine coordinate system $\bar{y}$ where $y_1 = 1$. Here T^0_α is defined by the equations: $\bar{y}_i\alpha_j(x) = \bar{y}_j\alpha_i(x)$, and $\bar{\alpha}_j(x) = \bar{y}_j\alpha_1(x)$ for $i, j \geq 2$ and $y_i\alpha_i(x) \geq 0$ for all i, whenever $\alpha(x) \neq 0$. Thus, $\overline{T}$ is semi-analytic in all coordinate charts as claimed. ▮

Remark 1.10. In [HL$_5$] a section was defined to be geometrically atomic if the set $T'_\alpha = \{(\frac{1}{t}\alpha(x), \alpha(x)) : 0 < t < \infty \text{ and } x \in X\}$ has locally finite $(\nu+1)$-measure in the fibre product $G \oplus G$. The two definitions are equivalent. To see this simply apply the diffeomorphism of $F : G \oplus H \to G \oplus H$ given by $F(P, a) = (P, a - \alpha(x))$.

Remark 1.11 Geometric atomicity is the key hypothesis of the theorems of this paper. There are analytic conditions which are equivalent to geometric atomicity. This equivalence will be presented in [HL$_6$]. However it might be useful to mention at least one special case in this paper. Suppose α is a smooth complex valued function (i.e. E and F are trivial **C**-line bundles). Then α is geometrically atomic if and only if $\alpha^*(\Theta)$ has locally Lebesgue integrable coefficients, where $\Theta = (-vdu + udv)/(u^2 + v^2) = d\theta$ is the standard angle form on $\mathbf{R}^2 \cong \mathbf{C}$.

Example 1 .12. (A map which is not GA). We noted in 1.2 above that any continuous real-valued function, considered as a bundle endomorphism (of any real bundle) is geometrically atomic. The corresponding statement for complex-valued functions is false even in the C^∞ case. This leads to the simplest maps which are not GA. For a specific example consider the smooth map $f : \mathbb{R}^n \to \mathbb{C}$, $n \geq 2$, defined by $f(x) = \exp(-\frac{1}{\|x\|^2} + i\frac{1}{\|x\|^{n-1}})$. Since $f^*\Theta = d(\frac{1}{\|x\|^{n-1}})$ is not locally Lebesgue integrable, f is not GA by Remark 1.11.

§2. The Operator Calculus.

The principal interest in the radial span of a flow stems from the fact that any current on the Grassmann compactification G determines a continuous operator from forms on G to generalized forms on X. The underlying kernel calculus is a simple adaptation of that given in [HP]. This calculus is particularly important for generalized Chern-Weil theory.

Let $\pi : G \to X$ be the Grassmann compactification of $H = \text{Hom}(E, F)$ as in §1. Following [deR] we denote by $\mathcal{E}^k(Y)$ the space of smooth differential k-forms on a manifold Y, and by $\mathcal{D}'^k(Y) \supset \mathcal{E}^k(Y)$ the generalized k-forms (or currents of degree k) on Y.

Definition 2.1. To each current T of dimension $\nu + r$ (where $\nu = \dim(X)$) we associate a continuous linear operator

$$\mathbb{T} : \mathcal{E}^*(G) \longrightarrow \mathcal{D}'^{*-r}(X)$$

by setting

$$\mathbb{T}(\omega) \equiv \pi_*\{\omega \wedge T\}.$$

Lemma 2.2. *Let T be as above. Then the operator corresponding to its boundary ∂T under the correspondence in Definition 2.1 is*

$$d \circ \mathbb{I} + (-1)^{r-1} \mathbb{I} \circ d$$

Proof. This is a straightforward adaptation of [HP]. See [HL$_4$, App. A] for example. ∎

Suppose now that T_α is the current on G associated to the radial span of a geometrically atomic bundle map $\alpha : E \to F$. (See Def. 1.1.) Then

$$\partial T_\alpha \ = \ [0] - P \tag{2.1}$$

where $[0]$ denotes the 0-section of $H \subset G$ and P is characterized in the next result.

Proposition 2.3. *Let $\alpha : E \to F$ be a geometrically atomic bundle map. Then for every smooth differential form ω on G the limit*

$$\mathbb{P}(\omega) \ = \ \lim_{t \to 0} (\tfrac{1}{t}\alpha)^*(\omega) \tag{2.2}$$

exists on X. Furthermore, the operator $\mathbb{I}_\alpha$ determined by the current T_α gives a chain homotopy

$$j^*(\omega) - \mathbb{P}(\omega) \ = \ d\mathbb{I}_\alpha(\omega) + \mathbb{I}_\alpha(d\omega) \qquad \text{for all } \ \omega \in \mathcal{E}^*(G) \tag{2.3}$$

where $j : X \hookrightarrow G$ is the inclusion as the zero section. The forms $\mathbb{I}_\alpha(\omega)$ (and $\mathbb{I}_\alpha(d\omega)$) are always L^1_{loc}.

Note that (2.3) is just the operator equation corresponding to the current equation (2.1) above.

Proof. To see more clearly what is going on, it is useful to "desingularize" T_α. Consider the currents on $\mathbb{R} \times G$ corresponding to the submanifolds-with-boundary

$$T_{\alpha,s,s'} \ \equiv \ \{(t, \tfrac{1}{t}\alpha(x)) : s \le t \le s' \text{ and } x \in X\}$$

oriented so that

$$\partial T_{\alpha,s,s'} \ = \ \{s'\} \times \Gamma(\tfrac{1}{s'}\alpha) - \{s\} \times \Gamma(\tfrac{1}{s}\alpha)$$

where $\Gamma(\beta)$ denotes the graph of a section β of $H \subset G$. Let $p : \mathbb{R} \times G \to G$ be projection and set $\pi' = \pi \circ p : \mathbb{R} \times G \to X$. These maps are proper on $T_{\alpha,s,s'}$ and one sees that

$$\pi'_* \left\{ p^*\omega \wedge \partial T_{\alpha,s,s'} \right\} \ = \ (\tfrac{1}{s'}\alpha)^*(\omega) - (\tfrac{1}{s}\alpha)^*(\omega). \tag{2.4}$$

However, since $\pi' = \pi \circ p$ we have

$$\pi'_* \left\{ p^*\omega \wedge \partial T_{\alpha,s,s'} \right\} \ = \ (\pi \circ p)_* \left\{ p^*\omega \wedge \partial T_{\alpha,s,s'} \right\} \ = \ \pi_* \left\{ \omega \wedge \partial (p_* T_{\alpha,s,s'}) \right\}. \tag{2.5}$$

Note that, since T_α is of finite volume, one has

$$\lim_{s'\to\infty}\lim_{s\to 0} p_* T_{\alpha,s,s'} \;=\; T_\alpha .$$

We conclude from (2.3) and (2.4) that

$$\lim_{s'\to\infty}(\tfrac{1}{s'}\alpha)^*(\omega) - \lim_{s\to 0}(\tfrac{1}{s}\alpha)^*(\omega) \;=\; \pi_*\left\{\omega\wedge\partial T_\alpha\right\} \;=\; d\pi_*\left\{\omega\wedge T_\alpha\right\} \;+\pi_*\left\{d\omega\wedge T_\alpha\right\}$$

which proves (2.2) and (2.3).

For the last assertion note that $\phi\equiv(\mathrm{Id}\times\pi)_*(p^*\omega\wedge T_{\alpha,s,s'})$ is a smooth form on $[s,s']\times X$. In fact it is exactly the pull-back of ω by the section $\frac{1}{t}\alpha$. Now $\pi'=\pi\circ p=\mathrm{pr}\circ(\mathrm{Id}\times\pi)$ where $\mathrm{pr}:[s,s']\times X\to X$ is the projection. Hence, $\pi'_*\left\{p^*\omega\wedge\partial T_{\alpha,s,s'}\right\}=\mathrm{pr}_*\phi$ is a smooth form on X since it is obtained by integration of a smooth form over the fibre of pr. Since T_α is a submanifold of finite volume, these smooth forms are converging locally in mass norm to the limit $\mathbb{T}_\alpha(\omega)=\pi_*\left\{\omega\wedge\partial T_\alpha\right\}$. However, on smooth forms the mass norm coincides with the L^1-norm. Hence the limit lies in L^1 locally. ∎

The importance of the above calculus comes from the following. Let $U\longrightarrow G$ denote the **tautological m-plane bundle** over the total space of the Grassmann bundle $\pi:G\to X$. Suppose that E and F are equipped with orthogonal connections (unitary connections in the complex case). Then from these connections and the splitting

$$\pi^*E\oplus\pi^*F \;=\; U\oplus U^\perp$$

we obtain connections on U and $U^\perp$. Suppose that Φ is an Ad-invariant polynomial on the Lie algebra of the structure group of E (either $O(m)$ or $U(m)$), and consider the characteristic form $\Phi(\Omega^U)$ on G. It is shown in [HL$_2$, I.8] that

$$(\tfrac{1}{t}\alpha)^*\Phi(\Omega^U) \;=\; \Phi(\overleftarrow{\Omega}_t) \tag{2.6}$$

where $\overleftarrow{\Omega}_t$ is the curvature of the *time-t pull-back connection on π^*E*. This is a family of connections constructed directly on E from the given data and a universal choice of smoothing function. In the limit as $t\to 0$, these connections tear and the curvature becomes concentrated along the singularities of α. Although the limiting connection is not everywhere defined, it is possible that the limit of the characteristic form $\Phi(\overleftarrow{\Omega}_t)$ exists as a current. When it does, it gives a localization of the Φ-characteristic class on the singularities of α. The analogous situation holds for invariant polynomials Ψ on the Lie algebra of the structure group of F.

Setting $\omega=\Phi(\Omega^U)$ or $\omega=\Psi(\Omega^{U^\perp})$ and applying Proposition 2.3 gives the following.

Theorem 2.4. *Suppose $\alpha:E\to F$ is geometrically atomic. Then for all Ad-invariant polynomial Φ and Ψ as above, the limits*

$$\mathbb{P}(\Phi)\;\equiv\;\lim_{t\to 0}\Phi(\overleftarrow{\Omega}_t) \qquad \textit{and} \qquad \mathbb{P}(\Psi)\;\equiv\;\lim_{t\to 0}\Psi(\overrightarrow{\Omega}_t)$$

exist in the space of currents on X. Furthermore, there are canonically defined forms T_Φ and T_Ψ with L^1_{loc}-coefficients on X such that

$$\Phi(\Omega^E) \;=\; I\!\!P(\Phi) + dT_\Phi \qquad \textit{and} \qquad \Psi(\Omega^F) \;=\; I\!\!P(\Psi) + dT_\Psi$$

The remainder of this paper is devoted to computing the operator $I\!\!P$ in various circumstances. This amounts to calculating the boundary of the current ∂T_α in G. It turns out that ∂T_α is closely tied to the primary singularities of the mapping α. We begin by recalling the calculation for "generic", or "normal", maps done in part I.

§**3. Normal Mappings.** In this section we review the notation, methods and results from [HL$_5$]. Let E and F be as above and recall that for each integer k, $0 \le k \le m = \text{rank}\, E$, we have the primary singularity set

$$\Sigma_k \;\equiv\; \{a \in \text{Hom}(E,F) : \dim\ker(a) = k\} \tag{3.1}$$

The closures of the Σ_k's give a filtration:

$$\{0\} = \overline{\Sigma}_m \subset \overline{\Sigma}_{m-1} \subset \overline{\Sigma}_{m-2} \subset \cdots \subset \overline{\Sigma}_0 = \text{Hom}(E,F)$$

These sets fit into a dynamical pattern on the Grassmann compactification $G = G_m(E \oplus F)$ as follows. Consider the multiplicative flow $\varphi_t : G \to G$ induced by $\widetilde{\varphi}_t : E \oplus F \to E \oplus F$ where $\widetilde{\varphi}_t(e,f) = (te, f)$. Note that the subset $H = \text{Hom}(E,F) \subset G$ is φ_t-invariant; in fact for $a \in H$

$$\varphi_t(a) \;=\; \tfrac{1}{t}a$$

The fixed-point set of this flow on G can be written as

$$F = \coprod_{k=0}^{m} F_k \tag{3.2}$$

where

$$F_k \;=\; G_k(E) \times G_{m-k}(F)$$

The stable and unstable manifolds of F_k in G are given respectively by

$$\Sigma_k \;\equiv\; \{P \in G : \dim(P \cap E) = k\} \qquad \text{and} \qquad \Upsilon_k \;\equiv\; \{P \in G : \dim(P \cap F) = m-k\}.$$

with projections

$$\Sigma_k \xrightarrow{p_1} F_k \xleftarrow{p_2} \Upsilon_k$$

given by

$$p_1(Q) = (Q \cap E,\, \text{pr}_F(Q)) \qquad \text{and} \qquad p_2(P) = (\text{pr}_E(P),\, P \cap F)$$

where $\mathrm{pr}_E : E \oplus F \to E$ and $\mathrm{pr}_F : E \oplus F \to F$ are the projections. Note that $\Upsilon_k \cap H = \emptyset$ for $k < m$.

Consider now the pull backs $\mathbf{E} \equiv \pi^* E$, $\mathbf{F} \equiv \pi^* F$ $\mathbf{H} \equiv \pi^* H$, $\mathbf{G} \equiv \pi^* G$ over $\pi : H \to X$, and note the diffeomorphisms: $\mathbf{E} \cong H \oplus E$, $\mathbf{F} \cong H \oplus F$, $\mathbf{H} \cong H \oplus H$, $\mathbf{G} \cong H \oplus G \subset G \oplus G$. For each k we define a sub-bundle

$$\pi_k : \mathbf{P}_k \to \Sigma_k$$

of the restriction $\mathbf{G}\big|_{\Sigma_k}$ by setting

$$\mathbf{P}_k \equiv \{(Q, P) \in \Sigma_k \oplus \Upsilon_k : p_1(Q) = p_2(P)\}. \tag{3.3}$$

where $\pi_k(Q, P) = Q$. This is a smooth fibre bundle whose fibre at a point $a \in \Sigma_k \cap H$ is

$$\pi_k^{-1}(a) \cong G_k(\ker a \oplus \operatorname{coker} a) \tag{3.4}$$

Over the total space of H there is a **tautological section** $\boldsymbol{\alpha} : \mathbf{E} \to \mathbf{F}$ given by $\boldsymbol{\alpha}(a) = a$.

Proposition 3.1. ([HL_5]) *The tautological section $\boldsymbol{\alpha}$ is geometrically atomic. Its radial span $\mathbf{T} \equiv T_{\boldsymbol{\alpha}}$ satisfies*

$$\partial \mathbf{T} = [0] - \sum_{k=0}^{m} \mathbf{P}_k$$

where $[0]$ is the current corresponding to the zero section of $\mathbf{H}$ and $\mathbf{P}_k$ is given as above.

Remark 3.2. In fact it is shown in [HL_5] that $\mathbf{T}$ extends to a current $\widetilde{\mathbf{T}}$ in the total space of the extended bundle $\mathbf{G} \cong G \oplus G \to G$, and that

$$\partial \widetilde{\mathbf{T}} = \sum_{k=0}^{m} \mathbf{P}'_k - \sum_{k=0}^{m} \mathbf{P}_k$$

where $\mathbf{P}'_k$ is a fibre bundle $\mathbf{P}'_k \to \Upsilon_k$ under projection to the other factor of $\mathbf{G} \cong G \oplus G$ In this setting the picture becomes much more symmetric. Each of the subvarieties $\mathbf{P}_k$ and $\mathbf{P}'_k$ is written as a fibre product over the fixed-point set F_k in the diagonal. Furthmore, under time-reversal in the flow the varieties $\mathbf{P}_k$ and $\mathbf{P}'_k$ exchange roles. This expanded picture enables one to write formulas of the type given below, for "meromorphic" bundle mappings (cf. [Z]).

Suppose now that Φ is an Ad-invariant form as above and consider the characteristic form $\omega = \Phi(\Omega^{\mathbf{U}})$ on $\mathbf{G}$ where $\mathbf{U} \to \mathbf{G}$ is the tautological m-plane bundle. The restriction of ω to the subset $\mathbf{P}_k$ can be written as $\omega\big|_{\mathbf{P}_k} = \Phi(\Omega^{\mathrm{Im}\boldsymbol{\alpha} \oplus \mathbf{U}_k})$ because of the canonical splitting

$$\mathbf{U}\big|_{\mathbf{P}_k} = \mathrm{Im}\boldsymbol{\alpha} \oplus \mathbf{U}_k, \tag{3.5}$$

where $\mathbf{U}_k \to G_k(\ker\boldsymbol{\alpha} \oplus \operatorname{coker}\boldsymbol{\alpha})$ is the tautological k-plane bundle along the fibres (3.4) of the projection π_k. The calculus of section 2 (with X replaced by H) now leads to:

Theorem 3.3 (Universal Case). *Let Φ and Ψ be Ad-invariant polynomials on the Lie algebras of the structure groups of E and F as in Theorem 2.4. Then there exist canonical L^1_{loc}-forms $\mathbf{T}_\Phi$ and $\mathbf{T}_\Psi$ on H satisfying the equations:*

$$\Phi(\Omega^{\mathbf{E}}) \;=\; \sum_{k=0}^{m} \mathrm{Res}_{k,\Phi}[\Sigma_k] + d\mathbf{T}_\Phi \qquad \textit{and} \qquad \Psi(\Omega^{\mathbf{F}}) \;=\; \sum_{k=0}^{m} \mathrm{Res}_{k,\Psi}[\Sigma_k] + d\mathbf{T}_\Psi$$

where $\mathrm{Res}_{k,\Phi}$ *and* $\mathrm{Res}_{k,\Psi}$ *are smooth residue forms on* Σ_k *given by*

$$\mathrm{Res}_{k,\Phi} \;=\; (\pi_k)_*\Phi\{\Omega^{\mathrm{Im}\boldsymbol{\alpha}\oplus\mathbf{U}_k}\} \qquad \textit{and} \qquad \mathrm{Res}_{k,\Psi} \;=\; (\pi_k)_*\Psi\{\Omega^{(\mathrm{ker}\boldsymbol{\alpha})^\perp\oplus\mathbf{U}_k^\perp}\}$$

where $\mathbf{U}_k \to G_k(\mathrm{ker}\boldsymbol{\alpha}\oplus\mathrm{coker}\boldsymbol{\alpha})$ *is the tautological k-plane bundle over the Grassmann compactification of the normal bundle* $\mathrm{Hom}(\mathrm{ker}\boldsymbol{\alpha},\mathrm{coker}\boldsymbol{\alpha})$ *to* Σ_k.

For certain generic sections of $\mathrm{Hom}(E,F)$ this result can be "pulled-back" to X.

Definition 3.4. A section α of $H = \mathrm{Hom}(E,F)$ is **normal** if it is transversal to each of the submanifolds $\Sigma_k \subset H$ for $k \geq 0$. Under this assumption each of the sets

$$\Sigma_k(\alpha) \equiv (\alpha)^{-1}(\Sigma_k) = \{x \in X : \dim(\ker\alpha_x) = k\}$$

is a smooth submanifold with locally finite volume and orientable normal bundle in X ([HL$_5$, 10.2]) and therefore defines a rectifiable current $[\Sigma_k(\alpha)]$.

Proposition 3.5. ([HL$_5$, 9.4]) *A normal bundle map α is geometrically atomic.*

Theorem 3.6. ([HL$_5$, 10.3]) *Let $\alpha : E \to F$ be a normal bundle map over a manifold X. Then for each Ad-invariant polynomial Φ as above, there exists a canonical L^1_{loc}-form T_Φ on X such that*

$$\Phi(\Omega^E) \;=\; \sum_{k=0}^{m} \mathrm{Res}_{k,\Phi}[\Sigma_k(\alpha)] + dT_\Phi \tag{3.6}$$

where $\mathrm{Res}_{k,\Phi}$ *is a smooth residue form on* $\Sigma_k(\alpha)$ *given by*

$$\mathrm{Res}_{k,\Phi} \;=\; (\pi_k)_*\Phi\{\Omega^{\mathrm{Im}\alpha\oplus U_k}\} \tag{3.7}$$

where $U_k \to G_k(\mathrm{ker}\alpha\oplus\mathrm{coker}\alpha)$ *is the tautological k-plane bundle over the Grassmann compactification of the normal bundle* $\mathrm{Hom}(\mathrm{ker}\alpha,\mathrm{coker}\alpha)$ *to* $\Sigma_k(\alpha)$.

Similarly, for each Ad-invariant polynomial Ψ as above, there is an L^1_{loc}-form T_Ψ with

$$\Psi(\Omega^F) \;=\; \sum_{k=0}^{m} \mathrm{Res}_{k,\Psi}[\Sigma_k(\alpha)] + dT_\Psi \tag{3.8}$$

where $\mathrm{Res}_{k,\Phi}$ *is a smooth residue form on* $\Sigma_k(\alpha)$ *given by*

$$\mathrm{Res}_{k,\Psi} \;=\; (\pi_k)_*\Psi\{\Omega^{(\mathrm{ker}\alpha)^\perp\oplus U_k^\perp}\} \tag{3.9}$$

This is a local form of MacPherson's Formula [Mac$_*$] at the level of forms and currents. The proof consists in showing that due to the transversality of α, the singularity stratification $\Sigma_*(\alpha)$ on X is modeled on the universal one, and the residue calculations are similar.

A analogous result holds for real analytic bundle maps whose singularity sets have the appropriate dimension. See [HL$_5$ §11] for details. The singular currents appearing in the formulas above are proved to have the following properties in [HL$_5$, §12].

Theorem 3.7. *Each of the currents* $[\Sigma_k(\alpha)]$, $\mathrm{Res}_{k,\Phi}[\Sigma_k(\alpha)]$, *and* $\mathrm{Res}_{k,\Psi}[\Sigma_k(\alpha)]$ *is d-closed on* X. *Furthermore, after changing* T_Φ *(or* T_Ψ*) by adding a flat current, we may assume that the connection used to compute the residue in (3.7) (or (3.9)) is a direct-sum connection.*

Many explicit examples are worked out in detail in [HL$_5$].

§4. Direct Sum Mappings.

A bundle map which can be expressed as a direct sum $\alpha \oplus \alpha' : E \oplus E' \to F \oplus F'$ is generally not normal as a map from $E \oplus E'$ to $F \oplus F'$, and MacPherson's Formula does not hold. In fact, generically the codimension of the singular set $\Sigma_k(\alpha \oplus \alpha')$ is much smaller than the predicted codimension for normal mappings. Nevertheless, the methods developed here apply to yield interesting formulas in this case.

Let E, F, $H \subset G$ be as in §1, and let E', F', $H' \subset G'$ denote a second such set-up over the same manifold X. Let

$$\pi : G \oplus G' \longrightarrow X$$

denote the fibre product of the Grassmann compactifications of H and H'. Consider the flow $\widetilde{\varphi}_t : E \oplus E' \oplus F \oplus F' \to E \oplus E' \oplus F \oplus F'$ given by $\widetilde{\varphi}_t(e, e', f, f') = (te, te', f, f')$, which can be considered the direct sum of the flows defined in §3 or the restriction of that flow for the bundle $\mathrm{Hom}(E \oplus E', F \oplus F')$. This induces a flow

$$\varphi_t : G \oplus G' \to G \oplus G'$$

whose fixed-point set is the sum of the fibre-products

$$\mathrm{Fix}(\varphi_t) \;=\; \coprod_{k,k'} F_k \oplus F'_{k'}$$

where F_k and $F'_{k'}$ are the fixed-point sets of the flows on each factor G and G' respectively (cf. §3). One sees that the stable and unstable manifolds are also fibre products

$$\Sigma(F_k \oplus F'_{k'}) \;=\; \Sigma_k \oplus \Sigma'_{k'} \qquad \text{and} \qquad \Upsilon(F_k \oplus F'_{k'}) \;=\; \Upsilon_k \oplus \Upsilon'_{k'}.$$

Our first result is an analogue of Theorem 3.1 with the components $\mathbf{P}_k$ replaced by fibre products of the stable and unstable manifolds above.

We consider first the "universal" case on the total space of the bundle

$$\pi : H \oplus H' \longrightarrow X. \tag{4.1}$$

We denote the pull-backs $\pi^*E, \pi^*E', \pi^*F, \pi^*F', \pi^*H \subset \pi^*G, ...$ by bold letters $\mathbf{E}, \mathbf{E}', \mathbf{F}, \mathbf{F}'$, $\mathbf{H} \subset \mathbf{G}$, etc. Note that over the total space of $H \oplus H'$ there is a tautological section $\boldsymbol{\alpha} \oplus \boldsymbol{\alpha}'$ of the bundle $\mathrm{Hom}(\mathbf{E} \oplus \mathbf{E}', \mathbf{F} \oplus \mathbf{F}')$.

Proposition 4.1. *The tautological section $\boldsymbol{\alpha} \oplus \boldsymbol{\alpha}'$ is geometrically atomic. Furthermore, its radial span $\mathbf{T} \equiv T_{\boldsymbol{\alpha}\oplus\boldsymbol{\alpha}'}$ satisfies the equation*

$$\partial \mathbf{T} \ = \ [0] - \sum_{k=0}^{m} \sum_{k'=0}^{m'} \mathbf{P}_k \oplus \mathbf{P}'_{k'} \tag{4.2}$$

where $\mathbf{P}_k \subset \mathbf{G}$ and $\mathbf{P}'_k \subset \mathbf{G}'$ are defined as in (3.3).

Proof. After taking local trivializations of the bundles E, F, E', F' one sees easily that the current $\mathbf{T}$ is independent of the X-coordinates, i.e., it is a trivial product of the base with a current defined in the fibre. Thus it suffices to consider the case where X is a point.

When $X = \mathrm{pt}$, one sees directly that $\mathbf{T}$ is algebraic, and so $\boldsymbol{\alpha} \oplus \boldsymbol{\alpha}'$ is geometrically atomic by 1.3. One also sees directly that $\partial\mathbf{T}$ is as claimed, namely, the product of the boundaries in each factor. To be more precise, note that $\mathbf{T} = T_{\boldsymbol{\alpha}\oplus\boldsymbol{\alpha}'}$ is the submanifold of codimension-1 in $T_{\boldsymbol{\alpha}} \times T_{\boldsymbol{\alpha}'}$ given by

$$\mathbf{T} \ = \ \{(\mathrm{gr}\tfrac{1}{t}a,\ \mathrm{gr}\tfrac{1}{t}a') \in \mathbf{G} \oplus \mathbf{G}' \ : \ 0 < t < \infty \ \text{ and } \ (a, a') \in H \oplus H'\}.$$

Now $(p, p') \in \mathrm{supp}\, \partial\mathbf{T}$ with $\pi(p, p') = (a, a')$ only if there exist sequences $(a_i, a'_i) \to (a, a')$ and $t_i \to 0$ or ∞, such that $(\mathrm{gr}\frac{1}{t_i}a_i,\ \mathrm{gr}\frac{1}{t_i}a'_i) \to (p, p')$. In particular, by [HL$_5$] we know that $p \in \mathbf{P}_k$ and $p' \in \mathbf{P}'_{k'}$ for some k, k'. Since dimensions are correct, the boundary of $\mathbf{T}$ is an integer linear combination of the submanifolds $\mathbf{P}_k \oplus \mathbf{P}'_{k'}$. Is is straightforward to see that these coefficients are are all 1. ∎

Note from the paragraph above that $\mathrm{supp}(\partial\mathbf{T}) \ \subset \ \mathbf{G} \oplus \mathbf{G}' \ \subset \ G_{m+m'}(E \oplus E' \oplus F \oplus F')$. Note also that each submanifold $\mathbf{P}_k \oplus \mathbf{P}'_{k'}$ admits a fibration

$$\pi_{k,k'} \equiv \pi_k \oplus \pi'_{k'} : \mathbf{P}_k \oplus \mathbf{P}'_{k'} \ \longrightarrow \ \Sigma_k \oplus \Sigma'_{k'}. \tag{4.3}$$

with fibre $G_{k,k'} = G_k(\ker \boldsymbol{\alpha} \oplus \mathrm{coker}\, \boldsymbol{\alpha}) \times G_{k'}(\ker \boldsymbol{\alpha}' \oplus \mathrm{coker}\, \boldsymbol{\alpha}')$

We now consider characteristic forms $\Phi(\Omega^{U\oplus U'}) = \Phi(\Omega^U \oplus \Omega^{U'})$ on $\mathbf{G} \oplus \mathbf{G}'$ where $U \to \mathbf{G}$ and $U' \to \mathbf{G}'$ denote the tautological bundles. Using the splittings (3.5), equation (4.3), and the operator calculus we conclude the following.

Theorem 4.2 (Universal Case). *Let E, E' F and F' be smooth complex vector bundles with connection on a manifold X and assume that $m \equiv \mathrm{rank}(E) \le \mathrm{rank}(F)$ and $m' \equiv \mathrm{rank}(E') \le \mathrm{rank}(F')$. Let Φ be an invariant polynomial on the Lie algebra of the structure group of $E \oplus E'$. Then there exists an L^1_{loc}-form T_Φ on the total space of $\mathrm{Hom}(E, F) \oplus \mathrm{Hom}(E', F')$ so that*

$$\Phi(\Omega^{E\oplus E'}) \ = \ \sum_{k=0}^{m} \sum_{k'=0}^{m'} \mathrm{Res}_{\Phi,k,k'}[\Sigma_k \oplus \Sigma'_{k'}] \ + \ dT_\Phi \tag{4.4}$$

$$\mathrm{Res}_{\Phi,k,k'} \ = \ (\pi_{k,k'})_* \Phi \left(\Omega^{\mathrm{Im}\, \boldsymbol{\alpha} \oplus U_k \oplus \mathrm{Im}\, \boldsymbol{\alpha}' \oplus U'_{k'}} \right)$$

where $\boldsymbol{\alpha} : \pi^* E \to \pi^* F$ *and* $\boldsymbol{\alpha}' : \pi^* E' \to \pi^* F'$ *denote the tautological bundle maps on* $\mathrm{Hom}(E, F)$ *and* $\mathrm{Hom}(E', F')$*, and* $(\pi_{k,k'})_*$ *denotes integration over the fibres of the compactification* $G_{k,k'} = G_k(\ker \boldsymbol{\alpha} \oplus \mathrm{coker}\, \boldsymbol{\alpha}) \oplus G_{k'}(\ker \boldsymbol{\alpha}' \oplus \mathrm{coker}\, \boldsymbol{\alpha}')$ *of the normal bundle* $\mathrm{Hom}(\ker \boldsymbol{\alpha}, \mathrm{coker}\, \boldsymbol{\alpha}) \oplus \mathrm{Hom}(\ker \boldsymbol{\alpha}', \mathrm{coker}\, \boldsymbol{\alpha}')$ *to* $\Sigma_k \oplus \Sigma_{k'}$ *(cf. (4.3)), and* U_k*,* $U'_{k'}$ *are the tautological bundles over* G_k *and* $G_{k'}$*.*

If $\mathrm{rank}(E) = \mathrm{rank}(F)$ *and* $\mathrm{rank}(E') = \mathrm{rank}(F')$*, then formula (4.4) becomes*

$$\Phi(\Omega^{E \oplus E'}) - \Phi(\Omega^{F \oplus F'}) \ = \ \sum_{k + k' > 0} \mathrm{Res}_{\Phi,k,k'}[\Sigma_k \oplus \Sigma'_{k'}] \ + \ dT_\Phi$$

If Ψ *is an invariant polynomial on the Lie algebra of the structure group of* $F \oplus F'$*, then there exists an* L^1_{loc}*-form* T_Ψ *on* $\mathrm{Hom}(E, F) \oplus \mathrm{Hom}(E', F')$ *so that*

$$\Psi(\Omega^{F \oplus F'}) \ = \ \sum_{k=0}^{m} \sum_{k'=0}^{m'} \mathrm{Res}_{\Psi,k,k'}[\Sigma_k \oplus \Sigma'_{k'}] \ + \ dT_\Psi \tag{4.5}$$

$$\mathrm{Res}_{\Psi,k,k'} \ = \ (\pi_{k,k'})_* \Psi \left(\Omega^{(\ker \boldsymbol{\alpha})^\perp \oplus U_k^\perp \oplus (\ker \boldsymbol{\alpha}')^\perp \oplus U'^{\perp}_{k'}} \right)$$

where $(\ker \boldsymbol{\alpha})^\perp$ *denotes the orthogonal complement of* $\ker \boldsymbol{\alpha}$ *in* E*,* $U_k^\perp$ *denotes the orthogonal complement of* U_k *in* $\ker \boldsymbol{\alpha} \oplus \mathrm{coker} \boldsymbol{\alpha}$*, etc.*

We now consider sections (α, α') of $\mathrm{Hom}(E, F) \oplus \mathrm{Hom}(E', F')$ over X.

Definition 4.3. A section (α, α') of $\mathrm{Hom}(E, F) \oplus \mathrm{Hom}(E', F')$ is called a **normal pair** if (α, α') is transversal to the submanifolds $\Sigma_k \oplus \Sigma'_{k'}$ for all k, k'.

In Appendix A we prove the following.

Proposition 4.4. *Normal pairs are residual (in particular dense) in the* C^1*-topology on sections of* $\mathrm{Hom}(E, F) \oplus \mathrm{Hom}(E', F')$*. Any normal pair of sections* (α, α') *is geometrically atomic. Furthermore, if* $\beta = (\alpha, \alpha')$ *is a normal pair, then for all* k, k'*, the submanifolds* $\Sigma_k(\alpha)$ *and* $\Sigma_{k'}(\alpha')$ *intersect transversely in* X*, and* $\beta^{-1}(\Sigma_k \oplus \Sigma_{k'}) = \Sigma_k(\alpha) \cap \Sigma_{k'}(\alpha')$*.*

Arguing as in the proof of [HL$_5$; Thm. 10.3] gives the following result.

Theorem 4.5. *Let* X, E, F, E', F', Φ *and* Ψ *be as in Theorem 4.2, and suppose that* $(\alpha, \alpha') : E \oplus E' \to F \oplus F'$ *is a direct-sum mapping which is a normal pair in the sense of 4.3. Then there exists an* L^1_{loc}*-form* T_Φ *on* X *so that*

$$\Phi(\Omega^{E \oplus E'}) \ = \ \sum_{k=0}^{m} \sum_{k'=0}^{m'} \mathrm{Res}_{\Phi,k,k'}[\Sigma_k(\alpha) \cap \Sigma'_{k'}(\alpha')] \ + \ dT_\Phi \tag{4.6}$$

$$\mathrm{Res}_{\Phi,k,k'} \ = \ (\pi_{k,k'})_* \Phi \left(\Omega^{\mathrm{Im}\, \alpha \oplus U_k \oplus \mathrm{Im}\, \alpha' \oplus U'_{k'}} \right)$$

where $(\pi_{k,k'})_*$ *denotes integration over the fibres of the compactification* $G_{k,k'} = G_k(\text{ker }\alpha \oplus \text{coker }\alpha) \oplus G_{k'}(\text{ker }\alpha' \oplus \text{coker }\alpha')$ *of the normal bundle to* $\Sigma_k(\alpha) \cap \Sigma'_{k'}(\alpha')$, *and where* $U_k \to G_k$ *and* $U'_{k'} \to G_{k'}$ *are the tautological bundles.*

If $\text{rank}(E) = \text{rank}(F)$ *and* $\text{rank}(E') = \text{rank}(F')$, *then formula (4.6) becomes*

$$\Phi(\Omega^{E\oplus E'}) - \Phi(\Omega^{F\oplus F'}) \ = \sum_{k+k'>0} \text{Res}_{\Phi,k,k'}[\Sigma_k(\alpha) \cap \Sigma'_{k'}(\alpha')] \ + \ dT_\Phi$$

If Ψ *is an invariant polynomial on the Lie algebra of the structure group of* $F \oplus F'$, *then there exists an* L^1_{loc}*-form* T_Ψ *on* X *so that*

$$\Psi(\Omega^{F\oplus F'}) \ = \ \sum_{k=0}^{m} \sum_{k'=0}^{m'} \text{Res}_{\Psi,k,k'}[\Sigma_k(\alpha) \cap \Sigma'_{k'}(\alpha')] \ + \ dT_\Psi \tag{4.7}$$

$$\text{Res}_{\Psi,k,k'} \ = \ (\pi_{k,k'})_* \Psi\left(\Omega^{(\text{ker }\alpha)^\perp \oplus U_k^\perp \oplus (\text{ker }\alpha')^\perp \oplus U'^\perp_{k'}}\right)$$

Remark 4.6. The analogue of Theorem 3.7 holds in this context. For generic (α, α') the currents $[\Sigma_k(\alpha) \cap \Sigma'_{k'}(\alpha')]$ and $\text{Res}_{\Phi,k,k'}[\Sigma_k(\alpha) \cap \Sigma'_{k'}(\alpha')]$ are each d-**closed**, and one may assume, after changing the transgression by a flat current, that the connection on $\text{Im}\,\alpha \oplus U_k \oplus \text{Im}\,\alpha' \oplus U'_{k'}$ is a direct sum connection. Thus if Φ is a **multiplicative sequence** in the sense of Hirzebruch, then the residue in (4.6) can be rewritten

$$\text{Res}_{\Phi,k,k'} \ = \ (\pi_{k,k'})_* \left\{\Phi\left(\Omega^{U_k}\right) \Phi\left(\Omega^{U'_{k'}}\right)\right\} \Phi\left(\Omega^{\text{Im }\alpha}\right) \Phi\left(\Omega^{\text{Im }\alpha'}\right) \tag{4.8}$$

Example 4.7. Diagonal maps. Let $\alpha_j : E_j \to F_j$ be a map of complex line bundles over X for $j = 1, ..., m$, and consider the direct sum mapping

$$\alpha = (\alpha_1, .., \alpha_m) : E_1 \oplus \cdots \oplus E_m \longrightarrow F_1 \oplus \cdots \oplus F_m.$$

Let $\Phi(\Omega) \equiv c(\Omega) \equiv \det(I + \frac{i}{2\pi}\Omega)$ be the total Chern class. Note that for a generic map $\alpha : E \to F$ of line bundles on X we have $\Sigma_1(\alpha) = \text{Div}(\alpha)$ (= the zero set of α as a section of $E^* \otimes F$) and $\overline{\Sigma}_0 = X$. Furthermore, $\pi_{k\,*} c\left(\Omega^{U_k}\right) = (-1)^k$ for $k = 0, 1$ and

$$c\left(\Omega^{\text{Im }\alpha}\right)\Big|_{\Sigma_k(\alpha)} \ = \ \begin{cases} 1 & \text{if } \ k = 1 \\ c(\Omega^F) = 1 + c_1(\Omega^F) & \text{if } \ k = 0. \end{cases}$$

From Theorem 4.5 and (4.8) we deduce the following formula for a generic diagonal map α:

$$\begin{aligned} c\left(\Omega^{E_1\oplus\cdots\oplus E_m}\right) \ &= \sum_{i_1<\cdots<i_k} (-1)^{m-k} c\left(\Omega^{F_{i_1}}\right) \dots c\left(\Omega^{F_{i_k}}\right) \text{Div}\left(\alpha_{i'_1}\right) \dots \text{Div}(\alpha_{i'_{m-k}}) + dT \\ &= \sum_I (-1)^{|I'|} \ c\left(\Omega^{F_I}\right) \text{Div}\left(\alpha_{I'}\right) \ + \ dT \\ &= \prod_{j=1}^{m} \left\{c\left(\Omega^{F_j}\right) - \text{Div}\left(\alpha_j\right)\right\} + \ dT \end{aligned}$$

where the first sum is over all k and $\{i_1, \ldots, i_k, i'_1, \ldots, i'_{m-k}\} = \{1, \ldots, m\}$.

§**5. Tensor Product Mappings.** In this section we consider the tensor product of two bundle mappings

$$\alpha : E \longrightarrow F \qquad \text{and} \qquad \alpha' : E' \longrightarrow F'$$

over a ν-manifold X. We shall see that for normal pairs (α, α') (cf. Def. 4.3), the section $\alpha \otimes \alpha'$ is geometrically atomic, and we shall derive a general formula which has many interesting special cases.

We begin be examining the universal case. Let $\pi : H \oplus H' \to X$ be as in (4.1) and note that over the total space of $H \oplus H'$ there is tautological pair of sections $(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$. As in §4 we denote π^*E, π^*E', π^*F and π^*F' by $\mathbf{E}, \mathbf{E}', \mathbf{F}$ and $\mathbf{F}'$ respectively. Then $\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}'$ gives a section of the bundle

$$\mathbf{Hom} \equiv \mathrm{Hom}(\mathbf{E} \otimes \mathbf{E}', \mathbf{F} \otimes \mathbf{F}') \ \longrightarrow \ H \oplus H' \tag{5.1}$$

which has compactification

$$\mathbf{G} \equiv G_{mm'}((\mathbf{E} \otimes \mathbf{E}') \oplus (\mathbf{F} \otimes \mathbf{F}')) \ \xrightarrow{\pi} \ H \oplus H' \tag{5.2}$$

where $m = \dim E$ and $m' = \dim E'$.

Proposition 5.1. *The universal section $\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}'$ over $H \oplus H'$ is geometrically atomic. Furthermore the boundary of $T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}$ is given by integration over a finite number of manifolds of finite volume in $\mathbf{G}$.*

Proof. Suppose that X is a point. Then $T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}$ is an irreducible real algebraic subset of the Zariski dense subset **Hom** in $\mathbf{G}$, and so its closure is an irreducible algebraic subset of $\mathbf{G}$. Hence, its regular points are a submanifold of finite volume in $\mathbf{G}$. Furthermore, its current boundary $\partial T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}$ is supported in its topological boundary $\overline{T}_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'} - T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}$ which is an algebraic variety of smaller dimension. By the Federer Support Theorem [F, 4.1.15] it follows directly that $\partial T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}$ is given by integration over the regular points of those components of $\overline{T}_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'} - T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}$ having dimension equal to $\dim(T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}) - 1$

For the general case, we consider local trivializations of the bundles over a domain $\Omega \subset X$ and note that over Ω the submanifold $T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}$ is a product $\Omega \times T_0$ in $\Omega \times \mathbf{G}_0$ where T_0 and $\mathbf{G}_0$ correspond to the case of a point considered above. ∎

We now want to compute the boundary of the current $\mathbf{T}$ given by integration over the submanifold $T_{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'}$. We shall do this by computing the contribution over each of the submanifolds $\Sigma_k \oplus \Sigma'_{k'} \subset H \oplus H'$. We begin by noting from the proof of Proposition 5.1 that $\partial\mathbf{T}$ consists of integration over a finite number of real algebraic varieties extended by local trivialization in the X-variables. Thus the X-variables play no essential role here; we can treat $\partial\mathbf{T}$ as an analytic chain. In particular, we can decompose

$$\partial\mathbf{T} \ = \ \mathbf{H} - \sum_{k=0}^{m} \sum_{k'=0}^{m'} \mathbf{P}_{kk'} \tag{5.3}$$

where $\mathbf{H} = [H \oplus H']$ is the zero-section and where $\mathbf{P}_{kk'}$ corresponds to the subvarieties Y among those comprising $\partial\mathbf{T}$ such that

$$Y \cap \pi^{-1}\left(\Sigma_k \oplus \Sigma'_{k'}\right) \quad \text{is dense in} \quad Y.$$

We shall see that in fact each $\mathbf{P}_{kk'}$ is an irreducible variety. To describe it we set some notation. Consider a point

$$(\alpha, \alpha') \in \Sigma_k \oplus \Sigma'_{k'}$$

and write

$$K = \ker\alpha, \quad I = \mathrm{Im}\alpha, \quad K' = \ker\alpha', \quad I' = \mathrm{Im}\alpha'$$

With respect to the decompositions $E = K \oplus K^\perp$ and $E' = K' \oplus K'^\perp$ we can write

$$\alpha = (0, a) \;\text{ where }\; a : K^\perp \to I \qquad \text{and} \qquad \alpha' = (0, a') \;\text{ where }\; a' : K'^\perp \to I' \tag{5.4}$$

Theorem 5.2. *Let*

$$\partial\mathbf{T} \;=\; \mathbf{H} - \sum_{k,k' \geq 0} \mathbf{P}_{kk'}$$

be the decomposition (5.3). Then $\mathbf{P}_{mm'} = 0$, *and for* $k + k' < m + m'$, *the current* $\mathbf{P}_{kk'}$ *is given by integration over a submanifold* $P_{kk'} \subset \mathbf{G}$ *with fibration*

$$\pi_{kk'} : P_{kk'} \longrightarrow \Sigma_k \oplus \Sigma'_{k'} \tag{5.5}$$

where $\pi_{kk'}$ *is the restriction of the projection in (5.2). At each* $(\alpha, \alpha') \in \Sigma_k \oplus \Sigma'_{k'}$ *the fibre of* $\pi_{kk'}$ *consists of all* mm'*-planes of the form*

$$(K \otimes K') \oplus (I \otimes I') \oplus \mathrm{gr}(a \otimes L') \oplus \mathrm{gr}(L \otimes a') \tag{5.6}$$

for $L \in \mathrm{Hom}(K, I^\perp)$ *and* $L' \in \mathrm{Hom}(K', I'^\perp)$, *where* a, a' *are given in (5.4). Thus*

$$\pi^{-1}_{kk'}(\alpha, \alpha') \;\cong\; \mathrm{Hom}(K, I^\perp) \oplus \mathrm{Hom}(K', I'^\perp)$$

and $P_{kk'}$ *can be written as a twisted fibre product*

$$P_{kk'} \;\cong\; \left\{\Sigma_k \oplus \Sigma'_{k'}\right\} \times_{F_k \times F'_{k'}} \left\{(\mathrm{gr}(a) \mathbin{\widetilde{\otimes}} \Upsilon'_{k'}) \oplus (\Upsilon_k \mathbin{\widetilde{\otimes}} \mathrm{gr}(a'))\right\} \tag{5.7}$$

where Υ_k, $\Upsilon'_{k'}$ *are the unstable manifolds of the fixed-point sets* F_k, $F'_{k'}$ *as in §2, and the operation* $\widetilde{\otimes}$ *is defined in Appendix B below.*

Proof of Theorem 5.2 Given $(\alpha, \alpha') \in \Sigma_k \oplus \Sigma'_{k'}$ we have that

$$\begin{aligned} \mathbf{K} \;&\equiv\; \ker(\alpha \otimes \alpha') \;=\; (K \otimes K') \oplus (K^\perp \otimes K') \oplus (K \otimes K'^\perp) \\ \mathbf{I} \;&\equiv\; \mathrm{Im}(\alpha \otimes \alpha') \;=\; I \otimes I'. \end{aligned}$$

Recall that

$$\lim_{t\to 0} \operatorname{gr}\left\{\tfrac{1}{t}\alpha\otimes\alpha'\right\} \;=\; \mathbf{K}\times\mathbf{I} \;\in\; G_R(E\otimes E')\times G_{M-R}(F\otimes F')$$

where $M=mm'$ and $R=(m-k)(m'-k')$. By [HL$_5$] we know that the points of $\pi_{kk'}^{-1}(\alpha,\alpha')$ must lie in the fibre

$$\operatorname{Hom}(\mathbf{K},\mathbf{I}^{\perp})$$

above $\mathbf{K}\times\mathbf{I}$ in this fixed-point set. To see which points occur, we must consider sequences of the form $\alpha_j\otimes\alpha'_j$ where

$$\alpha_j=(L_j,a_j)\ \xrightarrow[j\to\infty]{}\ (0,a) \qquad\text{and}\qquad \alpha'_j=(L'_j,a'_j)\ \xrightarrow[j\to\infty]{}\ (0,a')$$

for $L_j\in\operatorname{Hom}(K,I^{\perp})$, $L'_j\in\operatorname{Hom}(K',I'^{\perp})$, $a_j\in\operatorname{Hom}(K^{\perp},I)$ and $a'_j\in\operatorname{Hom}(K'^{\perp},I')$. For each j we have

$$gr\left(\tfrac{1}{t}\alpha_j\otimes\alpha'_j\right) \;=\;$$

$$\left\{(v_{00},v_{10},v_{01},v_{11},\tfrac{1}{t}L_j\otimes L'_j(v_{00}),\tfrac{1}{t}a_j\otimes L'_j(v_{10}),\tfrac{1}{t}L_j\otimes a'_j(v_{01}),\tfrac{1}{t}a_j\otimes a'_j(v_{11})\right\}$$

for all $(v_{00},v_{10},v_{01},v_{11})\in(K\otimes K')\oplus(K^{\perp}\otimes K')\oplus(K\otimes K'^{\perp})\oplus(K^{\perp}\otimes K'^{\perp})$. Thus

$$\begin{aligned} gr\left(\tfrac{1}{t}\alpha_j\otimes\alpha'_j\right) \;=\;& \left\{(v_{00},\tfrac{1}{t}L_j\otimes L'_j(v_{00})) : v_{00}\in K\otimes K'\right\}\\ &\oplus\left\{(v_{10},\tfrac{1}{t}a_j\otimes L'_j(v_{10})) : v_{10}\in K^{\perp}\otimes K'\right\}\\ &\oplus\left\{(v_{01},\tfrac{1}{t}L_j\otimes a'_j(v_{01})) : v_{01}\in K\otimes K'^{\perp}\right\}\\ &\oplus\left\{(v_{11},\tfrac{1}{t}a_j\otimes a'_j(v_{11})) : v_{11}\in K^{\perp}\otimes K'^{\perp}\right\} \end{aligned} \tag{5.8}$$

The support of the current $\mathbf{P}_{kk'}$ will lie the set of limit points of such sequences of graphs of $\frac{1}{t_j}\alpha_j\otimes\alpha'_j$ where $t_j\to 0$.

Setting $\alpha_j=(t_jL,a)$, $\alpha_j=(t_jL',a')$ and sending $t_j\to 0$ gives planes of the form (5.6). We shall show that all other limit points lie in a subanalytic set B with the property that

$$\dim(B)<\dim\{\operatorname{Hom}(K,I^{\perp})\times\operatorname{Hom}(K',I'^{\perp})\}=k(n-m+k)+k'(n'-m'+k'). \tag{5.9}$$

It then follows from the Federer Flat Support Theorem [F, 4.1.15] (cf. [HL$_4$, 2.7]) that

$$\mathbf{P}_{kk'}=n_{kk'}[P_{kk'}]$$

for some $n_{kk'}\in\mathbf{Z}$. A straightforward local calculation shows that $n_{kk'}=1$.

To establish (5.9) we proceed case-by-case as follows. Suppose first that

$$\lim_{j\to\infty} \frac{|L_j||L'_j|}{t_j} = \infty, \quad \lim_{j\to\infty} \frac{|L_j|}{t_j} = \infty, \quad \text{and} \quad \lim_{j\to\infty} \frac{|L'_j|}{t_j} = \infty$$

Without loss of generality we can assume that the limits

$$J = \lim_{j\to\infty} \operatorname{Im}(L_j) \qquad \text{and} \qquad J' = \lim_{j\to\infty} \operatorname{Im}(L'_j)$$

exist. Here the limit of the sequences of graphs (5.8) are of the form:

$$(J \otimes J') \oplus (J \otimes I') \oplus (I \otimes J') \oplus (I \otimes I').$$

Thus the space of such limits is the product of Grassmannians $G_m(I^\perp) \times G_{m'}(I'^\perp)$ whose dimension is $k(n-m) + k'(n'-m')$ and (5.9) holds.

The next case to consider is where

$$\lim_{j\to\infty} \frac{|L_j||L'_j|}{t_j} = c > 0, \quad \lim_{j\to\infty} \frac{|L_j|}{t_j} = \infty, \quad \text{and} \quad \lim_{j\to\infty} \frac{|L'_j|}{t_j} = \infty.$$

We may assume by passing to subsequences that $\frac{1}{t_j} L_j \otimes L'_j \to L_\infty \otimes L'_\infty$. We get limits of the form:

$$\operatorname{gr}(L_\infty \otimes L'_\infty) \oplus (\operatorname{Im} L_\infty \otimes I') \oplus (I \otimes \operatorname{Im} L'_\infty) \oplus (I \otimes I').$$

The space of such limits is parameterized by

$$\operatorname{Hom}(K, I^\perp) \times \operatorname{Hom}(K', I'^\perp)/\mathbf{k}$$

where $\mathbf{k} = \mathbb{C}$ or $\mathbb{R}$ depending on the case. We see that (5.9) holds. Note incidentally that this calculation shows that

$$\mathbf{P}_{mm'} = 0$$

since its support is too small.

Consider now the case where $\lim_{j\to\infty} \frac{|L_j||L'_j|}{t_j} = 0$. We must consider several possibilities. The first, where

$$\lim_{j\to\infty} \frac{|L_j|}{t_j} = c \quad \text{and} \quad \lim_{j\to\infty} \frac{|L'_j|}{t_j} = c',$$

is the generic case. In all other cases one sees that (5.9) holds. For example, suppose that

$$\lim_{j\to\infty} \frac{|L_j|}{t_j} = \infty \quad \text{and} \quad \lim_{j\to\infty} \frac{|L'_j|}{t_j} = c > 0.$$

Then limits are of the form:

$$(K \otimes K') \oplus (J \otimes a') \oplus \operatorname{gr}(a \otimes L'_\infty) \oplus (I \otimes I')$$

where J is as above and $L'_\infty = \lim_j \frac{1}{t_j} L'_j$. The dimension of all such limits is $k(n-m) + k'(n'-m'+k')$ and so (5.9) holds. The remaining cases are similar. ▮

We now consider the tautological bundle

$$\mathbf{U} \longrightarrow G_{mm'}(E \otimes E', \, F \otimes F')$$

and give $\mathbf{U}$ and $\mathbf{U}^\perp$ connections induced from the tensor product of given connections on E, E', F, F' via the splitting $\mathbf{U} \oplus \mathbf{U}^\perp = (E \otimes E') \oplus (F \otimes F')$. Then by (5.6) the restriction of $\mathbf{U}$ to the fibres of $P_{kk'}$ has the form

$$(5.10) \quad \mathbf{U}\Big|_{P_{kk'}} = \Big(\ker(\alpha) \otimes \ker(\alpha')\Big) \oplus \Big(\mathrm{Im}(\alpha) \otimes \mathrm{Im}(\alpha')\Big) \oplus \Big(\mathrm{gr}(a) \,\widetilde{\otimes}\, U'_{k'}\Big) \oplus \Big(U_k \,\widetilde{\otimes}\, \mathrm{gr}(a')\Big)$$

where

$$U_k \longrightarrow G_k(K \oplus I^\perp) \qquad \text{and} \qquad U'_{k'} \longrightarrow G_{k'}(K' \oplus I'^\perp)$$

are the tautological bundles restricted to the affine charts $\mathrm{Hom}(K, I^\perp)$ and $\mathrm{Hom}(K', I'^\perp)$ and $\widetilde{\otimes}$ is defined in Appendix B. This gives us the following theorem.

Theorem 5.3 (Universal Case). *Let Φ be an invariant polynomial on the Lie algebra of the structure group of $E \otimes E'$. Then the characteristic form $\Phi(\Omega^{E \otimes E'})$ satisfies the following equation on the total space of $H \oplus H'$:*

$$\Phi(\Omega^{E \otimes E'}) = \sum_{k=0}^{m} \sum_{k'=0}^{m'} \mathrm{Res}_{\Phi, kk'}[\Sigma_k \oplus \Sigma'_{k'}] + dT$$

where T is a canonical L^1_{loc}-form on $H \oplus H'$ and $\mathrm{Res}_{\Phi,kk'}$ is a smooth residue form on $\Sigma_k \oplus \Sigma'_{k'}$ given by $\mathrm{Res}_{\Phi,mm'} = 0$ and otherwise

$$(5.11) \qquad \mathrm{Res}_{\Phi,kk'} = \pi_{kk'*} \Phi \left\{ \Omega^{(\ker\alpha \otimes \ker\alpha') \oplus (\mathrm{Im}\,\alpha \otimes \mathrm{Im}\,\alpha') \oplus (\mathrm{gr}(a) \,\widetilde{\otimes}\, U'_{k'}) \oplus (U_k \,\widetilde{\otimes}\, \mathrm{gr}(a'))} \right\}$$

where

$$U_k \longrightarrow G_k(\ker\alpha \oplus \mathrm{Im}\alpha^\perp) \qquad \textit{and} \qquad U'_{k'} \longrightarrow G_{k'}(\ker\alpha' \oplus \mathrm{Im}\alpha'^\perp)$$

are the tautological bundles, and $\pi_{kk'}$ is the mapping (5.5).

Proof. Start with Theorem 5.2 and apply the operator calculus of §2. ▮

There is a companion result proved in the same manner.

Theorem 5.4 (Universal Case). *Let Ψ be an invariant polynomial on the Lie algebra of the structure group of $F \otimes F'$. Then the characteristic form $\Psi(\Omega^{F \otimes F'})$ satisfies the following equation on the total space of $H \oplus H'$:*

$$\Psi(\Omega^{F \otimes F'}) = \sum_{k=0}^{m} \sum_{k'=0}^{m'} \mathrm{Res}_{\Psi, kk'}[\Sigma_k \oplus \Sigma'_{k'}] + dT$$

where T is a canonical L^1_{loc}-form on $H \oplus H'$ and $\mathrm{Res}_{\Psi,kk'}$ *is a smooth residue form on* $\Sigma_k \oplus \Sigma'_{k'}$ *given by* $\mathrm{Res}_{\Psi,mm'} = 0$ *and otherwise*

$$(5.12) \quad \mathrm{Res}_{\Psi,kk'} \;=\; \pi_{kk'*}\Psi\left\{\Omega^{(\ker^\perp\alpha\otimes\ker^\perp\alpha')\oplus(\mathrm{Im}^\perp\alpha\otimes\mathrm{Im}^\perp\alpha')\oplus(\mathrm{gr}(a)\,\widetilde{\otimes}\,U'_{k'})^\perp\oplus(U_k\,\widetilde{\otimes}\,\mathrm{gr}(a'))^\perp}\right\}$$

where

$$U_k \longrightarrow G_k(\ker\alpha \oplus \mathrm{Im}^\perp\alpha) \qquad \text{and} \qquad U'_{k'} \longrightarrow G_{k'}(\ker\alpha' \oplus \mathrm{Im}^\perp\alpha')$$

are the tautological bundles, and $\pi_{kk'}$ *is the mapping (5.5). Here* $(gr(a)\,\widetilde{\otimes}\,U'_{k'})^\perp$ *denotes the orthogonal complement of* $gr(a)\,\widetilde{\otimes}\,U'_{k'}$ *in the subspace* $(\ker^\perp\alpha\otimes\ker\alpha')\oplus(\mathrm{Im}\alpha\otimes\mathrm{Im}^\perp\alpha')$. *Similarly,* $(U_k\,\widetilde{\otimes}\,\mathrm{gr}(a'))^\perp$ *denotes the complement in* $(\ker\alpha\otimes\ker^\perp\alpha')\oplus(\mathrm{Im}^\perp\alpha\otimes\mathrm{Im}\alpha')$.

The residue (5.12) can be re-expressed using adjoint transformations as follows

$$(5.12') \quad \mathrm{Res}_{\Psi,kk'} \;=\; \pi_{kk'*}\Psi\left\{\Omega^{(\mathrm{Im}\alpha^*\otimes\mathrm{Im}\alpha'^*)\oplus(\ker\alpha^*\otimes\ker\alpha'^*)\oplus(\mathrm{gr}(a^*)\,\widetilde{\otimes}\,U'^\perp_{k'})\oplus(U^\perp_k\,\widetilde{\otimes}\,\mathrm{gr}(a'^*))}\right\}$$

where a^* denotes the adjoint of a. This is derived using Lemma B.2(iii).

Note the coincidence of formulas (5.11) and (5.12'). This can be deduced directly from the fact that the family of push-forward connections for α coincides with the family of pull-back connections of the adjoint (cf. [HL$_2$]).

These results carry over to normal pairs. The following is proved in Appendix A.

Proposition 5.5. *Let (α, α') be a normal pair of sections of the bundle $H \oplus H'$ over X. Then the tensor product section $\alpha \otimes \alpha'$ over X is geometrically atomic.*

Theorem 5.6. *Let $\alpha : E \to F$ and $\alpha' : E' \to F'$ be smooth bundle maps over a manifold X with the property that (α, α') is a normal pair. The for any polynomial Φ as in Theorem 5.3, the characteristic form $\Phi(\Omega^{E\otimes E'})$ satisfies the following equation on X*

$$\Phi(\Omega^{E\otimes E'}) \;=\; \sum_{k=0}^{m}\sum_{k'=0}^{m'} \mathrm{Res}_{\Phi,kk'}[\Sigma_k(\alpha)\cap\Sigma'_{k'}(\alpha')] + dT_\Phi$$

where T_Φ is a canonical L^1_{loc}-form on X and $\mathrm{Res}_{\Phi,kk'}$ *is a smooth residue form on* $\Sigma_k(\alpha)\cap\Sigma'_{k'}(\alpha')$ *given as in Theorem 5.3.*

Similarly for any polynomial Ψ as in 5.4, the characteristic form $\Psi(\Omega^{F\otimes F'})$ satisfies the equation

$$\Psi(\Omega^{F\otimes F'}) \;=\; \sum_{k=0}^{m}\sum_{k'=0}^{m'} \mathrm{Res}_{\Psi,kk'}[\Sigma_k(\alpha)\cap\Sigma'_{k'}(\alpha')] + dT_\Psi$$

where T_Ψ is a canonical L^1_{loc}-form on X and $\mathrm{Res}_{\Psi,kk'}$ *is a smooth residue form on* $\Sigma_k(\alpha)\cap\Sigma'_{k'}(\alpha')$ *given by Theorem 5.4.*

Remark 5.7. The analogue of Theorem 3.7 holds also in this context (cf. Remark 4.6).

There are many interesting special cases. Note for example the simple case where we take the product $f\alpha$ where f is a regular function on X. Another interesting case is the tensor product $\sigma \otimes \sigma'$ of two sections $\sigma \in \Gamma(F)$ and $\sigma' \in \Gamma(F')$. We examine some of these in the next two sections.

§6. The Direct Sum of Cross Sections. The formulas for direct sum and tensor product mappings are particularly interesting in the case of sections. Consider cross-sections

$$\sigma : \mathbf{C} \longrightarrow F \qquad \text{and} \qquad \sigma' : \mathbf{C} \longrightarrow F'$$

where $\mathbf{C} = \mathbb{C} \times X$ denotes the **trivialized** line bundle, and suppose (σ, σ') is a normal pair (cf. Def. 4.3). Under this hypothesis each of the sections σ, σ' and $\sigma \oplus \sigma'$ has a smooth divisor. Furthermore, $\mathrm{Div}(\sigma)$ and $\mathrm{Div}(\sigma')$ meet transversely and $\mathrm{Div}(\sigma \oplus \sigma') = \mathrm{Div}(\sigma) \cap \mathrm{Div}(\sigma')$.

Now let $\mathbb{L}$ be a multiplicative series of characteristic polynomials in the sense of Hirzebruch [H]. Then Theorem 4.5 and Remark 4.6 give us:

$$\begin{aligned} \mathbb{L}(\Omega^{F\oplus F'}) \;&=\; \mathbb{L}(\Omega^{\sigma^\perp})\mathbb{L}(\Omega^{\sigma'^\perp}) + \mathrm{Res}_{\mathbb{L},\sigma}\, \mathbb{L}(\Omega^{\sigma'^\perp})\mathrm{Div}(\sigma) + \mathrm{Res}_{\mathbb{L},\sigma'}\, \mathbb{L}(\Omega^{\sigma^\perp})\mathrm{Div}(\sigma') \\ &\qquad + \mathrm{Res}_{\mathbb{L},\sigma\oplus\sigma'}\, \mathrm{Div}(\sigma) \cap \mathrm{Div}(\sigma') + dT_{\mathbb{L}} \end{aligned}$$

with

$$\mathrm{Res}_{\mathbb{L},\sigma} \;=\; \pi_* \mathbb{L}(\Omega^{U^\perp})$$

where U is the tautological line bundle over $\pi : \mathbb{P}(\mathbf{C} \oplus F) \to X$, and the other residues are defined similarly.

For example suppose $\mathbb{L} = c$, the total Chern class. Then

$$c(\Omega^{F\oplus F'}) \;=\; c(\Omega^{\sigma^\perp})c(\Omega^{\sigma'^\perp}) + c(\Omega^{\sigma'^\perp})\mathrm{Div}(\sigma) + c(\Omega^{\sigma^\perp})\mathrm{Div}(\sigma') + \mathrm{Div}(\sigma) \cap \mathrm{Div}(\sigma') + dT_c$$

Taking the top component of this equation gives the classical formula

$$c_{n+n'}(\Omega^{F\oplus F'}) \;=\; \mathrm{Div}(\sigma) \cap \mathrm{Div}(\sigma') + dT'$$

§7. The Tensor Product of Cross Sections. Let

$$\sigma : \mathbf{C} \longrightarrow F \qquad \text{and} \qquad \sigma' : \mathbf{C} \longrightarrow F'$$

be sections as in §6 and consider the tensor product section

$$\sigma \otimes \sigma' : \mathbf{C} \longrightarrow F \otimes F'$$

Section 5 gives the following equation of currents on the total space of $\mathbb{P}(\mathbf{C} \oplus (F \otimes F'))$:

$$\begin{aligned} \partial T \;&=\; [\mathbf{C}] - [\mathrm{Im}\sigma \otimes \mathrm{Im}\sigma'] \\ &\qquad - \left[\mathbb{P}\left\{\mathbf{C} \oplus (F \otimes \mathrm{Im}\sigma')\right\}\big|_{\mathrm{Div}(\sigma)}\right] - \left[\mathbb{P}\left\{\mathbf{C} \oplus (\mathrm{Im}\sigma \otimes F')\right\}\big|_{\mathrm{Div}(\sigma')}\right] \end{aligned}$$

where $[\mathbb{C}]$ denotes the embedding of X as the 0-section, and $[\mathrm{Im}\sigma \otimes \mathrm{Im}\sigma']$ is the submanifold given by graphing the section $[\sigma \otimes \sigma']$ of $\mathbb{P}(F \otimes F')$ over $X - \mathrm{Div}(\sigma) \cup \mathrm{Div}(\sigma')$.

If U denotes the tautological bundle over $\mathbb{P}(\mathbb{C} \oplus (F \otimes F'))$, one calculates that

$$\begin{aligned} U^{\perp}\big|_{[\mathbb{C}]} &= F \otimes F' \\ U^{\perp}\big|_{\mathrm{Im}\sigma \otimes \mathrm{Im}\sigma'} &= \mathbb{C} \oplus (\mathrm{Im}\sigma \otimes \mathrm{Im}\sigma')^{\perp} \qquad \text{and} \\ U^{\perp}\big|_{\mathbb{P}\{\mathbb{C} \oplus (F \otimes \mathrm{Im}\sigma')\}} &= U_1^{\perp} \oplus (F \otimes \mathrm{Im}(\sigma')^{\perp}) \end{aligned}$$

where $U_1^{\perp}$ is the Whitney dual of the tautological line bundle U_1 over the projective bundle

$$\pi : \mathbb{P}\{\mathbb{C} \oplus (F \otimes \mathrm{Im}\sigma')\} \longrightarrow \mathrm{Div}(\sigma).$$

Suppose now that Ψ is an invariant polynomial for the group $\mathrm{GL}_{nn'}$ where $n = \mathrm{dim} F$ and $n' = \mathrm{dim} F'$. Then we obtain the following formula on X:

$$\boxed{\Psi(\Omega^{F \otimes F'}) \;=\; \Psi(\Omega^{\mathbb{C} \oplus (\mathrm{Im}\sigma \otimes \mathrm{Im}\sigma')^{\perp}}) + \mathrm{Res}_{\sigma}\, \mathrm{Div}(\sigma) + \mathrm{Res}_{\sigma'}\, \mathrm{Div}(\sigma') + dT}$$

where

$$\mathrm{Res}_{\sigma} = \pi_* \Psi(\Omega^{U_1^{\perp} \oplus (F \otimes \mathrm{Im}(\sigma')^{\perp})})$$

and $\mathrm{Res}_{\sigma'}$ is defined similarly. In particular if $\Psi = \mathbb{L}$ is a multiplicative series in the sense of Hirzebruch [H], then

$$\mathbb{L}(\Omega^{F \otimes F'}) \;=\; \mathbb{L}(\Omega^{\{\sigma \otimes \sigma'\}^{\perp}}) + \mathrm{Res}'_{\sigma}\, \mathbb{L}(\Omega^{F \otimes \sigma'^{\perp}})\mathrm{Div}(\sigma) + \mathrm{Res}'_{\sigma'}\, \mathbb{L}(\Omega^{\sigma^{\perp} \otimes F'})\mathrm{Div}(\sigma') + dT$$

with

$$\mathrm{Res}'_{\sigma} \;=\; \pi_* \mathbb{L}(\Omega^{U_1^{\perp}}) \qquad \text{and} \qquad \mathrm{Res}'_{\sigma'} \;=\; \pi_* \mathbb{L}(\Omega^{U'^{\perp}_1})$$

For the top Chern class one gets the formula

$$c_{nn'}(\Omega^{F \otimes F'}) \;=\; c_{n(n'-1)}(\Omega^{F \otimes \sigma'^{\perp}})\mathrm{Div}(\sigma) + c_{n'(n-1)}(\Omega^{\sigma^{\perp} \otimes F'})\mathrm{Div}(\sigma') + dT$$

generalizing the classical line bundle case. Another interesting case is the Chern character:

$$\mathrm{ch}(\Omega^{F})\mathrm{ch}(\Omega^{F'}) \;=\; \mathrm{ch}(\Omega^{(\sigma \otimes \sigma')^{\perp}}) + \tfrac{(-1)^{n+1}}{n!}\mathrm{Div}(\sigma) + \tfrac{(-1)^{n'+1}}{n'!}\mathrm{Div}(\sigma') + dT$$

Note that these formulas are interesting in the simple case where σ' is just a complex-valued function f (a section of the trivial line bundle). Note that $\mathrm{Zero}(f\sigma) = \mathrm{Zero}(f) \cup \mathrm{Zero}(\sigma)$ will generically have a component of codimension-2.

§8. Mappings Given by Clifford Multiplication. The methods presented here apply directly to quaternion-linear maps $\alpha : E \to F$ between quaternionic bundles. More generally one can consider bundle mappings given by Clifford multiplication $e : S^{+}(V) \to S^{-}(V)$

where $S^{\pm}(V)$ are spinor bundles for a vector bundle V with spin structure, and where e is a cross section of V. This case is related to the Differentiable Riemann-Roch Theorem for embeddings and is covered extensively in [HL$_2$].

§9. **The General Method.** Each of the previous sections is simply an application of the following general method which applies to nearly any geometrically atomic bundle map $\alpha : E \to F$.

Step 1. Compute

$$\partial T_\alpha \ - [0] \ = \ P \ = \ \sum_i n_i [P_i]$$

where the P_i are oriented ν-dimensional manifolds of finite volume in G and the n_i are integers.

Step 2. For each i compute

$$U\big|_{P_i} \qquad \text{and/or} \qquad U^\perp\big|_{P_i}$$

Step 3. Given invariant polynomials Φ or Ψ as above, compute the current

$$\pi_* \left\{\Phi\left(\Omega^U\right) \wedge P_i\right\} \qquad \text{or} \qquad \pi_* \left\{\Psi\left(\Omega^{U^\perp}\right) \wedge P_i\right\}$$

for each i, where $\pi : G \longrightarrow X$ is the bundle projection. (In doing this use the fact that the tangent bundle to the fibres of π is $\mathrm{Hom}(U, U^\perp)$ and that $U \oplus U^\perp = \pi^*(E \oplus F)$.)

The Result. There are formulas on X of the form

$$\Phi(\Omega^E) \ = \ \sum_i \pi_* \left\{\Phi\left(\Omega^U\right) \wedge P_i\right\} + dT_\Phi \qquad \text{and} \qquad \Psi(\Omega^F) \ = \ \sum_i \pi_* \left\{\Psi\left(\Omega^{U^\perp}\right) \wedge P_i\right\} + dT_\Psi$$

Important Note. The first step is the crucial one and is often done as follows.

1. Compute the topological boundary $\partial_{\rm top} T_\alpha^0$ of the radial span of α.

2. Suppose $\partial_{\rm top} T_\alpha^0 = \bigcup_i P_i$ where P_i is a smooth orientable submanifold of dimension ν for $i > 0$ and P_0 is a set of Hausdorff ν-measure 0. Then by the Federer Flat Support Theorem [F, 4.1.15] we know that

$$\partial T_\alpha \ = \ \sum_{i>0} n_i [P_i]$$

for integers n_i and choices of orientations on the P_i. Of course one of the $[P_i]$ will be the zero-section [0].

3. Determine the integers n_i by local calculation at some convenient point of P_i.

Example 9.1. There are important bundle maps which do not lie in the generic classes of §§3-8. A good example, related to "excess intersection" theory [Fu], arises when $\sigma : \mathbb{C} \to F$ is a section of F whose zero set Z is of larger than expected dimension. Assume for simplicity that σ is normally non-degenerate, i.e., that the differential $d\sigma$ gives an isomorphism $d\sigma : N \cong F_0 \subset F$ of the normal bundle of Z with a sub-bundle of F along Z. Then direct calculation shows that

$$\partial T_\sigma - [0] \;=\; Q + \mathbb{P}(\mathbb{C} \oplus F_0)$$

where

$$Q \;=\; \{[\sigma(x)] \in \mathbb{P}(F_0) : x \in X - D\} \;\subset\; \mathbb{P}(\mathbb{C} \oplus F).$$

Write $F\big|_Z = F_0 \oplus F_1$ and let $n_k = \mathrm{dim} F_k$ with $n = n_0 + n_1$. Let U be the tautological line bundle on $\mathbb{P}(\mathbb{C} \oplus F)$ and $U^\perp$ its complementary n-plane bundle. Let $U_0^\perp$ be the corresponding object on $\mathbb{P}(\mathbb{C} \oplus F_0)$. Then one sees that

$$(9.1) \qquad U^\perp\big|_{\mathbb{P}(\mathbb{C} \oplus F_0)} \;=\; U_0^\perp \oplus F_1 \qquad \text{and} \qquad U^\perp\big|_Q \;=\; \mathbb{C} \oplus \sigma^\perp \oplus F_1$$

where $\sigma^\perp = F_0 \ominus \mathrm{Im}(\sigma)$. Let $c_n(\Omega^{U^\perp}) = \det(\Omega^{U^\perp})$ denote the top Chern form of $U^\perp$. Then from (9.1) we see that

$$c_n\left(\Omega^{U^\perp}\right)\big|_{\mathbb{P}(\mathbb{C} \oplus F_0)} \;=\; c_{n_0}\left(\Omega^{U_0^\perp}\right) \cdot c_{n_1}(\Omega^{F_1}) \qquad \text{and} \qquad c_n\left(\Omega^{U^\perp}\right)\big|_Q = 0.$$

From the fact that $\pi_*\, c_{n_0}\left(\Omega^{U_0^\perp}\right) = 1$ we obtain the equation

$$(9.2) \qquad \boxed{c_n(\Omega^F) \;=\; c_{n_1}(\Omega^{F_1}) \cdot [Z] + dT}$$

for the top Chern form of F on X.

Example 9.2. Consider $\sigma : \mathbb{C}^2 \to \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \cong \mathbb{C}^2$ given by $\sigma(z_1, z_2) = (z_1^2 z_2,\, z_1 z_2^2) = z_1 z_2 (z_1, z_2)$. Note that $\sigma = f\sigma_0$ where $\sigma_0(z) = z$ and f is the scalar function $f(z) = z_1 z_2$. Both σ_0 and f are atomic sections with $\mathrm{Div}(\sigma_0) = \{0\}$ and $\mathrm{Div}(f) = \{z_1\text{−axis}\} \cup \{z_2\text{−axis}\}$. However, these divisors do not meet in general position since they intersect in 0, and the results of §7 do not apply. Nevertheless one can compute ∂T_σ. In fact, setting $\overline{\mathbb{C}^n} = \mathbb{P}(\mathbb{C} \oplus \mathbb{C}^n)$, we have

$$T_\sigma \;=\; \{(z, \tfrac{1}{t} f(z) z) \in \mathbb{C}^2 \times \overline{\mathbb{C}^2} \;:\; 0 < t < \infty \;\text{ and }\; z \in \mathbb{C}^2\},$$

and one finds straightforwardly that

$$\partial T_\sigma - [0] \;=\; (0 \times \mathbb{C}) \times (\overline{\mathbb{C} \times 0}) \;+\; (\mathbb{C} \times 0) \times (\overline{0 \times \mathbb{C}}) \;+\; (0, 0) \times \overline{\mathbb{C}^2}.$$

Notice the contribution at $(0, 0)$ which does not appear in the case of normal pairs.

§**10. Non-linear Residue Integrals.** One might suspect that the boundary components P_i appearing in our general method are always linear (i.e., projective space) bundles over a subset of X. It is true that the fibres are always compactifications of homogeneous cones in H. However these cones may be non-trivial, as seen in the following.

Example 10.1 An example of a boundary component of the graph of a geometrically atomic bundle map which is not a vector bundle over $\Sigma_k(\alpha)$ but a bundle of homogeneous cones.

Consider the bundle mapping $f : \mathbb{C}^{k+2} \to \mathbb{C}^{k+2}$ over $\mathbb{C}^{n+2}$ given as follows

$$f(z, w, \zeta_1, ..., \zeta_n) \;=\; \begin{pmatrix} z^2 & zw & 0 \\ 0 & w^2 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

The ζ's play no essential role but are there to emphasize that the *generic* such map can have non-empty degeneracy sets $\Sigma_\ell(f)$ for $\ell = 1, 2$. (Recall that for normal maps the codimension of Σ_ℓ is ℓ^2.)

Note that

$$\begin{aligned} \Sigma_1(f) \;&=\; \{z = 0\} \cup \{w = 0\} - \{z = w = 0\}, \qquad \text{and} \\ \Sigma_2(f) \;&=\; \{z = w = 0\} \end{aligned}$$

Lemma 10.2. *Let T_f denote the radial span of f. The piece of the boundary ∂T_f which lies above $\Sigma_2(f)$ is a fibre bundle $\pi : B \to \Sigma_2(f)$ whose fibre above any point is the set of planes in the Grassmannian $G_{k+2}(\mathbb{C}^{k+2} \oplus \mathbb{C}^{k+2})$ of the form:*

$$\textit{graph} \begin{pmatrix} z^2 & zw \\ 0 & w^2 \end{pmatrix} \oplus \mathbb{C}^k \qquad\qquad \textit{for } (z, w) \in \mathbb{C}^2$$

where $\mathbb{C}^k$ =Im(Id) is in the second factor of $\mathbb{C}^{k+2} \oplus \mathbb{C}^{k+2}$.

Remark 10.3. Note that the above graph in $\mathbb{C}^2 \oplus \mathbb{C}^2$ is the 2-plane

$$\Gamma_{z,w} = \{(u, v, z^2u + zwv, w^2v) : u, v \in \mathbb{C}\}.$$

The union of these does not constitute a linear subspace of Hom$(\mathbb{C}^2, \mathbb{C}^2)$. The Plücker coordinates are

$$\begin{aligned} \Gamma_{z,w} &= (e_1 + z^2e_3) \wedge (e_2 + zwe_3 + w^2e_4) \\ &= e_1 \wedge e_2 + zwe_1 \wedge e_3 + w^2e_1 \wedge e_4 - z^2e_2 \wedge e_3 + z^2w^2e_3 \wedge e_4 \end{aligned}$$

which would be a **quadratic** curve if the planes could be put into a linear family.

Proof of Lemma 10.2. We have the radial span

$$T \;=\; T_f \;=\; \{(x, \tfrac{1}{t}f(x)) : x \in \mathbb{C}^{n+2} \textit{and } 0 < t < \infty\}.$$

We shall work at $x = 0$. (Other points of Σ_2 will be the same.) We have the short exact sequence

$$0 \to K \to \mathbb{C}^{k+2} \xrightarrow{f_0} I \to 0$$

where $I \equiv \mathrm{Im} f_0 \cong \mathbb{C}^k$ and $K \equiv \ker f_0 \cong \mathbb{C}^2$. The contribution to ∂T above $0 \in \Sigma_2$ will be supported in the subset

$$\mathrm{Hom}(K, I^\perp) \oplus (\{0\} \oplus I)$$

which in our case turns out to be

$$\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2) \oplus (\{0\} \oplus \mathbb{C}^k)$$

To see which points we get we examine the limits of the planes: $\mathrm{graph}(\frac{1}{t}f(x))$ as $t, x \to 0$. Write the Taylor expansion

$$f(0+h) = f_0 + df_0(h) + d^2 f_0(h) + d^3 f_0(h) + \dots$$

and observe that

$$\mathrm{graph}(\tfrac{1}{t}f(h)) \;=\; \{(v, \tfrac{1}{t}(f_0 + df_0(h) + d^2 f_0(h) + d^3 f_0(h) + \dots)(v)) \,:\, v \in \mathbb{C}^{k+2}\}$$

We now use the short exact sequence above to write

$$\mathbb{C}^{k+2} = K \oplus I$$

and express the graph as

$$\mathrm{graph}(\tfrac{1}{t}f(h)) \;=\; \{(k, i; \tfrac{1}{t}(f_0 + df_0(h) + d^2 f_0(h) + \dots)(k, i)) \,:\, (k, i) \in K \oplus I\}$$

On points of the form $(0, i)$ we have

$$\{(0, i;\, \tfrac{1}{t}(i + o(|h|\|i\|)) : i \in I\} \;=\; \{(0,\, ti\,;\, i + o(|h|)) : i \in I\}$$

and taking $t \to 0, |h| \to 0$ we get the subspace of points $\{0\} \oplus I \subset \{0\} \oplus \mathbb{C}^{k+2}$.

Now restrict to points of the form $(k, 0)$. Let $\widetilde{f}$ denote the restriction of f to K, and suppose that $d^\ell \widetilde{f}_0$ is the first non-vanishing term in the Taylor expansion of $\widetilde{f}$. (In our case $\ell = 2$.) Then setting $h = t^{1/\ell} h_0$

$$\begin{aligned}
\mathrm{graph}(\tfrac{1}{t}f(h)) &\supset \{(k, 0; \tfrac{1}{t}(df_0(h) + d^2 f_0(h) + \dots)(k, 0)) \,:\, (k, 0) \in K\} \\
&= \{(k,\, \tfrac{1}{t}(d^\ell \widetilde{f}_0(t^{1/\ell} h_0) + \dots)(k)) \,:\, k \in K\} \\
&= \{(k,\, (d^\ell \widetilde{f}_0(h_0) + o(t^{1/\ell}))(k)) \,:\, k \in K\} \\
&\xrightarrow[t \to 0]{} \mathrm{graph}\left\{ d^\ell \left(f\big|_K \right)(h_0) \right\}.
\end{aligned}$$

Thus if

$$P(h_0) \;\equiv\; \lim_{t \to 0}\, \mathrm{graph}\left\{ \tfrac{1}{t} f(t^{1/\ell} h_0) \right\},$$

then

$$P(h_0) \supset \{0\} \oplus I \qquad \text{and} \qquad P(h_0) \supset \text{ graph}\left\{d^\ell \left(f\big|_K\right)_0 (h_0)\right\}.$$

These two spaces are transversal, and so we conclude that

$$P(h_0) \ = \ (\{0\} \oplus I) \oplus \text{graph}\left\{d^\ell \left(f\big|_K\right)_0 (h_0)\right\}$$

Thus, if we let N_0 denote the normal space to $\Sigma_0(f)$ at 0, then the support of ∂T above 0 is contained in the set of planes $P(h_0)$ for $h_0 \in N_0$ i.e., compressing notation, we have

$$\text{supp}(\partial T) \cap \pi^{-1}(0) \ \subseteq \ \bigcup_{v \in N_0} \left(\text{gr}\left\{d^\ell \left(f\big|_K\right)_0 (v)\right\} + I\right) \tag{10.1}$$

In our case $\ell = 2$ and $d^2 \left(f\big|_K\right)_0 = f\big|_K$. ▮

Equation (10.1) is quite general and characterizes the fibre of the allowed support of ∂T above a degeneracy set.

§11. Final Remark - Toric Varieties.

The general methods introduced here seem well suited to the study of toric varieties. These are complex n-manifolds with an action of $G_n \equiv \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ (n-times) having a dense orbit. The generators of the action give a bundle map $\alpha : \mathbb{C}^n \longrightarrow TX$ which is geometrically atomic but rarely normal. However, our methods can be used to localize characteristic classes of X on the singular orbits of the action. Explicit calculations of the residues can be carried out as in §§9 and 10.

Among the simplest examples are the projective spaces $\mathbb{P}^n$ with an action of $G_n = G_{n+1}/D$, where D is the main diagonal subgroup, given by

$$\varphi_{t_0,\dots,t_n}([z_0 : \cdots : z_n]) \ = \ [e^{t_0} z_0 : \cdots : e^{t_n} z_n].$$

This gives a bundle map $\mathbb{C}^{n+1}/\mathbb{C} \cdot (1, 1, \dots, 1) \cong \mathbb{C}^n \longrightarrow T\mathbb{P}^n$ defined by

$$(t_0, \dots, t_n) \ \mapsto \ \pi_* \left\{ \sum_{k=0}^n t_k z_k \frac{\partial}{\partial z_k} \right\}$$

where $\pi : \mathbb{C}^{n+1} - \{0\} \ \to \ \mathbb{P}^n$ is the projection. (Note that $\pi_*(\sum z_k \partial/\partial z_k) = 0$). Now in each of the affine coordinate charts $U_k = \{[z] \in \mathbb{P}^n : z_k \neq 0\}$ the bundle map α can be written as a *diagonal* bundle mapping $\alpha_1 \oplus \cdots \oplus \alpha_n : \mathbb{C}^n \to \mathbb{C}^n$. To see this it suffices by symmetry to consider only the case $U_0 \cong \{[1 : z_1 : \cdots : z_n] : z \in \mathbb{C}^n\}$. For this we map $\mathbb{C}^n \to \mathbb{C}^{n+1}/\mathbb{C} \to T\mathbb{P}^n$ by

$$(t_1, ..., t_0) \ \mapsto \ \sum_{k=1}^n t_k z_k \partial/\partial z_k \cong (t_1 z_1, ..., t_n z_n).$$

(We can drop the π_* since these vector fields are tangent to the affine chart U_0.) Now the calculation of the residues is local, and in this chart we have the map written explicitly as a direct sum. Hence we can apply §4 to this case. In particular for the total Chern class we can apply Example 4.7 to compute the residues, and we find the following. Let $\Delta_k \equiv \{[z] \in \mathbb{P}^n : z_k = 0\}$ denote the k^{th} coodinate hyperplane, and set $\Delta_I = \Delta_{i_1} \cap \cdots \cap \Delta_{i_p}$ for $I = \{i_1 < \cdots < i_p\}$. The configuration $\{\Delta_I\}_I$ of linear subspaces gives $\mathbb{P}^n$ the combinatorial pattern of the n-simplex. Let $Sk_p = \cup\{\Delta_I : |i| = p\}$ denote its "codimension-p skeleton". Then applying 4.7 in each coordinate chart shows that for every integer p,

$$c_p(\mathbb{P}^n) \ = \ [Sk_p] + dT_p$$

for a canonical L^1_{loc}-form T_p. Our techniques apply similarly to more complicated and interesting toric varieties.

Appendix A. Normal Pairs Are Generic.

Lemma A.1. *Let $\pi : H \to X$ be a smooth vector bundle over a compact manifold, and suppose that $\Sigma \subset H$ is a submanifold such that the restriction $\pi : \Sigma \to X$ is a smooth fibration over X. Then the set of cross-sections $\alpha \in \Gamma(H)$ which are transversal to Σ is open and dense in the C^1-topology.*

Proof. Let $\alpha_1 \in \Gamma(H)$ be given and choose sections $\alpha_2, \ldots, \alpha_m$ such that

$$\text{span}\{\alpha_1(x), \ldots, \alpha_m(x)\} = H_x \qquad \text{for all } x \in X.$$

This gives a smooth bundle surjection $A : \mathbb{R}^m \times X \longrightarrow H$ defined by $A(t,x) = \sum t_i \alpha_i(x)$. From the Transversality Theorem for Families (cf. [HL$_1$]) we conclude that since A is transversal to Σ, the section $A_t(x) = A(t,x)$ is transversal to Σ for almost all $t \in \mathbb{R}^m$. This proves the density. The openness is clear. ∎

Lemma A.2. *Let $\pi : H \to X$ and $\Sigma \subset H$ be as above, and suppose that $\pi : H' \to X$, $\Sigma' \subset H$ is another such set-up over X. Then the set of pairs $(\alpha, \alpha') \in \Gamma(H) \times \Gamma(H')$ such that $\alpha \oplus \alpha'$ is transversal to $\Sigma \oplus \Sigma'$ in $H \oplus H'$, is open and dense in the C^1-topology.*

Proof. Given $(\alpha_1, \alpha'_1) \in \Gamma(H) \times \Gamma(H')$, choose (α_i, α'_i) for $i = 2, ..., m$ so that $\alpha_1, ..., \alpha_m$ and $\alpha'_1, ..., \alpha'_m$ are spanning set of sections as above. The argument now proceeds as in the proof of A.1. ∎

Corollary A.3. *Normal pairs are open and dense in the C^1-topology. If the manifold X is non-compact, then normal pairs are residual (a countable intersection of open dense subsets) in the C^1-topology.*

Proposition A.4. *For any normal pair (α, α'), the direct sum mapping $\alpha \oplus \alpha'$ and the tensor product mapping $\alpha \otimes \alpha'$ are geometrically atomic.*

Proof. Fix a point $p \in \Sigma_k(\alpha) \cap \Sigma'_{k'}(\alpha')$. By transversality there exist local coordinates $(x, a, a') \in \mathbb{R}^r \times \text{Hom}(\text{ker}a_p, \text{coker}a_p) \times \text{Hom}(\text{ker}a_p, \text{coker}a_p)$ for a neighborhood of p such that in these coordinates

$$\Sigma_k(\alpha) \cong \{a = 0\} \qquad \text{and} \qquad \Sigma'_{k'}(\alpha') \cong \{a' = 0\}$$

and for $\ell > k$ and $\ell' > k'$ we have

$$\text{(A.1)} \qquad \Sigma_\ell(\alpha) \cong \{\dim \ker a = \ell\} \qquad \text{and} \qquad \Sigma'_{\ell'}(\alpha') \cong \{\dim \ker a' = \ell'\}$$

Furthermore, in these coordinates the pair of bundle maps can be written

$$\text{(A.2)} \qquad (\alpha, \alpha') = (a \oplus b, a' \oplus b')$$

where $b = b(x, a, a')$ is an invertible $m \times m$-matrix at each point and similarly for b'. (Compare [HL$_5$, §10]). From (A.1) one sees easily that the submanifolds $\Sigma_\ell(\alpha) \cap \Sigma'_{\ell'}(\alpha')$ all have locally finite volume in X. Using (A.2) one concludes that along these submanifolds the finite volume property of the radial span of either $\alpha \oplus \alpha'$ or $\alpha \otimes \alpha'$ reduces to that of the universal case. ∎

Appendix B. The operator $\widetilde{\otimes}$. Consider vector spaces V, V', W, W' and the obvious projection

$$\mathrm{pr} : (V \oplus W) \otimes (V' \oplus W') \longrightarrow (V \otimes V') \oplus (W \otimes W').$$

Definition B.1. Given subspaces $S \subset V \oplus W$ and $S' \subset V' \oplus W'$ we define

$$S \,\widetilde{\otimes}\, S' \equiv \mathrm{pr}\,\{S \otimes S'\}.$$

Given a linear map $\alpha : V \to W$, we let $\mathrm{gr}(\alpha)$ denote its graph in $V \oplus W$.

Lemma B.2. *The operation $\widetilde{\otimes}$ has the following properties.*

(i) $\mathrm{gr}(\alpha \otimes \alpha') = \mathrm{gr}(\alpha) \,\widetilde{\otimes}\, \mathrm{gr}(\alpha')$ *for linear maps* $\alpha : V \to W$ *and* $\alpha' : V' \to W'$

(ii) $\dim(S \,\widetilde{\otimes}\, S') = \dim(S)\dim(S')$ *if* $S \cap V = S \cap W = \{0\}$.

(iii) If $S' \subset W'$, *then* $S \,\widetilde{\otimes}\, S' \subset W \otimes W'$.

Furthermore, if the spaces have inner products and $(\bullet)^*$ *denotes the adjoint, then*

(iv) $\mathrm{gr}(\alpha \otimes \alpha')^\perp = \mathrm{gr}(\alpha^*) \,\widetilde{\otimes}\, \mathrm{gr}(\alpha'^*)$ *for linear maps* $\alpha : V \to W$ *and* $\alpha' : V' \to W'$

Proof. Straightforward.

Corollary B.3. *Let $a : V \to W$ be an injective linear map. Then taking the projected tensor product with the graph of a gives a well-defined algebraic map*

$$G_{m'}(V' \oplus W') \xrightarrow{\mathrm{gr}(a)\,\widetilde{\otimes}\,\bullet} G_{mm'}((V \otimes V') \oplus (W \otimes W'))$$

where $m = \dim(V)$ and $m' = \dim(V')$. Furthermore, under this mapping one has that $G_{m'}(V') \to G_{mm'}(V \otimes V')$ and $G_{m'}(W') \to G_{mm'}(W \otimes W')$.

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