PLURISUBHARMONICITY IN A GENERAL GEOMETRIC CONTEXT

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ABSTRACT

Recently the authors have explored new concepts of plurisubharmonicity and pseudoconvexity, with much of the attendant analysis, in the context of calibrated manifolds. Here a much broader extension is made. This development covers a wide variety of geometric situations, including, for example, a notion of Lagrangian plurisubhamonicity and a notion of Lagrangian convexity. It also applies in a number of non-geometric situations. Results include: fundamental properties of $\mathcal{P}^+$-plurisubharmonic functions, plurisubharmonic distributions and regularity, $\mathcal{P}^+$-convex domains and $\mathcal{P}^+$-convex boundaries, topological restrictions on and construction of $\mathcal{P}^+$-convex domains, and a self-contained proof of existence and uniqueness of solutions to the Dirichlet problem for $\mathcal{P}^+$-harmonic functions. This last result includes classical results on the homogeneous Monge-Ampère equations in the real, complex and quaternionic cases.

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TABLE OF CONTENTS

1. Introduction.
2. Geometrically Defined Plurisubharmonic Functions.
3. More General PSH-Functions Defined by an Elliptic Cone $\mathcal{P}^+$.
4. $\mathcal{P}^+$-Plurisubharmonic Distributions.
5. Upper-Semi-Continuous $\mathcal{P}^+$-Plurisubharmonic Functions.
6. Some Classical Facts that Extend to $\mathcal{P}^+$-Plurisubharmonic Functions.
7. The Dirichlet Problem – Uniqueness.
8. The Dirichlet Problem – Existence.
9. $\mathcal{P}^+$-Convex Domains.
10. Topological Restrictions on $\mathcal{P}^+$-Convex Domains.
11. $\mathcal{P}_+$-Free Submanifolds.
12. $\mathcal{P}^+$-Convex Boundaries.

Appendix A: The Maximum Principle and Subaffine Functions.
Appendix B: Hessians of Plurisubharmonic Distributions.
Appendix C: Convex Elliptic Sets in $\mathrm{Sym}^2(\mathbf{R}^n)$.
Appendix D: The Dirichlet Problem for Convex Elliptic Sets.
Appendix E: Elliptic MA-operators / Gårding-Hyperbolic Polynomials on $\mathrm{Sym}^2(\mathbf{R}^n)$.

1. Introduction.

Recently the authors have shown that the concepts of plurisubharmonicity and pseudoconvexity from complex analysis carry over, along with many of the basic results, to other geometries, including calibrated and symplectic geometry. In this paper the same ideas and results are extended to a broad geometric context. The core concept is that of an *elliptic cone*. This is a closed convex cone $\mathcal{P}^+$ in the space $\mathrm{Sym}^2(\mathbf{R}^n)$ of symmetric $n \times n$-matrices, with the property that the relative interior of its polar dual $\mathcal{P}_+$ consists of positive definite matrices.

A function u of class C^2 on an open set $X \subset \mathbf{R}^n$ is defined to be $\mathcal{P}^+$-*plurisubharmonic* if $\mathrm{Hess}_x u \in \mathcal{P}^+$ at every point x.

Basic geometric examples are constructed as follows. Fix an integer p, $1 \leq p \leq n$, and denote by $G(p, \mathbf{R}^n)$ the grassmannian of p-planes in $\mathbf{R}^n$. Embed

$$G(p, \mathbf{R}^n) \ \subset \ \mathrm{Sym}^2(\mathbf{R}^n)$$

by associating to each p-plane ξ, the orthogonal projection $P_\xi : \mathbf{R}^n \to \xi \subset \mathbf{R}^n$. Now let $\mathbb{G} \subset G(p, \mathbf{R}^n)$ be any compact subset, and define $\mathcal{P}_+(\mathbb{G})$ (note the *lower* plus) to be the closed convex cone in $\mathrm{Sym}^2(\mathbf{R}^n)$ generated by $\mathbb{G}$. Then a function $u \in C^2(X)$ is $\mathcal{P}^+(\mathbb{G})$-plurisubharmonic (sometimes shortened to $\mathbb{G}$-*plurisubharmonic*) if and only if

$$\mathrm{tr}_\xi\{\mathrm{Hess}_x u\} \ \geq \ 0 \qquad \forall\, x \in X \ \text{ and } \ \forall\, \xi \in \mathbb{G}$$

where $\mathrm{tr}_\xi A \equiv \langle A, P_\xi \rangle$ denotes the trace of A on the p-plane ξ.

Important examples of this type are where $\mathbb{G} = \mathbb{G}(\phi)$ consists of the p-planes associated to a calibration ϕ of degree p (such as the Kähler, Quaternionic, Special Lagrangian, Associative, Coassociative or Cayley calibrations). Other interesting cases are where $\mathbb{G}$ is the set of all Lagrangian n-planes in $\mathbf{C}^n$, or where $\mathbb{G} = G(p, \mathbf{R}^n)$ is the full grassmannian.

This geometric case has the following interesting feature. A function $u \in C^2(X)$ is $\mathcal{P}^+(\mathbb{G})$-plurisubharmonic if and only if its restriction to every minimal $\mathbb{G}$-submanifold of X is subharmonic in the induced metric. (A $\mathbb{G}$-*submanifold* is a p-dimensional submanifold of X all of whose tangent planes lie in $\mathbb{G}$.)

Of course the concept of an elliptic cone $\mathcal{P}^+$ is much broader than the geometric case. Nevertheless, a surprising bulk of classical pluripotential theory carries over to this context. The notion of $\mathcal{P}^+$-plurisubharmonicity extends from C^2-functions to distributions, and every such distribution is actually in L^1_{loc} and has a unique upper semi-continuous representative with values in $[-\infty, \infty)$. The set $\mathrm{PSH}(X)$ of such functions has all the classical properties. For example, if $u, v \in \mathrm{PSH}(X)$, then $\max\{u, v\} \in \mathrm{PSH}(X)$. Also, $\mathrm{PSH}(X)$ is closed under decreasing limits and uniform limits. An important fact is that if $\mathcal{F} \subset \mathrm{PSH}(X)$ is a family which is locally bounded above, then (the upper semicontinuous regularization) of $\sup_{v \in \mathcal{F}} v$ is in $\mathrm{PSH}(X)$. This enables one to apply the Perron process.

There is a notion of $\mathcal{P}^+$-convexity generalizing the concept of pseudo-convexity in complex analysis. Given a compact set $K \subset X$, we define its $\mathcal{P}^+$-convex hull to be the set $\widehat{K}$ of points x with

$$u(x) \ \leq \ \sup_K u \qquad \text{for all smooth } u \in \mathrm{PSH}(X).$$

Then X is said to be $\mathcal{P}^+$-*convex* if for all $K \subset\subset X$ we have $\widehat{K} \subset\subset X$. It is proved that X *is* $\mathcal{P}^+$*-convex if and only if* X *admits a strictly* $\mathcal{P}^+$*-plurisubharmonic exhaustion function.*

Given a compact domain $\Omega \subset X$ with smooth boundary $\partial\Omega$, there is also a notion of $\mathcal{P}^+$-convexity (and strict $\mathcal{P}^+$-convexity) of the boundary. It is shown that if $\partial\Omega$ is strictly $\mathcal{P}^+$-convex, then Ω itself is $\mathcal{P}^+$-convex.

There is also a concept which generalizes the notion from complex geometry of being totally real. In §10 we introduce the notion of a linear subspace $V \subset \mathbf{R}^n$ which is $\mathcal{P}_+$-free. In the geometric case this means that V contains no $\mathbb{G}$-planes, that is, there are no $\xi \in \mathbb{G}$ with $\xi \subset V$. Then the *free dimension of* $\mathcal{P}_+$, denoted $\mathrm{fd}(\mathcal{P}_+)$, is defined to be the largest dimension of a $\mathcal{P}_+$-free subspace of $\mathbf{R}^n$, and we have the following generalization of the Andreotti-Frankel Theorem.

THEOREM 10.5. *Any* $\mathcal{P}^+$*-convex domain has the homotopy type of a CW-complex of dimension* $\leq \mathrm{fd}(\mathcal{P}_+)$.

The integer $\mathrm{fd}(\mathcal{P}_+)$ is often easily computable, particularly in the geometric cases. See §10 for examples.

A submanifold is said to be $\mathcal{P}^+$*-free* if all of its tangent planes are $\mathcal{P}_+$-free. This extends the notion of totally real submanifolds in complex geometry. In geometric cases any submanifold of dimension $\leq p$ is free. Generic submanifolds of dimension $\leq \mathrm{fd}(\mathcal{P}_+)$ are $\mathcal{P}_+$-free on an open dense subset. Therefore, examples of $\mathcal{P}_+$-free submanifolds are easy to construct. This leads to lots of $\mathcal{P}^+$-convex domains via the following analogue of the Grauert Tubular Neighborhood Theorem.

THEOREM 11.4. *Suppose* M *is a* $\mathcal{P}_+$*-free closed submanifold of* $X \subset \mathbf{R}^n$. *Then there exists a fundamental neighborhood system* $\mathcal{F}(M)$ *of* M *consisting of* $\mathcal{P}^+$*-convex domains. Moreover,*

a) M *is a deformation retract of each* $U \in \mathcal{F}(M)$.

b) Each compact subset $K \subset M$ *satisfies* $K = \widehat{K}_U$ *for all* $U \in \mathcal{F}(M)$.

The methods used in [$\mathrm{HW}_{1,2}$] to generalize the Grauert Theorem extend to prove this very general result.

Freeness of submanifolds and convexity of their tubular neighborhoods are related by the following fact. Let M be a closed submanifold of an open subset $X \subset \mathbf{R}^n$. Then M *is* $\mathcal{P}_+$*-free if and only if the square of the distance to* M *is strictly* $\mathcal{P}^+$*plurisubharmonic at each point of* M (and hence in a neighborhood of M). More generally we have the following result.

THEOREM 11.3. *Consider the two classes of closed sets.*

1) Closed subsets $Z \subset M$ *of a* $\mathcal{P}_+$*-free submanifold* $M \subset X$.

2) Zero sets $Z = \{f = 0\}$ *of non-negative strictly* $\mathcal{P}^+$*-plurisubharmonic functions* f.

Locally these two classes are the same.

An important feature of this paper is the proof of existence and uniqueness of solutions to the Dirichlet Problem for functions which are $\mathcal{P}^+$*-harmonic*. For functions $u \in C^2(X)$ this means that $\mathrm{Hess}_x u \in \partial\mathcal{P}^+$ for all $x \in X$. More generally for $u \in \mathrm{PSH}(X)$ this notion

is defined via a duality involving the subaffine functions, which are discussed in Appendix A. The main results are the following.

THEOREM 8.1. (The Dirichlet Problem – Existence). *Suppose Ω is a bounded domain in $\mathbf{R}^n$ with a strictly $\mathcal{P}^+$-convex boundary. Given $\varphi \in C(\partial\Omega)$, the function u on $\overline{\Omega}$ defined by taking the upper envelope:*

$$u(x) \;=\; \sup\{v(x) : v \in \mathcal{P}^+(\varphi)\} \qquad\qquad \text{where}$$

$$\mathcal{P}^+(\varphi) \;\equiv \{v \;:\; v \in \mathrm{USC}(\overline{\Omega}),\;\; v\big|_{\Omega} \in \mathrm{PSH}(\Omega) \;\text{ and }\; v\big|_{\partial\Omega} \leq \varphi\} \tag{8.1}$$

satisfies:

1) $u \in C(\overline{\Omega})$,

2) *u is $\mathcal{P}^+$-harmonic on Ω,*

3) $u\big|_{\partial\Omega} = \varphi$ *on* $\partial\Omega$.

THEOREM 7.1. (The Dirichlet Problem–Uniqueness). *Suppose $\mathcal{P}^+$ is an elliptic cone and that K is a compact subset of $\mathbf{R}^n$. If $u_1, u_2 \in C(K)$ are $\mathcal{P}^+$-harmonic on* Int$K$, *then*

$$u_1 \;=\; u_2 \;\text{ on } \partial K \qquad \Rightarrow \qquad u_1 \;=\; u_2 \;\text{ on } K$$

A rich source of examples outside the geometric case $\mathcal{P}^+(\mathbb{G})$ comes from the theory of Gårding hyperbolic polynomials. A homogeneous polynomial M of degree m on $\mathrm{Sym}^2(\mathbf{R}^n)$ is *hyperbolic with respect to the identity* I, if for each $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ the polynomial $p_A(t) = M(tI + A)$ has m real roots. Fundamental examples are the elementary symmetric functions $\sigma_k(A)$ in the eigenvalues of A. For any such polynomial with $M(I) > 0$, the connected component of the complement of $\{M = 0\}$ containing I is a convex cone $\mathcal{P}^+$, and our theory applies. This is discussed in detail in Appendix E.

Many of the results in this paper have been subsequently generalized by the authors. For example, in [HL$_4$] Theorems 10.5, 7.1 and 8.1 above have been established for fully non-linear, degenerate elliptic equations which are purely of second order. The main analytic tool used in [HL$_4$] is Slodkowski's deep result on the largest Hessian eigenvalue of convex functions [S]. The arguments also make use of subaffine functions and a certain duality intrinsic to these second order problems. Subaffine functions are introduced here in Appendix A and play an important role in the proof of the Uniqueness Theorem 7.1 above. We note that [HL$_4$] treats the Dirichlet Problem in highly non-convex cases, such as *all other branches* of the real, complex and quaternionic Monge-Ampere equations and all branches of the special Lagrangian potential equation.

The results of [HL$_4$] have subsequently been extended in two significant ways in [HL$_5$]. Here one works over a general riemannian manifold X, and the subequations considered are closed subsets $F \subset J^2(X)$ of the 2-jet bundle of functions on X (which depends locally on all the classical variables $(x, r, p, A) \in X \times \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n)$). The main analysis rests on the powerful Viscosity approach pioneered by Crandall, Ishii, Lions, Evans, Jensen and

others (cf. [CIL], [C]). Many of the results discussed here and in [HL_4] carry over to this setting. In particular, if F is a constant coefficient subequation on $\mathbf{R}^n$ which is invariant under a subgroup $G \subset \mathrm{O}_n$, then for domains in any riemannian manifold with a *topological* reduction of the structure group to G, there are existence and uniqueness results for the Dirichlet Problem. For example, in an almost complex manifold with a hermitian metric one can treat all branches of the homogeneous complex Monge-Ampère equation.

While these latter papers largely subsume the results here, we feel that this article has valuable features. The exposition is less technical. The cases covered here include many of basic geometric interest. Finally, since the basic sets $\mathcal{P}^+$ are convex cones, we are able to use convolution (for smooth approximation) and classical distribution theory. This makes the analytic part of the paper more widely accessible.

Conventions:

1. Throughout this paper X shall denote a connected open subset of $\mathbf{R}^n$. We note that almost all of the analysis done here carries over to much more general riemannian manifolds X.

2. Whenever $C \subset V$ is a convex cone in a finite dimensional vector space V we shall denote by $\mathrm{Int}C$ the interior of C in the vector subspace $W = \mathrm{span}\, C$.

2. Geometrically Defined Plurisubharmonic Functions

In this section we discuss a notion of plurisubharmonicity, for C^∞-functions, based on a distinguished subset $\mathbb{G}$ of the grassmannian. We shall begin with some definitions and notation. Let $G(p, \mathbf{R}^n)$ denote the grassmannian of unoriented p-planes through the origin in $\mathbf{R}^n$. Let $\mathrm{Sym}^2(\mathbf{R}^n)$ denote the vector space of quadratic forms (functions) on $\mathbf{R}^n$. We identify $G(p, \mathbf{R}^n)$ with a subset of $\mathrm{Sym}^2(\mathbf{R}^n)$ by associating to each $\xi \in G(p, \mathbf{R}^n)$ the quadratic form P_ξ corresponding to orthogonal projection of $\mathbf{R}^n$ onto ξ. The natural inner product on $\mathrm{Sym}^2(\mathbf{R}^n)$ is given by the trace: $\langle A, B\rangle = \mathrm{tr}AB$. Let $\mathcal{P}$ denote the set of non-negative quadratic forms, $A \geq 0$. This is a closed convex cone with vertex at the origin in $\mathrm{Sym}^2(\mathbf{R}^n)$. The interior, $\mathrm{Int}\mathcal{P}$, consists of the positive definite quadratic forms, $A > 0$. The extreme rays in $\mathcal{P}$ are generated by the rank-1 projections $G(1, \mathbf{R}^n)$.

The *polar* of a closed convex cone $\mathcal{C}$ with vertex at the origin is defined by

$$\mathcal{C}^0 \equiv \mathrm{polar}\,\mathcal{C} \equiv \{A : \langle A, B\rangle \geq 0 \ \text{ for all } \ B \in \mathcal{C}\}. \tag{2.1}$$

The Bipolar Theorem states that $(\mathcal{C}^0)^0 = \mathcal{C}$. Note that the convex cone $\mathcal{P}$ is self-polar, that is $\mathcal{P}^0 = \mathcal{P}$, since $A \geq 0$ if and only if $\langle A, P_\xi\rangle \geq 0$ for all $\xi \in G(1, \mathbf{R}^n)$. (If $x \in \mathbf{R}^n$ is a unit vector and ξ is the line through x, then $\langle A, P_\xi\rangle = \langle Ax, x\rangle$.)

Given $\xi \in G(p, \mathbf{R}^n)$ and $A \in \mathrm{Sym}^2(\mathbf{R}^n)$, the *$\xi$-trace of A*, defined by

$$\mathrm{tr}_\xi A = \langle A, P_\xi\rangle = \mathrm{tr}\left(A\big|_\xi\right), \tag{2.2}$$

is central to our development.

Given a function $u \in C^\infty(X)$, its hessian at a point $x \in X$ will be denoted by $\mathrm{Hess}_x\, u$. This is a quadratic form on $\mathbf{R}^n$, i.e., $\mathrm{Hess}_x\, u \in \mathrm{Sym}^2(\mathbf{R}^n)$.

Definition 2.1. Suppose $\mathbb{G}$ is a non-empty closed subset of $G(p, \mathbf{R}^n)$. A function $u \in C^\infty(X)$ is called *$\mathbb{G}$-plurisubharmonic* if

$$\mathrm{tr}_\xi\,(\mathrm{Hess}_x\, u) \geq 0 \qquad \text{for each } \xi \in \mathbb{G}, x \in X. \tag{2.3}$$

Let $\mathrm{PSH}^\infty(X, \mathbb{G})$ denote this space of $\mathbb{G}$-plurisubharmonic functions.

Suppose W is an affine p-plane through x with tangent space $TW = \xi$. Then

$$\mathrm{tr}_\xi\,(\mathrm{Hess}_x\, u) = \mathrm{tr}\left(\mathrm{Hess}_x\, u\big|_\xi\right) = \mathrm{tr}\left(\mathrm{Hess}_x\, u\big|_W\right) = \Delta\left(\, u\big|_W\right). \tag{2.4}$$

Call W an *affine $\mathbb{G}$-plane* if $TW = \xi \in \mathbb{G}$. This proves the following.

Lemma 2.2. *A function $u \in C^\infty(X)$ is $\mathbb{G}$-plurisubharmonic if and only if*

$$\Delta\left(\, u\big|_{X\cap W}\right) \geq 0 \qquad \textit{for each affine } \mathbb{G}\textit{−plane } W. \tag{2.3$'$}$$

That is:

A function $u \in \mathrm{PSH}^\infty(X, \mathbb{G})$ iff its restriction to every affine $\mathbb{G}$-plane W is subharmonic.

A submanifold M of $\mathbf{R}^n$ is a $\mathbb{G}$-*submanifold* if $T_xM \in \mathbb{G}$ at each point $x \in M$.

THEOREM 2.3. *Suppose $M \subset X$ is a $\mathbb{G}$-submanifold which is minimal. For each $u \in C^\infty(X)$ which is $\mathbb{G}$-plurisubharmonic, the restriction of u to M is subharmonic in the induced riemannian metric on M.*

Proof. Recall the classical fact (cf. §1 in [HL$_2$]) that if $u \in C^\infty(X)$, then for a minimal submanifold M, the Laplace Beltrami operator of M is given at $x \in M$ by

$$\Delta_M \left(u\big|_M\right) \;=\; \mathrm{tr}\left\{\mathrm{Hess}_x\, u\big|_{T_xM}\right\} \;=\; \mathrm{tr}_{T_xM}\left\{\;\mathrm{Hess}_x\, u\big|_{T_xM}\right\}. \qquad \blacksquare$$

$\mathbb{G}$-Pluriharmonic and $\mathbb{G}$-Harmonic Functions. In tandem with the concept of $\mathbb{G}$-plurisubharmonicity it is natural to define a function $u \in C^\infty(X)$ to be $\mathbb{G}$-*pluriharmonic* if

$$\mathrm{tr}_\xi \mathrm{Hess}_x u \;=\; 0 \qquad \text{for each } \xi \in \mathbb{G} \text{ and each } x \in X. \tag{2.5}$$

That is, u is $\mathbb{G}$-pluriharmonic if and only if the restriction of u to each affine $\mathbb{G}$-plane is harmonic. As in the proof of Theorem 2.3, if M is a $\mathbb{G}$-submanifold which is minimal and u is $\mathbb{G}$-pluriharmonic, then $u\big|_M$ is harmonic in the induced riemannian metric on M. Unfortunately, with rare exceptions, the space of $\mathbb{G}$-pluriharmonic functions is small (finite dimensional). See the examples below.

A weakening of the definition of $\mathbb{G}$-pluriharmonicity provides a much larger class.

DEFINITION 2.4. Suppose $u \in C^\infty(X)$. Then

1) u is called $\mathbb{G}$-*harmonic* if $u \in \mathrm{PSH}^\infty(X, \mathbb{G})$ and for each $x \in X$, the trace

$$\mathrm{tr}_\xi \mathrm{Hess}_x u \;=\; 0 \qquad \text{for some } \xi \in \mathbb{G}.$$

2) u is called *strictly* $\mathbb{G}$-*plurisubharmonic* if for each $x \in X$,

$$\mathrm{tr}_\xi \mathrm{Hess}_x u \;>\; 0 \qquad \text{for all } \xi \in \mathbb{G}.$$

Examples. There are many geometrically interesting cases of $\mathbb{G}$-plurisubharmonic functions to which our general theory will apply. This wealth of examples is one of the important features of this paper.

A rich source is the theory of calibrations [HL$_{1,2}$]. Let ϕ be a constant coefficient p-form on $\mathbf{R}^n$ with the property that $\phi(\xi) \le 1$ for all $\xi \in G(p, \mathbf{R}^n)$. Then we define the ϕ-grassmannian to be the set

$$\mathbb{G}(\phi) \;=\; \{\xi \in G(p, \mathbf{R}^n) : \phi(\xi) = 1\}$$

All the following examples but 1,3, and 13 can be constructed this way.

In sections 4 and 5 we shall generalize the notion of $\mathcal{P}^+$-plurisubharmonicity to distributions, and show that any such distribution is an upper semi-continuous function. For $\mathcal{P}^+ = \mathcal{P}^+(\mathbb{G})$ the space of such functions is denoted $\mathrm{PSH}(X, \mathbb{G})$, and we have that

$u \in \mathrm{PSH}(X, \mathbb{G})$ if and only if the restriction $u\big|_W$ to every affine $\mathbb{G}$-plane W is subharmonic (or $\equiv -\infty$). The following is a list of basic examples in the geometric case.

1. $\mathbb{G} = G(1, \mathbf{R}^n)$. $\mathrm{PSH}(X, \mathbb{G})$ is the set of convex functions on X.

2. $\mathbb{G} = G(n, \mathbf{R}^n) = \{I\}$ with $I \in \mathrm{Sym}^2(\mathbf{R}^n)$ the identity. $\mathrm{PSH}(X, \mathbb{G})$ is the set of subharmonic functions on X.

3. $\mathbb{G} = G(p, \mathbf{R}^n)$ for $1 < p < n$. $\mathrm{PSH}(X, \mathbb{G})$ is called the set of *real p-plurisubharmonic functions* on X. The defining property is that they are subharmonic on every affine p-plane.

4. $\mathbb{G} = \mathbf{P}^{n-1}(\mathbf{C}) = G_{\mathbf{C}}(1, \mathbf{C}^n) \subset G(2, \mathbf{R}^{2n})$ gives the set of standard plurisubharmonic functions in complex analysis.

5. $\mathbb{G} = \mathbf{P}^{n-1}(\mathbf{H}) = G_{\mathbf{H}}(1, \mathbf{H}^n) \subset G(4, \mathbf{R}^{4n})$ gives the set of quaternionic plurisubharmonic functions on quaternionic n-space $\mathbf{H}^n$ (cf. [Al], [AV]).

6. $\mathbb{G} = G_{\mathbf{C}}(p, \mathbf{C}^n)$ for $1 < p < n$ gives complex p-plurisubharmonic functions on $\mathbf{C}^n$. The defining property is that they are subharmonic on every complex affine p-plane.

7. $\mathbb{G} = G_{\mathbf{H}}(p, \mathbf{H}^n)$ for $1 < p < n$ gives quaternionic p-plurisubharmonic functions on $\mathbf{H}^n$. The defining property is that they are subharmonic on every quaternionic affine p-plane.

8. $\mathbb{G} = \{x_1\text{-axis}\} \subset G(1, \mathbf{R}^n)$ gives the *horizontally convex functions*, i.e., the functions which are convex in the variable x_1.

9. $\mathbb{G} = \mathrm{SLAG} \subset G(n, \mathbf{C}^n)$, the set of (unoriented) special Lagrangian n-planes in $\mathbf{C}^n$.

10. $\mathbb{G} = \mathrm{ASSOC} \subset G(3, \mathbf{R}^7)$, the set of (unoriented) associative 3-planes in $\mathrm{Im}\mathbf{O} \cong \mathbf{R}^7$, the imaginary octonions.

11. $\mathbb{G} = \mathrm{COASSOC} \subset G(4, \mathbf{R}^7)$, the set of (unoriented) coassociative 4-planes in $\mathrm{Im}\mathbf{O}$.

12. $\mathbb{G} = \mathrm{CAY} \subset G(4, \mathbf{R}^8)$, the set of (unoriented) Cayley 4-planes in the octonions $\mathbf{O} \cong \mathbf{R}^8$.

13. $\mathbb{G} = \mathrm{LAG} \subset G(n, \mathbf{C}^n)$, the set of Lagrangian n-planes in $\mathbf{C}^n$.

REMARK 2.5. As noted in the introduction, for expository reasons the discussion in this paper is confined to $\mathbf{R}^n$ with $\mathbb{G}$ parallel. However, all of the examples above can be carried over to general riemannian manifolds equipped with some additional structure. Note for example that 4,6 and 13 make sense on any symplectic manifold with a compatible riemannian metric. A quite general analysis on riemannian manifolds is carried out in [HL_5].

Elliptic Subsets $\mathbb{G}$ of the Grassmannian.

In this subsection a notion of ellipticity is discussed which puts a very natural restriction on the subsets $\mathbb{G} \subset G(p, \mathbf{R}^n)$.

Let $\mathcal{P}_+(\mathbb{G})$ denote the closed convex cone in $\mathrm{Sym}^2(\mathbf{R}^n)$, with vertex at the origin, determined by the compact set $\mathbb{G} \subset \mathrm{Sym}^2(\mathbf{R}^n)$. Let $\mathcal{P}^+(\mathbb{G})$ denote the polar of $\mathcal{P}_+(\mathbb{G})$. (Recall that the *polar* of a convex cone C with vertex at the origin in a vector space V is

the set $C^0 = \{v \in V^* : \langle v, w\rangle \geq 0 \ \forall\, w \in C\}$.) Note that since $\mathcal{P} = \mathcal{P}_+(G(1, \mathbf{R}^n))$ contains all the grassmannians $G(p, \mathbf{R}^n)$, we have

$$\mathcal{P}_+(\mathbb{G}) \subset \mathcal{P}, \qquad \text{and hence} \qquad \mathcal{P} \subset \mathcal{P}^+(\mathbb{G}),$$

for any $\mathbb{G} \subset G(p, \mathbf{R}^n)$. Set

$$S(\mathbb{G}) = \operatorname{span} \mathbb{G} = \operatorname{span} \mathcal{P}_+(\mathbb{G}).$$

As one can see from the examples, $S(\mathbb{G})$ is usually a proper vector subspace of $\mathrm{Sym}^2(\mathbf{R}^n)$, and, in particular, $\mathcal{P}_+(\mathbb{G})$ has no interior in $\mathrm{Sym}^2(\mathbf{R}^n)$. However, considered as a subset of $S(\mathbb{G})$, the interior of $\mathcal{P}_+(\mathbb{G})$ has closure equal to $\mathcal{P}_+(\mathbb{G})$. By $\mathrm{Int}\mathcal{P}_+(\mathbb{G})$ we shall always mean the interior of $\mathcal{P}_+(\mathbb{G})$ in $S(\mathbb{G})$, not in $\mathrm{Sym}^2(\mathbf{R}^n)$. In particular, $\mathrm{Int}\mathcal{P}_+(\mathbb{G})$ is never empty.

Definition 2.6. A closed subset $\mathbb{G} \subset G(p, \mathbf{R}^n)$ is *elliptic* if each $A \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$ is positive definite.

The following conditions on a closed subset $\mathbb{G} \subset G(p, \mathbf{R}^n)$ are equivalent.

1) Given $x \in \mathbf{R}^n$, if $x \lrcorner \xi = 0$ for all $\xi \in \mathbb{G}$, then $x = 0$.

2) For each unit vector $e \in \mathbf{R}^n$, P_e is never orthogonal to $S(\mathbb{G}) = \operatorname{span} \mathbb{G}$.

3) There does not exist a hyperplane $W \subset \mathbf{R}^n$ with $\mathbb{G} \subset \mathrm{Sym}^2(W) \subset \mathrm{Sym}^2(\mathbf{R}^n)$.

To see that 1) and 2) are equivalent, note that $\langle P_e, P_\xi\rangle = |e \lrcorner \xi|^2$. If $e \perp W$, then $\mathbb{G} \subset \mathrm{Sym}^2(W)$ if and only if $e \lrcorner \xi = 0$ for all $\xi \in \mathbb{G}$, so that 2) $\Leftrightarrow$ 3).

Definition 2.7. A closed subset $\mathbb{G} \subset G(p, \mathbf{R}^n)$ is said *to involve all the variables in* $\mathbf{R}^n$ if one of the equivalent conditions 1), 2), 3) holds.

Proposition 2.8. *Suppose $\mathbb{G}$ is a closed subset of $G(p, \mathbf{R}^n)$. Then $\mathbb{G}$ is elliptic if and only if $\mathbb{G}$ involves all of the variables in $\mathbf{R}^n$.*

Proof. Suppose there exists $A \in S(\mathbb{G})$ which is positive definite. Then $\mathbb{G}$ must involve all of the variables in $\mathbf{R}^n$ because otherwise by 3) $\mathbb{G}$, and so also $\mathcal{P}_+(\mathbb{G})$ and $S(\mathbb{G})$, would be contained in $\mathrm{Sym}^2(W)$ for some hyperplane W. This is a contradiction since $A \in \mathrm{Sym}^2(W)$ can never be positive definite.

If $\mathbb{G}$ involves all the variables in $\mathbf{R}^n$, then, by 2), we have the following. Under the orthogonal decomposition

$$P_e \;=\; E_e + S_e \qquad \text{with } S_e \in S(\mathbb{G}) \;\text{ and }\; E_e \perp S(G), \tag{2.6}$$

the component S_e is never zero. Now choose $A \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$. Since $S_e \in S(\mathbb{G})$, it follows that for small $\epsilon > 0$ we have $A - \epsilon S_e \in \mathrm{Int}\mathcal{P}_+(\mathbb{G}) \subset \mathcal{P}$. Therefore, $0 \leq \langle P_e, A - \epsilon S_e\rangle = \langle P_e, A\rangle - \epsilon |S_e|^2$ proving that $\langle P_e, A\rangle > 0$ for all unit $e \in \mathbf{R}^n$, i.e. $A > 0$. ∎

Remark . The proof also shows that each $A \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$ is positive definite if and only if $\exists\, A \in S(\mathbb{G})$ which is positive definite.

Each $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ determines a constant coefficient linear second-order operator $\langle \mathrm{Hess}\, u, A\rangle$. This linear operator is elliptic if and only if A is positive definite. In this case

$$\Delta_A u = \langle \mathrm{Hess}\, u, A\rangle$$

will be called the *A-Laplacian*.

DEFINITION 2.9. Suppose $\mathbb{G}$ is elliptic. Then for each $A \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$, the A-Laplacian Δ_A will be called a *mollifying Laplacian* for $\mathbb{G}$-plurisubharmonic functions.

MOLLIFYING LEMMA 2.10. *Suppose $\mathbb{G}$ is elliptic and $u \in C^\infty(X)$. Then u is $\mathbb{G}$-plurisubharmonic if and only if u is Δ_A-subharmonic for each mollifying Laplacian Δ_A.*

Proof. This follows from the fact that $\mathbb{G} \subset \mathcal{P}_+(\mathbb{G})$ and that $\mathcal{P}_+(\mathbb{G})$ is the closure of $\mathrm{Int}\mathcal{P}_+(\mathbb{G})$. ∎

3. More General Plurisubharmonic Functions Defined by an Elliptic Cone $\mathcal{P}^+$

The basic properties of geometrically defined plurisubharmonic functions remain valid in much greater generality. Suppose $\mathcal{P}^+$ is a closed convex cone in $\mathrm{Sym}^2(\mathbf{R}^n)$ with vertex at the origin. Let $\mathcal{P}_+$ denote the polar cone. Let $S(\mathcal{P}_+)$ denote the span of $\mathcal{P}_+$, and let $\mathrm{Int}\mathcal{P}_+$ denote the relative interior of $\mathcal{P}_+$ in the vector subspace $S(\mathcal{P}_+)$ of $\mathrm{Sym}^2(\mathbf{R}^n)$. The subspace $E(\mathcal{P}^+) = S(\mathcal{P}_+)^\perp \subset \mathcal{P}^+$ is called the *edge* of $\mathcal{P}^+$.

Definition 3.1.

1) $\mathcal{P}^+$ is a *positive cone* if each $A \in \mathcal{P}_+$ is positive, i.e. $A \geq 0$.
2) $\mathcal{P}^+$ is an *elliptic cone* if each $A \in \mathrm{Int}\mathcal{P}_+$ is positive definite.

Remark. Of course in the geometric case $\mathcal{P}_+ = \mathcal{P}_+(\mathbb{G})$, the *positivity condition* $\mathcal{P}_+ \subset \mathcal{P}$ is automatic.

If $\mathcal{P}^+$ is an elliptic cone (and, to a lesser extent, if $\mathcal{P}^+$ is a positive cone), it is reasonable to investigate $\mathcal{P}^+$-plurisubharmonic functions, even though they have no direct geometric interpretation.

Definition 3.2. A function $u \in C^\infty(X)$ is *$\mathcal{P}^+$-plurisubharmonic* if

$$\mathrm{Hess}_x u \in \mathcal{P}^+ \qquad \text{for each } x \in X. \tag{3.1}$$

Remark. If $\mathrm{Hess}_x u \in \partial\mathcal{P}^+$, then u is *$\mathcal{P}^+$-harmonic*. If $\mathrm{Hess}_x u \in \mathrm{Int}\mathcal{P}^+$, then u is *strictly $\mathcal{P}^+$-plurisubharmonic*. Finally, if $\mathrm{Hess}_x u \perp S(\mathcal{P}_+)$, then u is *$\mathcal{P}^+$-pluriharmonic*.

A main point is that the Mollifying Lemma remains valid.

Mollifying Lemma 3.3. *Suppose $\mathcal{P}^+$ is an elliptic cone and $u \in C^\infty(X)$. Then u is $\mathcal{P}^+$-plurisubharmonic if and only if u is Δ_A-subharmonic for each mollifying Laplacian.*

Remark 3.4. There is an analogue of (2.3)′. Let $\mathbb{G}$ denote the extreme points in the compact convex base $\mathcal{B}_+ = \mathcal{P}_+ \cap \{\mathrm{tr} = 1\}$. Then u is $\mathcal{P}^+$-plurisubharmonic at x if and only if

$$\langle \mathrm{Hess}_x\, u, A\rangle \;=\; \mathrm{tr}\,\{(\mathrm{Hess}_x\, u)A\} \;\geq 0 \qquad \forall A \in \mathbb{G}.$$

However, this is not particularly interesting or useful unless the set of extreme points of the base $\mathcal{B}_+ = \mathcal{P}_+ \cap \{\mathrm{tr} = 1\}$ are known. It is easy to see in the geometric case where $\mathbb{G} \subset G(p, \mathbf{R}^n)$ and $\mathcal{P}_+ = \mathcal{P}_+(\mathbb{G})$, that the set of extreme points of $\mathcal{P}_+ \cap \{\mathrm{tr} = p\}$ is exactly $\mathbb{G}$.

Reformulating Ellipticity for a Convex Cone $\mathcal{P}^+$.

The positivity and ellipticity conditions have useful reformulations. First, the

Positivity Condition for $\mathcal{P}_+$: $\mathcal{P}_+ \subset \mathcal{P}$, that is, every $A \in \mathcal{P}_+$ is ≥ 0

can be stated in the equivalent dual form since $\mathcal{P}$ is self-polar.

Positivity Condition for $\mathcal{P}^+$: $\mathcal{P} \subset \mathcal{P}^+$, that is, every $A \geq 0$ belongs to $\mathcal{P}^+$.

In terms of functions u, this says that each convex function is $\mathcal{P}^+$-plurisubharmonic .

As noted, there are many cases where $S(\mathcal{P}_+) = \operatorname{span}\mathcal{P}_+$ is not all of $\operatorname{Sym}^2(\mathbf{R}^n)$. However, there is a different kind of incompleteness that should be ruled out. We say that $\mathcal{P}_+$ *only involves the variables in a proper subspace* $W \subset \mathbf{R}^n$ if

$$\mathcal{P}_+ \ \subset \ \operatorname{Sym}^2(W). \tag{3.2}$$

Otherwise we say that $\mathcal{P}_+$ *involves all the variables in* $\mathbf{R}^n$.

Completeness Condition for $\mathcal{P}_+$: The convex cone $\mathcal{P}_+$ involves all the variables in $\mathbf{R}^n$.

This can be expressed in dual form as follows. Suppose e is a unit vector and W is the orthogonal hyperplane in $\mathbf{R}^n$. We say that $\mathcal{P}^+$ *can be defined using the variables in* W if

$$\mathcal{P}^+ \ = \ \operatorname{Sym}^2(W)^\perp \oplus \left(\mathcal{P}^+ \cap \operatorname{Sym}^2(W)\right). \tag{3.2$'$}$$

It is easy to see that (3.2) and (3.2)$'$ are equivalent.

Completeness Condition for $\mathcal{P}^+$: The convex cone $\mathcal{P}^+$ can not be defined using the variables in a proper subspace $W \subset \mathbf{R}^n$.

PROPOSITION 3.5. The convex cone $\mathcal{P}^+$ is elliptic if and only if the positivity condition and the completeness condition are both satisfied.

Proof. First note that if $\mathcal{P}^+$ is elliptic, then $\mathcal{P}_+ = \overline{\operatorname{Int}\mathcal{P}_+} \subset \mathcal{P}$, i.e., the Positivity Condition is satisfied. The Completeness Condition must also be satisfied, since $\mathcal{P}_+ \subset \operatorname{Sym}^2(W)$ excludes the possibility of $\mathcal{P}_+$ containing $A > 0$.

The following fact is basic to our discussion.

If the Positivity Condition $\mathcal{P}_+ \subset \mathcal{P}$ is satisfied, then for each $A \in \mathcal{P}_+$ and $W = e^\perp$:

$$\langle A, P_e\rangle \ = \ A(e,e) = 0, \ \text{ if and only if } \ A \in \operatorname{Sym}^2(W) \subset \operatorname{Sym}^2(\mathbf{R}^n) \tag{3.3}$$

This follows because if $A \geq 0$ and $A(e,e) = 0$, then $0 \leq A(te+u, te+u) = 2tA(e,u)+A(u,u)$ for all $t \in \mathbf{R}$ and all $u \in W = e^\perp$. Hence, $A(e,u) = 0$ for all $u \in W$, i.e., $A \in \operatorname{Sym}^2(W)$.

If $\mathcal{P}_+$ involves all the variables in $\mathbf{R}^n$and the Positivity Condition $\mathcal{P}_+ \subset \mathcal{P}$ is satisfied, then because of (3.3), $\langle A, P_e\rangle$ cannot vanish for all $A \in \mathcal{P}_+$, i.e., S_e is never zero. (Recall the decomposition (2.6).) This forces $A \in \operatorname{Int}\mathcal{P}_+$ to be positive definite exactly as in the last paragraph of the proof of Proposition 2.8. ∎

Assuming positivity there is is a simple test for completeness, which yields a useful test for ellipticity only in terms of $\mathcal{P}^+$ (not $\mathcal{P}_+$).

PROPOSITION 3.6. *The convex cone* $\mathcal{P}^+$ *is elliptic if and only if* $P_e \in \mathcal{P}^+$ *but* $-P_e \notin \mathcal{P}^+$ *for each unit vector* $e \in \mathbf{R}^n$.

Proof. Consider $E(\mathcal{P}^+) = S(\mathcal{P}_+)^\perp$, the edge of $\mathcal{P}^+$. Note that $\mathcal{P}^+ \cap S(\mathcal{P}_+)$ is the polar of $\mathcal{P}_+$ in the vector space $S(\mathcal{P}_+)$. It is easy to see that:

$$\mathcal{P}^+ \ = \ (\mathcal{P}^+ \cap S(\mathcal{P}_+)) \oplus E(\mathcal{P}^+).$$

Since the P_e's generate the extreme rays in $\mathcal{P}$, positivity for $\mathcal{P}^+$ is equivalent to $P_e \in \mathcal{P}^+$ for all e. Assuming positivity, we have that (3.3) above is true and says that:

$$\exists\, P_e \in E(\mathcal{P}^+) \quad \Longleftrightarrow \quad \text{Completeness for } \mathcal{P}_+ \text{ fails.}$$

Finally note that $P_e \in E(\mathcal{P}^+)$ if and only if both $\pm P_e \in \mathcal{P}^+$. ∎

REMARK . The edge $E(\mathcal{P}^+)$ can also be described, without using the inner product, as $\{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A + \mathcal{P}^+ = \mathcal{P}^+\}$.

Smoothing Maxima. As we shall discuss, many of the facts concerning classical subharmonic functions on $\mathbf{R}^n$ extend, once we have a suitable definition of (non-smooth) $\mathcal{P}^+$-plurisubharmonic functions. However, limiting the discussion to smooth $\mathcal{P}^+$-plurisubharmonic functions, there are still several interesting facts that extend. One of the most basic is smoothing the maximum of two $\mathcal{P}^+$-plurisubharmonic functions. Let $M(t) \equiv \max\{t_1, ..., t_m\}$ for $t \in \mathbf{R}^m$. Suppose $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbf{R}^m)$, $\varphi \geq 0$, $\int \varphi = 1$, with $\varphi(-t) = \varphi(t)$ and $\mathrm{spt}\,\varphi \subset \{t : |t| \leq 1\}$.

Since M is a convex function, $M_\epsilon = M * \varphi_\epsilon$ is convex and decreases to M as $\epsilon \to 0$. Also, $\sum_{j=1}^m \frac{\partial M}{\partial t_j} = 1$ implies $\sum_{j=1}^m \frac{\partial M_\epsilon}{\partial t_j} = 1$, or equivalently, $M(t + \lambda e) = M(t) + \lambda$ implies $M_\epsilon(t + \lambda e) = M_\epsilon(t) + \lambda$, where $e = (1, ..., 1)$. Moreover, $M_\epsilon(t) - \epsilon \leq M(t) \leq M_\epsilon(t)$. Finally note that $\frac{\partial M}{\partial t_j} \geq 0$ implies that $\frac{\partial M_\epsilon}{\partial t_j} \geq 0$.

Given $u^1, ..., u^m \in C^\infty(X)$ and Ψ a smooth function of m variables, the chain rule implies that

$$\mathrm{Hess}\Psi(u^1, ..., u^m) \;=\; \sum_{j=1}^m \frac{\partial \Psi}{\partial t_j} \mathrm{Hess} u^j + \sum_{i,j=1}^m \frac{\partial^2 \Psi}{\partial t_i \partial t_j} \nabla u^i \circ \nabla u^j \tag{3.4}$$

Maxima Property: Suppose $\mathcal{P}^+$ is a positive cone (not necessarily elliptic). Given $u^1, ..., u^m \in \mathrm{PSH}^\infty(X)$, one has that:

1) $M_\epsilon(u^1, ..., u^m) \in \mathrm{PSH}^\infty(X)$,
2) $M_\epsilon(u^1, ..., u^m) - \epsilon \leq M(u^1, ..., u^m) \leq M_\epsilon(u^1, ..., u^m)$,
3) $M_\epsilon(u^1, ..., u^m)$ decreases to $M(u^1, ..., u^m)$ as $\epsilon \to 0$.

Proof. Properties 2) and 3) are properties of $M(t)$ and $M_\epsilon(t)$. To prove 1) consider a more general function Ψ and apply (3.3). The value of the quadratic form $B \equiv \sum \frac{\partial^2 \Psi}{\partial t_i \partial t_j} \nabla u^i \circ \nabla u^j$ on $\xi = (\xi_1, ..., \xi_n) \in \mathbf{R}^n$ is $\sum \frac{\partial^2 \Psi}{\partial t_i \partial t_j} \langle \nabla u^i, \xi \rangle \langle \nabla u^j, \xi \rangle$, which is ≥ 0 if Ψ is convex. If each $\frac{\partial \Psi}{\partial t_j} \geq 0$ and $\sum_j \frac{\partial \Psi}{\partial t_j} = 1$, then the quadratic form $A = \sum_j \frac{\partial \Psi}{\partial t_j} \mathrm{Hess} u^j$ is a convex combination of the quadratic forms $\mathrm{Hess} u^j$, $j = 1, ..., m$. These assumptions are valid for $\Psi = M_\epsilon$. The convexity of $\mathcal{P}^+$ and the positivity condition imply that $\mathrm{Hess}\Psi(u^1, ..., u^m) = A + B \in \mathcal{P}^+$, which proves 1). ∎

EXERCISE 3.6. Suppose $\psi' \geq 0$ and $\psi'' \geq 0$. Show that $u \in \mathrm{PSH}^\infty(X)$ implies $\psi(u) \in \mathrm{PSH}^\infty(X)$.

4. $\mathcal{P}^+$-Plurisubharmonic Distributions.

Throughout this section we assume that $\mathcal{P}^+ \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is an elliptic cone (with vertex at the origin).

DEFINITION 4.1. A distribution $u \in \mathcal{D}'(X)$ is *$\mathcal{P}^+$-plurisubharmonic* if

$$(\mathrm{Hess}\, u)(\varphi A) \geq 0 \quad \text{for all} \quad A \in \mathcal{P}_+ \tag{4.1}$$

and all test functions $\varphi \in C^\infty_{\mathrm{cpt}}(X)$ with $\varphi \geq 0$,

It is easy to see Definition 4.1 is compatible with Definition 2.1 for $u \in C^\infty(X) \subset \mathcal{D}'(X)$.

NOTE . Let $\mathrm{PSH}^{\mathrm{dist}}(X)$ denote the set of $u \in \mathcal{D}'(X)$ which are $\mathcal{P}^+$-plurisubharmonic distributions on X. Obviously $\mathrm{PSH}^{\mathrm{dist}}(X)$ is a closed convex cone in $\mathcal{D}'(X)$.

The condition (4.1) for distributional $\mathcal{P}^+$-plurisubharmonicity can be modified as follows. The test function φ can be eliminated since we have

$$(\mathrm{Hess}\, u)(\varphi A) \;=\; (\langle \mathrm{Hess}\, u, A\rangle)(\varphi).$$

where $\mathrm{Hess}\, u$ is a symmetric matrix with entries in $\mathcal{D}'(X)$. Therefore, for a given $A \in \mathrm{Sym}^2(\mathbf{R}^n)$, condition (4.1) is equivalent to the statement that

$$\Delta_A u \;=\; \langle \mathrm{Hess}\, u, A\rangle \;\geq\; 0 \quad \text{(i.e., } \langle \mathrm{Hess}\, u, A\rangle \text{ is a nonnegative measure.)} \tag{4.1}'$$

The Mollifying Lemma 3.3 extends to distributions.

MOLLIFYING LEMMA 4.2. *Suppose $\mathcal{P}^+$ is an elliptic cone. A distribution $u \in \mathcal{D}'(X)$ is $\mathcal{P}^+$-plurisubharmonic if and only if u is a Δ_A-subharmonic distribution for each mollifying Laplacian Δ_A (i.e., each $A \in \mathrm{Int}\mathcal{P}^+$).*

Proof. This is essentially the equivalence of (4.1) and (4.1)′. Also note that each $A \in \mathcal{P}_+$ can be approximated by elements in $\mathrm{Int}\mathcal{P}_+$. ∎

$\mathbb{G}$-Plurisubharmonic Distributions. Assume that $\mathcal{P}^+ = \mathcal{P}^+(\mathbb{G})$ is geometrically defined by an elliptic subset $\mathbb{G}$ of the grassmannian $G(p, \mathbf{R}^n)$. For each $\xi \in \mathbb{G}$, consider the degenerate Laplacian defined by $A = P_\xi$, i.e.,

$$\Delta_\xi u \;=\; \langle \mathrm{Hess}\, u, P_\xi\rangle \tag{4.2}$$

Equation (2.3)′ has an extension from $u \in C^\infty(X)$ to $u \in \mathcal{D}'(X)$.

PROPOSITION 4.3. *Suppose $u \in \mathcal{D}'(X)$. Then $u \in \mathrm{PSH}(X)$ if and only if*

$$\Delta_\xi u \;\geq\; 0 \qquad \textit{for all } \xi \in \mathbb{G}. \tag{4.3}$$

Proof. Each $A \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$ is a finite positive linear combination of projections P_ξ with $\xi \in \mathbb{G}$. Hence, (4.3) implies that $\Delta_A u \geq 0$ for each $A \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$, so that $u \in \mathrm{PSH}^{\mathrm{dist}}(X)$

by the mollifying Lemma 4.2. Conversely, if $u \in \mathrm{PSH}^{\mathrm{dist}}(X)$, then $\Delta_A u \geq 0$ for each $A \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$. If $\xi \in \mathbb{G}$ and $t > 0$, then for $A' \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$, one has $A = P_\xi + tA' \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$ since $\frac{1}{t}P_\xi + A' \in \mathrm{Int}\mathcal{P}_+(\mathbb{G})$ for t large. Hence, $\Delta_\xi u + t\Delta_{A'} u \geq 0$ for all $t > 0$, which proves that $\Delta_\xi u \geq 0$ if $\xi \in \mathbb{G}$. ∎

Many of the classical facts about subharmonic distributions immediately carry over to $\mathcal{P}^+$-plurisubharmonic distributions because of the Mollifying Lemma 4.2. We list these classical facts in §6.

5. Upper-Semi-Continuous $\mathcal{P}^+$-Plurisubharmonic Functions.

Throughout this section we assume that $\mathcal{P}^+ \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is an elliptic cone (with vertex at the origin). Let $\mathrm{USC}(X)$ denote the space of $[-\infty, \infty)$-valued function on X which are upper-semi-continuous, and not $\equiv -\infty$ on any component of X.

Definition 5.1. A function $u \in L^1_{\mathrm{loc}}(X)$ is called L^1_{loc}-*upper-semi-continuous* if the essential limit superior

$$\widetilde{u}(x) \;=\; \operatorname{ess\,lim\,sup}_{y \to x} u(y) \tag{5.1}$$

satisfies the conditions:

(i) $\widetilde{u} \in \mathrm{USC}(X)$, and
(ii) $\widetilde{u}$ lies in the $L^1_{\mathrm{loc}}(X)$-equivalence class of u.

Proposition 5.2. *Each $\mathcal{P}^+$-plurisubharmonic distribution u is L^1_{loc}-upper-semi-continuous.*

The proof of the proposition will be given below.

Definition 5.3. If $u \in \mathrm{PSH}^{\mathrm{dist}}(X)$, the associated canonical representative $\widetilde{u} \in \mathrm{USC}(X)$ is said to be an *upper-semi-continuous $\mathcal{P}^+$-plurisubharmonic function.* Let $\mathrm{PSH}^{\mathrm{u.s.c.}}(X)$ denote the space of upper-semi-continuous $\mathcal{P}^+$-plurisubharmonic functions on X.

Corollary 5.4. *The map sending $u \in \mathrm{PSH}^{\mathrm{dist}}(X)$ to $\widetilde{u} \in \mathrm{PSH}^{\mathrm{u.s.c.}}(X)$ is an isomorphism.*

Proof. The map is surjective by definition, and injectivity is obvious. ∎

We shall denote these equivalent spaces $\mathrm{PSH}^{\mathrm{dist}}(X) \cong \mathrm{PSH}^{\mathrm{u.s.c.}}(X)$ simply by $\mathrm{PSH}(X)$ when no confusion will arise.

Classical potential theory applies to each Laplacian Δ_A with A positive definite. Since Δ_A is obtained from the standard Laplacian Δ by a linear change of coordinates, any result for the standard Laplacian Δ that is independent of choice of linear coordinates applies to each Δ_A as well.

Let $\mathrm{SH}^{\mathrm{u.s.c.}}_A(X)$ denote the space of classical Δ_A-subharmonic functions. That is, $u \in \mathrm{SH}^{\mathrm{u.s.c.}}_A(X)$ if $u \in \mathrm{USC}(X)$ and for each compact subset K of X and each Δ_A-harmonic function h on a neighborhood of K,

$$u \leq h \;\text{ on }\; \partial K \quad \text{implies} \quad u \leq h \;\text{ on }\; K \tag{5.2}$$

Let $\mathrm{SH}^{\mathrm{dist}}_A(X)$ denote the space of Δ_A-distributions on X. That is, $u \in \mathrm{SH}^{\mathrm{dist}}_A(X)$ if $u \in \mathcal{D}'(X)$ and $\Delta_A u \geq 0$ is a non-negative regular Borel measure on X.

For the standard Laplacian Δ on $\mathbf{R}^n$ there are many references for the fact that $\mathrm{SH}^{\mathrm{dist}}(X)$ and $\mathrm{SH}^{\mathrm{u.s.c.}\cdot}(X)$ can be identified. More specifically, with $A = I$:

1) $u \in \mathrm{SH}_A^{\mathrm{dist}}(X)$ implies $u \in L^1_{\mathrm{loc}}(X)$.

2) $u \in \mathrm{SH}_A^{\mathrm{dist}}(X)$ implies that $\widetilde{u} \in \mathrm{SH}_A^{\mathrm{u.s.c.}\cdot}(X)$ and $\widetilde{u}$ lies in the $L^1_{\mathrm{loc}}(X)$-class of u.

3) $u \in \mathrm{SH}_A^{\mathrm{u.s.c.}\cdot}(X)$ implies $u \in L^1_{\mathrm{loc}}(X)$.

4) $u \in \mathrm{SH}_A^{\mathrm{u.s.c.}\cdot}(X)$ implies $\Delta_A u \geq 0$.

Note that 1) and 2) provide an injective map $\mathrm{SH}^{\mathrm{dist}}(X) \to \mathrm{SH}^{\mathrm{u.s.c.}\cdot}(X)$ given by $u \mapsto \widetilde{u}$, while 3) and 4) assert the surjectivity of this map.

These properties 1)—4) carry over to any $A > 0$ by the appropriate linear coordinate change on $\mathbf{R}^n$. This proves that

$$\mathrm{SH}_A^{\mathrm{dist}}(X) \;\cong\; \mathrm{SH}_A^{\mathrm{u.s.c.}\cdot}(X). \tag{5.3}$$

The L^1_{loc}-upper-semi-continuity Condition 2) can be proved as follows for $\Delta_A = \Delta$. Let $B_r(x)$ denote the ball of radius r about x and $|B_r(x)|$ the volume of $B_r(x)$. By the mean value inequality

$$\widetilde{u}(x) \;\leq\; \frac{1}{|B_r(x)|}\int_{B_r(x)} u \;\leq\; \operatorname*{ess\,sup}_{B_r(x)} u \;\leq\; \sup_{B_r(x)} \widetilde{u}. \tag{5.4}$$

Since $\widetilde{u}$ is u.s.c., we have $\limsup_{y\to x} \widetilde{u}(y) = \widetilde{u}(x)$ forcing the essential lim sup to equal $\widetilde{u}(x)$. ∎

Stated differently, we have shown that if $u \in \mathcal{D}'(X)$ is both Δ_{A_1}-subharmonic and Δ_{A_2}-subharmonic, then the two classical representatives $\widetilde{u_1}, \widetilde{u_2} \in \mathrm{USC}(X)$ are equal. Thus there is no ambiguity in the u.s.c. function $\widetilde{u}$ representing a $\mathcal{P}^+$-plurisubharmonic function.

Proof of Proposition 5.2. If $u \in \mathrm{PSH}^{\mathrm{dist}}(X)$, then for some $A > 0$, $u \in \mathrm{SH}_A^{\mathrm{dist}}(X)$ and Condition 2) is valid. ∎

As a corollary, the Mollifying Lemma can be stated for u.s.c. $\mathcal{P}^+$-plurisubharmonic functions.

Mollifying Lemma 5.5. *A function $u \in \mathrm{USC}(X)$ is u.s.c. $\mathcal{P}^+$-plurisubharmonic if and only if u is u.s.c. Δ_A-subharmonic for each mollifying Laplacian Δ_A, i.e., each $A \in \mathrm{Int}\mathcal{P}_+$.*

Upper-Semi-Continuous $\mathbb{G}$-Plurisubharmonic Functions. Suppose that $\mathcal{P}^+ = \mathcal{P}^+(\mathbb{G})$ is geometrically defined by an elliptic subset $\mathbb{G}$ of the grassmannian $G(p, \mathbf{R}^n)$. Theorem 2.3 about C^∞ $\mathbb{G}$-plurisubharmonic functions, has only a weak extension (Proposition 4.3) to $\mathbb{G}$-plurisubharmonic distributions. However, it has a strong extension to upper-semi-continuous $\mathbb{G}$-plurisubharmonic functions.

Theorem 5.6. *Suppose $\mathcal{P}^+$ is geometrically defined by an elliptic subset $\mathbb{G}$ of $G(p, \mathbf{R}^n)$. Let $u \in \mathrm{PSH}^{\mathrm{u.s.c.}\cdot}(X)$ and suppose W is an affine $\mathbb{G}$-plane with $W \cap X$ connected. Then either $u\big|_{W\cap X} \equiv -\infty$ or*

$$u\big|_{W\cap X} \text{ is subharmonic.}$$

More generally, suppose M is any connected $\mathbb{G}$-submanifold of X, i.e., $T_xM \in \mathbb{G}$ for all $x \in M$. If M is a minimal submanifold, then either $u\big|_M \equiv -\infty$ or

$$u\big|_M \text{ is subharmonic.}$$

in the induced riemannian metric on M.

Proof. Assume that $u \in \mathrm{PSH}^{\mathrm{u.s.c.}}(X)$ and u is not $\equiv -\infty$ on M. As noted in §6, there exists a sequence $\{u_j\}$ of smooth $\mathcal{P}^+$-plurisubharmonic functions on X decreasing to u. By Theorem 2.3 each $u_j\big|_M$ is subharmonic. Hence, the decreasing limit $u\big|_M$ is subharmonic. ■

Theorem 5.6 has a converse.

PROPOSITION 5.7. *Suppose that u is a $[-\infty,\infty)$-valued u.s.c. function on a ball $B \subset \mathbf{R}^n$ with the property that for every affine $\mathbb{G}$-plane W in $\mathbf{R}^n$, either $u\big|_{W\cap B} \equiv -\infty$ or $u\big|_{W\cap B}$ is subharmonic. If $u \in L^1_{\mathrm{loc}}(B)$, then $u \in \mathrm{PSH}^{\mathrm{u.s.c.}}(B)$.*

Proof. It suffices to show that $u \in \mathrm{PSH}^{\mathrm{dist}}(B)$ by Corollary 5.4. By Proposition 4.3 it suffices to show that $\Delta_\xi u \geq 0$ for each $\xi \in \mathbb{G}$. Choose coordinates so that ξ is the first axis p-plane in $\mathbf{R}^n$ and (x,y) belongs to $\mathbf{R}^p \times \mathbf{R}^{n-p} = \mathbf{R}^n$. It suffices to show that $\int_{\mathbf{R}^n} u\Delta_x\varphi \geq 0$ for all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbf{R}^n)$, $\varphi \geq 0$. Now $U(y) \equiv \int_{\mathbf{R}^p} u(x,y)\Delta_x\varphi(x,y) \in L^1_{\mathrm{loc}}(\mathbf{R}^{n-p})$, and $U \geq 0$ a.e. by hypothesis. Hence, by Fubini's Theorem $\int_{\mathbf{R}^n} u\Delta_x\varphi = \int_{\mathbf{R}^{n-p}} U(y)dy \geq 0$.

6. Some Classical Facts that Extend to $\mathcal{P}^+$-Plurisubharmonic Functions.

In this section we list other useful properties of PSH(X)-functions.

Some of the standard results for Δ_A-subharmonic functions immediately extend to $\mathcal{P}^+$-plurisubharmonic functions by the Mollifying Lemma 5.5. Other facts require more discussion. In what follows, $u \in \mathrm{PSH}(X)$ is always the canonical, u.s.c. representative.

Facts that follow immediately from the Mollifying Lemma.

(1) (Maxima) $\max\{u^1, ..., u^N\} \in \mathrm{PSH}(X)$ if $u^1, ..., u^N \in \mathrm{PSH}(X)$.

(2) If ψ is a convex non-decreasing function, then $\psi(u) \in \mathrm{PSH}(X)$ for each $u \in \mathrm{PSH}(X)$

(3) (Maximum Principle) If K is a compact subset of X and $u \in \mathrm{PSH}(X)$, then

$$u(x) \ \leq \ \sup_{\partial K} u \qquad \text{for all} \ \ x \in K.$$

(4) (Decreasing Limits) If $\{u_j\}_{j=0}^{\infty}$ is a decreasing (i.e., $u_j \geq u_{j+1}$) sequence of functions in PSH(X) and X is connected, then unless $u = \lim_{j\to\infty} u_j$ is identically $-\infty$, one has $u \in \mathrm{PSH}(X)$ and $\{u_j\}$ converges to u in $L^1_{\mathrm{loc}}(X)$.

(5) (Increasing Limits) Suppose $\{u_j\}_{j=0}^{\infty}$ is an increasing (i.e., $u_j \leq u_{j+1}$) sequence of functions in PSH(X). If the limit $u = \lim_{j\to\infty} u_j$ is locally bounded above, then the u.s.c. regularization $u^*(x) = \limsup_{y\to x} u(y)$ of u belongs to PSH(X) with $u^* = u$ a.e. and $\{u_j\}$ converging to u in $L^1_{\mathrm{loc}}(X)$.

(5)′ (Families Locally Bounded Above) Suppose $\mathcal{F} \subset \mathrm{PSH}(X)$ is locally uniformly bounded above. Then the upper envelope $v = \sup_{f\in\mathcal{F}} f$ has u.s.c. regularization $v^* \in \mathrm{PSH}(X)$ and $v^* = v$ a.e.. Moreover, there exists a sequence $\{u_j\} \subset \mathcal{F}$ with $v^j = \max\{u_1, ..., u_j\}$ converging to v^* in $L^1_{\mathrm{loc}}(X)$.

(6) (Viscosity Plurisubharmonic) $u \in \mathrm{PSH}(X)$ if and only if $u \in \mathrm{USC}(X)$ and for each point $x \in X$ and each function $\varphi \in C^2$ near x with $u - \varphi$ having a local maximum at x, one has

$$\mathrm{Hess}_x\varphi \in \mathcal{P}^+.$$

Facts that do not follow immediately from the Mollifying Lemma.

(7) For each $u \in \mathrm{PSH}^{\mathrm{u.s.c.}}(X)$, there exists a decreasing sequence of smooth functions $\{u_j\} \in \mathrm{PSH}^{\infty}(X_j)$ with $u = \lim_{j\to\infty} u_j$, where $X_j = \{x \in X : \mathrm{dist}(x, \partial X) \geq 1/j\}$.

(8) If $u^1, ..., u^m \in \mathrm{PSH}(X)$ have the property that $\mathrm{Hess}u^j - \Lambda$ is $\mathcal{P}^+$-positive, where $\Lambda : X \to \mathrm{Sym}^2(\mathbf{R}^n)$ is continuous, then $\mathrm{Hess}\{M_\epsilon(u^1, ..., u^m)\} - \Lambda$ is $\mathcal{P}^+$-positive.

(9) (Richberg) Suppose $u \in C(X) \cap \mathrm{PSH}(X)$ has the property that $\mathrm{Hess}u - \Lambda$ is $\mathcal{P}^+$-positive on X where $\Lambda : X \to \mathrm{Sym}^2(\mathbf{R}^n)$ is continuous. Given $\lambda \in C(\overline{X})$, $\lambda > 0$ on X, there exists $\widetilde{u} \in C^\infty(X) \cap \mathrm{PSH}(X)$ with

$$u \ \leq \ \widetilde{u} \ \leq \ u + \lambda \qquad \text{on} \ \ \Omega$$

such that $\mathrm{Hess}\widetilde{u} - (1 - \lambda)\Lambda$ is $\mathcal{P}^+$-positive on X.

Some Comments:

By the "classical case of (k)" we will mean statement (k) with $\mathrm{PSH}(X)$ replaced by $\mathrm{SH}_A(X)$ with $A > 0$.

(5)′: The classical case of (5)′ follows from the classical case of (1) and (5) because of Choquet's Lemma, which says that for any family $\mathcal{F} \subset \mathrm{USC}(X)$ which is uniformly bounded above, there exists a sequence $\{u_j\} \subset \mathcal{F}$ such that the upper envelopes $v(x) = \sup_{f\in\mathcal{F}} f(x)$ and $u(x) = \sup_j u_j(x)$ satisfy the inequalities $u \le v \le v^* \le u^*$ which forces $u^* = v^*$.. Note that (5)′ also follows directly from (1) and (5) by using Choquet's Lemma.

(6): There is an ϵ-strict version of (6). See Definition 12.6. See a) and b) below.

(7): This statement can be proved as follows. If $u_\epsilon = \varphi_\epsilon * u$ is a convolution smoothing, then $\Delta_A u_\epsilon = \varphi_\epsilon * (\Delta_A u)$ so that each u_ϵ is in PSH^∞ on a subset of X a distance ϵ away from the boundary. If $I \in \mathcal{P}_+$, then the convolutions $u_\epsilon = \varphi_\epsilon * u$ with $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\frac{|x|}{\epsilon})$ based on a radial function $\varphi(|x|)$, decrease monotonically to u as $\epsilon \to 0$. Since Δ_A is equivalent to Δ under a linear coordinate change, we can also find φ such that $u_\epsilon = \varphi_\epsilon * u \searrow u$ if u is Δ_A-subharmonic. ∎

(8) and (9): A matrix of distributions, such as $\mathrm{Hess}u - \Lambda$, is defined to be $\mathcal{P}^+$-*positive* if $\langle \mathrm{Hess}u - \Lambda, A\rangle \ge 0$ is a non-negative measure for all $A \in \mathcal{P}_+$. The proofs of (8) and (9) are the same as in the several complex variable case. See Richberg [R] and [D] Lemma 5.18 e) for (8) and Theorem 5.21 for (9).

Pluriharmonicity and Strict Plurisubharmonicity. It is straightforward to extend the definition of pluriharmonicity to distributions.

1) A distribution $u \in \mathcal{D}'(X)$ is $\mathcal{P}^+$-*pluriharmonic* if $\Delta_A u = 0$ for all $A \in \mathcal{P}_+$, or equivalently (see Appendix B) the $S(\mathcal{P}_+)$-Hessian of u is identically zero.

The appropriate extensions of partial and strict are more problematic. Uniform strictness can be put in a satisfactory state.

Suppose $u \in \mathrm{PSH}(X)$ and $\epsilon > 0$. Then u is ϵ-*strict* is either of the following two equivalent conditions are satisfied. (The proof of this equivalence is omitted.)

a) $u - \epsilon|x|^2 \in \mathrm{PSH}(X)$.

b) For each point $x \in X$ and each function $\varphi \in C^2$ near x which is "superior" to u in the sense that $u - \varphi$ has a local maximum at x, one has $\mathrm{Hess}_x\varphi - \epsilon I \in \mathcal{P}^+$.

It is convenient to extend strictness from C^2 functions to general plurisubharmonic functions as follows.

2) $u \in \mathrm{PSH}(X)$ is said to be *strict* if u is ϵ-strict for some $\epsilon > 0$.

The major defect of this definition is best understood by the following example.

EXAMPLE 6.1. Note that the negation of strictness is no longer the appropriate notion of being harmonic. For the standard Laplacian Δ on $\mathbf{R}^n$, u is strictly subharmonic if the absolutely continuous part of the measure Δu is bounded below a.e. by some $\epsilon > 0$. Hence, u being subharmonic but not strict does not imply that u is harmonic.

In the next section we examine the more difficult notion of $\mathcal{P}^+$-harmonic functions.

7. The Dirichlet Problem – Uniqueness.

Here we consider the Dirichlet problem for functions which are "$\mathcal{P}^+$-harmonic" (see Def. 7.7). A full discussion of this concept is given below. However, for C^2-functions u on X this simply means that $\mathrm{Hess}_x u \in \partial\mathcal{P}^+$ for each $x \in X$, and if, furthermore, $\mathcal{P}^+ = \mathcal{P}^+(\mathbb{G})$ is geometrically defined, it means that (u is $\mathbb{G}$-psh and) at each x, $\mathrm{tr}_\xi \mathrm{Hess}_x u = 0$ for some $\xi \in \mathbb{G}$. The main result of this section is the following.

THEOREM 7.1. (Uniqueness for the Dirichlet Problem). *Suppose $\mathcal{P}^+$ is an elliptic cone and that K is a compact subset of $\mathbf{R}^n$. If $u_1, u_2 \in C(K)$ are $\mathcal{P}^+$-harmonic on* Int$K$, *then*

$$u_1 \;=\; u_2 \;\text{ on } \partial K \qquad \Rightarrow \qquad u_1 \;=\; u_2 \;\text{ on } K$$

In order to formulate our definition for non-C^2 functions, it is useful to study functions v with $-\mathrm{Hess}_x v \notin \mathrm{Int}F$. These are in some sense (to be made precise) dual to the $\mathcal{P}^+$-plurisubharmonic functions.

DEFINITION 7.2. Given a closed subset $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$, the *Dirichlet dual* is defined to be

$$\widetilde{F} \;=\; -(\sim \mathrm{Int}F) \;= \sim(-\mathrm{Int}F).$$

Note that

$$\partial F \;=\; F \cap (-\widetilde{F}).$$

LEMMA 7.3. *Suppose $\mathcal{P}^+$ satisfies the positivity condition. Then*

$$B \in \widetilde{\mathcal{P}^+} \quad \Longleftrightarrow \quad A + B \in \widetilde{\mathcal{P}} \qquad \text{for all } \; A \in \mathcal{P}^+$$

Proof. Since $\mathrm{Int}\mathcal{P}^+ = \mathcal{P}^+ + \mathrm{Int}\mathcal{P}$, we have that

$$B' \notin \mathrm{Int}\mathcal{P}^+ \;\Longleftrightarrow\; B' - A \notin \mathrm{Int}\mathcal{P} \;\text{ for all } A \in \mathcal{P}^+.$$

Set $B = -B'$. Then

$$B \notin -\mathrm{Int}\mathcal{P}^+ \;\Longleftrightarrow\; B + A \notin -\mathrm{Int}\mathcal{P} \;\text{ for all } A \in \mathcal{P}^+.$$

■

In Appendix A we introduce the class of *subaffine functions* SA(X) *on* X, and we refer the reader there for a full discussion. We mention, however, that a function $w \in C^2(X)$ is *subaffine* if and only if for all $x \in X$, $\mathrm{Hess}_x w \in \widetilde{\mathcal{P}}$, i.e., $-\mathrm{Hess}_x w \notin \mathrm{Int}\mathcal{P}$, i.e., $\mathrm{Hess}_x w$ has at least one eigenvalue ≥ 0. The following concept is basic to uniqueness and enables us to define $\mathcal{P}^+$-harmonic functions (see Def. 7.7).

DEFINITION 7.4. A function $v \in \mathrm{USC}(X)$ is said to be of *type* $\widetilde{\mathcal{P}}^+$ *on* X if

$$A + v \in \mathrm{SA}(X) \;\text{ for all quadratic functions } A \in \mathcal{P}^+.$$

Let $\widetilde{\mathrm{PSH}}(X)$ denote the space of all such functions.

If $v \in C^2(X)$, then

$$v \text{ is of type } \widetilde{\mathcal{P}}^+ \quad \Longleftrightarrow \quad \mathrm{Hess}_x v \in \widetilde{\mathcal{P}}^+ \text{ for all } x \in X. \tag{7.1}$$

This follows since, as remarked above, $A + v \in \mathrm{SA}(X)$ if and only if $A + \mathrm{Hess}_x v \in \widetilde{\mathcal{P}}$ for all $x \in X$, which by Lemma 7.3 is true for all $A \in \mathcal{P}^+$ if and only if $\mathrm{Hess}_x v \in \widetilde{\mathcal{P}}^+$.

REMARK 7.5. If $\mathcal{P}^+ = \mathcal{P}^+(\mathbb{G})$ is geometrically defined, then

$$\widetilde{\mathcal{P}}^+(\mathbb{G}) \;=\; \{B \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{tr}_\xi B \geq 0 \text{ for some } \xi \in \mathbb{G}\}.$$

To see this first note that

$$\mathrm{Int}\mathcal{P}^+(\mathbb{G}) \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{tr}_\xi A > 0 \text{ for all } \xi \in \mathbb{G}\},$$

that is,

$$\sim \mathrm{Int}\mathcal{P}^+(\mathbb{G}) \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{tr}_\xi A \leq 0 \text{ for some } \xi \in \mathbb{G}\}.$$

Now set $B = -A$ and apply the definition of $\widetilde{\mathcal{P}}^+(\mathbb{G})$.

To establish the basic properties of this class it is useful to have alternative definitions of type $\widetilde{\mathcal{P}}^+$ functions.

LEMMA 7.6. *Fix $v \in \mathrm{USC}(X)$. Then $v \in \widetilde{\mathrm{PSH}}(X)$ if and only if*

$$u + v \in \mathrm{SA}(X) \text{ for all } u \in C^2(X) \text{ which are } \mathcal{P}^+\text{-plurisubharmonic .}$$

Moreover, $v \notin \widetilde{\mathrm{PSH}}(X)$ if and only if $\exists\, A \in \mathcal{P}^+$, a affine, $x_0 \in X$, and $\epsilon > 0$ such that

$$\begin{aligned} a + A + v \;&\leq\; -\epsilon|x - x_0|^2 \quad && \text{for } x \text{ near } x_0 \\ &=\; 0 && \text{at } x = x_0. \end{aligned} \tag{7.2}$$

Proof. If $u + v \notin \mathrm{SA}(X)$ with $u \in C^2(X)$ of type $\mathcal{P}^+$, then by Lemma A.2 there exist $x_0 \in X$, $\epsilon > 0$, and a' affine with

$$\begin{aligned} a' + u + v \;&\leq\; -2\epsilon|x - x_0|^2 \quad && \text{for } x \text{ near } x_0 \\ &=\; 0 && \text{at } x = x_0. \end{aligned} \tag{7.3}$$

Now since $u \in C^2(X)$, we have $A = \frac{1}{2}\mathrm{Hess}_{x_0} u \in \mathcal{P}^+$. Using the Taylor series for u about x_0 it is easy to see that (7.3) implies (7.2). Now (7.2) implies that there exists $A \in \mathcal{P}^+$ with $A + v \notin \mathrm{SA}(X)$ (i.e., (7.2) implies $v \notin \widetilde{\mathrm{PSH}}(X)$). The last implication needed is trivial from Definition 7.4. Namely, if $v \notin \widetilde{\mathrm{PSH}}(X)$, then $\exists\, u \in C^2(X)$ of type $\mathcal{P}^+$ with $u + v \notin \mathrm{SA}(X)$. ∎

DEFINITION 7.7. A function u such that $u \in \mathrm{PSH}(X)$ and $-u \in \widetilde{\mathrm{PSH}}(X)$ will be called *$\mathcal{P}^+$-harmonic on X*.

Note that for such functions u, since both u and $-u$ are upper semi-continuous on X, one has $u \in C(X)$. Furthermore, since $\partial\mathcal{P}^+ = \mathcal{P}^+ \cap (-\widetilde{\mathcal{P}}^+)$, if $u \in C^2(X)$, then u is $\mathcal{P}^+$-harmonic if and only if $\mathrm{Hess}_x u \in \partial\mathcal{P}^+$ for each $x \in X$.

Because of the Maximum Principle in Appendix A, Theorem 7.1 is an immediate consequence of the next result.

THEOREM 7.8. (The Subaffine Theorem). *Suppose $\mathcal{P}^+$ is an elliptic cone. If $u \in \mathrm{PSH}(X)$ and $v \in \widetilde{\mathrm{PSH}}(X)$, then $u + v \in \mathrm{SA}(X)$.*

Proof. Fact (7) above says that u is the decreasing limit of smooth functions u_j which are $\mathcal{P}^+$-plurisubharmonic. By the first part of Lemma 7.6, $u_j + v$ is subaffine. Finally, the decreasing limit of subaffine functions is again subaffine. ∎

8. The Dirichlet Problem – Existence.

We now investigate the existence of solutions to the natural Dirichlet problem associated with $\mathcal{P}^+$-plurisubharmonic functions on a smoothly bounded domain Ω. For the existence question, we assume the boundary $\partial\Omega$ is strictly $\mathcal{P}^+$-convex, a concept introduced and discussed in detail in §12. A principle result, Theorem 12.4, states that if $\partial\Omega$ is strictly $\mathcal{P}^+$-convex, then there exists a smooth, strictly $\mathcal{P}^+$-plurisubharmonic function on a neighborhood of $\overline{\Omega}$, which is a defining function for $\partial\Omega$. It is this result that will be used below, and the reader can, for the moment, take its conclusion as the working assumption.

As before we assume $\mathcal{P}^+$ is an elliptic cone.

THEOREM 8.1. (The Dirichlet Problem – Existence). *Suppose Ω is a bounded domain in $\mathbf{R}^n$ with a strictly $\mathcal{P}^+$-convex boundary. Given $\varphi \in C(\partial\Omega)$, the function u on $\overline{\Omega}$ defined by taking the upper envelope:*

$$u(x) \;=\; \sup\{v(x) : v \in \mathrm{PSH}(\varphi)\}$$

over the family

$$\mathrm{PSH}(\varphi) \;\equiv \{v \;:\; v \text{ is u.s.c. on } \;\overline{\Omega},\;\; v\big|_{\Omega} \in \mathrm{PSH}^{\mathrm{u.s.c.}}(\Omega) \;\text{ and }\; v\big|_{\partial\Omega} \leq \varphi\} \tag{8.1}$$

satisfies:

1) $u \in C(\overline{\Omega})$,

2) *u is $\mathcal{P}^+$-harmonic on Ω,*

3) $u\big|_{\partial\Omega} = \varphi$ *on* $\partial\Omega$.

Proof. By the Maximum Principle the family $\mathrm{PSH}(\varphi)$ is uniformly bounded above on $\overline{\Omega}$ by $\sup_{\partial\Omega}\varphi < \infty$. Hence by 5′) in §6, the u.s.c. regularization u^* of the upper envelope u of $\mathrm{PSH}(\varphi)$, belongs to $\mathrm{PSH}^{\mathrm{u.s.c.}}(\Omega)$. That is

$$u^*\big|_{\Omega} \in \mathrm{PSH}^{\mathrm{u.s.c.}}(\Omega). \tag{8.2}$$

Let h denote the unique Δ_A-harmonic solution to the Dirichlet problem for some mollifying Laplacian Δ_A. Then $h \in C(\overline{\Omega})$, $u \leq h$ on $\overline{\Omega}$ and $h = \varphi$ on $\partial\Omega$. Hence, $u^* \leq h$ on $\overline{\Omega}$ so that

$$u^*\big|_{\partial\Omega} \;\leq\; \varphi. \tag{8.3}$$

This proves

PROPOSITION 8.2. *$u^* \in \mathrm{PSH}(\varphi)$ and therefore*

$$u^* \;=\; u \qquad \text{on } \;\overline{\Omega}. \tag{8.4}$$

The following classical barrier argument is taken from Bremermann [B].

LEMMA 8.3. *The function u on $\overline{\Omega}$ is continuous at each point of $\partial\Omega$, and $u\big|_{\partial\Omega} = \varphi$ on $\partial\Omega$.*

Proof. It suffices to show that

$$\liminf_{x\to x_0} u(x) \;\geq\; u(x_0) \qquad \text{for all } \; x_0 \in \partial\Omega.$$

because of (8.3) above. Fix $x_0 \in \partial\Omega$ and choose a smooth function $\psi \geq 0$ with $\psi(x_0) = 0$ and $\psi(x) > 0$ for $x \neq x_0$. Replacing ψ by a sufficiently small scalar multiple of ψ we may assume that $\rho - \psi$ is strictly plurisubharmonic on $\overline{\Omega}$, where ρ is the defining function for $\partial\Omega$ given by Theorem 12.4. Now for each $\epsilon > 0$, there exists $C >> 0$ so that the function

$$v(x) \;\equiv\; C(\rho(x) - \psi(x)) + \varphi(x_0) - \epsilon$$

satisfies

$$v \;=\; -C\psi + \varphi(x_0) - \epsilon \;\leq\; \varphi \qquad \text{on } \; \partial\Omega.$$

Thus $v \in \mathrm{PSH}(\varphi)$. Consequently, $v \leq u$ on $\overline{\Omega}$, and so

$$\liminf_{x\to x_0} u(x) \;\geq\; \lim_{x\to x_0} v(x) \;=\; \varphi(x_0) - \epsilon.$$

■

We now apply an argument of Walsh [W] to prove interior continuity.

PROPOSITION 8.5. $u \in C(\overline{\Omega})$.

Proof. Let $\Omega_\delta \equiv \{x \in \overline{\Omega} : \mathrm{dist}(x, \partial\Omega) < \delta\}$ denote an interior δ-neighborhood of the boundary $\partial\Omega$. Suppose $\epsilon > 0$ is given. By the continuity of u at points of $\partial\Omega$ and the compactness of $\partial\Omega$, it follows easily that there exists a $\delta > 0$ such that

$$\text{If } \; x \in \Omega_{2\delta}, \; |y| < \delta \text{ and } \; x + y \in \overline{\Omega}, \;\; \text{then } \; u(x+y) - u(x) \;<\; \epsilon. \tag{8.5}$$

Now for $|y| < \delta$ fixed, consider the function

$$f_y(x) \equiv \max\{u(x+y) - \epsilon, u(x)\} \qquad \text{on } \; \Omega - \overline{\Omega_\delta}.$$

Note that $f_y \in \mathrm{PSH}(\Omega - \overline{\Omega_\delta})$ by 1) in Section 6.

Now consider the restriction of f_y to $\Omega_{2\delta} - \overline{\Omega_\delta}$. Then $x \in \Omega_{2\delta}$, $|y| < \delta$, and $x + y \in \overline{\Omega}$, so that (8.5) implies that

$$f_y(x) \;=\; u(x) \qquad \text{on } \; \Omega_{2\delta} - \overline{\Omega_\delta}.$$

We extend f_y to all of $\overline{\Omega}$ by setting $f_y = u$ on $\Omega_{2\delta}$. The function f_y now belongs to the family $\mathrm{PSH}(\varphi)$. Hence, $f_y \leq u$. For $x \in \Omega - \overline{\Omega_\delta}$ this yields

$$u(x+y) - \epsilon \;\leq\; u(x) \qquad \text{if } |y| < \delta.$$

Replacing y by $-y$ and x by $x + y$ yields

$$u(x) - \epsilon \;\leq\; u(x+y) \qquad \text{if } |y| < \delta \text{ and } x \in \Omega - \overline{\Omega_{2\delta}}.$$

This proves that

$$|u(x+y)-u(x)| \;<\; \epsilon \qquad \text{if } |y|<\delta \text{ and } x\in\Omega-\overline{\Omega_{2\delta}}.$$

■

Finally, to complete the proof of Theorem 8.1 we must show that the Perron function u defined by (8.1) is $\mathcal{P}^+$-harmonic on Ω. We already have $u\in \mathrm{PSH}(\Omega)$. Hence, we must show that $-u\in \widetilde{\mathrm{PSH}}(\Omega)$. Suppose $K\subset\Omega$ is compact and let w be a polynomial of degree two which is $\mathcal{P}^+$ plurisubharmonic with $w\le u$ on ∂K. We must show that $w\le u$ on K. However, this must hold, since otherwise one could change u to $\max\{w,u\}$ on K and violate the maximality of the Perron function u. ■

REMARK 8.6. Suppose $\mathcal{P}_0^+\subset\mathcal{P}_1^+$ are elliptic cones. Then if a boundary $\partial\Omega$ is strictly $\mathcal{P}_0^+$-convex, it is also strictly $\mathcal{P}_1^+$-convex. Furthermore, if u is $\mathcal{P}_0^+$-plurisubharmonic , then it is also $\mathcal{P}_1^+$-plurisubharmonic. It follows that if $\Omega\subset\mathbf{R}^n$ is a bounded domain with strictly $\mathcal{P}_0^+$-convex boundary, and $\varphi\in C(\partial\Omega)$ is given, then the unique solutions u_0 and u_1 to the Dirichlet Problem for $\mathcal{P}_0^+$ and $\mathcal{P}_1^+$ respectively, given by Theorems 7.1 and 8.1, satisfy

$$u_0 \;\le\; u_1 \qquad \text{on } \Omega$$

9. $\mathcal{P}^+$-Convex Domains

In this section we introduce the notion of $\mathcal{P}^+$-convex domains and give several characterizations of them. We then establish topological restrictions on any such domain. In many cases these restrictions are known to be sharp.

We assume throughout this section X is a connected open subset of $\mathbf{R}^n$, and that $\mathcal{P}^+ \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is a convex cone which satisfies the Positivity Condition: $\mathcal{P}^+ \subset \mathcal{P}$, but not necessarily the full Ellipticity Condition (i.e., not the Completeness Condition).

DEFINITION 9.1. Given a compact subset $K \subset X$, we define the $\mathrm{PSH}^\infty(X)$-*hull of* K to be the set

$$\widehat{K} \equiv \widehat{K}_{\mathcal{P}^+,X} \equiv \{x \in X : u(x) \leq \sup_K u \text{ for all } u \in \mathrm{PSH}^\infty(X)\}.$$

If $\widehat{K} = K$, then K is called $\mathcal{P}^+$-*convex*.

LEMMA 9.2. *Suppose K is a compact subset of X. A point x is not in $\widehat{K}$ if and only if there exists $u \in \mathrm{PSH}^\infty(X)$ with $u \geq 0$ on X and $u = 0$ on a neighborhood of K but $u(x) >> 0$; and with u strict at x.*

Proof. Suppose $x_0 \notin \widehat{K}$. Then there exits $v \in \mathrm{PSH}^\infty(X)$ with $\sup_K v < 0 < v(x_0)$. Multiplying v by a large constant, we may assume that $v(x_0)$ is large. Replacing v by $v + \epsilon|x|^2$, we may assume that v is strict at x_0. An ϵ-approximation $u = \max_\epsilon\{0, v\}$ to the maximum $\max\{0, v\}$ satisfies all the conditions. ∎

PROPOSITION 9.3. *The following two conditions are equivalent.*

1) $K \subset\subset X \;\Rightarrow\; \widehat{K} \subset\subset X$.

2) There exists a C^∞ proper exhaustion function u for X which is strictly $\mathcal{P}^+$-psh.

DEFINITION 9.4. If the equivalent conditions of Proposition 9.3 are satisfied, then X is a $\mathcal{P}^+$-*convex domain* in $\mathbf{R}^n$.

Proof of Proposition 9.3. We first show that 2) $\Rightarrow$ 1). If $K \subset X$ is compact, then $c = \sup_K u$ is finite and $\widehat{K}$ is contained in the compact prelevel set $\{u \leq c\}$.

To see that 1) $\Rightarrow$ 2), choose compact $\mathrm{PSH}^\infty(X)$-convex sets $K_1 \subset K_2 \subset \cdots$ with $K_m \subset K_{m+1}^o$ and $X = \bigcup_m K_m$. By the Lemma above and the compactness of $K_{m+2} - K_{m+1}^o$ we may find $u^1, ..., u^N \in \mathrm{PSH}^\infty(X)$, which are non-negative and vanish on a neighborhood of K_m, with $u_m = \max_\epsilon\{u^1, ..., u^N\} > m$ on $K_{m+2} - K_{m+1}^o$. The maximum $u = \max\{u_1, u_2, ...\}$ satisfies 2), except for strictness. To obtain strictness, replace u by $u + \frac{1}{2}|x|^2$, which is strict because I is an interior point of $\mathcal{P} \subset \mathcal{P}^+$. ∎

REMARK 9.5. Condition 2) in Proposition 9.3 can be weakened in several ways.

First, the exhaustion u need only be $\mathcal{P}^+$-plurisubharmonic, not strict, since one can always replace u with $u + |x|^2$.

Second, u only needs to be defined near ∞ in the one point compactification of X. More precisely, if there exists $u \in \mathrm{PSH}^\infty(X - K)$, where K is compact, u is bounded near K, and $\lim_{x\to\infty} u(x) = \infty$, then 2) holds. To see this, note that for large c, $v = u + |x|^2$ is a smooth strictly $\mathcal{P}^+$plurisubharmonic function outside the compact subset $\{v \leq c - 1\}$. Pick a convex increasing function $\varphi \in C^\infty(\mathbf{R})$ with $\varphi = c$ on a neighborhood of $(-\infty, c-1]$

and with $\varphi(t) = t$ on $(c+1, \infty)$. Then $\varphi(v(x)) \in \mathrm{PSH}^\infty(X)$ and equals $v(x)$ outside the compact set $\{v \leq c+1\}$.

10. Topological Restrictions on $\mathcal{P}^+$-Convex Domains

We begin our discussion of the topology of $\mathcal{P}^+$-convex domains with the following definition. Note that for any linear subspace, $W \subset \mathbf{R}^n$ there is a natural embedding $\mathrm{Sym}^2(W) \subset \mathrm{Sym}^2(\mathbf{R}^n)$.

DEFINITION 10.1.

a) A linear subspace $W \subset \mathbf{R}^n$ is *$\mathcal{P}_+$-free* if $\mathcal{P}_+ \cap \mathrm{Sym}^2(W) = \{0\}$. In the geometric case where $\mathcal{P}_+ = \mathcal{P}_+(\mathbb{G})$, this means that W does not contain any p-planes $\xi \in \mathbb{G}$. In this case we say that W is *$\mathbb{G}$-free*.

b) A linear subspace $N \subset \mathbf{R}^n$ is *$\mathcal{P}^+$-strict* if $P_N \in \mathrm{Int}\mathcal{P}^+$.

LEMMA 10.2. *Suppose that $\mathbf{R}^n = N \oplus W$ is an orthogonal decomposition. Then W is $\mathcal{P}_+$-free if and only if N is $\mathcal{P}^+$-strict.*

Proof. If N is not strict, then by the Positivity Condition $P_N \in \partial\mathcal{P}^+$. Hence, there exists $A \in \mathcal{P}_+$, $A \neq 0$, with $\langle P_N, A\rangle = 0$. By the positivity assumption $\mathcal{P}_+ \subset \mathcal{P}$ and the basic fact (3.3), it follows easily that $\langle P_N, A\rangle = 0$ if and only if $A \in \mathrm{Sym}^2(W)$. Thus, $\mathcal{P}_+ \cap \mathrm{Sym}^2(W) \neq \{0\}$, contradicting W being free. On the other hand, if P_N is strict, then for all $A \in \mathcal{P}_+$, $\langle P_N, A\rangle > 0$ unless $A = 0$, proving that $\mathcal{P}_+ \cap \mathrm{Sym}^2(W) = \{0\}$. ∎

REMARK 10.3.

$$P_N \text{ is strict if and only if } \mathrm{Int}\mathcal{P}^+ \cap \mathrm{Sym}^2(N) \neq \emptyset.$$

Proof. Note that if P_N is strict, then $P_N \in \mathrm{Int}\mathcal{P}^+ \cap \mathrm{Sym}^2(N)$. For the converse, suppose there exists $H \in \mathrm{Int}\mathcal{P}^+ \cap \mathrm{Sym}^2(N)$, then $H \neq 0$ and $\langle H, A\rangle > 0$ for all non-zero $A \in \mathcal{P}_+$. However, $\langle H, A\rangle = 0$ for all $A \in \mathrm{Sym}^2(W)$ proving that W is free. Hence N is strict by Lemma 10.2.

DEFINITION 10.4. The *free dimension of $\mathcal{P}_+$*, denoted by free-dim$(\mathcal{P}_+)$ (or free-dim$(\mathbb{G})$ in the geometric case), is the maximal dimension of a $\mathcal{P}_+$-free subspace of $\mathbf{R}^n$. By Lemma 10.2 this equals the maximal codimension of a $\mathcal{P}^+$-strict subspace.

Somewhat surprisingly the Andreotti-Frankel Theorem in complex analysis has a very general extension. The usual proof of the Andreotti-Frankel result is quite specific to complex analysis, relying on canonical forms.

THEOREM 10.5. *Let X be a $\mathcal{P}^+$-convex domain in $\mathbf{R}^n$. Then X has the homotopy-type of a CW-complex of dimension $\leq$ free-dim$(\mathcal{P}_+)$.*

Proof. Let $u \in C^\infty(X)$ be a strictly $\mathcal{P}^+$-plurisubharmonic proper exhaustion function. By standard approximation theorems (cf. [MS]) we may assume that all critical points of u are non-degenerate. The theorem will follow if we can show that each critical point x_0 of u in X has index $\leq$ free-dim$(\mathcal{P}^+)$.

Since u is $\mathcal{P}^+$-plurisubharmonic , we have $\mathrm{Hess}_{x_0} u \in \mathcal{P}^+$, that is

$$\langle \mathrm{Hess}_{x_0} u, A \rangle \ \geq \ 0 \ \text{ for all } A \in \mathcal{P}_+. \tag{10.1}$$

Suppose now that the index of $\mathrm{Hess}_{x_0} u$ is $>$ free-dim$(\mathcal{P}_+)$. Then there exists a subspace $W \subset \mathbf{R}^n$ with $\dim(W) >$ free-dim$(\mathcal{P}_+)$ such that

$$\mathrm{Hess}_{x_0} \left(u\big|_W \right) \ < \ 0. \tag{10.2}$$

Now by definition of free-dim$(\mathcal{P}_+)$ there exists a non-zero $A \in \mathrm{Sym}^2(W) \cap \mathcal{P}_+$. Hence, $\langle A, \mathrm{Hess}_{x_0} u \rangle = \langle A, \mathrm{Hess}_{x_0} \left(u\big|_W \right) \rangle < 0$. ∎

REMARK 10.6. The free dimension of $\mathcal{P}_+$ is $n-1$ if and only if each hyperplane W is free, i.e., $\mathcal{P}_+ \cap \mathrm{Sym}^2(W) = \{0\}$, or equivalently each $P_e \in \mathrm{Int}\mathcal{P}^+$ for $0 \neq e \in \mathbf{R}^n$. Otherwise the free dimension is $< n-1$. In this case $\partial\Omega$ is connected for every $\mathcal{P}^+$-convex domain. (This is the case $k = 0$ in the next Corollary.) A special case of this connectedness appears as Lemma A in [CNS].

COROLLARY 10.7. *Let $\Omega \subset\subset X$ be a strictly $\mathcal{P}^+$-convex domain with smooth boundary $\partial\Omega$, and let D be the free dimension of $\mathcal{P}_+$. Then*

$$H_k(\partial\Omega; \mathbf{Z}) \ \cong \ H_k(\Omega; \mathbf{Z}) \qquad \text{for all } k < n - D - 1$$

and the map $H_{n-D-1}(\partial\Omega; \mathbf{Z}) \to H_{n-D-1}(\Omega; \mathbf{Z})$ is surjective.

Proof. This follows from the exact sequence

$$H_{k+1}(\Omega, \partial\Omega; \mathbf{Z}) \ \to \ H_k(\partial\Omega; \mathbf{Z}) \ \to \ H_k(\Omega; \mathbf{Z}) \ \to \ H_k(\Omega, \partial\Omega; \mathbf{Z}),$$

Lefschetz Duality: $H_k(\Omega, \partial\Omega; \mathbf{Z}) \cong H^{n-k}(\Omega; \mathbf{Z})$, and Theorem 10.5. ∎

Geometric Examples. Consider the geometric case $\mathcal{P}^+ = \mathcal{P}^+(\mathbb{G})$. Set fd$(\mathbb{G})$ = free-dim$(\mathbb{G})$. The following facts were shown in [HL$_2$].

1. $\mathbb{G} = G(1, \mathbf{R}^n)$ (Convex geometry). fd$(\mathbb{G}) = 0$.
2. $\mathbb{G} = G(n, \mathbf{R}^n)$ (PSH$(X, \mathbb{G})$ = subharmonic functions on X). fd$(\mathbb{G}) = n - 1$.
3. $\mathbb{G} = G(p, \mathbf{R}^n)$ for $1 < p < n$. fd$(\mathbb{G}) = p - 1$.
4. $\mathbb{G} = \mathbf{P}^{n-1}(\mathbf{C}) = G_{\mathbf{C}}(1, \mathbf{C}^n) \subset G(2, \mathbf{R}^{2n})$ (Complex psh-functions). fd$(\mathbb{G}) = n$.
5. $\mathbb{G} = \mathbf{P}^{n-1}(\mathbf{H}) = G_{\mathbf{H}}(1, \mathbf{H}^n) \subset G(4, \mathbf{R}^{4n})$ (Quaternionic psh-functions). fd$(\mathbb{G}) = 3n$.
6. $\mathbb{G} = G_{\mathbf{C}}(p, \mathbf{C}^n)$ for $1 < p < n$. fd$(\mathbb{G}) = n + p - 1$.
7. $\mathbb{G} = G_{\mathbf{H}}(p, \mathbf{H}^n)$ for $1 < p < n$. fd$(\mathbb{G}) = 3n + p - 1$.
8. $\mathbb{G} = \{x_1\text{-axis}\} \subset G(1, \mathbf{R}^n)$. fd$(\mathbb{G}) = n - 1$.
9. $\mathbb{G} = \mathrm{SLAG} \subset G(n, \mathbf{C}^n)$, the special Lagrangian n-planes in $\mathbf{C}^n$. fd$(\mathbb{G}) = 2n - 2$

10. $\mathbb{G} = \mathrm{ASSOC} \subset G(3, \mathbf{R}^7)$, the associative 3-planes in $\mathrm{Im}\mathbf{O} \cong \mathbf{R}^7$. $\mathrm{fd}(\mathbb{G}) = 4$.

11. $\mathbb{G} = \mathrm{COASSOC} \subset G(4, \mathbf{R}^7)$, the coassociative 4-planes in $\mathrm{Im}\mathbf{O}$. $\mathrm{fd}(\mathbb{G}) = 4$.

12. $\mathbb{G} = \mathrm{CAY} \subset G(4, \mathbf{R}^8)$, the Cayley 4-planes in the octonions $\mathbf{O} \cong \mathbf{R}^8$. $\mathrm{fd}(\mathbb{G}) = 4$.

13. $\mathbb{G} = \mathrm{LAG} \subset G(n, \mathbf{C}^n)$, the set of Lagrangian n-planes in $\mathbf{C}^n$. $\mathrm{fd}(\mathbb{G}) = 2n - 2$.

Some Non-Geometric Examples. Let $\sigma_k(A) : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathbf{R}$ be the kth elementary symmetric function of the eigenvalues defined by the equation $\det(I + tA) = \sum_k \sigma_k(A) t^k$ Consider the closed convex cone $\mathcal{P}^+(\sigma_k)$ whose interior is the connected component, containing I, of the set $\{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \sigma_k(A) > 0\}$. Then

14. $\mathrm{fd}(\mathcal{P}^+(\sigma_k)) = n - k$.

Note that every k-plane N is $\mathcal{P}^+(\sigma_k)$-strict because $\sigma_k(P_N) = 1$. On the other hand $\sigma_k(P_N) = 0$ for any $(k-1)$-plane. Thus, the strict dimension of $\mathcal{P}^+(\sigma_k)$ is k or equivalently, the free dimension of $\mathcal{P}_+(\sigma_k)$ is $n - k$.

11. $\mathcal{P}_+$-Free Submanifolds

We have seen in §10 that there are sometimes quite strong restrictions on the homotopy dimension of $\mathcal{P}^+$-convex domains. In this section we show that within these restrictions the topology of such domains can be quite complicated. One of the main results, Theorem 11.4, is that any submanifold $M \subset X$, which is $\mathcal{P}_+$-free, has a fundamental system of strictly $\mathcal{P}^+$-convex neighborhoods homotopy equivalent to M.

Most proofs in this section are omitted since they carry over by direct generalization from [HL$_2$]. The reader can consult [HL$_2$] for further results and details.

DEFINITION 11.1. A closed submanifold $M \subset X \subset \mathbf{R}^n$ is *$\mathcal{P}_+$-free* if the tangent space $T_x M$ is $\mathcal{P}_+$-free at each point $x \in M$. (In the geometric case where $\mathcal{P}_+ = \mathcal{P}_+(\mathbb{G})$ this means that there are no $\mathbb{G}$-planes which are tangential to M.)

THEOREM 11.2. *Suppose M is a closed submanifold of $X \subset \mathbf{R}^n$. Then M is $\mathcal{P}_+$-free if and only if the square of the distance to M is strictly $\mathcal{P}^+$-plurisubharmonic at each point in M (and hence in a neighborhood of M in X).*

Proof. Given $x_0 \in M$, let $N = (T_{x_0} M)^\perp$ denote the normal to M at x_0. Let $f_M(x) = \frac{1}{2}\mathrm{dist}^2_M(x)$ denote half the square of the distance to M. One can calculate that

$$\mathrm{Hess}_{x_0} f_M \;=\; P_N.$$

(See, [HL$_2$, (6.3)].) Now the theorem is immediate from Lemma 10.2. ∎

THEOREM 11.3. *Consider the two classes of closed sets.*

1) Closed subset $Z \subset M$ of a $\mathcal{P}_+$-free submanifold $M \subset X$.

2) Zero sets $Z = \{f = 0\}$ of non-negative strictly $\mathcal{P}^+$-plurisubharmonic functions f.

Locally these two classes are the same.

Proof. Suppose $Z \subset M$ is as in 1). Choose $\psi \in C^\infty(X)$ with $\psi \geq 0$ and $\{\psi = 0\} = Z$. Then for $\epsilon > 0$ small, the function $f_M + \epsilon\psi$ is strictly $\mathcal{P}^+$-plurisubharmonic and $Z = \{f_M + \epsilon\psi = 0\}$.

Assume $Z = \{f = 0\}$ is as in 2). At $x_0 \in Z$ choose coordinates $x = (z, y)$ in a neighborhood of x_0 so that

$$\mathrm{Hess}_{x_0} f \;=\; \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix}$$

where Λ is a diagonal matrix with non-zero diagonal entries:

$$\frac{\partial^2 f}{\partial y_1^2}(x_0),\ \ldots\ , \frac{\partial^2 f}{\partial y_r^2}(x_0).$$

Set

$$M \;=\; \left\{ \frac{\partial f}{\partial y_1} = \cdots = \frac{\partial f}{\partial y_r} = 0 \right\}.$$

This defines a submanifold M in a neighborhood of x_0, since $\nabla\frac{\partial f}{\partial y_1}, ..., \nabla\frac{\partial f}{\partial y_r}$ are linearly independent at x_0. At x_0 the normal space to M is $N \;=\; \{(0, y) : y \in \mathbf{R}^r\}$. Strict plurisubharmonicity implies $\mathrm{Hess}_{x_0} f \in \mathrm{Int}\mathcal{P}^+ \cap \mathrm{Sym}^2(N)$ so that $T_{x_0}M = N^\perp$ is $\mathcal{P}_+$-free by Lemma 10.2. Since the freeness condition is open, the manifold M is $\mathcal{P}_+$-free in a neighborhood of x_0. Since $f \geq 0$, $\nabla f = 0$ at all points of $Z = \{f = 0\}$, and so $Z \subset M$. ∎

THEOREM 11.4. *Suppose M is a $\mathcal{P}_+$-free closed submanifold of $X \subset \mathbf{R}^n$. Then there exists a fundamental neighborhood system $\mathcal{F}(M)$ of M consisting of $\mathcal{P}^+$-convex domains. Moreover,*

a) M is a deformation retract of each $U \in \mathcal{F}(M)$.

b) Each compact subset $K \subset M$ is $\mathrm{PSH}^\infty(U, \mathcal{P}^+)$-convex for each $U \in \mathcal{F}(M)$.

The proof of this theorem is exactly as in [HL_2, Thm. 6.6] and is omitted.

12. $\mathcal{P}^+$-Convex Boundaries

In this section we introduce the notion of $\mathcal{P}^+$-convexity for smooth boundaries of domains in $\mathbf{R}^n$. We show, for bounded domains, that if the boundary is strictly $\mathcal{P}^+$-convex at each point, then there exists a global defining function ρ for the domain which is strictly $\mathcal{P}^+$-plurisubharmonic on its closure. It is then easy to see that $-\log(-\rho)$ is a strictly $\mathcal{P}^+$-plurisubharmonic exhaustion, and so the domain is $\mathcal{P}^+$-convex.

Fix a domain $\Omega \subset\subset \mathbf{R}^n$ with smooth boundary $\partial\Omega$. By a *defining function for* $\partial\Omega$ we mean a smooth function ρ defined in a neighborhood of $\partial\Omega$ such that in this neighborhood $\Omega = \{x \in \mathbf{R}^n : \rho(x) < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$.

An element $A \in \mathcal{P}_+$ is said to be *tangential at* $x \in \partial\Omega$ if $A \in \mathrm{Sym}^2(T_x\partial\Omega)$. In terms of the 2×2 blocking induced by the decomposition $\mathbf{R}^n = N_x(\partial\Omega) \oplus T_x(\partial\Omega)$, this means $A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$

DEFINITION 12.1. We say that $\partial\Omega$ is strictly $\mathcal{P}^+$-*convex* at a point $x \in \partial\Omega$ if

$$\langle \mathrm{Hess}_x\rho, A\rangle \;>\; 0 \qquad \text{for all non zero } A \in \mathcal{P}_+ \text{ which are tangential at } x. \tag{12.1}$$

If $\langle \mathrm{Hess}_x\rho, A\rangle \geq 0$ for all tangential $A \in \mathcal{P}_+$, then $\partial\Omega$ is said to be $\mathcal{P}^+$-*convex at* x.

NOTE 12.2. These notions are independent of choice of defining function ρ. If $\widetilde{\rho} = f\rho$ with $f > 0$ in $C^\infty(\partial\Omega)$, then $\mathrm{Hess}\widetilde{\rho} = f\mathrm{Hess}\rho + \rho\mathrm{Hess}f + 2\nabla\rho \circ \nabla f$, and so $\langle \mathrm{Hess}_x\widetilde{\rho}, A\rangle = f\langle \mathrm{Hess}_x\rho, A\rangle$ for $A \in \mathcal{P}_+$ which are tangential at x.

REMARK 12.3. (**The Geometric Case**) If $\mathcal{P}_+ = \mathcal{P}_+(\mathbb{G})$, where $\mathbb{G}$ is a closed subset of the grassmannian $G(p, \mathbf{R}^n)$, note that $A \in \mathcal{P}_+(\mathbb{G})$ is tangential if and only if $A = \sum_j \lambda_j P_{\xi_j}$ with each $\lambda_j > 0$ and each $\xi_j \in \mathbb{G}$ *tangential* in the sense that $\mathrm{span}\,\xi_j \subset T_x\partial\Omega$. To show this let n denote a unit normal to $\partial\Omega$ at x. If $A \in \mathcal{P}_+(\mathbb{G})$, then by definition $A = \sum_j \lambda_j P_{\xi_j}$ with each $\lambda_j > 0$ and each $\xi_j \in \mathbb{G}$. If A is tangential to $\partial\Omega$ at x, then $0 = \langle A, P_n\rangle = \sum_j \lambda_j \langle P_{\xi_j}, P_n\rangle$ and hence each $\langle P_{\xi_j}, P_n\rangle = |n \lrcorner \xi_j|^2$ vanishes, which implies that $\mathrm{span}\,\xi_j \subset T_x\partial\Omega$. Consequently, *$\partial\Omega$ is strictly $\mathcal{P}^+$-convex at $x \in \partial\Omega$ if and only if*

$$\mathrm{tr}_\xi \mathrm{Hess}_x\rho \;=\; \langle \mathrm{Hess}_x\rho, P_\xi\rangle \;>\; 0 \;\text{ for all } \xi \in \mathbb{G} \text{ which are tangential at } x \tag{12.2}$$

(and $\partial\Omega$ is $\mathcal{P}^+$-convex at x iff $\mathrm{tr}_\xi \mathrm{Hess}_x\rho \geq 0$ for all $\xi \in \mathbb{G}$ tangential at x).

THEOREM 12.4. *Suppose that Ω has a strictly $\mathcal{P}^+$-convex boundary. Then there exists a strictly $\mathcal{P}^+$-plurisubharmonic function on a neighborhood of $\overline{\Omega}$ which is a defining function for $\partial\Omega$.*

Proof. Fix $C > 0$ and consider $\widetilde{\rho} = \rho + \frac{1}{2}C\rho^2$. This is also a defining function for $\partial\Omega$. At $x \in \partial\Omega$

$$\mathrm{Hess}_x\widetilde{\rho} \;=\; \mathrm{Hess}_x\rho + C(\nabla\rho \circ \nabla\rho). \tag{12.3}$$

LEMMA 12.5. *For C sufficiently large, $\widetilde{\rho}$ is strictly $\mathcal{P}^+$-plurisubharmonic at each point $x \in \partial\Omega$.*

Proof. Since $\mathcal{P}_+ \subset \mathcal{P}$, condition (3.3) states that the tangential condition

$$A \in \mathcal{P}_+ \cap \mathrm{Sym}^2(T_x\partial\Omega) \quad \text{is equivalent to} \quad A \in \mathcal{P}_+ \ \text{ and } \ \langle \nabla\rho(x) \circ \nabla\rho(x), A\rangle = 0. \tag{12.4}$$

Now restrict attention to the compact base $\mathcal{B}_+ = \{A \in \mathcal{P}_+ : \mathrm{tr}A = 1\}$ for $\mathcal{P}_+$. Consider the open subsets of $\partial\Omega \times \mathcal{B}_+$ defined by

$$U_\delta \ = \ \{(x, A) \in \partial\Omega \times \mathcal{B}_+ : \langle \nabla\rho(x) \circ \nabla\rho(x), A\rangle < \delta\}. \tag{12.5}$$

Because of (12.4) these sets U_δ form a fundamental neighborhood system, in $\partial\Omega \times \mathcal{B}_+$, for the compact set

$$K \ = \ \{(x, A) \in \partial\Omega \times \mathcal{B}_+ : A \text{ is tangential to } \partial\Omega \text{ at } x\}.$$

The hypothesis that $\partial\Omega$ is strictly $\mathcal{P}_+$-convex implies that, for $\epsilon > 0$ sufficiently small, $N(K) = \{(x, A) \in \partial\Omega \times \mathcal{B}_+ : \langle \mathrm{Hess}_x\rho, A\rangle > \epsilon\}$ contains K. This proves that there exist $\epsilon, \delta > 0$ such that for each $(x, A) \in \partial\Omega \times \mathcal{B}_+$

$$\langle \nabla\rho(x) \circ \nabla\rho(x), A\rangle < \delta \quad \Rightarrow \quad \langle \mathrm{Hess}_x\rho, A\rangle > \epsilon. \tag{12.6}$$

Choose $M > 0$ so that $-M < \langle \mathrm{Hess}_x\rho, A\rangle$ for all $(x, A) \in \partial\Omega \times \mathcal{B}_+$. Then, for $\langle \nabla\rho(x) \circ \nabla\rho(x), A\rangle \geq \delta$, one has

$$\langle \mathrm{Hess}_x\widetilde{\rho}, A\rangle \ = \ \langle \mathrm{Hess}_x\rho + C(\nabla\rho(x) \circ \nabla\rho(x)), A\rangle \ \geq \ C\delta - M, \tag{12.7}$$

while for $\langle \nabla\rho(x) \circ \nabla\rho(x), A\rangle < \delta$, one has

$$\langle \mathrm{Hess}_x\widetilde{\rho}, A\rangle \ \geq \ \langle \mathrm{Hess}_x\rho, A\rangle \ > \ \epsilon \tag{12.8}$$

by (12.6). Choose $C > M/\delta$. ∎

Since $\widetilde{\rho}$ is strictly $\mathcal{P}^+$-plurisubharmonicat each point $x \in \partial\Omega$, the same is true in a neighborhood $\{-2t < \widetilde{\rho} < 2t\}$ of $\partial\Omega$.

To complete the proof of the theorem, it remains to extend $\widetilde{\rho}$ to all of $\overline{\Omega}$. The function $\max\{\widetilde{\rho}, -t\}$ is a $\mathcal{P}^+$-plurisubharmonic extension, but it is not smooth or strict. However, replacing $-t$ by $a|x|^2 - t$, where $a > 0$ is chosen small enough so that $a|x|^2 - t < \widetilde{\rho}$ on $\{-\frac{t}{2} < \widetilde{\rho} < 0\}$, and then smoothing, we have that for $\epsilon > 0$ sufficiently small,

$$\widehat{\rho} \ = \ \max_\epsilon\{\widetilde{\rho}, a|x|^2 - t\}$$

is a C^∞ strictly $\mathcal{P}^+$-plurisubharmonic function on a neighborhood of $\overline{\Omega}$ which agrees with $\widetilde{\rho}$ on a neighborhood of $\partial\Omega$. ∎

REMARK 12.6. In the non-geometric cases, where $\mathcal{P}^+$ is given but $\mathcal{P}_+$ may be difficult to determine explicitly, the proof of Lemma 12.5 (see (12.3)) provides a convenient criterion for strict boundary convexity. Namely:

$$\begin{gathered} \partial\Omega \ \text{ is strictly } \mathcal{P}^+ \text{ convex at } x \in \partial\Omega \qquad \Leftrightarrow \\ \mathrm{Hess}_x\rho + C(\nabla\rho(x) \circ \nabla\rho(x)) \in \mathrm{Int}\mathcal{P}^+ \quad \text{for } C > 0 \text{ sufficiently large} \end{gathered} \tag{12.9}$$

The corresponding statement for $\mathcal{P}^+$-convexity is false. Consider $\mathcal{P}^+ = \mathcal{P}$ and $n = 2$ with $T_x\partial\Omega = \operatorname{span} e_2$. Then $H = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$ is ≥ 0 and tangential at x, but $H + Ce_1 \circ e_1 = \begin{pmatrix} C & a \\ a & 0 \end{pmatrix}$ is never in $\mathcal{P}^+ = \mathcal{P}$.

We now consider convexity of $\partial\Omega$ in terms of its second fundamental form II with respect to the outward pointing normal. Let ρ denote the signed distance function to $\partial\Omega$, i.e., $\rho(x) = -\text{dist}(x, \partial\Omega)$ for $x \in \Omega$ and $\rho(x) = \text{dist}(x, \partial\Omega)$ for $x \notin \Omega$ (so ρ is a defining function for $\partial\Omega$). One computes (see [HL$_2$, §5]) that for points $x \in \partial\Omega$

$$\text{Hess}_x\, \rho \;=\; \begin{pmatrix} 0 & 0 \\ 0 & -II \end{pmatrix} \tag{12.10}$$

with respect to the orthogonal decomposition

$$T_x\mathbf{R}^n \;=\; N_x\partial\Omega \oplus T_x\partial\Omega. \tag{12.11}$$

PROPOSITION 12.7. *Suppose $\Omega \subset\subset \mathbf{R}^n$ has a smooth boundary, and denote by II the second fundamental form of $\partial\Omega$ with respect to the outward pointing normal $n = \nabla\rho$. Then $\partial\Omega$ is strictly $\mathcal{P}^+$-convex at a point $x \in \partial\Omega$ if and only if*

$$\langle II, A\rangle \;<\; 0 \;\text{ for all } \textit{non zero } A \in \mathcal{P}_+ \textit{ which are tangential at } x$$

or equivalently

$$-II + Cn \circ n \in \text{Int}\mathcal{P}^+ \qquad \text{for } C > 0 \;\textit{ sufficiently large.} \tag{12.12}$$

Proof. Since ρ is a defining function for $\partial\Omega$, the first assertion follows immediately from (12.10). The proof of (12.12) is discussed in Remark 12.6. ∎

REMARK 12.8. (The Geometric Case). The boundary $\partial\Omega$ is strictly $\mathcal{P}^+(\mathbb{G})$-convex at $x \in \partial\Omega$ if and only if

$$\text{tr}_\xi II \;<\; 0 \quad \text{for each } \xi \in \mathbb{G} \text{ which is tangential at } x.$$

Finally we discuss the relationship of boundary convexity to the convexity of the domain itself.

PROPOSITION 12.9. *Suppose that $\Omega \subset\subset \mathbf{R}^n$ has a smooth, strictly $\mathcal{P}^+$-convex boundary. Then Ω is a $\mathcal{P}^+$-convex domain.*

Proof. If ρ is a strictly $\mathcal{P}^+$-convex defining function (such as the one given by Theorem 12.4), then $-\log(\delta)$, with $\delta = -\rho$, is a strictly $\mathcal{P}^+$-plurisubharmonicexhaustion function. One computes that

$$\langle \text{Hess}(-\log\delta), A\rangle \;=\; \frac{1}{\delta}\langle \text{Hess}\,\rho, A\rangle + \frac{1}{\delta^2}\langle \nabla\rho \circ \nabla\rho, A\rangle. \tag{12.13}$$

The right hand side is > 0 for all non-zero $A \in \mathcal{P}_+$. ∎

In general it is not true that boundaries of $\mathcal{P}^+$-convex domains are $\mathcal{P}^+$-convex. (See [HL$_2$, §5] for examples). However, we have the following.

THEOREM 12.10. *Let $\delta = \operatorname{dist}(\bullet, \partial\Omega)$ denote the distance to $\partial\Omega$ in Ω. If $-\log\delta$ is $\mathcal{P}^+$-plurisubharmonic near $\partial\Omega$, then $\partial\Omega$ is $\mathcal{P}^+$-convex.*

Proof. If $\partial\Omega$ is not $\mathcal{P}^+$-convex, then there exists $x \in \partial\Omega$ and $A \in \mathcal{P}_+ \cap \operatorname{Sym}^2(T_x\partial\Omega)$ with $\langle II, A\rangle > 0$. Since A is tangential, we have $\langle \nabla\delta \circ \nabla\delta, A\rangle = 0$. Let ℓ denote the line segment in Ω which emanates from x normally to the boundary, i.e., in the direction $\nabla\delta$. It follows from (12.10) and (12.13) that $\langle \operatorname{Hess}(-\log\delta), A\rangle = -\frac{1}{\delta}\langle \operatorname{Hess}\delta, A\rangle < 0$ at all points of ℓ near to x. Consequently, $-\log\delta$ is not $\mathcal{P}^+$-plurisubharmonic in any neighborhood of $\partial\Omega$. ∎

Appendix A.
The Maximum Principle and Subaffine Functions.

An upper semi-continuous function $u : X \to [-\infty, \infty)$ satisfies the *maximum principle* if for each compact subset $K \subset X$

$$\sup_K u \ \leq \ \sup_{\partial K} u. \tag{A.1}$$

A function u may locally satisfy the maximum principle without satisfying the maximum principle on all of X. (Consider, for example, a function u on $\mathbf{R}$ with compact support, $0 \leq u \leq 1$, $u \equiv 1$ on a neighborhood of the origin and otherwise monotone.) However, this situation is easily remedied. First note that (A.1) is equivalent to the condition that:

$$u \ \leq \ c \ \text{ on } \partial K \qquad \Rightarrow \qquad u \ \leq \ c \ \text{ on } K \qquad \text{for all constants } c, \tag{A.1}'$$

i.e., u is *sub-constants*. Replacing the constant functions by the affine functions, consider the condition:

$$u \ \leq \ a \ \text{ on } \partial K \qquad \Rightarrow \qquad u \ \leq \ a \ \text{ on } K \qquad \text{for all affine functions } a \tag{A.2}$$

Definition A.1. A function $u \in \mathrm{USC}(X)$ satisfying (A.2) for all compact subsets $K \subset X$ will be called *subaffine* on X. Let $\mathrm{SA}(X)$ denote the space of all $u \in \mathrm{USC}(X)$ that are locally subaffine on X, i.e., for all $x \in X$ there exists a neighborhood B of x with $u\big|_B$ sub-affine.

Note that if u is subaffine on X, then the restriction to any open subset is also subaffine.

Lemma A.2. *If u is locally subaffine on X, then u is subaffine on X. In fact, u is not subaffine on X if and only if*

$$\begin{aligned} &\textit{There exist } \ x_0 \in X, \ \textit{a affine, and } \epsilon > 0 \textit{ such that} \\ &\qquad (u-a)(x) \ \leq \ -\epsilon|x-x_0|^2 \ \textit{ near } x_0, \ \textit{ and} \\ &\qquad (u-a)(x_0) \ = \ 0 \end{aligned} \tag{A.3}$$

Proof. Subaffine implies locally subaffine, which implies (A.3) is impossible. Hence, it remains to show that if (A.3) is false, then u is subaffine, or equivalently, if u is not subaffine on X, then (A.3) is true. If u is not subaffine on X, then for some compact set $K \subset X$ and some affine function b, the difference $w = u - b$ has an interior maximum point for K. For $\epsilon > 0$ sufficiently small, the same is true for $w = u + \epsilon|x|^2 - b$. Choose a maximum point $x_0 \in \mathrm{Int}K$ for w and let $M = w(x_0)$ denote the maximum value on K. Then $u+\epsilon|x|^2-b-M \leq 0$ on K and equals zero at x_0. Since $\epsilon|x|^2$ and $\epsilon|x-x_0|^2$ differ by the affine function, this proves that there is an affine function a such that $u+\epsilon|x-x_0|^2-a \leq 0$ on K and is equal to zero at x_0, i.e., (A.3) is true. ∎

THEOREM A.3. (Maximum Principle). *Suppose $K \subset \mathbf{R}^n$ is compact and $u \in \mathrm{USC}(K)$. If $u \in \mathrm{SA}(\mathrm{Int}K)$, then*

$$\sup_K u \ \leq \ \sup_{\partial K} u.$$

Proof. Exhaust $\mathrm{Int}K$ by compact sets K_ϵ. Since $u \in \mathrm{SA}(\mathrm{Int}K)$, $\sup_{K_\epsilon} u \leq \sup_{\partial K_\epsilon} u$. Since $u \in \mathrm{USC}(K)$, each $U_\delta = \{x \in K : u(x) < \sup_{\partial K} u + \delta\}$, for $\delta > 0$, is an open neighborhood of ∂K in K. Therefore, there exits $\epsilon > 0$ with $\partial K_\epsilon \subset U_\delta$ which implies that $\sup_{\partial K_\epsilon} u \leq \sup_{\partial K} u + \delta$. ∎

For functions which are C^2 (twice continuously differentiable), the subaffine condition is a condition on the hessian of u at each point.

PROPOSITION A.4. *Suppose $u \in C^2(X)$. Then*

$$u \in \mathrm{SA}(X) \quad \Longleftrightarrow \quad \mathrm{Hess}_x u \ \textit{has at least one eigenvalue} \ \geq 0 \ \textit{at each point} \ x \in X.$$

Proof. Suppose $\mathrm{Hess}_{X_0} u < 0$ (negative definite) at some point $x_0 \in X$. Then the Taylor expansion of u about x_0 implies (A.3) Therefore, since $u(x_0) = 0$, $u \notin \mathrm{SA}(X)$.

Conversely, if $u \notin \mathrm{SA}(X)$, then (A.3) is valid for some point $x_0 \in X$ which implies that $\mathrm{Hess}_{x_0} u + \epsilon I \leq 0$. So $\mathrm{Hess}_x u < 0$ is negative definite. ∎

EXAMPLE (n=1). Suppose I is an open interval in $\mathbf{R}$. Then

$$u \in \mathrm{SA}(I) \quad \Longleftrightarrow \quad \text{either} \ \ u \in \mathrm{Convex}(I) \text{ or } u \equiv -\infty.$$

Proof. Suppose $u \in \mathrm{SA}(I)$ equals $-\infty$ at one point $\alpha \in I$ but u is finite at another point $\beta \in I$. Choose a to be the affine function with $a(\alpha) = -N$ and $a(\beta) = u(\beta)$. By (A.2), we have $u \leq a$ on $[\alpha, \beta]$, which implies (by letting $N \to \infty$) that $u \equiv -\infty$ on $[\alpha, \beta)$. Hence u is either $\equiv -\infty$ or it is finite-valued on all of I (and therefore convex). The converse is immediate. ∎

Appendix B.
Hessians of Plurisubharmonic Distributions.

The decomposition

$$\mathrm{Sym}^2(\mathbf{R}^n) \;=\; E \oplus S \tag{B.1}$$

induces a decomposition of each $\mathrm{Sym}^2(\mathbf{R}^n)$-valued test function on X, and hence of each $\mathrm{Sym}^2(\mathbf{R}^n)$-valued distribution on X. Applying this to $\mathrm{Hess}\, u$, with $u \in \mathcal{D}'(X)$, we have

$$\mathrm{Hess}\, u \;=\; (\mathrm{Hess}\, u)_E + (\mathrm{Hess}\, u)_S. \tag{B.2}$$

LEMMA B.1. *If $u \in \mathrm{PSH}^{\mathrm{dist}}(X)$, then $(\mathrm{Hess}\, u)_S$ is an S-valued measure on X.*

Proof. Since the interior of $\mathcal{P}_+$ in S is non-empty, we may choose a basis $A_1, ..., A_N \in \mathrm{Int}\mathcal{P}_+$ for S. the dual basis $A_1^*, ..., A_N^*$ for S will have the property that $(\mathrm{Hess}\, u)_S = u_1 A_1 + \cdots + u_n A_N$ with each $u_j \in \mathcal{D}'(X)$. Given $\varphi \in C^\infty_{\mathrm{cpt}}(X)$,

$$u_j(\varphi) \;=\; (\mathrm{Hess}\, u)_S(\varphi A_j). \tag{B.3}$$

If u is a $\mathcal{P}^+$-plurisubharmonic distribution, (B.3) implies that each $u_j \geq 0$ is a non-negative measure. ∎

Note that using any basis for $\mathrm{Sym}^2(\mathbf{R}^n)$ (for example the standard basis), $(\mathrm{Hess}\, u)_S$ will have measure coefficients.

LEMMA B.2. *Suppose H is an S-valued measure on X. Then there exists a measure $\|H\| \geq 0$ and a function $\overrightarrow{H} : X \to S$ which is in L^1_{loc} on X with respect to the measure $\|H\|$, and $|\overrightarrow{H}(x)| = 1$, $\|H\|$-a.e., such that*

$$H(\Phi) \;=\; \int_X \langle \overrightarrow{H}, \Phi \rangle \, \|H\|$$

for each S-valued test form Φ on X. Also, $\|H\|$ and $\overrightarrow{H}$ are unique.

Proof. This is a standard fact about vector-valued measures.

THEOREM B.3. *Suppose $u \in \mathrm{PSH}^{\mathrm{dist}}(X)$ and abbreviate $(\mathrm{Hess}\, u)_S$ by H_u. Then*

$$(\mathrm{Hess}\, u)_S \;=\; \overrightarrow{H_u} \, \|H_u\| \tag{B.4}$$

with $\|H_u\| \geq 0$ and $|\overrightarrow{H_u}(x)| = 1$ for $\|H_u\|$ a.a. $x \in X$.

Appendix C.
Convex Elliptic Sets in $\mathrm{Sym}^2(\mathbf{R}^n)$.

Suppose F is an unbounded closed convex set in a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$, and assume that F has interior but $F \neq V$. We can associate with F two closed convex cones with vertex at the origin, $\mathcal{P}^+(F)$ and $\mathcal{P}_+(F)$ which are polars of each other.

$\mathcal{P}^+(F)$ — The Ray Cone of F: Pick $a \in F$. Consider the set of directions $\overrightarrow{u}$ such that the ray from a in the direction $\overrightarrow{u}$ in contained in F. This coincides with the compact subset

$$\bigcap_{r>0} \tfrac{1}{r}\partial B_r \cap (F - a)$$

of the unit sphere. The cone on this compact set is called the *ray cone of* F and is denoted by $\mathcal{P}^+(F)$. Since $F - a$ is convex, $\mathcal{P}^+(F)$ is convex. If $b \in F$ is any point in F, it is easy to see that the ray $\{a + tv : t \geq 0\}$ is contained in F if and only if the ray $\{b + tv : t \geq 0\}$ is contained in F. That is, $\mathcal{P}^+(F)$ is independent of the choice of point $a \in F$.

$\mathcal{P}_+(F)$ — The Cone of Supporting Directions for F: For each non-zero $u \in V$ and each $\lambda \in \mathbf{R}$, consider the closed half-space

$$H(u, \lambda) \;=\; \{v \in V : u \cdot v \geq \lambda\}$$

If $F \subset H(u, \lambda)$ for some $\lambda \in \mathbf{R}$, then u is a *supporting direction vector for* F. Let $\mathcal{P}_+(F)$ denote the closure of the set of these supporting direction vectors. Obviously, $\mathcal{P}_+(F)$ is a closed set of rays at the origin in V. If $F \subset H(u, \lambda)$ and $F \subset H(u', \lambda')$ and $0 \leq s \leq 1$, then it is easy to see that $F \subset H(su + (1-s)u', s\lambda + (1-s)\lambda')$. Thus $\mathcal{P}_+(F)$ is convex.

Proposition C.1. *Suppose F is an unbounded closed convex subset of V with* $\operatorname{span} F = V$ *but $F \neq V$. Then $\mathcal{P}^+(F)$ and $\mathcal{P}_+(F)$ are polars of each other (with* $\operatorname{span} \mathcal{P}^+(F) = V$ *and $\mathcal{P}^+(F) \neq V$).*

Proof. Suppose $v \in \mathcal{P}^+(F)$ and u is a supporting direction vector. Then for $a \in F$, the ray $\{a + tv : t \geq 0\} \subset F$ and there exists $\lambda \in \mathbf{R}$ with $F \subset H(u, \lambda)$. Therefore, $\lambda \leq \langle u, a + tv \rangle = \langle u, a \rangle + t\langle u, v \rangle$ for all $t \geq 0$ which implies that $\langle u, v \rangle \geq 0$. This proves that each of $\mathcal{P}^+(F)$ and $\mathcal{P}_+(F)$ is contained the the polar of the other.

Suppose v is in the polar of $\mathcal{P}_+(F)$, i.e., $\langle u, v \rangle \geq 0$ if $F \subset H(u, \lambda)$ for some λ. Consider the ray $\{a + tv : t \geq 0\}$ through $a \in F$. This ray is contained in $H(u, \lambda)$ since $\langle a + tv, u \rangle = \langle a, u \rangle + t\langle v, u \rangle \geq \lambda + t\langle v, u \rangle \geq \lambda$ if $t \geq 0$. By the Hahn-Banach Theorem this ray must be contained in F. Hence, $v \in \mathcal{P}^+(F)$. Thus $\mathcal{P}^+(F)$ is the polar of $\mathcal{P}_+(F)$. The reverse follows from the bipolar theorem

The Edge of F. The set $E(F) = \{v \in V : \pm v \in \mathcal{P}^+(F)\}$ consisting of those $v \in V$ such that the full affine line $\{a + tv : t \in \mathbf{R}\}$ through $a \in F$ is contained in F, is called the *linearity of* F or the *edge of* F. Set $S(F) \equiv E(F)^\perp$. Then

$$F \;=\; E(F) \times (F \cap S(F))$$

is a tube with base $F \cap S(F)$. In this case the ray cone $\mathcal{P}^+(F)$ is also a tube

$$\mathcal{P}^+(F) \;=\; E(F) \times (\mathcal{P}^+(F) \cap S(F))$$

with the same edge as F.

Note that $\operatorname{span} \mathcal{P}^+(F) = V$ since F is assume to have interior, but

$$\operatorname{span} \mathcal{P}_+(F) \;=\; S(F).$$

Convex Elliptic Sets.

DEFINITION C.2. A closed convex subset $F \subset \operatorname{Sym}^2(\mathbf{R}^n)$ which satisfies

(1) $F + \mathcal{P} \;\subset\; F$.

(2) F can not be defined using the variables in a proper subspace $W \subset \mathbf{R}^n$,

will be called a *convex elliptic set*.

PROPOSITION C.3. *A closed convex subset $F \subset \operatorname{Sym}^2(\mathbf{R}^n)$ is elliptic if and only if its ray cone $\mathcal{P}^+(F)$ is an elliptic cone.*

Proof. It is easy to see that F satisfies the positivity condition (1) if and only if $\mathcal{P}^+(F)$ does. It remains to show that F can be defined using the variables in a proper subspace $W \subset \mathbf{R}^n$ if and only if the ray cone $\mathcal{P}^+(F)$ can be defined using the variables in W

We must show that $\operatorname{Sym}^2(W)^\perp \subset F \iff \operatorname{Sym}^2(W)^\perp \subset \mathcal{P}^+(F)$. One way is trivial. For the other, suppose $\operatorname{Sym}^2(W)^\perp \subset \mathcal{P}^+(F)$. We may assume $0 \in F$. Then $A \in \mathcal{P}^+(F)$ if and only if the ray $\{tA : t \geq 0\} \subset F$. Hence, $\operatorname{Sym}^2(W)^\perp \subset F$. ∎

COROLLARY C.4. *Suppose F is a closed convex set in $\operatorname{Sym}^2(\mathbf{R}^n)$ with $F + \mathcal{P} \subset F$. Then F cannot be defined using fewer of the variables in $\mathbf{R}^n$ if and only if each $A \in \operatorname{Int}\mathcal{P}_+(F)$ is positive definite.*

Appendix D.
The Dirichlet Problem for Convex Elliptic Sets.

The main results of this paper carry over from elliptic cones to convex elliptic sets F. Suppose

$$H \;=\; \{B \in \mathrm{Sym}^2(\mathbf{R}^n) : \langle A, B\rangle \geq c\}$$

is a supporting half-space for F with $A \in \mathrm{Int}\mathcal{P}_+(F)$. By Corollary C.4, A is positive definite. Pick $B_0 \in \partial H$, i.e., $\langle A, B_0\rangle = c$. Then the replacement for the mollifying condition $\Delta_A u \geq 0$ is $\Delta_A u \geq \Delta_A B_0 = \langle A, B_0\rangle = c$. The Mollifying Lemma 4.2 remains valid for F-plurisubharmonic distributions. The notion of being u.s.c. F-plurisubharmonic carries over in a straightforward manner. For both these concepts a function u is of type F if and only if it is of type H for all supporting half-spaces H. The key approximation property (7) is section 6 remains valid, with standard convolution providing the proof.

The equivalent definitions of type $\widetilde{F}$ carry over from those of type $\widetilde{\mathcal{P}}^+$.

Finally, the Dirichlet Problem is solvable in this context. See Theorem 7.1 (Uniqueness) and Theorem 8.1 (Existence). In the existence statement the boundary $\partial\Omega$ must be strictly $\mathcal{P}^+(F)$-convex.

EXAMPLE D.1. A simple but illuminating example of a convex elliptic set is

$$F \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A \geq 0 \text{ and } \det A \geq c\}$$

for a constant $c \geq 0$. One sees that the ray cone $\mathcal{P}^+(F) = \mathcal{P}$. The corresponding classical equation is: $\det\{\mathrm{Hess}\, u\} = c$.

EXAMPLE D.2. A more interesting example is

$$F \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A \geq 0 \text{ and } \mathrm{Trace}\{\arctan(A)\} \geq k\pi\} \tag{D.1}$$

where $n = 2k+1$ or $2k+2$. The corresponding equation

$$\mathrm{Im}\{\det(I + iA)\} \;=\; 0 \tag{D.2}$$

arises in the study of Special Lagrangian submanifolds, and the Dirichlet problem for (D.2) was studied in depth by Caffarelli, Nirenberg and Spruck [CNS]. In fact the locus of (D.2) has k connected components and [CNS] treats only the "outermost" component, which corresponds to the boundary of the set F defined in (D.1). In [CNS] the authors show that the ray cone

$$\mathcal{P}^+(F) \;=\; \begin{cases} \mathcal{P} & \text{if } n \text{ is odd} \\ \mathcal{Q} & \text{if } n \text{ is even} \end{cases}$$

where $\mathrm{Int}\mathcal{Q}$ is the component of the set $\{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \sigma_{n-1}(A) > 0\}$ which contains the identity I (and σ_{n-1} denotes the $(n-1)$st elementary symmetric function). In [HL$_4$] existence and uniqueness of continuous solutions to the Dirichlet Problem are established for all branches of the equation (D.2). However, the smoothness of the solutions for smooth boundary data remains largely open (see [Y] however).

Appendix E.

Elliptic MA-operators / Gårding-Hyperbolic Polynomials on $\mathrm{Sym}^2(\mathbf{R}^n)$.

For each polynomial P on the vector space $\mathrm{Sym}^2(\mathbf{R}^n)$ consider the associated (non-linear) partial differential operator defined by $\mathbf{P}(u) = P(\mathrm{Hess}u)$. If P is the determinant, the associated operator is the real Monge-Ampère operator.

DEFINITION E.1. Let M be a homogeneous polynomial of degree m on $\mathrm{Sym}^2(\mathbf{R}^n)$. Suppose that the identity I is a hyperbolic direction for M in the sense of Gårding [G]. That is, $M(I) > 0$ and for each $A \in \mathrm{Sym}^2(\mathbf{R}^n)$, the polynomial $p_A(t) = M(tI + A)$ has exactly m real zeros on $\mathbf{R}$. Then the operator

$$\mathbf{M}(u) \;=\; M(\mathrm{Hess}u) \tag{$E.1$}$$

will be called a *Monge-Ampère or* MA-*operator*, and the polynomial M will be called an MA-*polynomial*.

The negatives of the roots of p_A are called the M-*eigenvalues of* A and are denoted $\lambda_1(A), ..., \lambda_m(A)$ so that $M(tI + A) = M(I)\prod_{j=1}^m (t + \lambda_j(A))$.

Gårding's beautiful theory of hyperbolic polynomials states that the set

$$\Gamma(M) \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \lambda_j(A) > 0,\ j = 1, ..., m\} \tag{$E.2$}$$

is an open convex cone in $\mathrm{Sym}^2(\mathbf{R}^n)$ equal to the connected component of $\{M > 0\}$ containing I. The closed convex cone

$$\mathcal{P}^+(M) \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \lambda_j(A) \geq 0,\ j = 1, ..., m\} \tag{$E.3$}$$

is the closure of $\Gamma(M)$, and $\Gamma(M) = \mathrm{Int}\mathcal{P}^+(M)$. Gårding also shows that the edge of $\mathcal{P}^+(M)$ is given by $\lambda_1(A) = \cdots = \lambda_m(A) = 0$.

The Positivity Condition on $\mathcal{P}^+(M)$ (from §3) can be stated in several equivalent ways in terms of M.

First the condition that $\mathrm{Int}\mathcal{P} \subset \mathcal{P}^+(M)$, (i.e., $A > 0 \;\;\Rightarrow\;\; \lambda_j(A) \geq 0$ for all j) is stated directly in terms of M as:

1) $M(tI + A) \neq 0$ if $t > 0$ and $A > 0$.

Second, the condition that $P_e \in \mathcal{P}^+(M)$ for all unit vectors e in $\mathbf{R}^n$, (i.e., $\lambda_j(P_e) \geq 0$ for all j and e) is expressed directly in term of M as

1)′ $M(tI + P_e) \neq 0$ for all $t > 0$ and all unit vectors e.

By Proposition 3.6, assuming positivity, $\mathcal{P}^+(M)$ is elliptic if and only if $-P_e \notin \mathcal{P}^+(M)$ for all unit vectors e. Equivalently, $P_e \notin E(\mathcal{P}^+(M))$, the edge of $\mathcal{P}^+(M)$. That is,

2) $M(I + sP_e)$ is not constant in s for each unit vector e.

PROPOSITION E.2. *The convex cone $\mathcal{P}^+(M)$ defined by an* MA-*polynomial is an elliptic cone if and only if for each unit vector $e \in \mathbf{R}^n$,*

1)′ $M(tI + P_e) \neq 0$ for all $t > 0$, and

2) $M(I + sP_e)$ is non-constant in s.

M-Elementary Symmetric Functions. Given an MA-polynomial, expanding out $M(tI + A)$ as a polynomial of degree m in t yields

$$M^{(0)}(A)t^m + M^{(1)}(A)t^{m-1} + \cdots + M^{(m)}(A) \tag{E.4}$$

where $M^{(0)}(A) = M(I)$, $M^{(m)}(A) = M(A)$, and

$$M^{(k)}(A) \;=\; \frac{1}{(m-k)!}\frac{d^{m-k}}{dt^{m-k}}M(tI + A)\big|_{t=0}. \tag{E.5}$$

By Rolle's Theorem the roots of

$$M^{(k-1)}(tI + A) \;=\; \frac{1}{(m-k)}\frac{d}{dt}M^{(k)}(tI + A) \tag{E.6}$$

strictly separate the roots of $M^{(k)}(tI + A)$.

PROPOSITION E.3. *If M is an elliptic MA-polynomial on $\mathrm{Sym}^2(\mathbf{R}^n)$, then for each k, $1 \leq k \leq m$, $M^{(k)}$ is also an elliptic MA-polynomial.*

Proof. . Using induction and (E.6) is suffices to prove the Proposition for $k = m - 1$. By Rolle's Theorem the $M^{(m-1)}$-eigenvalues of A separate the M-eigenvalues of A. Hence, if there are m of the M-eigenvalues, then there must be at least $m - 1$ of the $M^{(m-1)}$-eigenvalues. Moreover, if all the M-eigenvalues of A are ≥ 0, then all the $M^{(m-1)}$-eigenvalues of A are ≥ 0. Thus if M satisfies positivity, so does $M^{(m-1)}$. Since $P_e \geq 0$, its M-eigenvalues must be ≥ 0. Since M is elliptic, P_e does not belong to the edge of $\mathcal{P}^+(M)$. Thus P_e must have at least one M-eigenvalue > 0. By the separating property, this implies that P_e must have at least one $M^{(m-1)}$-eigenvalue > 0. Therefore, $P_e \notin E(\mathcal{P}^+(M^{(m-1)}))$. ■

Examples: The basic examples are given by taking M to be the determinant. There are four cases corresponding to $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathbf{O}$. In all four cases the cone $\mathcal{P}^+(M)$ is elliptic.

1. The determinant on $\mathrm{Sym}^2(\mathbf{R}^n)$.
2. The determinant on $\mathrm{Herm}_{\mathbf{C}}\mathrm{Sym}^2(\mathbf{C}^n) \subset \mathrm{Sym}^2(\mathbf{R}^{2n})$.
3. The determinant on $\mathrm{Herm}_{\mathbf{H}}\mathrm{Sym}^2(\mathbf{H}^n) \subset \mathrm{Sym}^2(\mathbf{R}^{4n})$.
4. The determinant on $\mathrm{Herm}_{\mathbf{O}}\mathrm{Sym}^2(\mathbf{O}^2) \subset \mathrm{Sym}^2(\mathbf{R}^{16})$.

The quaternionic case is perhaps best understood as a polar action [DK]. Namely, Sp_n acts on $\mathrm{Herm}_{\mathbf{H}}\mathrm{Sym}^2(\mathbf{H}^n)$ with cross-section given by the space D of diagonal matrices. The polynomial $\lambda_1 \cdots \lambda_n$ on D extends to an Sp_n-invariant polynomial, $\det_{\mathbf{H}}$, on $\mathrm{Herm}_{\mathbf{H}}\mathrm{Sym}^2(\mathbf{H}^n)$ (cf. [AV]).

In each of these cases the inhomogeneous equation has been been treated: the real case by Taylor-Rauch, the complex case by Bedford-Taylor, the quaternionic case by Alesker-Verbitsky and the octonian case also by Alesker-Verbitsky.

Notice that in each of these cases the polynomial $M^{(k)}(A)$ is exactly the kth elementary symmetric function $\sigma_k^K(A)$ of the eigenvalues of the K hermitian symmetric part of A for $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$.

Certain versions of the inhomogeneous Monge-Ampère can be treated by the methods in [HL$_{4,5}$]. For example one can insert a function $f(x, u)$ with $f_u \geq 0$. One can also address all other branches of the determinant in this inhomogeneous form.

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