GEOMETRIC RESIDUE THEOREMS

Reese Harvey and H. Blaine Lawson, Jr.*

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Introduction.

Roughly speaking residue theorems in geometry are results which associate topological invariants to the singularities of geometric objects. The discovery and use of such theorems has a history dating back at least to Riemann. A classical example is Hopf's theorem relating the singularities of vector fields to the Euler characteristic.

A somewhat different topic in modern geometry is Chern-Weil Theory. This associates to a smooth bundle with connection a canonical family of differential forms which represent characteristic classes of the bundle. The forms are written explicitly as universal polynomials in curvature. Furthermore, for two distinct connections ω, ω' on a bundle, the difference of the characteristic forms can be written as a coboundary $p(\Omega) - p(\Omega') = dT$ where $T = T(\omega, \omega')$ is also *canonically* expressed in terms of the connections. These transgression forms T lead to important secondary invariants (cf. [CS], [ChS]).

Recently the authors developed a generalized Chern-Weil Theory for singular connections [HL$_2$] where characteristic forms are replaced by characteristic currents written in terms of curvature and the singularities of some given geometric object. In this paper we shall use our theory to systematically deduce a wide variety of geometric residue theorems. Our formulas refine the classical ones in several ways. To begin they are derived *canonically* at the level of differential forms and currents. For example, for a mapping α between bundles with connection there are formulas

$$p(\Omega) - \Sigma(\alpha) \;=\; dT$$

where: $p(\Omega)$ is a canonical characteristic form as above, $\Sigma(\alpha)$ is a current defined purely in terms of the singularities of α, and T is a **canonical transgression** form (with L^1_{loc}-coefficients). This enables us to define secondary invariants for certain connections and singularities.

Furthermore, our theory generates canonical smooth families

$$p(\Omega) - p(\Omega_s) \;=\; dT_s \qquad \text{for} \;\; 0 < s \leq \infty$$

where $T_\infty = 0$ and where one has convergence

$$T_s \to T \qquad \text{everywhere in } L^1_{\text{loc}}$$

as $s \to 0$. In particular the families of smooth characteristic forms $p(\Omega_s)$ converge to the singular current, i.e.,

$$p(\Omega_s) \to \Sigma(\alpha)$$

as $s \to 0$. In certain "approximation modes" $p(\Omega_s)$ will be supported in the s-tubular neighborhood of $\Sigma(\alpha)$. The virtue of the explicit nature of these formulas was seen in [HL$_2$] where the procedure gave simple explicit formulas for the Thom class of a bundle with connection. In fact it gave families interpolating between the pull-back of the Euler (or top Chern) form and the current given by zero-section of the bundle.

The emphasis in this paper will be as much on the general method as on the detailed structure of various formulas. The authors hope to provide the reader with the techniques for explicitly relating singularities of maps, sections of bundles etc. to characteristic forms in the manner above whenever the need arises. However, we shall also derive here a series of such explicit formulas in a broad range of fields. These will include: Thom-Porteous formulas at the level of forms and currents, formulas of Poincaré-Lelong type between Chern/Pontjagin forms and linear dependency currents of families of cross-sections of a bundle, residue theorems relating degeneracies of maps between manifolds and characteristic forms, residue theorems for singularities of CR-structures, new invariants for pairs of complex structures, invariants for pairs of plane fields, higher self-intersection formulas for tangent plane fields, higher order contact currents for pairs of foliations and relations to characteristic forms. In a subsequent paper we shall similarly establish various determinental formulas and, in particular, explicit Poincaré-Lelong equations for Shubert cells on Grassmann manifolds.

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A notational convention: Throughout this paper X will denote a manifold which is *oriented* unless it is stated otherwise.

§1. Divisors and Atomicity.

In this section we review briefly the theory of atomic sections and divisors introduced in [HS]. This material enhances the range and applicability of the subsequent results, but it is not necessary for understanding their proofs. The reader could skip this section and simply replace "atomicity" everywhere by "non-degenerate vanishing".

Let $f : U \to \mathbf{R}^p$ be a C^∞-map where $U \subseteq \mathbf{R}^n$ is an open set.

Definition 1.1. The map f is **atomic** if

$$f^*\left(\frac{dy^I}{|y|^{|I|}}\right) \in L^1_{\text{loc}}$$

for all $I = (i_1, \ldots, i_p)$ such that $|I| = \Sigma i_k < p$.

Let $\rho : \mathbf{R}^p - \{0\} \longrightarrow S^{p-1}$ denote radial projection onto the unit sphere, and define

$$\begin{aligned}(1.2) \qquad \Theta &= \frac{1}{c_p}\rho^*(d\,\mathrm{vol}_{S^{p-1}}) \\ &= \frac{1}{c_p}\left(\sum_{k=1}^{p} y_k \frac{\partial}{\partial y_k}\right) \llcorner \left(\frac{dy_1 \wedge \cdots \wedge dy_p}{|y|^p}\right) \\ &= \frac{1}{c_p}\sum_{k=1}^{p}(-1)^{k-1}\frac{y_k}{|y|^p} dy_1 \wedge \cdots \wedge \widehat{dy}_k \wedge \cdots \wedge dy_p\end{aligned}$$

where $c_p = \mathrm{vol}(S^{p-1})$. The coefficients of this form are integrable in bounded neighborhoods of 0, and Θ satisfies the current equation

$$d\Theta = [0] \qquad \text{in } \mathbf{R}^p.$$

Note that if $f : U \to \mathbf{R}^p$ is atomic, then $f^*\Theta \in L^1_{\mathrm{loc}}$ on U.

Definition 1.3. Let $f : U \to \mathbf{R}^p$ be atomic. Then the **divisor** of f is the current of degree p (and dimension $n - p$) on U given by taking the exterior derivative of the potential $d(f^*\Theta)$, i.e.,

$$\mathrm{Div}(f) = d(f^*\Theta).$$

This current has the following properties.

$$d\,\mathrm{Div}(f) = 0 \tag{1.4}$$

$$\mathrm{supp}\,\mathrm{Div}(f) \subseteq \{x \in U : f(x) = 0\} \stackrel{\mathrm{def}}{=} Z(f) \tag{1.5}$$

$$\text{If } 0 \text{ is a regular value of } f, \text{ then } \quad \mathrm{Div}(f) = [Z(f)] \tag{1.6}$$

where $[Z(f)]$ is the current given by integration over the manifold $Z(f)$. The first two properties are obvious. The last is straightforward to verify. Note that the definition of $[Z(f)]$ involves a choice of orientation on $Z(f)$.

Theorem 1.7. ([HS]) *Let $\tilde{f}(x) = g(x)f(\phi(x))$ where $\phi : U \to U$ is a diffeomorphism and $g : U \to GL_p(\mathbf{R})$ is a smooth map. Then $\tilde{f}$ is atomic if and only if f is atomic. Furthermore, if $\det(g) > 0$ on U and ϕ is orientation preserving, then*

$$\phi_*\,\mathrm{Div}(\tilde{f}) = \mathrm{Div}(f).$$

As an immediate corollary the concepts of atomicity and divisor extend to sections of a vector bundle.

Definition 1.8. Let $E \to X$ be a smooth vector bundle over an n-manifold X. A smooth section $\mu \in \Gamma(E)$ is said to be **atomic** if each point $x \in X$ has a neighborhood with local coordinates and a local trivialization of E with respect to which μ is an atomic $\mathbf{R}^p$-valued function.

If E and X are oriented, and if $\mu \in \Gamma(E)$ is atomic, then $\mathrm{Div}(\mu)$ is a well defined current of degree p (and dimension $n - p$) on X, called the **divisor** of μ.

Remark 1.9. Note that $\mathrm{Div}(\mu)$ is well-defined in the non-orientable case provided that the first Stiefel-Whitney classes satisfy

$$w_1(E) = w_1(X)$$

in $H^1(X;\, \mathbf{Z}_2)$. This condition guarantees that we can choose local trivializations of E over a coordinate covering so that the changes of trivialization $g_{\alpha\beta}$ and the Jacobian matrices of the changes of local coordinates $\phi_{\alpha\beta}$ satisfy $\det(g_{\alpha\beta})\cdot\det(\phi_{\alpha\beta}) > 0$. (See [Z].)

In [HS] effective criteria are established which guarantee atomicity.

Theorem 1.10. ([HS]) *Let $f : U \to \mathbf{R}^p$ be real analytic. If $\dim Z(f) \leqq n - p$, then f is atomic.*

Theorem 1.11. ([HS]) *Suppose $f : U \to \mathbf{R}^p$ satisfies:*

(1) *There are constants $c > 0$, $N > 0$ such that*

$$\|f(x)\| \ \geqq \ c\,\mathrm{dist}(x, Z(f))^N,$$

(2) *The Minkowski dimension of $Z(f)$ is $< n - p + 1$.*

Then f is atomic.

It is also proved in [HS] that if f is atomic, its divisor is integrally flat. Hence one has the following "regularity".

Theorem 1.12. ([HS]) *Let $f : U \to \mathbf{R}^p$ be atomic. If the mass of* $\mathrm{Div}(f)$ *is locally finite, then* $\mathrm{Div}(f)$ *is locally rectifiable.*

§2. Degeneracy currents.

In this section we introduce the notion of the k^{th} degeneracy current of a bundle map. This is a current associated to the drop in rank of the map to rank $\leqq k$.

For the definitions we must fix some notation. Let $E \to X$ and $F \to X$ be smooth vector bundles over an oriented manifold X, where E and F are either both complex or both real, and let

$$m \;=\; \operatorname{rank} E \quad \text{and} \quad n \;=\; \operatorname{rank} F.$$

Fix an integer k with $0 \leqq k \leqq \min\{m, n\}$ and set

$$r \;=\; m - k.$$

Let

$$\pi : G_r(E) \longrightarrow X \tag{2.1}$$

be the smooth bundle whose fibre at $x \in X$ is the set of all r-dimensional linear subspaces of E_x (the fibre of E at x). Over $G_r(E)$ there is a tautological vector bundle U of rank r whose fibre at $P \in G_r(E)$ consists of all vectors $v \in P$. There is a natural bundle embedding

$$U \subset \pi^* E \tag{2.2}$$

and if we introduce a metric in E, this gives a natural splitting

$$\pi^* E \cong U \oplus U^{\perp}. \tag{2.3}$$

Suppose now that we are given a smooth bundle map

$$\alpha : E \longrightarrow F.$$

Then this lifts to a mapping $\pi^*\alpha : \pi^*E \to \pi^*F$ over $G_r(E)$, and composing with j gives a map

$$\hat{\alpha} \overset{\text{def}}{=} \pi^*\alpha\big|_U : U \longrightarrow \pi^* F. \tag{2.4}$$

Definition 2.5. The bundle map α is said to be **k-atomic** if $\hat{\alpha}$ is an atomic section of the bundle $\operatorname{Hom}(U, \pi^*F) = U^* \otimes \pi^*F$ over $G_r(E)$.

Definition 2.6. For a bundle map α which is k-atomic, we define its $\mathbf{k}^{\mathbf{th}}$ **degeneracy current** on X to be

$$\mathbb{D}_k(\alpha) \;=\; \pi_* \operatorname{Div}(\hat{\alpha})$$

where π_* denotes the push-forward of currents by $\pi : G_r(E) \to X$.

Note that when E and F are real bundles, $\mathrm{Div}(\hat{\alpha})$, and therefore $\mathbb{D}_k(\alpha)$, only make sense when $w_1(U^* \otimes \pi^* F) = w_1(G_r(E))$ (cf. 1.9). This condition holds when $m \equiv n \equiv k \pmod 2$ (See Appendix A, A.6 - A.10).

The codimension of $\mathrm{Div}(\hat{\alpha})$ is rn (or $2rn$ in the complex case), and the fibre dimension of $G_r(E)$ is rk (or $2rk$ respectively). Hence, we have

$$\operatorname{codim} \mathbb{D}_k \;=\; r(n-k) \;=\; (m-k)(n-k)$$

in the real case, and

$$\operatorname{codim} \mathbb{D}_k \;=\; 2(m-k)(n-k)$$

in the complex case.

Lemma 2.7. *For any k-atomic section α, one has*

$$\operatorname{supp} \mathbb{D}_k(\alpha) \subseteq \{x \in X : \operatorname{rank} \alpha_x \leqq k\}$$

Proof. If $x \in \operatorname{supp} \mathbb{D}_k(\alpha)$, then there exist a subspace $U \subset E_x$ of dimension r such that $\alpha_x \big|_U = 0$. Hence, $\operatorname{rank} \alpha_x \leqq m - r = k$. $\square$

Note that if $\operatorname{rank} \alpha_x = k$, then there is exactly one subspace of dimension r (namely $\ker \alpha_x$) on which $\alpha_x = 0$. That is, above each point of X where $\operatorname{rank} \alpha = k$, there is exactly one point in the zero set $Z(\hat{\alpha})$ of $\hat{\alpha}$.

Proposition 2.8. *Suppose $\hat{\alpha}$ vanishes non-degenerately. Then*

$$RK_k(\alpha) \stackrel{\text{def}}{=} \{x \in X : \operatorname{rank} \alpha_x \;=\; k\}$$

is a locally rectifiable set, and

$$\mathbb{D}_k(\alpha) \;=\; [RK_k(\alpha)]$$

i.e., $\mathbb{D}_k(\alpha)$ is the current given by integration over this set.

Proof. By hypothesis we know that $Z(\hat{\alpha})$ is a smooth proper submanifold of $G_r(E)$, and that $\mathrm{Div}(\hat{\alpha}) = [Z(\hat{\alpha})]$. Therefore, $\mathbb{D}_k(\alpha) = \pi_*[Z(\hat{\alpha})]$, i.e., $\mathbb{D}_k(\alpha)$ is the d-closed locally rectifiable current given by the push-forward of the manifold $Z(\hat{\alpha})$. This current has dimension $N = \dim X - (m-k)(n-k)$ ($N = \dim X - 2(m-k)(n-k)$ in the complex case). The Federer-Sard Theorem [Fe] implies that the set of critical values of the map $\pi \big|_{Z(\hat{\alpha})}$, from $Z(\hat{\alpha})$ to X, has Hausdorff N-dimensional measure zero. Hence $\pi_*[Z(\hat{\alpha})] = p[R]$ where R is the set of regular values and p is an integer. It remains to show that

$$R \subset RK_k(\alpha), \tag{2.9}$$

since, as noted above,

$$\pi : Z^0(\hat{\alpha}) \;=\; \{u \in Z(\hat{\alpha}) : \pi(U) \in RK_k(\alpha)\} \longrightarrow RK_k(\alpha)$$

is one to one. To see (2.9), we observe that if $\operatorname{rank} \alpha_x = k - p$, then

$$\{U \subset E_x : \operatorname{rank} U \;=\; r \quad \text{and} \quad U \subset \ker \alpha_x\} \;=\; \pi^{-1}(x) \cap Z(\hat{\alpha})$$

is a submanifold of $G_r(E_x)$ diffeomorphic to the Grassmannian of r-planes in $(r+p)$-space. Thus, the preimage under $\pi \big|_{Z(\hat{\alpha})}$ of each point x with $\operatorname{rank} \alpha_x < k$ is a smooth submanifold of positive dimension. $\square$

It is appropriate here to point out that bundle maps are generically k-atomic, in fact they generically satisfy the hypothesis of Proposition 2.8. Recall that a smooth cross-section of a bundle is said to **vanish non-degenerately** if its graph is transversal to the zero-section. The following Propsition is a minor modification of the standard Thom Transversality Theorem [GG].

Proposition 2.10. *Suppose X is compact with (possibly empty) boundary. Then the set of smooth bundle maps α for which $\hat{\alpha}$ vanishes non-degenerately is open and dense in the C^1-topology. Consequently, for any manifold X the set of such α is residual (i.e., contains the intersection of a countable family of open dense subsets.)*

Proof. Openness is clear. To prove density we fix a section α and a point $x_0 \in X$. Choose trivializations of E and F in a neighborhood $\mathcal{U}$ of x_0. Then we have a family of sections of $\mathrm{Hom}(E, F)$ over $\mathcal{U}$ given by

$$\alpha_L \;=\; \alpha\,\Big|_U + L$$

for $L \in \mathrm{Hom}(E_{x_0}, F_{x_0})$. This gives a family of sections $\hat{\alpha}_L$ of $\mathrm{Hom}(U, \pi^*F)$ over $\widetilde{\mathcal{U}} \stackrel{\text{def}}{=} \pi^{-1}(\mathcal{U})$. We think of this as a map of manifolds

$$\begin{aligned} \widetilde{\mathcal{U}} \times V &\longrightarrow \mathrm{Hom}(U, \pi^*F)\,\Big|_{\widetilde{U}} \\ (u, L) &\longmapsto \hat{\alpha}_L(u) \end{aligned} \tag{2.11}$$

where $V = \mathrm{Hom}(E_{x_0}, F_{x_0})$. This map (2.11) is actually a submersion. To see this note first that $\pi \circ \hat{\alpha}_L(u) = u$, and so the image of $T_u\widetilde{\mathcal{U}} \times \{0\} \subset T_{(u,L)}(\widetilde{\mathcal{U}} \times V)$ is a transversal to the fibre of π at all points. However, the map $\{u\} \times V \to \mathrm{Hom}(U, \pi^*F)_u$ is a surjective linear map at each u (which sends L to "$L\,\big|_U$"). This shows that (2.11) is a submersion.

Using a partition of unity and standard constructions one can globalize to a submersion

$$G_r(E) \times W \xrightarrow{\Psi} \mathrm{Hom}(U, \pi^*F) \tag{2.12}$$

where W is a finite-dimensional vector space, where

$$\Psi_w \stackrel{\text{def}}{=} \Psi\,\Big|_{G_r(E)\times\{w\}} \;=\; \hat{\alpha}_w$$

for a section $\alpha_w \in \Gamma(\mathrm{Hom}(E, F))$, and where $\alpha_0 = \alpha$. (Here W will be a direct sum of V's as above.) Since Ψ is a submersion, it is transversal to the zero-section of $\mathrm{Hom}(U, \pi^*F)$. By a standard argument using Sard's Theorem for families (cf. [HL_1]) we conclude that Ψ_w is transversal to the zero section of $\mathrm{Hom}(U, \pi^*F)$ for almost all $w \in W$. $\square$

In general k-atomicity is much weaker than requiring $\hat{\alpha}$ to vanish non-degenerately. One useful criterion for k-atomicity is the following.

Lemma 2.13. *Let α be a real analytic bundle map. If the set $Z(\hat{\alpha}) \subset G_r(E)$ has codimension $\geqq (m-k)n$, then α is k-atomic.*

Proof. This is an immediate application of Theorem 1.10 above. $\square$

Proposition 2.14. *Suppose α is real analytic, and satisfies the condition that*

$$\begin{aligned} \operatorname{codim} RK_{k-p} \ &\geqq\ (m-k)(n-k+p), \\ (\operatorname{codim} RK_{k-p} \ &\geqq\ 2(m-k)(n-k+p) \quad \textit{in the complex case}), \end{aligned}$$

for all p, $0 \leq p \leq k$. Then α is k-atomic.

Proof. If $\operatorname{rank} \alpha_x = k-p$, then $\dim\ker \alpha_x = r+p$ and $\pi^{-1}(x) \cap Z(\hat{\alpha})$ is a Grassmannian of r-dimensional subspaces of $(r+p)$-space. This is a manifold of dimension rp ($2rp$ in the complex case). It follows that

$$\dim\{\pi^{-1}(RK_{k-p}) \cap Z(\hat{\alpha})\} \ =\ \dim(RK_{k-p}) + rp$$

for each p. Let $d = \dim X$. Then the condition of 2.13 will be satisfied if $\dim(RK_{k-p}) + rp \leqq d - (m-k)(n-k)$ for all p, i.e., if $\operatorname{codim}(RK_{k-p}) \geqq (m-k)(n-k) + rp = (m-k)(n-k+p)$ for all p (with appropriate changes in the complex case). $\square$

We now address the problem of real bundles. If E and F are real, then there will be two cases of interest for our discussion. In neither case is E or F assumed to be orientable.

2.15. Real Bundles; Case I. Here we assume that m, n and k are all even integers. In this case $\operatorname{Hom}(U, \pi^*F)$ is canonically oriented even when U and F are not. Furthermore the fibre of π, which is the Grassmannian $G_r(\mathbf{R}^m)$ of unoriented r-planes in $\mathbf{R}^m$ ($r = m-k$), is also oriented. See Corollary A.6 and Corollary A.8 in Appendix A for proofs.

2.16. Real Bundles; Case II. Here we assume that m, n and k are all odd integers. In this case neither $\operatorname{Hom}(U, \pi^*F)$ nor $G_r(E)$ are oriented, but they have the same first Stiefel-Whitney class. (See Corollary A.6 and Corollary A.8 in Appendix A.) Consequently, $\operatorname{Div}(\hat{\alpha})$ is defined. The fibre $G_r(\mathbf{R}^m)$ of π is also not oriented, so one must be careful when integrating over the fibre. (See Proposition A.11 and it's proof.)

In the remaining cases a direct analysis shows that either $w_1(G_r(E)) \neq w_1(\operatorname{Hom}(U, \pi^*F))$, and so $\operatorname{Div}(\hat{\alpha})$ is not defined as a current, or the fibre integral of the Euler form $\chi(\operatorname{Hom}(U, \pi^*F))$ is zero, and so results of the type obtained in §5 are uninteresting.

§3. Poincaré-Lelong families.

In this section we briefly recall a basic result of [HL$_2$] which provides the first step in all of our subsequent constructions. Let $V \to X$ be a smooth complex vector bundle of rank m with a complex connection D and a metric $< \cdot, \cdot >$ (which is in general unrelated to the connection), and let μ be an atomic section of V.

From this data we introduce a family of connections $\overrightarrow{D}_s$ on V which become singular as $s \searrow 0$ precisely along the zeros of μ. Taking the determinant of the curvature $\overrightarrow{R}_s$ of $\overrightarrow{D}_s$ gives a family of smooth $2m$-forms which converge as $s \to 0$ to $\mathrm{Div}(\mu)$. Moreover, the transgression forms for this family also converge and provide a canonical and functorial coboundary between $c_m(R^V)$ and $\mathrm{Div}(\mu)$, where $R^V = D^2$ is the curvature of the given connection.

To begin the process we fix an **approximation mode** by choosing a function $\chi \in C^\infty([0,\infty])$ with $\chi(0) = 0$, $\chi(\infty) = 1$, and $\chi' \geqq 0$. We then define the family of connections $\overrightarrow{D}_s$ by setting

$$\overrightarrow{D}_s \nu \;=\; D\nu - \chi_s \frac{< \nu, \mu >}{|\mu|^2} D\mu \tag{3.1}$$

on sections $\nu \in \Gamma(V)$, where $\chi_s = \chi(|\mu|^2/s^2)$. This is a smooth family of smooth connections on F for all $s > 0$. Let

$$\overrightarrow{R}_s \;=\; (\overrightarrow{D}_s)^2 \tag{3.2}$$

denote the curvature of $\overrightarrow{D}_s$, and consider the family of forms

$$\tau_s \overset{\text{def}}{=} \det\left(\tfrac{i}{2\pi}\overrightarrow{R}_s\right) \;=\; c_m(\overrightarrow{R}_s). \tag{3.3}$$

Note that $\tau_\infty = c_m(R^V)$ is the top Chern form of the given connection. For all $s > 0$ there is a transgression form

$$\sigma_s \;=\; \left(\tfrac{i}{2\pi}\right)^m \int_s^\infty \det\left(\dot{\overrightarrow{D}}_t \;;\; \overrightarrow{R}_t\right) dt \tag{3.4}$$

where

$$\det(A;B) \;=\; \frac{d}{dt}\det(B+tA)\,\Big|_{t=0} \;=\; \mathrm{tr}(\widetilde{B}A)$$

where $\widetilde{B}$ is the transposed matrix of cofactors of B, and where $\dot{\overrightarrow{D}}_t = (d/dt)\overrightarrow{D}_t$. These $(2m-1)$-forms σ_s have the property that

$$d\sigma_s \;=\; c_m(R^V) - \tau_s. \tag{3.5}$$

Our main result is the following.

Theorem 3.6. ([HL$_2$]) *Let μ be an atomic section of the bundle V with connection D as above. Then the limit $\sigma = \lim_{s\to 0} \sigma_s$ exists in the space of forms on X with $L^1_{\mathrm{loc}}(X)$-coefficients. This limit is independent of choice of approximation mode and satisfies the current equation*

$$c_m(R^V) - \mathrm{Div}(\mu) \;=\; d\sigma. \tag{3.7}$$

In particular, the family of L^1_{loc}-forms $\rho_s \overset{\mathrm{def}}{=} \sigma - \sigma_s$ satisfies the equation

$$\tau_s - \mathrm{Div}(\mu) \;=\; d\rho_s \tag{3.8}$$

and has the property that

$$\lim_{s\to 0} \rho_s \;=\; 0 \quad \textit{and} \quad \lim_{s\to\infty} \rho_s \;=\; \sigma \tag{3.9}$$

in L^1_{loc}.

Equation (3.7) generalizes the classical Poincaré-Lelong formula for line bundles and for this reason we call τ_s the **Poincaré-Lelong family**. This family provides canonical smoothings of the divisor $\mathrm{Div}(\mu)$. If χ has the property that $\chi(t) = 1$ for all $t \geqq 1$, then τ_s has the additional property that

$$\mathrm{supp}(\tau_s) \subseteq \{x \in X \;:\; \|\mu_x\| \leqq s\}$$

for all $s > 0$.

There is a companion result when V is a real oriented bundle of rank $2m$ with an orthogonal connection D, and an atomic section μ. We fix any χ as above and introduce a smooth family of orthogonal connections defined on a section $\nu \in \Gamma(V)$ by

$$\overrightarrow{D}_s \nu \;=\; D\nu - \chi_s \frac{<\nu, \mu>}{|\mu|^2} D\mu + \chi_s \frac{<D\nu, \mu>}{|\mu|^2}\mu \tag{3.10}$$

where $\chi_s \;=\; \chi(|\mu|^2/s^2)$ as above. Let $\overrightarrow{R}_s \;=\; (\overrightarrow{D}_s)^2$, and consider the family of closed $2m$-forms

$$\tau^{\mathbf{R}}_s \overset{\mathrm{def}}{=} \mathrm{Pf}\left(-\tfrac{1}{2\pi}\overrightarrow{R}_s\right) \tag{3.11}$$

where $\mathrm{Pf}(A)$ denotes the Pfaffian of a skew-symmetric matrix A. We also define transgression forms

$$\sigma^{\mathbf{R}}_s \;=\; \left(\tfrac{-1}{2\pi}\right)^m \int_s^\infty \mathrm{Pf}(\overrightarrow{D}_t \;;\; \overrightarrow{R}_t)dt \tag{3.12}$$

where

$$\mathrm{Pf}(A\;;\;B) \;=\; \frac{d}{dt}\mathrm{Pf}(B + tA)\,\Big|_{t=0}.$$

These forms satisfy the equation

$$d\sigma^{\mathbf{R}}_s \;=\; \boldsymbol{\chi}(R^V) - \tau^{\mathbf{R}}_s \qquad \text{for} \quad s > 0$$

where $\boldsymbol{\chi}(R^V) = \mathrm{Pf}\left(-\frac{1}{2\pi}R^V\right)$ is the Chern-Euler form of the connection D.

Theorem 3.13. ([HL$_2$]) *Let μ be an atomic section of the oriented bundle V with orthogonal connection D as above. Then the limit $\sigma^{\mathbf{R}} = \lim_{s\to 0} \sigma_s^{\mathbf{R}}$ exists in $L^1_{\rm loc}(X)$ and is independent of the choice of approximation mode. It satisfies the current equation*

$$\chi(R^V) - \mathrm{Div}(\mu) \ = \ d\sigma^{\mathbf{R}}. \tag{3.14}$$

In particular, the $L^1_{\rm loc}$-forms $\rho_s^{\mathbf{R}} = \sigma^{\mathbf{R}} - \sigma_s^{\mathbf{R}}$ satisfy

$$\tau_s^{\mathbf{R}} - \mathrm{Div}(\mu) \ = \ d\rho_s^{\mathbf{R}} \tag{3.15}$$

on X and have the property that

$$\lim_{s\to 0} \rho_s^{\mathbf{R}} \ = \ 0 \quad and \quad \lim_{s\to \infty} \rho_s^{\mathbf{R}} \ = \ \sigma^{\mathbf{R}} \tag{3.16}$$

in $L^1_{\rm loc}(X)$.

We will call $\tau_s^{\mathbf{R}}$ the **Euler family** associated to D, μ and χ.

The results above enhance the fundamental work of of [Ch] and [ChB], [ChB$_2$] where the potentials σ and σ_s were introduced in the special case of the tautological section over the total space of V. In this universal case our family τ_s (and $\tau_s^{\mathbf{R}}$) provide families of canonical Thom forms for $s > 0$ which converge to the zero section as $s \to 0$. See Appendix C for simple explicit formulas for τ_s, σ_s, σ, $\tau_s^{\mathbf{R}}$, $\sigma_s^{\mathbf{R}}$, and $\sigma^{\mathbf{R}}$ taken from [HL$_2$].

§4. Thom-Porteous families (complex case).

In this section we generalize the results of §3 to arbitrary degeneracy currents. Let us fix smooth complex vector bundles $E \to X$ and $F \to X$ equipped with connections D^E and D^F and with hermitian metrics (not necessarily related to the connections). We suppose that $\mathrm{rank}(E) = m$ and $\mathrm{rank}(F) = n$ and we assume that X is oriented.

In terms of the given connections we shall derive characteristic forms which are cohomologous to the degeneracy currents (cf. [T][P][M1][R]). To do this we must introduce the Shur polynomials. Suppose

$$\xi \;=\; 1 + \xi_1 + \xi_2 + \cdots$$

is a differential form on X where each ξ_k is homogeneous of degree $2k$. Then for non-negative integers a and b we define the **Shur polynomial** in ξ by

$$\Delta_a^{(b)}(\xi) \;=\; \det\big(\!\big(\xi_{a-i+j}\big)\!\big)_{\substack{1\le i\le b\\ 1\le j\le b}} \tag{4.1}$$

i.e., $\Delta_a^{(b)}(\xi)$ is the homogeneous form given by the determinant of the $b \times b$ matrix whose $(i,j)^{\text{th}}$ entry is ξ_{a-i+j}. These polynomials satisfy the fundamental identity (cf. [Fu, pg. 264])

$$\Delta_a^{(b)}(\xi) \;=\; (-1)^{ab}\Delta_b^{(a)}(\xi^{-1}) \tag{4.2}$$

where ξ^{-1} is defined by the relation $\xi \cdot \xi^{-1} = 1$, and also the identity

$$\Delta_a^{(b)}(\xi) \;=\; (-1)^{ab}\Delta_a^{(b)}(\tilde{\xi}) \tag{4.3}$$

where $\tilde{\xi} \overset{\text{def}}{=} 1 - \xi_1 + \xi_2 - \xi_3 + \cdots$.

Consider now the **total Chern form** of the connection D^F given by

$$c(R^F) \;=\; \det\big(I + \tfrac{i}{2\pi}R^F\big) \;=\; 1 + c_1(R^F) + \cdots + c_n(R^F)$$

where $R^F = (D^F)^2$. The form $c(R^E)$ is defined similarly, and there is the inverse form $c(R^E)^{-1}$ determined by $c(R^E)c(R^E)^{-1} = 1$. The **Thom-Porteous form** of type (a,b) is then defined to be the form

$$\Delta_a^{(b)}\{c(R^F)c(R^E)^{-1}\}.$$

We consider now a smooth bundle map

$$\alpha : E \longrightarrow F$$

and we fix an approximation mode χ as in §3. Let k be an integer with $0 \le k < \min\{m,n\}$ and set $N = 2(m-k)(n-k)$. Then we have the following.

Theorem 4.4. *Suppose that α is k-atomic. Then there exist a canonical smooth family of smooth N-forms $\mathbb{TP}_s$ and a smooth family of L^1_{loc} $(N-1)$-forms S_s on X, for $0 < s \leqq \infty$, such that*

$$\mathbb{TP}_s - \mathbb{D}_k(\alpha) \;=\; dS_s \tag{4.5}$$

for all s, and

$$\lim_{s\to 0} S_s \;=\; 0$$

in L^1_{loc}. Furthermore, the transgression form $S = S_\infty$ is independent of the choice of approximation mode χ and satisfies the current equation

$$\Delta^{(m-k)}_{n-k}\{c(R^F)c(R^E)^{-1}\} - \mathbb{D}_k(\alpha) \;=\; dS. \tag{4.6}$$

Proof. Consider the (atomic) section $\hat{\alpha}$ of the bundle $H \overset{\text{def}}{=} \text{Hom}(U, \pi^*F)$ introduced in §2. The metrics and connections on E and F induce a natural metric and connection on H. Thus, using χ, we can define a Poincaré-Lelong family of smooth $2(m-k)n$-forms τ_s and canonical L^1_{loc}-forms ρ_s associated to $\hat{\alpha}$. They satisfy the equation

$$\tau_s - \text{Div}(\hat{\alpha}) \;=\; d\rho_s \tag{4.7}$$

for all $0 < s \leqq \infty$. Applying π_* to (4.7) gives the equation

$$\mathbb{TP}_s - \mathbb{D}_k(\alpha) \;=\; dS_s \tag{4.8}$$

where $\mathbb{TP}_s \overset{\text{def}}{=} \pi_*\tau_s$ (called the **Thom-Porteous family**) and $S_s \overset{\text{def}}{=} \pi_*\rho_s$ are the smooth and L^1_{loc} forms respectively obtained by integration over the fibre of the smooth bundle $\pi : G_r(E) \to X$. From 3.6 we know that $\sigma = \rho_\infty$ satisfies the equation

$$c_M(R^H) - \text{Div}(\hat{\alpha}) \;=\; d\sigma \tag{4.9}$$

on $G_r(E)$, where $M = (m-k)n$. Therefore, applying π_* to (4.9) gives the formula

$$\pi_* c_M(R^H) - \mathbb{D}_k(\alpha) \;=\; dS \tag{4.10}$$

on X, where $S = \pi_*\rho_\infty = \pi_*\sigma$.

It remains to compute $\pi_* c_M(R^H)$. To begin we recall from Appendix A that the Shur polynomials are exactly what is needed to compute the top Chern form of a tensor product connection such as that on $H = U^* \otimes \pi^*F$. The formula is

$$c_M(R^H) \;=\; \Delta^{(m-k)}_n\{\pi^* c(R^F)\cdot c(R^U)^{-1}\}. \tag{4.11}$$

We now recall the splitting

$$\pi^*E \;=\; U \oplus U^\perp. \tag{4.12}$$

There are two connections on π^*E: the induced one π^*D^E, and the direct sum connection $D^U \oplus D^{U^\perp}$ induced from π^*D^E by projection onto the factors in (4.12). Taking the canonical convex family of connections joining these two and using the standard transgression formula [HL$_2$, I.1.19] gives an equation of smooth forms

$$c(R^U)c(R^{U^\perp}) \;=\; \pi^*c(R^E) + d\eta$$

on $G_r(E)$. This can be rewritten as

$$c(R^U)^{-1} \;=\; \pi^*c(R^E)^{-1}\cdot c(R^{U^\perp}) + d\eta'$$

where $\eta' = c(R^U)^{-1}\pi^*c(R^E)^{-1}\eta$. Plugging this into (4.9) gives the formula

$$c_M(R^H) \;=\; \Delta_n^{(m-k)}\left\{\pi^*\left(c(R^F)c(R^E)^{-1}\right)c(R^{U^\perp})\right\} + d\eta'' \tag{4.13}$$

for a smooth form η'' on $G_r(E)$.

We now observe that $c(R^{U^\perp})$ is of the form

$$c(R^{U^\perp}) \;=\; 1 + c_1(R^{U^\perp}) + \cdots + c_k(R^{U^\perp}).$$

Furthermore, the fibre dimension of π is $2(m-k)k$, and so π_* has the property that

$$\pi_*\{(\pi^*\phi)\psi\} \;=\; \phi(\pi_*\psi_0)$$

for smooth forms ϕ on X and ψ on $G_r(E)$, where ψ_0 is the homogeneous component of ψ of degree $2(m-k)k$. Consequently, applying π_* to (4.13) gives

$$\pi_*c_M(R^H) \;=\; \Delta_{n-k}^{(m-k)}\{c(R^F)c(R^E)^{-1}\}\pi_*(c_k(R^{U^\perp})^{(m-k)}) + dS' \tag{4.14}$$

where $S' = \pi_*\eta''$ is a smooth form on X. To see this observe that the only $(m-k)$-fold product of Chern classes of $U^\perp$ which has degree $2(m-k)k$ is $c_k(U)^{m-k}$. Hence all other terms in the determinant can be dropped when integrating over the fibre. Now $\pi_*c_k(R^{U^\perp})^{m-k}$ is a closed 0-form, i.e., a constant. This constant is determined by the topological class which is computed in Proposition B.1 of Appendix B. We prove there that

$$\pi_*(c_k(U^\perp)^{m-k}) \;=\; 1.$$

Hence, we have that

$$\pi_*c_M(R^H) \;=\; \Delta_{n-k}^{(m-k)}\{c(R^F)c(R^E)^{-1}\} + dS'.$$

Replacing S_s by $S_s - \left(\frac{s}{1+s}\right)S'$ and $\mathbb{P}_s$ by $\mathbb{P}_s - \left(\frac{s}{1+s}\right)dS'$ in equation (4.8) now gives the result. $\square$

Remark 4.15 The adjoint problem. Let $\alpha : E \longrightarrow F$ be as above and consider the adjoint mapping

$$\alpha^* : E^* \longrightarrow F^*$$

of the dual bundles. If α^* is k-atomic, the degeneracy current $\mathbb{D}_k(\alpha^*)$ is defined and Theorem 4.4 applies. In this case equation (4.6) becomes

$$\Delta_{m-k}^{(n-k)}\{c(R^{E^*})c(R^{F^*})^{-1}\} - \mathbb{D}_k(\alpha^*) \;=\; dS. \tag{4.16}$$

We note from the Shur relations (4.2) and (4.3) and the fact that $c(E^*) = \tilde{c}(E) = 1 - c_1(E) + c_2(E) - \ldots$, that

$$\Delta_{m-k}^{(n-k)}\{c(R^{E^*})c(R^{F^*})^{-1}\} \;=\; (-1)^d\Delta_{m-k}^{(n-k)}\{c(R^{E})c(R^{F})^{-1}\} \;=\; \Delta_{n-k}^{(m-k)}\{c(R^{F})c(R^{E})^{-1}\}$$

where $d = (m-k)(n-k)$. Hence the left hand sides of (4.6) and (4.16) coincide. Furthermore one can show that

$$\operatorname{supp} \mathbb{D}_k(\alpha^*) \;=\; \operatorname{supp} \mathbb{D}_k(\alpha),$$

and that for generic maps

$$\mathbb{D}_k(\alpha^*) \;=\; \mathbb{D}_k(\alpha). \tag{4.17}$$

One conjectures that (4.17) holds for general k-atomic bundle maps.

Remark 4.18. Holomorphic Case. In the special case where X, E, F and α are all holomorphic and where the D^E and D^F are canonical hermitian connections, equation (4.7) and therefore also its push-forward (4.8) can be written as $\partial\bar{\partial}$-equations, that is, the right hand sides can be replaced by a $\partial\bar{\partial}T$ where T is a current of bidegree q, q for appropriate q. Consequently in this case Theorem 4.4 has $\partial\bar{\partial}$-refinement.

§5. Thom-Porteous families (real case).

In this section we shall establish the analogue of Theorem 4.2 for morphisms $\alpha : E \to F$ of real vector bundles. To state the result in its proper generality we need some preliminary discussion.

Throughout this section X will be an oriented manifold and $E \to X$ and $F \to X$ will be smooth, real vector bundles furnished with orthogonal connections D^E and D^F respectively. We assume $\operatorname{rank} E = m$ and $\operatorname{rank} F = n$ and we fix an integer k with $0 \leq k < \min\{m, n\}$. We do *not* assume that E and F are orientable. There are two cases of interest.

Case I. The integers $m = 2m_0$, $n = 2n_0$, and $k = 2k_0$ are all even.

Case II. The integers $m = 2m_0 + 1$, $n = 2n_0 + 1$ and $k = 2k_0 + 1$ are all odd.

To state the theorem we need to consider the Shur polynomials of Pontrjagin forms. Suppose

$$\eta \;=\; 1 + \eta_1 + \eta_2 + \cdots$$

is a differential form on X where each η_k is homogeneous of degree $4k$. Then for a, $b \geqq 0$ we define

$$\widetilde{\Delta}_a^{(b)}(\eta) \;=\; \det\left(\!\left(\eta_{a-i+j}\right)\!\right)_{\substack{1\leq i\leq b\\ 1\leq j\leq b}}. \tag{5.1}$$

Associated to the connection D^F is the **total Pontrjagin form**

$$p(R^F) = 1 + p_1(R^F) + \cdots + p_{n_0}(R^F),$$

where $p(A) = \det\left(I + \frac{1}{4\pi^2}A^2\right)$. The form $p(R^E)$ is given similarly, and we define the **Thom-Porteous form** of type (a, b) in the Pontrjagin classes to be the homogeneous form

$$\widetilde{\Delta}_a^{(b)}\{p(R^F)p(R^E)^{-1}\}.$$

We now consider a smooth bundle map

$$\alpha : E \longrightarrow F,$$

and we fix an approximation mode χ as in §3.

Theorem 5.2. *Let E, F and k be as above (either Case I or Case II), and suppose that α is k-atomic. Let $N = (m-k)(n-k) = 4(m_0-k_0)(n_0-k_0)$. Then there exists a canonical smooth family of smooth N-forms $\mathbb{P}_s$, $s > 0$, and a smooth family of L^1_{loc} $(N-1)$-forms S_s on X, for $0 < s \leqq \infty$ such that*

$$\mathbb{P}_s - \mathbb{D}_k(\alpha) \;=\; dS_s \tag{5.3}$$

for all s, and

$$\lim_{s\to 0} S_s \;=\; 0$$

in L^1_{loc}. Furthermore, the transgression form $S = S_\infty$ is independent of approximation mode and satisfies the equation

$$\widetilde{\Delta}_{n_0-k_0}^{(m_0-k_0)}\{p(R^F)p(R^E)^{-1}\} - \mathbb{D}_k(\alpha) \;=\; dS \tag{5.4}$$

of currents on X.

Proof. Let $\pi : G_r(E) \to X$ be the fibre bundle whose fibre at x is the Grassmannian of unoriented r-planes in E_x, where

$$r = m - k = 2(m_0 - k_0) = 2r_0.$$

Let $U \to G_r(E)$ be the tautological r-plane bundle, and let $\hat{\alpha}$ be the section of

$$H = \mathrm{Hom}(U, \pi^* F) \cong U^* \otimes F$$

defined as in (2.4). By assumption $\hat{\alpha}$ is atomic. At this point our discussion breaks into the two cases above.

We begin with Case I. Here both the bundle $\mathrm{Hom}(U, \pi^*F)$ and the maninfold $G_r(E)$ are oriented. Using the approximation mode we can define the Euler family of smooth forms $\tau_s^{\mathbf{R}}$ and the L^1_{loc}-forms $\rho_s^{\mathbf{R}}$ associated to $\hat{\alpha}$ as in Theorem 3.11. They satisfy the equation

$$\tau_s^{\mathbf{R}} - \mathrm{Div}(\hat{\alpha}) = d\rho_s^{\mathbf{R}} \tag{5.5}$$

on $G_r(E)$ for all $0 < s \leqq \infty$. Applying π_* to (5.5) gives the equation

$$\mathbb{P}_s - \mathbb{D}_k(\alpha) = dS_s \tag{5.6}$$

where $\mathbb{P}_s \stackrel{\mathrm{def}}{=} \pi_* \tau_s^{\mathbf{R}}$ (called the **Thom-Porteous family**) and $S_s \stackrel{\mathrm{def}}{=} \pi_* \rho_s^{\mathbf{R}}$ are smooth and L^1_{loc} forms respectively on X. From 3.11 we know that $\sigma^{\mathbf{R}} = \rho_\infty^{\mathbf{R}}$ satisfies the equation

$$\boldsymbol{\chi}(R^H) - \mathrm{Div}(\hat{\alpha}) = d\sigma^{\mathbf{R}} \tag{5.7}$$

on $G_r(E)$ where $H = \mathrm{Hom}(U, \pi^*F)$, and where $\boldsymbol{\chi}(R^H) = \mathrm{Pf}\left(-\frac{1}{2\pi} R^H\right)$ is the Chern-Euler class of H. Applying π_* in (5.7) gives the equation

$$\pi_* \boldsymbol{\chi}(R^H) - \mathbb{D}_k(\alpha) = dS \tag{5.8}$$

on X, where $S = \pi_* \sigma$.

It remains to compute $\pi_* \boldsymbol{\chi}(R^H)$. To begin we recall from Theorem A.17 in Appendix A the formula

$$\boldsymbol{\chi}(R^H) = \widetilde{\Delta}_{n_0}^{(m_0 - k_0)} \{\pi^* p(R^F) \cdot p(R^U)^{-1}\}. \tag{5.9}$$

Transgressing between the pullback connection and its projection onto the splitting

$$\pi^* E = U \oplus U^\perp$$

gives the equation

$$p(R^U) p(R^{U^\perp}) = \pi^* p(R^E) + d\eta \tag{5.10}$$

which can be rewritten as

$$p(R^U)^{-1} \;=\; \pi^*p(R^E)^{-1}p(R^{U^\perp}) + d\eta'. \tag{5.11}$$

Plugging this into (5.9) gives the formula

$$\chi(R^H) \;=\; \widetilde{\Delta}_{n_0}^{(m_0-k_0)}\left\{\pi^*\left(p(R^F)p(R^E)^{-1}\right)p(R^{U^\perp})\right\} + d\eta'' \tag{5.12}$$

of smooth forms on $G_r(E)$. Arguing exactly as in the proof of Theorem 4.2, we now see that

$$\pi_*\chi(R^H) \;=\; \widetilde{\Delta}_{n_0-k_0}^{(m_0-k_0)}\left\{p(R^F)p(R^E)^{-1}\right\}\pi_*\left(p_{k_0}(R^{U^\perp})^{m_0-k_0}\right) + dS' \tag{5.13}$$

where $S' = \pi_*\eta''$ is a smooth form on X.

The term $\pi_*p_{k_0}(R^{U^\perp})^{m_0-k_0}$ is a d-closed 0-form, i.e., a constant. In Proposition B.5 of Appendix B it is proved that

$$\pi_*p_{k_0}(U^\perp)^{m_0-k_0} \;=\; 1, \tag{5.14}$$

and so this constant is 1. Replacing S_s by $S_s - \frac{s}{1+s}S'$ and $\mathbb{P}_s$ by $\mathbb{P}_s - \frac{s}{1+s}dS'$ in equation (5.6) now completes the proof for Case I.

The argument for Case II is highly analogous. The main difference comes from the fact that while X is oriented, $G_r(E)$ is not because the fibre of π is not orientable. Consequently, when passing through the proof, one must keep in mind the following points. Let O denote the orientation bundle for the manifold $G_r(E)$. Tensoring the exterior powers of the cotangent bundle of $G_r(E)$ by the real line bundle O yields bundles whose sections are called twisted differential forms.

A. Currents of dimension k on the d dimensional manifold $G_r(E)$ include twisted forms of degree $d-k$ which can be allowed to have $L^1_{\rm loc}$ coefficients.
B. Twisted forms with $L^1_{\rm loc}$ coefficients can be integrated over the fibre and this corresponds to current push forward π_*.
C. $\chi(R^U)$ is a twisted form on $G_r(E)$.

The main calculational difference in the argument is that equation (5.9) must be replaced by

$$\chi(R^H) \;=\; (-1)^{(n_0+1)r_0}\chi(R^U)\widetilde{\Delta}_{n_0}^{(m_0-k_0)}\{\pi^*p(R^F)\cdot p(R^U)^{-1}\} \tag{5.15}$$

which is proved in Theorem A.20 in Appendix A, and (5.14) is replaced by

$$\pi_*\{\chi(U)p_{m_0}(U^\perp)^{m_0-k_0}\} \;=\; 1, \tag{5.16}$$

which is proved in Proposition B.11 of Appendix B. $\square$

The formulas which appear in Theorem 5.2 were computed at the cohomology level by R. MacPherson in his Harvard University Thesis in 1970 (cf. [M1][M2]).

Remark 5.17 The adjoint problem. Let $\alpha : E \longrightarrow F$ be as above and consider the adjoint mapping

$$\alpha^* : E^* \longrightarrow F^*$$

of the dual bundles. If α^* is k-atomic, the degeneracy current $\mathbb{D}_k(\alpha^*)$ is defined and Theorem 5.2 applies. In this case equation (5.4) becomes

$$\widetilde{\Delta}^{(n_0-k_0)}_{m_0-k_0}\{p(R^E)p(R^F)^{-1}\} - \mathbb{D}_k(\alpha^*) \;=\; dS. \tag{5.18}$$

From (4.2) we see that

$$\widetilde{\Delta}^{(n_0-k_0)}_{m_0-k_0}\{p(R^E)p(R^F)^{-1}\} \;=\; (-1)^d\widetilde{\Delta}^{(m_0-k_0)}_{n_0-k_0}\{p(R^F)p(R^E)^{-1}\}$$

where $d = (m_0 - k_0)(n_0 - k_0)$. It can be shown that $\operatorname{supp}\mathbb{D}_k(\alpha^*) \;=\; \operatorname{supp}\mathbb{D}_k(\alpha)$ and that for generic maps

$$\mathbb{D}_k(\alpha^*) \;=\; (-1)^d\mathbb{D}_k(\alpha). \tag{5.19}$$

One conjectures that (5.19) holds for general k-atomic bundle maps.

Remark 5.20. A version of Theorem 5.2 holds for any pair of bundles E and F whenever $k = 0$, provided that $E^* \otimes F = \mathrm{Hom}(E, F)$ is oriented or, more generally, that $w_1(E^* \otimes F) = w_1(X)$. In this version the characteristic form in equation (5.4) is just the Euler form of $\chi(E^* \otimes F)$ and $\mathbb{D}_k(\alpha)$ is just the divisor of α considered as a section of $\mathrm{Hom}(E, F)$. This result is merely an instance of the general results discussed in §3.

§6. Characteristic forms and the degeneracies of k-frame fields.

Among the classical theorems in topology are those which relate the characteristic classes of a bundle to the singularities of fields of k-frames in the bundle. We shall derive such formulas, at the level of forms and currents, as direct consequences of the general theory. The results generalize the classical Chern formula $c_n(F) = \mathrm{Div}(\alpha) + dT$ and our families (cf. §2) to all Chern and Pontryagin classes.

Let $F \to X$ be a smooth complex vector bundle of rank n with connection D^F over an oriented manifold X, and consider a set of $k+1$ cross-sections $\alpha_0, \ldots, \alpha_k \in \Gamma(F)$ where $k < n$. This is equivalent to a bundle map

$$\alpha : \underline{\mathbf{C}}^{k+1} \longrightarrow F$$

from the trivial bundle $\underline{\mathbf{C}}^{k+1}$ given by setting $\alpha_x(t_0, \ldots, t_k) = \Sigma t_i \alpha_i(x)$ for $x \in X$. We shall say that this frame field $(\alpha_0, \ldots, \alpha_k)$ has a **good dependency locus** if the map α is k-atomic. In this case we can define the **linear dependency current**

$$\mathbb{LD}(\alpha_0, \ldots, \alpha_k) \stackrel{\mathrm{def}}{=} \mathbb{D}_k(\alpha). \tag{6.1}$$

Theorem 6.2. *Let $F \to X$ be a complex vector bundle with connection over an oriented manifold, and suppose that $\alpha_0, \ldots, \alpha_k \in \Gamma(F)$ are $k+1$ smooth sections with a good dependency locus. Then there exists an L^1_{loc}-form S on X such that*

$$c_{n-k}(R^F) \;=\; \mathbb{LD}(\alpha_0, \ldots, \alpha_k) + dS \tag{6.3}$$

where $n = \mathrm{rank}(F)$. Furthermore, there exist smooth families of smooth forms $\mathbb{P}_s$ and L^1_{loc}-forms S_s, $0 < s \leqq \infty$, with $\mathbb{P}_\infty = c_{n-k}(R^F)$ and $S_\infty = S$, such that

$$\mathbb{P}_s \;=\; \mathbb{LD}(\alpha_0, \ldots, \alpha_k) + dS_s \tag{6.4}$$

and $\lim_{s \to 0} S_s = 0$ in L^1_{loc}.

Proof. We endow the trivialized bundle $\underline{\mathbf{C}}^{k+1}$ over X with the canonical metric and flat connection. We introduce on F a smooth hermitian metric (not necessarily related to D^F), and we choose an approximation mode χ. In terms of this data, Theorem 4.4 provides the families $\mathbb{P}_s$ and S_s for the current $\mathbb{D}_k(\alpha)$. Formula (4.6) translates directly into (6.3) above. $\square$

Note. In algebraic geometry these linear dependency classes are used to *define* the Chern classes in the Chow ring of a smooth variety (cf. [Fu]).

Theorem 6.2 gives a direct proof of the basic fact that if $\alpha_0, \ldots, \alpha_k$ are linearly dependent on a small set, i.e., one of dimension $< n - k$, and if they vanish algebraically as in Theorem 1.11 (1), then $[c_{n-k}] = 0$ in $H^{2(n-k)}(X)$.

Theorem 6.2 can be generalized to higher dependencies. Fix integers $\ell > 0$ and k with $0 \le k < \text{rank}(F)$, and consider $k+\ell$ smooth cross-sections $\alpha_1, \dots, \alpha_{k+\ell} \in \Gamma(F)$. They give rise to bundle map

$$\alpha : \underline{\mathbf{C}}^{k+\ell} \longrightarrow F$$

as above. We say that the frame field $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{k+\ell})$ has a **good ℓ-dependency locus** if α is k-atomic. In this case we define the **ℓ-dependency current**

$$\mathbb{LD}_\ell(\boldsymbol{\alpha}) \;=\; \mathbb{D}_k(\alpha) \tag{6.5}$$

which measures in a rigorous way the set of points x where at least ℓ of the vectors $\alpha_1(x), \dots, \alpha_{k+\ell}(x)$ become linearly dependent on the remaining ones.

Theorem 6.6. *Let F be as above and suppose $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{k+\ell})$ are $k+\ell$ sections of F with a good ℓ-dependency locus. Then there is an L^1_{loc}-form S on X such that*

$$\det{}_{\ell\times\ell}\left(\!\left(c_{n-k-i+j}(R^F)\right)\!\right) \;=\; \mathbb{LD}_\ell(\boldsymbol{\alpha}) + dS. \tag{6.7}$$

Furthermore there exist families $\mathbb{P}_s$ and S_s, $0 < s \leqq \infty$, with properties analogous to those in Theorem 6.2.

Proof. One applies Theorem 4.4 as in the proof above. $\square$

Of course one retrieves Theorem 6.2 from 6.6 as the special case where $\ell = 1$. At the other extreme we can take $k = 0$. Note that $\mathbb{LD}_\ell(\alpha_1, \dots, \alpha_\ell)$ measures the simultaneous vanishing of ℓ generic sections of F. Here we get the predictable formula

$$c_n(R^F)^\ell \;=\; \mathbb{LD}_\ell(\alpha_1, \dots, \alpha_\ell) + dS. \tag{6.8}$$

There are corresponding theorems in the real case. Let $F \to X$ be a smooth real vector bundle of rank n with orthogonal connection D^F. Consider smooth cross-sections $\alpha_1, \dots, \alpha_{k+\ell} \in \Gamma(F)$ where $0 \leqq k < n$ and ℓ is even. We say that $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{k+\ell})$ has a **good ℓ-dependency locus** if the bundle map

$$\alpha : \underline{\mathbf{R}}^{k+\ell} \longrightarrow F$$

given by $\alpha_x(t_1, \dots, t_{k+\ell}) = \Sigma t_i \alpha_i(x)$ is k-atomic. In this case we define the **ℓ-dependency current**

$$\mathbb{LD}_\ell(\boldsymbol{\alpha}) \;=\; \mathbb{D}_k(\alpha).$$

Since $\ell = 2\ell_0$ is even, there are two cases to consider:

Case I. $\quad n = 2n_0 \quad$ and $\quad k = 2k_0$
Case II. $\quad n = 2n_0 + 1 \quad$ and $\quad k = 2k_0 + 1$

for integers n_0, $k_0 \geqq 0$. In Case II we assume F to be orientable.

Theorem 6.9. *Let F and $\boldsymbol{\alpha}$ be as above (either Case I or Case II) and suppose that $\boldsymbol{\alpha}$ has a good ℓ-dependency locus. Then there is an L^1_{loc}-form S on X such that*

$$\det{}_{\ell_0\times\ell_0}\left(\!\left(p_{n_0-k_0-i+j}(R^F)\right)\!\right) \;=\; \mathbb{LD}_\ell(\boldsymbol{\alpha}) + dS. \tag{6.10}$$

Furthermore there exist families $\mathbb{P}_s$ and S_s, $0 < s \le \infty$, with properties analogous to those in Theorem 6.2.

Proof. Introduce the canonical flat orthogonal connection on the trivialized bundle $\underline{\mathbf{R}}^{k+\ell}$ and apply Theorem 5.2. $\square$

Setting $\ell_0 = 1$ gives the following analogue of the formula in Theorem 6.2:

$$p_{n_0-k_0}(R^F) \;=\; \mathbb{LD}_2(\alpha_1,\ldots,\alpha_{k+2}) + dS \tag{6.11}$$

where $k = 2k_0$ or $2k_0+1$ depending on which of the Cases I or II we are considering. If we set $k_0 = 0$, we obtain

$$p_{n_0}(R^F)^{\ell_0} \;=\; \mathbb{LD}_{2\ell_0}(\alpha_1,\ldots,\alpha_{2\ell_0}) + dS \tag{6.12}$$

where $\mathbb{LD}_{2\ell_0}(\alpha_1,\ldots,\alpha_{2\ell_0})$ is measuring the simultaneous vanishing of the $2\ell_0$ sections.

Theorem 6.9 has some interesting applications in the next section.

§7. Singularities of projections.

Let X be a smooth oriented m-manifold with an immersion

$$j : X \looparrowright \mathbf{R}^N$$

into euclidean N-space. Fix an integer $n < N$ and consider the set of linear maps from $\mathbf{R}^N$ to $\mathbf{R}^n$. Each $P \in \mathrm{Hom}(\mathbf{R}^N, \mathbf{R}^n)$ restricts to give a smooth mapping

$$\widehat{P} \;=\; P \circ j : X \longrightarrow \mathbf{R}^n.$$

We are interested in studying the singularities of these projections. In particular for fixed $k < \min\{m, n\}$ and generic P we want to understand the locus where the differential

$$d\widehat{P} : TX \longrightarrow \underline{\mathbf{R}}^n$$

has rank $\leqq k$.

The projection P is called **k-atomic** on X if the bundle map $d\widehat{P}$ is k-atomic, and under this hypotheses one can define the **$\mathbf{k}^{\text{th}}$ degeneracy current** of the projection P on X to be

$$\mathbb{D}_k(P) \stackrel{\text{def}}{=} \mathbb{D}_k(d\widehat{P}).$$

Theorem 7.1. *Let $j : X \looparrowright \mathbf{R}^N$ be a C^∞ immersion of a smooth oriented m-manifold into euclidean space. Fix integers k and n with $k < \min\{m, n\}$ and with $k \equiv m \equiv n \pmod 2$. Then for almost all $P \in \mathrm{Hom}(\mathbf{R}^N, \mathbf{R}^n)$ the projection P is k-atomic on X and the following holds. There is a canonical L^1_{loc}-form $S = S(P, k, n)$ such that*

$$\det{}_{\ell_0 \times \ell_0}\big(\!\big(p_{m_0 - k_0 - i + j}(X)\big)\!\big) \;=\; \mathbb{D}_k(P) + dS \tag{7.2}$$

where $m_0 = [m/2]$, $n_0 = [n/2]$, $k_0 = [k/2]$, $\ell_0 = n_0 - k_0$, and where

$$p(R^X) \;=\; 1 + p_1(R^X) + p_2(R^X) + \cdots$$

is the total Pontrjagin form of X for its induced riemannian connection. Furthermore, there are smooth families of smooth forms $\mathbb{P}_s$ and L^1_{loc}-forms S_s, $0 < s \leqq \infty$, with $\mathbb{P}_\infty = \widetilde{\Delta}^{(n_0 - k_0)}_{m_0 - k_0}\{p(R^X)\}$ and $S_\infty = S$, such that

$$\mathbb{P}_s \;=\; \mathbb{D}_k(P) + dS_s \tag{7.3}$$

and $\lim_{s \to 0} S_s = 0$ in L^1_{loc}.

In particular, if $n_0 = k_0 + 1$ then there is a canonical cohomology

$$p_{m_0 - k_0}(R^X) \;=\; \mathbb{D}_k(P) + dS \tag{7.4}$$

between the $(m_0 - k_0)^{\text{th}}$ Pontrjagin form and the current which measures where the differential of the projection $\widehat{P} : X \to \mathbf{R}^{k+2}$ drops rank by 2.

Proof. To prove that P is k-atomic for almost all P we mimick the proof of Proposition 2.10. Set $W = \mathrm{Hom}(\mathbf{R}^N, \mathbf{R}^n)$ and let $\pi : G_r(\underline{\mathbf{R}}^n) \to X$ be the bundle projection and $\underline{\mathbf{U}} \to G_r(\underline{\mathbf{R}}^n)$ the tautological r-plane bundle, where $r = n - k$. Then there is a C^∞ map

$$(7.5) \qquad W \times G_r(\underline{\mathbf{R}}^n) \;=\; W \times G_r(\mathbf{R}^n) \times X \xrightarrow{\Psi} \mathrm{Hom}(\underline{\mathbf{U}}, \pi^* TX)$$

(into the total space of the bundle $\mathrm{Hom}(\underline{\mathbf{U}}, \pi^*TX)$ over $G_r(\underline{\mathbf{R}}^n)$) given by

$$\Psi(P, U, x) \;=\; \widetilde{P}^* \Big|_U \qquad \text{where} \quad \widetilde{P} \;=\; P \Big|_{T_x X}.$$

We claim that Ψ is transversal to the zero section of this bundle. To see this suppose that $\Psi(P, U, x) = 0$ and let $Q_0 \in \mathrm{Hom}(U, T_x X)$ be given. Extend Q_0 to $Q \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^N)$ by defining Q to be zero on $U^\perp$. Consider the curve

$$P_t \;=\; P + tQ^* \qquad \text{in } \; W.$$

Then

$$\frac{d}{dt}\Psi(P_t, U, x)\Big|_{t=0} \;=\; \left(Q^* \,\big|_{T_x X}\right)^* \Big|_U \;=\; Q\Big|_U \;=\; Q_0.$$

Hence $d\Psi$, restricted to $W \cong T_p W \subset T_{(P,U,x)}\,(W \times G_r(\mathbf{R}^n) \times X)$, maps onto the fibre of $\mathrm{Hom}(\underline{\mathbf{U}}, \pi^*TX)$ and is therefore transversal to the zero-section as claimed.

It now follows from Sard's Theorem for Families ([HL_1]) that for almost all $P \in W$ the restriction of Ψ to $\{P\} \times G_r(\underline{\mathbf{R}}^m)$ is transversal to the zero section of $\mathrm{Hom}(\underline{\mathbf{U}}, \pi^*TX)$. All such P are k-atomic. This proves the first assertion of the theorem. The remaining assertions are straightforward consequences of Theorem 6.9 applied to $d\widehat{P} : TX \longrightarrow \underline{\mathbf{R}}^n$ where TX carries the riemannian connection for the metric induced by j and where $\underline{\mathbf{R}}^n$ carries the canonical flat orthogonal connection. $\square$

§8. Singularities of maps.

We shall now considerably generalize the results of the last section. Let X and Y be smooth riemannian manifolds of dimensions m and n respectively (where Y need not be orientable) and consider a smooth mapping

$$f : X \longrightarrow Y.$$

Fix an integer $k < \min\{m, n\}$ and assume that $m \equiv n \equiv k \pmod 2$. Then the map f is said to be **k-atomic** if the differential

$$df : TX \longrightarrow f^*TY$$

is k-atomic. Arguing as in sections 2 and 7, one can show that generic smooth maps have this property. Whenever f is k- atomic, we can define its **$\mathbf{k^{th}}$ degeneracy current** $\mathbb{D}_k(f) \overset{\text{def}}{=} \mathbb{D}_k(df)$.

Theorem 8.1. *Let $f : X \to Y$, m, n and k be as above, and suppose that f is k-atomic. Set $m_0 = [m/2]$, $n_0 = [n/2]$ and $k_0 = [k/2]$. Then there is a canonical L^1_{loc}-form S on X such that*

$$\widetilde{\Delta}^{(m_0-k_0)}_{n_0-k_0}\left\{f^*p(R^Y)/p(R^X)\right\} \;=\; \mathbb{D}_k(f) + dS \tag{8.2}$$

where

$$p(R^X) \;=\; 1 + p_1(R^X) + p_2(R^X) + \cdots \quad \textit{and} \quad p(R^Y) \;=\; 1 + p_1(R^Y) + p_2(R^Y) + \cdots$$

are the total Pontrjagin forms in the riemannian curvatures of X and Y. Furthermore there are smooth families of smooth forms $\mathbb{P}_s$ and L^1_{loc}-forms S_s, $0 < s \leqq \infty$, with $\mathbb{P}_\infty = \Delta^{(m_0-k_0)}_{n_0-k_0}\{f^*p(R^Y)/p(R^X)\}$ and $S_\infty = S$, such that

$$\mathbb{P}_s \;=\; \mathbb{D}_k(f) + dS_s$$

and $\lim_{s\to 0} S_s = 0$ in L^1_{loc}.

Proof. This is a straightforward application of Theorem 5.2. □

An interesting special case occurs when $\dim X = \dim Y = 4$ and $k = 2$. Here (8.2) has the form

$$f^*p_1(Y) - p_1(X) \;=\; \Sigma n_i x_i + dS \tag{8.3}$$

where $\mathbb{D}_2(f) = \Sigma n_i x_i$ is a discrete sum of points with integer coefficients. Let $N_f = \Sigma n_j$ denote the **total 2-degeneracy number** of f.

This yields a 4-dimensional analogue of the classical Riemann-Hurwitz Theorem [M1], [M2], [R].

Corollary 8.4. *Suppose that $f : X \to Y$ is a smooth map between compact oriented 4-manifolds with isolated points of 2-degeneracy. Then*

$$M_f p_Y - p_X \;=\; N_f$$

where p_X and p_Y are the first Pontrjagin numbers of the manifolds X and Y respectively, M_f is the degree of the map f, and N_f is the total 2-degeneracy number of f.

In his thesis [S] Robert Stingley gives reinterpretations of N_f and methods of computing it in terms of the local geometry of the map f.

§9. Milnor currents.

There is a complete analogue of the formulas of §8 for the complex case. Such results are not new; they can be deduced from work by Fulton, Bismut, Gillet and Soulé. However we add them here for interest, and remark that the holomorphic assumptions here can be considerably relaxed to statements about smooth almost complex manifolds.

Let X and Y be complex manifolds of dimensions m and n respectively, and consider a holomorphic map

$$f : X \longrightarrow Y.$$

Introduce complex connections and hermitian metrics on X and Y. Fix $k < \min\{m, n\}$. We say that f is **k-atomic** if

$$\operatorname{codim}_{\mathbf{C}}\{x \in X : \operatorname{rank}(df_x) \leqq k\} \geqq (m-k)(n-k).$$

By 1.10 and 2.7 this hypothesis implies the existence of the k^{th} **degeneracy current** $\mathbb{D}_k(f) \overset{\text{def}}{=} \mathbb{D}_k(df)$.

Theorem 9.1. *Let $f : X \to Y$ be a holomorphic k-atomic map as above. Then there exists a canonical L^1_{loc}-form S on X such that*

$$\Delta_{n-k}^{(m-k)}\{f^* c(Y)/c(X)\} \;=\; \mathbb{D}_k(f) + dS \tag{9.2}$$

where $d = (m-k)(n-k)$ and where $c(X) = \det(1 + (i/2\pi)R^X)$ and $c(Y) = \det(1 + (i/2\pi)R^Y)$ are the total Chern forms of X and Y in their given connections. Furthermore, there are smooth families $\mathbb{P}_s$, τ_s, $0 < s \leqq \infty$ with properties analogous to those in 8.1.

Proof. This is a direct consequence of Theorem 4.4 and (4.2). $\square$

Remark. When X and Y are given the canonical hermitian connections associated to the metrics, the term dR in formula (9.2) can be replaced by $\partial\bar{\partial}T$ as noted in Remark 4.18.

An interesting case occurs when X and Y are compact, $\dim Y = n = 1$, and $k = 0$. Then f is a map to a complex curve, and it will be 0-atomic iff it has isolated singular points, say $x_1, \ldots, x_\ell \in X$. We define the **Milnor current** to be

$$\mathbb{M}(f) \overset{\text{def}}{=} \mathbb{D}_k(f) \;=\; \sum_{i=1}^{\ell} m_i[x_i]$$

where the integers m_i are the local *Milnor numbers*

$$m_i \;=\; \dim\left\{\mathcal{O}_{x_i}\Big/\left\langle \tfrac{\partial f}{\partial t_1}, \ldots, \tfrac{\partial f}{\partial t_m}\right\rangle\right\}.$$

Equation (9.2) has the form

$$(-1)^m\{c_m(R^X) - c_{m-1}(R^X) f^* c_1(R^Y)\} \;=\; \mathbb{M}(f) + dS. \tag{9.3}$$

When $n = 1$ this equation integrates to give the classical Riemann-Hurwitz Formula.

Another interesting case arises when $X \subset \mathbf{P}^N$ is a projective m-manifold and $f : X \to \mathbf{P}^m$ is given by linear projection $\mathbf{P}^N - \mathbf{P}^{N-m-1} \to \mathbf{P}^m$, where $X \cap \mathbf{P}^{N-m-1} = \emptyset$ (Noether normalization). Let ω be the Kähler form of the metric induced on X. Then (9.2) implies formulas of the following type. If f is $(m-k)$-atomic for $k^2 \leqq m$, then

$$\Delta_k^{(k)}\{c(X)(1+\omega)^{-(m+1)}\} \ = \ (-1)^{k^2}\mathbb{D}_{m-k}(f) + dS_k \tag{9.4}$$

for an L^1_{loc}-form S_k. Every such f is $(m-1)$-atomic, and $\mathbb{D}_{m-1}(f) = \mathbb{B}(f)$ is the **branching divisor** of f. This gives the formula:

$$(m+1)\omega - c_1(X) \ = \ \mathbb{B}(f) - dS. \tag{9.5}$$

For example if X is a curve of degree d and genus g, then (9.5) implies that $2(d+g-1) = |\mathbb{B}| =$ the total order of branching of f.

There are of course many many such formulas coming from methods of enumerative geometry.

§10. CR-singularities.

The methods introduced above yield some interesting results in CR geometry. Consider an immersion

$$f : X \looparrowright Z$$

of a real manifold X into a complex manifold Z, where

$$m \;=\; \dim_{\mathbf{R}}(X) \;\geq\; n \;=\; \dim_{\mathbf{C}}(Z).$$

Then the differential $df : TX \to f^*TZ$ extends to a *complex* bundle map

$$df_{\mathbf{C}} : TX \otimes_{\mathbf{R}} \mathbf{C} \longrightarrow f^*TZ. \tag{10.1}$$

Definition 10.2. Assume that the bundle map $df_{\mathbf{C}}$ is k-atomic where $0 \leq k < n$, and let $r = n - k$. Then we define the r^{th} **complex tangency current** of f to be

$$\mathbb{C}r_r(f) \;=\; \mathbb{D}_k(df_{\mathbf{C}}).$$

Roughly speaking this current corresponds to the locus of points x where f_*T_xX contains a complex subspace having r "excess" dimensions, i.e., more complex tangency (by r) than expected. Specifically we have:

Lemma 10.3. *The support of* $\mathbb{C}r_r(f)$ *satisfies*

$$\operatorname{supp} \mathbb{C}r_r(f) \subseteqq \{x \in X : \dim_{\mathbf{C}}(T_xX \cap JT_xX) \geqq m - n + r\}$$

where J denotes the almost complex structure of Z (and where for notational convenience we have identified T_xX with f_*T_xX).

Proof. By (2.7) we have $\operatorname{supp} \mathbb{C}r_r(f) \subseteq \{x \in X : \operatorname{rank}(df_{\mathbf{C}}) \leqq k\}$. Note that at $x \in X$,

$$\begin{aligned}
\ker(df_{\mathbf{C}}) \;&=\; \{V + iW : V, W \in T_xX \;\text{ and }\; V + JW = 0\} \\
&=\; \{V + iJV : V, JV \in T_xX\} \\
&=\; [(T_xX \cap JT_xX) \otimes \mathbf{C}]^{0,1}.
\end{aligned}$$

Since $\operatorname{rank}(df_{\mathbf{C}}) = m - \dim_{\mathbf{C}} \ker(df_{\mathbf{C}})$ we have $\operatorname{rank}(df_{\mathbf{C}}) \leqq k$ iff $\dim_{\mathbf{C}}(T_xX \cap JT_xX) \geqq m - k = m - n + r$. $\square$

We now suppose that X carries a riemannian metric and that Z carries a hermitian metric and a complex connection. Define $c(Z)$ as in (9.1) and set

$$\tilde{p}(X) = 1 - p_1(R^X) + p_2(R^X) - \cdots + (-1)^{[m/4]} p_{[m/4]}(R^X)$$

and

$$\tilde{c}(Z) = 1 - c_1(R^Z) + c_2(R^Z) - \cdots + (-1)^n c_n(R^Z)$$

where $p_i(R^X)$ and $c_i(R^Z)$ are the i^{th} Pontrjagin and Chern forms of X and Z respectively.

Theorem 10.4. *Let $f : X \to Z$ be an immersion of a real m-manifold into a complex n-manifold with $n \leqq m$ as above, and assume that $df_{\mathbf{C}}$ is $(n-r)$-atomic where $0 < r \leqq n$. Then there is a canonical L^1_{loc} form S on X such that*

$$\begin{aligned} (10.5) \qquad \Delta^{(r)}_{m-n+r}\{\tilde{p}(X)/f^*\tilde{c}(Z)\} &= \Delta^{(m-n+r)}_{r}\{f^*c(Z)/\tilde{p}(X)\} \\ &= \mathbb{C}r_r(f) + dR. \end{aligned}$$

Furthermore there are smooth families $\mathbb{P}_s$, S_s $0 < s \leqq \infty$ as in Theorems 6.2, 8.1, 9.1 etc.

Note 10.6. The term on the left is computed by writing $\tilde{p}(X)/f^*\tilde{c}(Z) = 1 + \xi_1 + \xi_2 + \cdots$, where ξ_i is a $2i$-form, and then applying formula (4.1).

Note 10.7. In Theorem 10.4 it suffices that Z be an almost complex manifold. Furthermore f need not be an immersion; but in this case Lemma 10.3 does not apply.

Proof of Theorem 10.4. Apply Theorem 4.4 to the bundle map $df_{\mathbf{C}}$ and note that

$$c(R^{T^*X\otimes\mathbf{C}}) \;=\; c(R^{TX\otimes\mathbf{C}}) \;=\; \tilde{p}(X).$$

This establishes the second equality in (10.5). The first equality is a direct consequence of the Shur relations (4.2) and (4.3). $\square$

Consider for example the case where $Z = \mathbf{C}^n$ and $r = 1$. This gives the following.

Corollary 10.8. *Let $f : X \looparrowright \mathbf{C}^{k+1}$ be an immersion of a smooth m-manifold with the property that $df_{\mathbf{C}}$ is k- atomic. Suppose $m - k = 2\ell > 0$. Then there is an L^1_{loc}-form S on X with*

$$\mathbb{C}r_1(f) \;=\; (-1)^\ell p_\ell(X) + dS.$$

Example 10.9. Suppose X^4 is a smooth 4-manifold and

$$f : X^4 \looparrowright \mathbf{C}^3$$

an immersion as above. Then we have that

$$p_1(X) \;=\; \Sigma n_i x_i + dS$$

where $x_1, \ldots, x_N$ are the points of complex tangency of the immersion and where $n_1, \ldots, n_N$ are integers computed from the local CR geometry of f.

Example 10.10. Consider the case where $m = n$. Let

$$f : X^n \looparrowright Z^n$$

be an immersion of a real n-manifold into a complex n-manifold. If $df_{\mathbf{C}}$ is $(n-r)$-atomic, we have

$$\mathbb{C}r_r(f) \;=\; \Delta_r^{(r)}\{\tilde{p}(X)/\tilde{c}(Z)\} + dS$$

(where $\tilde{c}(Z)$ is assumed to be pulled back via f to X). When $r = 1$ we have

$$\mathbb{C}r_1(f) \;=\; c_1(Z) + dS.$$

When $r = 2$, a calculation yields

$$\mathbb{C}r_2(f) \;=\; \Delta_2^{(2)}c(Z) - p_1(Z)p_1(X) + p_1(X)^2 + dS.$$

Example 10.11. Consider an immersion

$$f : S^m \looparrowright Z$$

and give S^m the standard Riemannian connection for which $p(S^m) \equiv 1$. Then if $df_{\mathbf{C}}$ is $(n-r)$-atomic we have

$$\mathbb{C}r_r(f) \;=\; \Delta_r^{(m-n+r)}\{c(Z)\} + dS.$$

One case of interest is where $m = n = 2r^2$. Then we have

$$\mathbb{C}r_r(f) \;=\; \begin{vmatrix} c_r(Z) & c_{r+1}(Z) & \cdots & c_{2r}(Z) \\ c_{r-1}(Z) & c_r(Z) & & \\ \vdots & & & \\ c_1(Z) & & & c_r(Z) \end{vmatrix} + dS.$$

There is a counterpart to all of the discussion above for the case where $m \leqq n$.

Theorem 10.12. *Let $f : X \looparrowright Z$ be an immersion of a real m-manifold into a complex n-manifold where $m \leqq n$. Assume that the map $df_{\mathbf{C}}$ of (10.1) is $(m-r)$-atomic. For given connections and metrics on X and Z as above, there is a canonical L^1_{loc}-form S such that*

$$\mathbb{C}r_r(f) \stackrel{\text{def}}{=} \mathbb{D}_{m-r}(df_{\mathbf{C}}) \;=\; \Delta_{(n-m+r)}^{(r)}\{c(Z)/\tilde{p}(X)\} - dS \tag{10.13}$$

with approximating families $\mathbb{P}_s$ and S_s as in (10.4).

The proof of Lemma 10.3 shows that in this case

$$\operatorname{supp} \mathbb{C}r_r(f) \subseteq \{x \in X : \dim_{\mathbf{C}}(T_xX \cap JT_xX) \geqq r\}.$$

Remark 10.14. Note that if $f : X \looparrowright Z$ is a totally real immersion, then Theorem 10.12 implies that

$$\Delta^{(r)}_{n-m+r}\{c(Z)/\tilde{p}(X)\} \;=\; 0 \quad \text{in} \;\; H^*(X) \tag{10.15}$$

for all $r > 0$. However in this case $TZ\,\big|_X = (TX \oplus \mathbb{C}) \oplus \nu$ where ν is a complex bundle of dimension $n - m$. Hence, on X we have

$$c(Z) \;=\; c(TX \oplus \mathbb{C})c(\nu) \;=\; \tilde{p}(X)c(\nu)$$

and so

$$c(Z)/\tilde{p}(X) \;=\; c(\nu) \;=\; 1 + c_1(\nu) + \cdots + c_{n-m}(\nu)$$

and (10.15) follows trivially.

On the other hand if f is Lagrangian (in the sense that $f_*T_xX) \perp J(f_*T_xX)$ for all x), then for a natural choice of connections on X and Z one has $\Delta^{(r)}_{n-m+r}\{c(Z)/\tilde{p}(X)\} \equiv 0$ on X, and via (10.15) a secondary invariant $[T] \in H^{\mathrm{odd}}(X; \mathbf{R})$ is defined.

Some of the results in this section are related to work of Lai [Lai], Webster [W1,2,3], and Wolfson [Wo].

§11. Invariants for pairs of complex structures.

In this section we shall introduce characteristic invariants which measure the relative singularities of a pair of complex structures. Consider a real C^∞ vector bundle $E \to X$ of rank $2n$, and suppose that J_1 and J_2 are smooth almost complex structures on E. These structures induce decompositions

$$E \otimes_{\mathbf{R}} \mathbf{C} \;=\; E_1^{1,0} \oplus E_1^{0,1} \;=\; E_2^{1,0} \oplus E_2^{0,1} \tag{11.1}$$

into the $\pm i$ eigenspaces of J_1 and J_2 respectively. ($E_k^{1,0}$ is the $+i$ eigenspace.) Consider the bundle map α given by the composition

$$E_1^{1,0} \hookrightarrow E \otimes_{\mathbf{R}} \mathbf{C} \twoheadrightarrow E_2^{0,1}$$

Lemma 11.2. *At each $x \in X$, there is an isomorphism*

$$\begin{aligned}\ker \alpha_x \;&\cong \{V \in E_x : J_1 V \;=\; J_2 V\} \\ &= \textit{the maximal subspace of } E_x \textit{ which is simultaneously} \\ &\quad J_1 \textit{ and } J_2 \textit{ complex.}\end{aligned}$$

Proof. Note that $E_1^{1,0} = \{V - iJ_1 V : V \in E_x\}$. Now for $V \in E_x$, we have

$$\begin{aligned} v \;&\stackrel{\text{def}}{=}\; V - iJ_1V \in \ker\alpha \iff \pi_2^{1,0}(v) \;=\; 0 \\ &\iff v + iJ_2 v \;=\; V - iJ_1V + iJ_2(V - iJ_1V) \;=\; 0 \\ &\iff J_2V \;=\; J_1V. \quad \square \end{aligned}$$

Definition 11.3. Fix r, $1 \le r \le n$. The structures J_1, J_2 will be called ***r*-transversal** if the bundle map α is $(n-r)$-atomic. Under this hypothesis we define the r^{th} **coincidence current** of the pair J_1, J_2 to be

$$\mathbb{Q}_r(J_1, J_2) \;=\; \mathbb{D}_{n-r}(\alpha).$$

From 11.2 and 2.7 we have that

$$\operatorname{supp} \mathbb{Q}_r(J_1, J_2) \subseteqq \{x \in X : J_1 = J_2 \;\text{ on a subspace }\; W \subset E_x \;\text{ with }\; \dim W \geqq r\}.$$

Theorem 11.4. *Suppose J_1 and J_2 are r- transversal. Then given complex connections on $E_1 \equiv (E, J_1)$ and $E_2 \equiv (E, J_2)$, there exists an L^1_{loc}-form S such that*

$$\mathbb{Q}_r(J_1, J_2) \;=\; \Delta_r^{(r)} c(E_2 - E_1) + dS$$

where $c(E_2 - E_1) = c(R^{E_2})c(R^{E_1})^{-1}$. Furthermore, there are smooth families $\mathbb{P}_s$ and S_s, $0 < s \leqq \infty$ as in 6.2, 8.1, 9.1.

Proof. Apply Theorem 4.4. $\square$

Example 11.5. Fix $E_1 = X \times \mathbf{C}^n = (X \times \mathbf{R}^{2n}, J_0)$. The product bundle with the canonical flat connection, and consider $E_2 = (X \times \mathbf{R}^{2n}, J)$ where J is any other almost complex structure on this trivialized $\mathbf{R}^{2n}$-bundle. For those J which are r-transversal to the given flat structure we have

$$Q_r(J) \stackrel{\text{def}}{=} \mathbb{Q}_r(J_0, J) \;=\; \Delta_r^{(r)} c(R^J) + dS_r$$

where $c(R^J)$ are the Chern forms of a J-complex connection. For example,

$$\begin{aligned}
Q_1(J) &= c_1(R^J) + dR_1 \\
Q_2(J) &= \det \begin{pmatrix} c_2 & c_3 \\ c_1 & c_2 \end{pmatrix} + dR_2 \\
&= c_2^2 - c_1 c_3 + dT_2 \\
Q_3(J) &= \det \begin{pmatrix} c_3 & c_4 & c_5 \\ c_2 & c_3 & c_4 \\ c_1 & c_2 & c_3 \end{pmatrix} + dR_3.
\end{aligned}$$

Note. There are cohomology relations

$$c(E)c(\overline{E}) \;=\; 1 \quad \text{and} \quad c_n(E) \;=\; 0$$

on any complex bundle which is trivial as a real bundle. Nevertheless, E may still be non-trivial as a complex bundle. Consider for example the complex line bundle $L \to S^1 \times S^1$ of Chern class 2, and set $E = L \oplus \mathbf{C}$. Then

$$c_1(E) \;=\; c_1(L) \;=\; 2$$

and so E is non-trivial. However E *is* trivial as an $\mathbf{R}^4$-bundle. To see this note first that the classifying map $S^1 \times S^1 \longrightarrow \mathrm{BU}_2 \longrightarrow \mathrm{BSO}_4$ can be lifted to a map $S^1 \times S^1 \longrightarrow \mathrm{BSpin}_4$ since the only obstruction to this lifting is $w_2(E) = c_1(E)$ (mod 2) $= 0$. However, BSpin_4 is 3-connected, so this lift, and therefore also the map to BSO_4, are contractible.

Example 11.6. (Diffeomorphisms) Let $f : X \to X$ be a diffeomorphism of an almost complex manifold with almost complex structure J. For generic f, the structures J and f^*J will be r-transversal for all r, and so given a J-compatible connection ∇ on X, we can take the f^*J-compatible connection $f^*\nabla$ and obtain L^1_{loc}-forms S_r with

$$\mathbb{Q}_{r,J}(f) \stackrel{\text{def}}{=} \mathbb{Q}_r(J, f^*J) \;=\; \Delta_r^{(r)} \{f^*c(R^J)/c(R^J)\} + dS_r.$$

§12. Invariants for plane fields and foliations.

This discussion given in §10 for mappings can be applied to plane fields, and in particular to foliations. Even more generally suppose $F \to X$ is a smooth complex vector bundle of complex dimension n, and consider a real sub-bundle

$$j : P \hookrightarrow F$$

of (real) dimension m. Then there is a natural extension of j to a complex bundle map

$$j_{\mathbf{C}} : P \otimes_{\mathbf{R}} \mathbf{C} \longrightarrow F. \tag{12.1}$$

Fix an integer r, with $0 < r \leqq \min\{m, n\}$, and set $k = \min\{m, n\} - r$. We say that P has **good complex r-tangencies** if $j_{\mathbf{C}}$ is k-atomic. Under this hypotheses the complex tangency current

$$\mathbb{C}r_r(P) \stackrel{\text{def}}{=} \mathbb{D}_k(j_{\mathbf{C}}) \tag{12.2}$$

is defined. In analogy with 10.3 and (10.12) we have

$$\operatorname{supp} \mathbb{C}r_r(P) \subseteq \{x \in X \;\; : \;\; \dim_{\mathbf{C}}(P_x \cap J_x P_x) \geqq r + \max\{0, m-n\}\}.$$

This is the subset of X where the dimension of the maximal complex subspace of P is greater by at least r than the "expected" or "generic" dimension.

Theorem 12.3. *Let $j : P \hookrightarrow F$ be a real m-dimensional subbundle with good complex r-tangencies in a complex n-dimensional bundle F. Let P be equipped with a real connection and metric, and let F be given a complex connection and hermitian metric. (No relation among the four is assumed.)*

Then there is a canonically defined L^1_{loc}-form S on X such that

$$\mathbb{C}r_r(P) \;=\; \begin{cases} \Delta^{(r)}_{m-n+r}\{\tilde{p}(P)/\tilde{c}(F)\} + dS, & \text{when } \; m \geqq n \\ \Delta^{(r)}_{n-m+r}\{c(F)/\tilde{p}(P)\} + dS, & \text{when } \; m \leqq n \end{cases} \tag{12.4}$$

where $c(F)$, $\tilde{c}(F)$ and $\tilde{p}(P)$ are defined as in §10. Furthermore there are approximating families Σ_s, S_s, $0 < s \leqq \infty$, as in previous theorems.

Note. The equations in (12.4) can be rewritten by using the elementary Shur relations (4.2) and (4.3). In particular, with $k = \min\{m, n\} - r$, we have

$$\Delta^{(m-k)}_{n-k}\{c(F)/\tilde{p}(P)\} \;=\; \Delta^{(n-k)}_{m-k}\{\tilde{p}(P)/\tilde{c}(F)\}.$$

Example 12.5. (Plane fields and foliations). A case of geometric interest occurs when

$$j : P \hookrightarrow TZ$$

is a real m-plane field on an almost complex manifold Z. This arises for example when P is the tangent plane field $T\mathcal{F}$ of an m-dimensional foliation $\mathcal{F}$ of Z. In this case the currents $\mathbb{C}r_r(\mathcal{F}) \equiv \mathbb{C}r_r(T\mathcal{F})$ correspond to the *excess complex tangencies* of the foliation. The formulas in (12.4) give cohomological obstructions to finding an isotopy of $\mathcal{F}$ to a foliation without complex tangencies of dimension $r + \max\{0, m - n\}$.

§13. Higher self-intersections of plane fields and invariants for pairs of foliations.

Let $F \to X$ be a smooth bundle of rank n and consider two subbundles

$$A \overset{i}{\hookrightarrow} F \quad \text{and} \quad B \overset{j}{\hookrightarrow} F$$

of ranks a and b respectively. We assume all bundles to be simultaneous real or simultaneously complex. From this data we get a bundle map

$$A \xrightarrow{\alpha} F/B$$

given by restricting the projection $\pi : F \to F/B$ to A. Note that

$$\ker \alpha_x \;=\; A_x \cap B_x$$

whose "expected" or "generic" dimension is

$$e \;=\; \max\{a + b - n, 0\}.$$

Fix integers $r > 0$ and $k \geq 0$ with $e+r = a-k$, or equivalently $r+k = \min\{a, n-b\}$. Note that $\mathrm{rank}(\alpha_x) = a - \dim(A_x \cap B_x) \leq k$ if and only if $\dim(A_x \cap B_x) \geq a - k = e + r$. We say that A and B make **good r-contact** if α is k-atomic. Under this assumption we define the **r-contact current**

$$\mathbb{C}t_r(A, B) \;=\; \mathbb{D}_k(\alpha)$$

and note that

$$\mathrm{supp}\, \mathbb{C}t_r(A, B) \subseteq \{x \in X : \dim(A_x \cap B_x) \geqq e + r\}.$$

This current measures the contact degeneracies of rank r, i.e., the set where A meets B in at least r dimensions more than expected. Setting $e^* = \max\{n - (a + b), 0\}$ and applying §§4 and 5 give the following.

Theorem 13.1. *Suppose A, B and F are complex and that F is provided with a complex connection and metric. If A and B make good r-contact, then there is a canonical L^1_{loc}-form T such that*

$$\mathbb{C}t_r(A, B) \;=\; \Delta^{(e+r)}_{e^*+r}\{c(R^F)/c(R^A)c(R^B)\} + dS.$$

Theorem 13.2. *Suppose A, B and F are real bundles of rank $2a_0$, $2b_0$ and $2n_0$ respectively and let $e = 2e_0$, $e^* = 2e_0^*$. Suppose F is provided with an orthogonal connection. Then if A and B make good r-contact where $r = 2r_0$ is even, there is a canonical L^1_{loc}-form S such that*

$$\mathbb{C}t_r(A, B) \;=\; \widetilde{\Delta}^{(e_0+r_0)}_{e_0^*+r_0}\{p(R^F)/p(R^A)p(R^B)\} + dS.$$

Remark 13.3. One could equivalently formulate the problem by considering the adjoint map $\alpha^* : (F/B)^* \to A^*$. Note that at $x \in X$, we have

$$\ker(\alpha_x^*) \;=\; A_x^\perp \cap B_x^\perp$$

whose expected dimension is

$$e^* \;=\; \max\{n-(a+b), 0\}.$$

Applying 4.4 gives us "dual" formulas which are equivalent to those of 13.1 and 13.2. To see this, observe that by the Shur relations (4.3) and (4.2) we have

$$\Delta_{e^*+r}^{(e+r)}\{c(R^F)/c(R^A)c(R^B)\} \;=\; \Delta_{e+r}^{(e^*+r)}\{c(R^{A^*})c(R^{B^*})/c(R^{F^*})\}$$

and

$$\widetilde{\Delta}_{e_0^*+r_0}^{(e_0+r_0)}\{p(R^F)/p(R^A)p(R^B)\} \;=\; \widetilde{\Delta}_{e_0+r_0}^{(e_0^*+r_0)}\{p(R^A)p(R^B)/p(R^F)\}.$$

Example 13.4. (**Higher self intersection classes**). Given $A \subset F$ as above one can take B to be a generic displacement of A (which will be r-atomic for all relevant r) and compute the "higher self intersections" of the plane field. Consider for example the complex line field $\lambda \subset T\mathbf{P}^3$ on complex projective 3-space which is tangent to the fibres of the twistor map $\mathbf{P}^3 \to S^4$. Now we have $c(\lambda) = 1 + 2\omega$ where $\omega \in H^2(\mathbf{P}^3, \mathbf{Z}) \cong \mathbf{Z}$ is the canonical generator. The self-intersection class coming from taking $A = B = \lambda$, $F = T\mathbf{P}^3$ and $r = 1$ in 13.1 is

$$\Delta_2^{(1)}\{c(T\mathbf{P}^3)/c(\lambda)^2\} \;=\; \Delta_2^{(1)}\{(1+\omega)^4/(1+2\omega)^2\} \;=\; 2\omega^2.$$

This implies that a generic deformation of λ (as a smooth complex subbundle) in $T\mathbf{P}^3$ will coincide with λ along a cycle in $\mathbf{P}^3$ which represents the homology class $2[\mathbf{P}^1] \in H_2(\mathbf{P}^3\ ;\ \mathbf{Z}) \cong \mathbf{Z}$.

There are analogous complex line fields λ on every $\mathbf{P}_{\mathbf{C}}^{2n+1}$ tangent to the fibres of the bundle $\pi : \mathbf{P}_{\mathbf{C}}^{2n+1} \to \mathbf{P}_{\mathbf{H}}^{n}$. Here the generic self-intersection locus satisfies

$$\mathbb{C}t_1(\lambda) \;=\; \Delta_{2n}^{(1)}\{(1+\omega)^{2n+2}/(1+2\omega)^2\} + dS \;=\; (n+1)\omega^{2n} + dS.$$

Example 13.5. (**Relative foliation cycles**). Suppose we are given two foliations $\mathcal{F}$ and $\mathcal{F}'$ of dimensions $2a_0$ and $2b_0$ respectively in a Riemannian manifold of dimension $2n_0$. Then we can apply 13.2 to $A = T\mathcal{F}$ and $B = T\mathcal{F}'$ to produce canonical cohomologies between the current of $(e+r)$-dimensional contact points of $\mathcal{F}$ and $\mathcal{F}'$, i.e., between

$$\mathbb{C}t_r(\mathcal{F}, \mathcal{F}') \stackrel{\text{def}}{=} \mathbb{C}t_r(T\mathcal{F}, T\mathcal{F}'),$$

and the characteristic forms defined in 13.2.

APPENDIX A Orientations and tensor products

In this appendix we discuss questions of orientations and characteristic classes of tensor product bundles. We begin with some elementary definitions. Let $V \to Y$ be a real vector bundle of rank n over a manifold Y. By definition there is an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of Y and local trivializations

$$f_\alpha : V\Big|_{U_\alpha} \xrightarrow{\approx} U_\alpha \times \mathbf{R}^n.$$

The compositions give transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_n(\mathbf{R})$$

by setting $g_{\alpha\beta}(x) = f_\alpha \circ f_\beta^{-1}(x, \cdot)$. They have the property that $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$ and satisfy the **cocycle condition**

$$\text{(A.1)} \qquad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} \equiv 1 \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

In terms of this data we define a Čech 1-cocycle $\{w_{\alpha\beta}\}$ on the cover $\mathcal{U}$ by setting

$$w_{\alpha\beta} \stackrel{\text{def}}{=} \mathrm{sgn}\{\det(g_{\alpha\beta})\} : U_\alpha \cap U_\beta \longrightarrow \{1, -1\} \cong \mathbf{Z}_2.$$

From (A.1) we see that $w_{\alpha\beta}$ satisfies the cocycle condition

$$\text{(A.2)} \qquad w_{\alpha\beta}w_{\beta\gamma}w_{\gamma\alpha} \equiv 1 \qquad \text{in } U_\alpha \cap U_\beta \cap U_\gamma$$

and therefore determines a class

$$w_1(V) \in H^1(Y\ ;\ \mathbf{Z}_2)$$

which can be shown to be independent of the choice of local trivializations for V. It is called the **first Stiefel-Whitney class** of V. Note that the cocycle $w_{\alpha\beta}$, considered as transition functions on $\mathcal{U}$, determines a two-fold covering (i.e., a $\mathbf{Z}_2$-bundle) over Y, called the **orientation bundle** $Or(V)$ of V. One can naturally identity the fibre of $Or(V)$ at x with the two possible orientations of V_x. If Y is a connected manifold then $\mathrm{Cov}_2(Y) \cong \mathrm{Hom}(\pi_1 Y, \mathbf{Z}_2) \cong H^1(Y; \mathbf{Z}_2)$ (where $\mathrm{Cov}_2(Y)$ denotes the equivalence classes of 2-fold coverings of Y). This gives an equivalent definition of $w_1(V)$ and shows that

$$\text{(A.3)} \qquad V \text{ is orientable} \iff w_1(V) = 0.$$

Proposition A.4. *Let $U \to Y$ and $F \to Y$ be real vector bundles with* $\mathrm{rank}(U) = m$ *and* $\mathrm{rank}(F) = n$*, and set $H = Hom(U, F) = U^* \otimes F$. Then*

$$\text{(A.5)} \qquad w_1(H) = nw_1(U) + mw_1(F).$$

Proof. Let $A : \mathbf{R}^m \to \mathbf{R}^m$ and $B : \mathbf{R}^n \to \mathbf{R}^n$ be linear maps, and consider $A \otimes B : \mathbf{R}^{nm} \to \mathbf{R}^{nm}$. Then

$$\det(A \otimes B) \;=\; (\det A)^n (\det B)^m.$$

Now let $\{a_{\alpha\beta}\}$ and $\{b_{\alpha\beta}\}$ be transition functions for U and F respectively over an open covering $\mathcal{U}$ of Y. Then H has transition functions $h_{\alpha\beta} = a^*_{\alpha\beta} \otimes b_{\alpha\beta}$, and so

$$\det(h_{\alpha\beta}) \;=\; \det(a^*_{\alpha\beta})^n \det(b_{\alpha\beta})^m \;=\; \det(a_{\alpha\beta})^n \det(b_{\alpha\beta})^m.$$

Hence,

$$\operatorname{sgn}(\det h_{\alpha\beta}) \;=\; (\operatorname{sgn}\det a_{\alpha\beta})^n (\operatorname{sgn}\det b_{\alpha\beta})^m.$$

Rewriting this additively gives the result. $\square$

Corollary A.6. *The bundle $H = Hom(U, F)$ is orientable if any one of the following conditions holds:*

(i) $\operatorname{rank} U$ *and* $\operatorname{rank} F$ *are both even.*
(ii) F *is orientable and* $\operatorname{rank} F$ *is even.*
(iii) U *is orientable and* $\operatorname{rank} U$ *is even.*

Corollary A.7. *Suppose that* $\operatorname{rank} F$ *is odd and that either F is orientable or* $\operatorname{rank} U$ *is even. Then*

$$w_1(H) \;=\; w_1(U).$$

Let $G_r(\mathbf{R}^m)$ denote the Grassmannian of (unoriented) r- planes in $\mathbf{R}^m$, and let $U \longrightarrow G_r(\mathbf{R}^m)$ denote the tautological r-plane bundle. There is a natural embedding $U \hookrightarrow \underline{\mathbf{R}}^m$ into the trivialized m-plane bundle, and this gives a splitting

$$\underline{\mathbf{R}}^m \;=\; U \oplus U^\perp.$$

Corollary A.8. *If r and m are both even, then $G_r(\mathbf{R}^m)$ is orientable.*

Proof. Apply A.6, part (i), to $TG_r(\mathbf{R}^m) = \operatorname{Hom}(U, U^\perp)$. $\square$

Consider a smooth real vector bundle $E \to X$ of rank m, and let $\pi : G_r(E) \to X$ be the Grassmann bundle whose fibre at $x \in X$ consists of all (unoriented) r-planes in E_x. Let $U \to G_r(E)$ be the tautological r-plane bundle with canonical embedding $U \hookrightarrow \pi^*E$. After a choice of metric in E we have a splitting

$$\pi^*E \;=\; U \oplus U^\perp$$

and there is a bundle equivalence

$$TG_r(E) \cong \pi^*TX \oplus \operatorname{Hom}(U, U^\perp). \tag{A.9}$$

Corollary A.10. *Suppose X is orientable. If r and m are even, then $G_r(E)$ is orientable. If r is even and m is odd, then*

$$w_1(U) \;=\; w_1(G_r(E)).$$

Proof. By A.9 and Proposition A.4 we have

$$w_1(G_r(E)) \;=\; \pi^* w_1(X) + w_1(U^* \otimes U^{\perp}) \;=\; (m-r)w_1(U) + r w_1(U^{\perp}). \quad \square$$

We now consider integration over the fibres of π. To begin recall that an **r-form twisted by the orientation bundle** on a manifold Y is a section of the bundle $\Lambda^r T^*Y \otimes_{\mathbf{Z}_2} Or(Y)$ where $Or(Y) = Or(TY)$, and where $\mathbf{Z}_2 = \{1, -1\}$ acts multiplicatively. A **density** on Y is a d-form twisted by $Or(Y)$ where $d = \dim(Y)$. Densities with compact support can be integrated over Y (cf [St]).

Proposition A.11. *Let X, $G_r(E)$ and U be as in Corollary A.10, and assume that r is even and m is odd. Then for any orthogonal connection on U with curvature R^U, the Euler form of U on $G_r(E)$*

$$\chi(R^U) \;=\; \mathrm{Pfaff}\left(-\tfrac{1}{2\pi} R^U\right)$$

is an r-form twisted by the orientation bundle of $G_r(E)$. For any q-form ω on $G_r(E)$ the product $\omega \wedge \chi(R^U)$ can be integrated over the fibres of $\pi : G_r(E) \to X$, yielding a smooth form on X.

Proof. Let $r = 2r_0$. For a skew-symmetric transformation $A : \mathbf{R}^r \to \mathbf{R}^r$ we have the equation

$$\left(\tfrac{1}{2} e^t A e\right)^{r_0} \;=\; r_0!\, \mathrm{Pfaff}(A) e_1 \wedge \cdots \wedge e_r \tag{A.12}$$

in $\wedge^r \mathbf{R}^r$, where $e_1, \ldots, e_r$ denotes any orthonormal basis of $\mathbf{R}^r$, and where $e^t A e = \Sigma A_{ij} e_i \wedge e_j$. The left hand side of (A.12) is independent of orientation. Hence from (A.12) we see that the Pfaffian of R^U is twisted by $Or(U)$. However, from (A.9) we have

$$Or(G_r(E)) \;=\; \pi^* Or(X) \otimes Or(U^* \otimes U^{\perp}) \;=\; Or(U^*) \;=\; Or(U)$$

under our assumption that X is orientable and r is even. Now any $Or(G_r(E))$-twisted form α on $G_r(E)$ (for example, $\chi(R^U) \wedge \omega$) defines a current on $G_r(E)$ and hence can be pushed forward by π as a current to X. To see this push forward is well defined let β be a form with compact support on X of degree $d' = \dim(G_r(E)) - \deg \alpha$. Then

$$[\pi_* \alpha](\beta) \;=\; \int_{G_r(E)} \alpha \wedge \pi^* \beta$$

which is well-defined since $\alpha \wedge \pi^* \beta$ is a density on $G_r(E)$. Finally since α is smooth this push forward can be computed using fibre integration yielding a smooth untwisted form on X, since X is oriented. $\square$

We have observed above that if U and F are real vector bundles of even rank, then $U^* \otimes F$ is orientable. However, it remains to choose a canonical orientation.

Convention A.13. Let V and W be finite dimensional vector spaces with ordered bases $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_n)$ respectively. Then the **canonical orientation** on $V \otimes W$ with respect to these bases is given by

$$(e_1 \otimes f_1,\ e_2 \otimes f_1, \ldots, e_m \otimes f_1,\ e_1 \otimes f_2, \ldots, e_m \otimes f_2, \ldots, e_m \otimes f_n).$$

This depends only on the orientations of V and W determined by $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_m)$. If n is even, it is independent of the orientation of V (and if m is even it is independent of the orientation of W). If $n \equiv m \equiv 0 \pmod 2$, it is independent of the orientations of both V and W.

Remark A.14. Given an inner product on V there are natural isomorphisms $V \otimes W \cong V^* \otimes W = \mathrm{Hom}(V, W)$ which transfer the canonical orientation to $\mathrm{Hom}(V, W)$ (independently of the choice of inner product).

Suppose $\dim_{\mathbf{R}}(V) = \dim_{\mathbf{R}}(W) = 2$ and both are oriented and equipped with inner products so that $V \cong W \cong \mathbb{C}$. Then we have a natural splitting

$$\mathrm{Hom}_{\mathbf{R}}(V, W) \;=\; \mathrm{Hom}_{\mathbb{C}}(V, W) \oplus \mathrm{Hom}_{\mathbb{C}}(V, \overline{W}) \tag{A.15}$$

given by writing

$$A \;=\; \tfrac{1}{2}(A - J_W A J_V) + \tfrac{1}{2}(A + J_W A J_V)$$

where J_V, J_W are the complex structures corresponding to the orientations on V and W.

Lemma A.16. *The canonical orientation on* $\mathrm{Hom}_{\mathbf{R}}(V, W)$ *is opposite to the one corresponding to the complex structure induced via* (A.15).

Proof. Let (v_1, v_2), (w_1, w_2) be oriented bases for V and W, and let $h_{ij} = v_j^* \oplus w_i$ $1 \le i,\ j \le 2$ be the corresponding basis of $H = \mathrm{Hom}_{\mathbf{R}}(V, W)$. Then $\mathrm{Hom}_{\mathbb{C}}(V, W)$ has an oriented basis (ϵ_1, ϵ_2) where $\epsilon_1 = h_{11} + h_{22}$ and $\epsilon_2 = J(\epsilon_1) = \epsilon_1 \circ J_V = h_{21} - h_{12}$. Similarly $\mathrm{Hom}(V, \overline{W})$ has basis $\epsilon_1' = h_{11} - h_{22}$ and $\epsilon_2' = J(\epsilon_1') = \epsilon_1' \circ J_V = -h_{21} - h_{12}$. Taking the wedge product over $\mathbf{R}$ gives

$$\begin{aligned}
\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 &= (h_{11} + h_{22}) \wedge (h_{21} - h_{12}) \wedge (h_{11} - h_{22}) \wedge (-h_{21} - h_{12}) \\
&= -4 h_{11} \wedge h_{21} \wedge h_{12} \wedge h_{22}. \quad \square
\end{aligned}$$

We shall now compute the Euler form of the bundle $\mathrm{Hom}(U, F)$ over a manifold Y. In practice we will set $Y = G_r(E)$ and take U to be the tautological bundle.

Theorem A.17. *Let $U \to Y$ and $F \to Y$ be smooth vector bundles with orthogonal connections. Give $H \stackrel{\text{def}}{=} Hom(U, F)$ the induced tensor product connection. Suppose*

$$\operatorname{rank}(U) \;=\; 2r_0 \quad \textit{and} \quad \operatorname{rank}(F) \;=\; 2n_0$$

for integers r_0, $n_0 > 0$. Then the Euler form of H for its canonical orientation is given by

$$\chi(R^H) \;=\; \widetilde{\Delta}_{n_0}^{(r_0)}\{p(R^F)p(R^U)^{-1}\} \tag{A.18}$$

where $\widetilde{\Delta}_a^{(b)}$ is the Shur polynomial introduced in §5.

Proof. We begin by proving equation A.18 at the cohomology level. Consider first the case where $\dim U = \dim F = 2$ and both U and F are oriented. Set

$$a \;=\; \chi(U) \;=\; c_1(U) \quad \text{and} \quad b \;=\; \chi(F) \;=\; c_1(F)$$

in $H^2(Y; \mathbf{R})$. Then from the equation

$$\operatorname{Hom}_{\mathbf{R}}(U, F) \;=\; \operatorname{Hom}_{\mathbb{C}}(U, F) \oplus \operatorname{Hom}_{\mathbb{C}}(U, \overline{F})$$

and Lemma A.16 we get

$$\begin{aligned} \chi(H) \;&=\; -\chi(U^* \otimes_{\mathbb{C}} F)\chi(U^* \otimes_{\mathbb{C}} \overline{F}) \\ &=\; -(b-a)(-b-a) \\ &=\; b^2 - a^2 \\ &=\; p_1(F) - p_1(U). \end{aligned}$$

Suppose now that U and F are oriented and apply the Splitting Principle to write them formally as

$$U \;=\; U_1 \oplus \cdots \oplus U_{r_0} \quad \text{and} \quad F \;=\; F_1 \oplus \cdots \oplus U_{n_0}$$

where U_i and F_j are oriented 2-plane bundles. We formally set

$$a_i \;=\; \chi(U_i) \quad \text{and} \quad b_j \;=\; \chi(F_j)$$

for all i, j. Then

$$\operatorname{Hom}(U, F) \;=\; \bigoplus_{j=1}^{n_0} \bigoplus_{i=1}^{r_0} \operatorname{Hom}(U_i, F_j)$$

and so

$$\chi(\operatorname{Hom}(U, F)) \;=\; \prod_{j=1}^{n_0} \prod_{i=1}^{r_0} (b_j^2 - a_i^2) \;=\; \widetilde{\Delta}_{n_0}^{(r_0)}\{1 + \eta_1 + \eta_2 + \cdots\} \tag{A.19}$$

where $\eta_k \in H^{4k}(Y; \mathbf{R})$ is defined by

$$\frac{p(F)}{p(U)} \;=\; \frac{\prod_{j=1}^{n_0}(1+b_j^2)}{\prod_{i=1}^{r_0}(1+a_i^2)} \;=\; 1+\eta_1+\eta_2+\cdots .$$

See [Fu, page 419] for the second equality in (A.19). This establishes formula (A.18) at the cohomology level when U and F are orientable.

To establish the formula at the level of forms we recall that by [NR] any k-plane bundle **with connection** over a manifold Y is induced from the universal bundle $\mathbf{U}_k$ with its connection over $G_k(\mathbf{R}^N)$, for N sufficiently large, by a smooth map $f : Y \to G_k(\mathbf{R}^N)$. Suppose now that $f : Y \to G_{2r_0}(\mathbf{R}^N)$ and $g : Y \to G_{2n_0}(\mathbf{R}^N)$ classify U and F with their connections, i.e., $f^*(\mathbf{U}_{2r_0}^*) \cong U^*$ and $g^*(\mathbf{U}_{2n_0}) \cong F$. Then $f \times g : Y \to G_{2r_0}(\mathbf{R}^N) \times G_{2n_0}(\mathbf{R}^N)$ has the property that

$$(f\times g)^*(\mathbf{U}_{2r_0}^* \otimes \mathbf{U}_{2n_0}) \cong U^* \otimes F$$

as bundles with connection. Consequently if formula (A.18) holds for $\mathbf{U}_{2r_0}^* \otimes \mathbf{U}_{2n_0}$, it holds in general, since

$$\begin{aligned}
\chi\left(R^{U^*\otimes F}\right) \;&=\; (f\times g)^*\chi\left(R^{\mathbf{U}_{2r_0}^*\otimes \mathbf{U}_{2n_0}}\right) \\
&=\; (f\times g)^*\Delta_{n_0}^{(r_0)}\left\{p(R^{\mathbf{U}_{2n_0}})p(R^{\mathbf{U}_{2r_0}})^{-1}\right\} \\
&=\; \widetilde{\Delta}_{n_0}^{(r_0)}\left\{p(R^F)p(R^U)^{-1}\right\}.
\end{aligned}$$

Hence it suffices to prove (A.18) in the universal case. However, here (A.18) is a consequence of the cohomology calculation. To see this note that both $\chi(R^{\mathbf{U}_{2r_0}^*\otimes \mathbf{U}_{2n_0}})$ and $\widetilde{\Delta}_{n_0}^{(r_0)}\{p(R^{\mathbf{U}_{2n_0}})p(R^{\mathbf{U}_{2r_0}})^{-1}\}$ are invariant under the full isometry group of $G_{2r_0}(\mathbf{R}^N) \times G_{2n_0}(\mathbf{R}^N)$ and are therefore harmonic by a standard result in the theory of symmetric spaces.

When U and F are possibly non-orientable we pass to the 2-fold or 4-fold covering where they are orientable. The equation of forms (A.18) is invariant under the covering group and therefore descends to Y. $\square$

Theorem A.20. *Let U, F and $H = Hom(U,F)$ be as in Theorem A.17 except that*

$$\mathrm{rank}(U) \;=\; 2r_0 \quad \textit{and} \quad \mathrm{rank}(F) = 2n_0+1$$

for integers r_0, $n_0 > 0$. Then the canonical orientation gives an isomorphism $Or(H) \cong Or(U)$, and the $Or(H)$-twisted Euler form of H is equal to the following $Or(U)$-twisted form

$$\chi(R^H) \;=\; \chi(R^U)\widetilde{\Delta}_{n_0}^{(r_0)}\left\{p(R^F)p(R^U)^{-1}\right\}. \tag{A.21}$$

Proof. Note that by Proposition A.4 we have $w_1(H) = w_1(U)$, and so $Or(H) \cong Or(U)$ under the identification $\mathrm{Cov}_2(Y) \cong H^1(Y; \mathbf{Z}_2)$. We lift to the 2-fold covering of Y where H and U are orientable and establish (A.21) there. Since both sides of (A.21) transform by -1 under the (non-trivial) deck-transformation, the result will follow.

We assume therefore that U is orientable. We may assume also that F is orientable since if not we pass to a 2-fold covering where it is, and then observe that the equation is invariant, as in the proof of A.17. The proof now proceeds as before. We apply the Splitting Principle and formally write

$$U \;=\; U_1 \oplus \cdots \oplus U_{r_0} \quad \text{and} \quad F \;=\; F_1 \oplus \cdots \oplus F_{n_0} \oplus \mathbf{R}$$

where U_i, F_j are oriented 2-plane bundles with $a_i = \chi(U_i)$ and $b_j = \chi(F_j)$. Then

$$H \;=\; U^* \otimes F \;=\; U^* \oplus \bigoplus_{j=0}^{n_0} \bigoplus_{i=0}^{r_0} U_i^* \otimes F_j$$

and so

$$\chi(H) \;=\; \chi(U) \prod_{j=1}^{n_0} \prod_{i=1}^{r_0} (b_j^2 - a_i^2) \;=\; \chi(U) \Delta_{n_0}^{(r_0)} \{p(F)p(U)^{-1}\}.$$

As above, this cohomology formula implies the formula at the level of forms via [NR] and the uniqueness of invariant forms in the cohomology of $G_{2r_0}(\mathbf{R}^N) \times G_{2n_0}(\mathbf{R}^N)$. $\square$

We conclude this section by presenting the analogue of the last two theorems for the complex case.

Theorem A.22. *Let $U \to Y$ and $F \to Y$ be smooth complex vector bundles with complex connections and with*

$$\mathrm{rank}(U) \;=\; r \quad \textit{and} \quad \mathrm{rank}(F) \;=\; n.$$

Give $H = Hom(U, F) = U^* \oplus F$ the induced tensor product connection. Then the top Chern form of H is given by

$$c_{nr}(R^H) \;=\; \Delta_n^{(r)} \{c(R^F)c(R^U)^{-1}\} \tag{A.23}$$

where $c(R) = \det\left(1 + \frac{i}{2\pi} R\right)$ denotes the top Chern form of the connection.

Proof. At the cohomology level this result is well known (cf. [Fu, 14.4.12]), and one can deduce the formula at the level of forms by passing to the classifying spaces as in the proof of A.17. $\square$

Alternatively one can prove (A.23) directly by establishing the following matrix identity. Let $M_{k\times k}$ denote the vector space of complex $k\times k$ matrices. For any commutative ring R with unit, and for integers a, $b>0$, let

$$\Delta_b^{(a)} : R[[t]] \longrightarrow R \tag{A.24}$$

be the function given by

$$\Delta_b^{(a)}\left(\sum_{k\geq 0} r_k t^k\right) \;=\; \det_{a\times a}\left(\!\left(r_{b-i+j} \right)\!\right)$$

where $\left(\!\left(r_{b-i+j} \right)\!\right)$ is the $a\times a$ matrix whose $(i,j)^{\text{th}}$ entry is r_{b-i+j} when $r-i+j \geqq 0$ and 0 otherwise.

Lemma A.25. *For $A\in M_{a\times a}$ and $B\in M_{b\times b}$, consider the matrix $A\oplus 1_b - 1_a\oplus B \in M_{ab\times ab}$. Then*

$$\det\left\{A\oplus 1_b - 1_a \oplus B\right\} \;=\; \Delta_b^{(a)}\left\{\det(1_b+tB)\det(1_a+tA)^{-1}\right\}.$$

Proof. It suffices to restrict attention to the Zariski open dense subset of diagonalizable matrices (A,B) in $M_{a\times a}\times M_{b\times b}$. Thus we may assume that $A=\operatorname{diag}(x_1,\dots,x_a)$ and $B=\operatorname{diag}(y_1,\dots,y_b)$. The lemma is now an immediate consequence of the following.

Fact A.26. ([Fu, page 419]) *In the polynomial ring $R=\mathbb{C}[x_1,\dots,x_a,y_1,\dots,y_b]$ in $(a+b)$ indeterminants, one has the identity*

$$\prod_{j=1}^{b}\prod_{i=1}^{a}(y_j-x_i) \;=\; \Delta_b^{(a)}\left\{\frac{\prod\limits_{j=1}^{b}(1+ty_j)}{\prod\limits_{i=1}^{a}(1+tx_i)}\right\}.$$

Note. Using A.26 one can give alternative proofs of Propositions A.17 and A.20 above.

APPENDIX B Integration over the fibres in $\mathbf{G_r}(\mathbf{E})$

In this appendix we establish the topological formulas required to complete the computations in §§4 and 5.

Proposition B.1. *Let $E \to X$ be a smooth hermitian vector bundle of* rank m. *Consider the Grassmann bundle $\pi : G_r(E) \to X$ of r-planes in E, where $0 < r < m$, and let $U \to G_r(E)$ be the tautological complex r-plane bundle. Write*

$$\pi^* E \;=\; U \oplus U^\perp,$$

and let $c_{m-r}(U^\perp)$ be the top Chern class of $U^\perp$. Then under the Gysin map $\pi_* : H^{2r(m-r)}(G_r(E)) \to H^0(X)$ we have

$$\pi_* \left\{ c_{m-r}(U^\perp)^r \right\} \;=\; 1. \tag{B.2}$$

Proof. It suffices to prove (B.2) in the case that X is a point, i.e., to prove that

$$\left\langle c_{m-r}(\mathbf{U}^\perp)^r,\; G_r(\mathbf{C}^m) \right\rangle \;=\; 1. \tag{B.3}$$

To see this we consider the Poincaré dual of $c_{m-r}(\mathbf{U}^\perp)$ i.e., the divisor $\mathrm{Div}(\alpha_v)$ of an atomic section $\alpha_v \in \Gamma(\mathbf{U}^\perp)$ defined by fixing a vector $v \in \mathbf{C}^m$ and setting

$$\alpha_v(U) \;=\; \pi_{U^\perp}(v)$$

at $U \in G_r(\mathbf{C}^m)$ where $\pi_{U^\perp} : \mathbf{C}^m = U \oplus U^\perp \to U^\perp$ is orthogonal projection. Now α_v vanishes non-degenerately and

$$\mathrm{Div}(\alpha_v) \;=\; \{U \in G_r(\mathbf{C}^m) : v \in U\}.$$

We now choose r vectors $v_1, \ldots, v_r \in \mathbf{C}^m$ which are linearly independent. Then these divisors meet transversely and

$$\begin{aligned} \mathrm{Div}(\alpha_{v_1}) \cap \cdots \cap \mathrm{Div}(\alpha_{v_r}) \;&=\; \{U \in G_r(\mathbf{C}^m) : v_1, \ldots, v_r \in U\} \\ &=\; \{\mathrm{span}(v_1, \ldots, v_r)\}. \end{aligned} \tag{B.4}$$

Under Poincaré duality cup product followed by evaluation on the fundamental class becomes intersection product. Hence, (B.4) $\Rightarrow$ (B.3). $\square$

Proposition B.5. *Let $E \to X$ be a smooth riemannian vector bundle of* rank m. *Consider the Grassmann bundle $\pi : G_r(E) \to X$ of r-planes in E, where $0 < r < m$, and let $U \to G_r(E)$ be the tautological real r-plane bundle. Suppose*

$$r \;=\; 2r_0 \quad \text{and} \quad m \;=\; 2m_0$$

for positive integers r_0 and m_0. Write

$$\pi^* E \;=\; U \oplus U^\perp$$

and let $p_{m_0 - r_0}(U^\perp)$ be the top Pontrjagin class of $U^\perp$. Then under the Gysin map $\pi_* : H^{4r_0(m_0 - r_0)}(G_{2r_0}(E)) \to H^0(X)$ we have

$$\pi_* \left\{ p_{m_0 - r_0}(U^\perp)^{r_0} \right\} \;=\; 1. \tag{B.6}$$

Proof. It suffices to consider the case where X is a point, i.e., it suffices to prove that

$$\left\langle p_{m_0-r_0}(\mathbf{U}^\perp)^{r_0},\ G_{2r_0}(\mathbf{R}^{2m_0})\right\rangle \ =\ 1. \tag{B.7}$$

To do this we lift to the Grassmannian $\widetilde{G}_{2r_0}(\mathbf{R}^{2m_0})$ of oriented $2r_0$-planes with tautological bundle $\widetilde{\mathbf{U}}$, and prove that

$$\left\langle p_{m_0-r_0}(\widetilde{\mathbf{U}}^\perp)^{r_0},\ \widetilde{G}_{2r_0}(\mathbf{R}^{2m_0})\right\rangle \ =\ 2. \tag{B.8}$$

Since $\widetilde{G}_{2r_0}(\mathbf{R}^{2m_0}) \to G_{2r_0}(\mathbf{R}^{2m_0})$ is a 2-sheeted covering whose deck transformation preserves the orientation of $\widetilde{G}_{2r_0}(\mathbf{R}^{2m_0})$ and the class $p_{m_0-r_0}(\widetilde{\mathbf{U}}^\perp)$, we see that (B.8) $\Rightarrow$ (B.7).

To prove (B.8) we first establish the following.

Lemma B.9.

$$p_{m_0-r_0}(\widetilde{\mathbf{U}}^\perp)\ =\ \chi(\widetilde{\mathbf{U}}^\perp)^2.$$

Proof. We apply the Splitting Principle and write $\widetilde{\mathbf{U}}^\perp$ formally as a direct sum of oriented 2-plane bundles

$$\widetilde{\mathbf{U}}^\perp \ =\ U_1\oplus\cdots\oplus U_{k_0} \tag{B.10}$$

where $k_0 = m_0 - r_0$, and set $a_j = \chi(U_j)$. Then by definition

$$\begin{aligned} p_{k_0}(\widetilde{\mathbf{U}}^\perp) \ &=\ \sigma_{k_0}(a_1^2,\dots,a_{k_0}^2) \\ &=\ a_1^2\cdots a_{k_0}^2 \ =\ (a_1\cdots a_{k_0})^2 \\ &=\ \chi(\widetilde{\mathbf{U}}^\perp)^2. \quad \square \end{aligned}$$

We now proceed as in the proof of (B.1). The Poincaré dual of $\chi(\widetilde{\mathbf{U}}^\perp)$ is represented by the divisor of the atomic section $\alpha_v \in \Gamma(\widetilde{\mathbf{U}}^\perp)$ defined by fixing a vector $v\in\mathbf{R}^{2m_0}$ and setting

$$\alpha_v(\widetilde{U})\ =\ \pi_{\widetilde{U}^\perp}(v)$$

at $\widetilde{U}\in\widetilde{G}_{2r_0}(\mathbf{R}^{2m_0})$ where $\pi_{\widetilde{U}^\perp}:\mathbf{R}^{2m_0}=\widetilde{U}\oplus\widetilde{U}^\perp\to\widetilde{U}^\perp$ is orthogonal projection. In fact α_v vanishes non-degenerately and

$$\mathrm{Div}(\alpha_v)\ =\ \{\widetilde{U}\in\widetilde{G}_{2r_0}(\mathbf{R}^{2m_0}) : v\in\widetilde{U}\}.$$

We choose $2r_0$ linearly independent vectors $v_1,\dots,v_{2r_0}\in\mathbf{R}^{2m_0}$. Then the oriented submanifolds $\mathrm{Div}(\alpha_{v_j})$ meet transversely and

$$\begin{aligned} \mathrm{Div}(\alpha_{v_1})\cap\cdots\cap\mathrm{Div}(\alpha_{v_{2r_0}}) \ &=\ \left\{\widetilde{U}\in\widetilde{G}_{2r_0}(\mathbf{R}^{2m_0}) : v_1,\dots,v_{2r_0}\in\widetilde{U}\right\} \\ &\cong\ \{v_1\wedge\cdots\wedge v_{2r_0},\ -v_1\wedge\cdots\wedge v_{2r_0}\}, \end{aligned}$$

i.e., this intersection consists exactly of the two points corresponding to the plane $\text{span}\{v_1, \ldots, v_{2r_0}\}$ with its two possible orientations. Using invariance under the deck transformation group and a local calculation one concludes that

$$\begin{aligned}\left\langle p_{m_0-r_0}(\widetilde{\mathbf{U}}^\perp)^{r_0},\ \widetilde{G}_{2r_0}(\mathbf{R}^{2m_0})\right\rangle &= \left\langle \chi(\widetilde{\mathbf{U}}^\perp)^{r_0},\ \widetilde{G}_{2r_0}(\mathbf{R}^{2m_0})\right\rangle \\ &= \#\{\text{Div}(\alpha_{v_1}) \cap \cdots \cap \text{Div}(\alpha_{v_{2r_0}})\} \\ &= 2.\end{aligned}$$

This establishes (B.8) and proves (B.5). □

We now recall that if r is even and m is odd, then $G_r(\mathbf{R}^m)$ is non-orientable and $Or(G_r(\mathbf{R}^m)) = Or(U)$. More generally if $E \to X$ is a oriented bundle of odd rank over an oriented manifold, and if r is even, then $G_r(E)$ is non-orientable and $Or(G_r(E)) = Or(U)$, where U is the tautological bundle as above. As seen in (A.11), integration over the fibre gives a Gysin map $\pi_* : H^k(G_r(E), Or(U)) \to H^{k-r(m-r)}(X; \mathbf{R})$ where $H^k(G_r(E); Or(U))$ denotes the cohomology of $Or(U)$-twisted forms, i.e., cohomology with coefficients in the local system $Or(U)$. As seen also in (A.11), $\chi(U) \wedge \Omega$ is an $Or(U)$-twisted class for any $\Omega \in H^*(G_r(E); \mathbf{R})$.

Proposition B.11. *Let $E \to X$ be an oriented riemannian bundle of* rank m *over an oriented manifold. Let $G_r(E)$, U and $U^\perp$ be as in Proposition B.5, but assume that*

$$r = 2r_0 \quad \textit{and} \quad m = 2m_0 + 1$$

for positive integers r_0 and m_0. Then under the Gysin map $\pi_* : H^{r(m-r)}(G_r(E); Or(U)) \to H^0(X; \mathbf{R})$ we have

$$\pi_* \left\{\chi(U) p_{m_0-r_0}(U^\perp)^{r_0}\right\} = 1. \tag{B.12}$$

Proof. It suffices to consider the case where X is a point. Fix $v_0 \in \mathbf{R}^m$ and let β be the cross-section of $\mathbf{U}$ given by

$$\beta(U) = \pi_U(v_0)$$

where $\pi_U : \mathbf{R}^m = U \oplus U^\perp \to U$ is orthogonal projection. Now the Poincaré dual of $\chi(\mathbf{U})$ is represented by the oriented cycle

$$\begin{aligned}\text{Div}(\beta) &= \text{The Grassmannian of } r\text{-planes in } v_0^\perp \\ &\cong G_{2r_0}(\mathbf{R}^{2m_0}).\end{aligned}$$

By the standard formula for the Poincaré dual we have

$$\begin{aligned}\left\langle \chi(\mathbf{U}) p_{m_0}(\mathbf{U}^\perp)^{r_0},\ G_r(\mathbf{R}^m)\right\rangle &= \left\langle p_{m_0}(\mathbf{U}^\perp)^{r_0},\ \chi(\mathbf{U}) \cap G_r(\mathbf{R}^m)\right\rangle \\ &= \left\langle p_{m_0}(\mathbf{U}^\perp)^{r_0},\ \text{Poincaré dual of } \chi(\mathbf{U})\right\rangle \\ &= \left\langle p_{m_0}(\mathbf{U}^\perp)^{r_0},\ G_{2r_0}(\mathbf{R}^{2m_0})\right\rangle \\ &= 1\end{aligned}$$

where the last equality comes from (B.5). □

APPENDIX C Explicit formulae

Here we collect together explicit algebraic expressions for the various forms occuring in the equations of Section 3. These explicit expressions are taken from [HL_2]. Suppose that V is either a complex vector bundle of complex rank n or a real vector bundle of even rank 2n which is oriented. Equip V with an inner product $\langle\ ,\ \rangle_V$ and a connection D_V. The equations are:

Complex Case:

$$\begin{aligned} c_n(\Omega_V) - \tau_s &= d\sigma_s \\ \tau_s - [X] &= dr_s \\ c_n(\Omega_V) - [X] &= d\sigma. \end{aligned}$$

Real Case:

$$\begin{aligned} \chi(\Omega_V) - \tau_s &= d\sigma_s \\ \tau_s - [X] &= dr_s \\ \chi(\Omega_V) - [X] &= d\sigma. \end{aligned}$$

In both cases σ_s and r_s are related by

$$r_s = \sigma - \sigma_s$$

and

$$\lim_{s\to 0} r_s = 0 \qquad \lim_{s\to\infty} \sigma_s = 0,$$

in $L^1_{\mathrm{loc}}(V)$. These are current equations on the total space of V with $X \subset V$ the zero section. Alternatively, given a smooth atomic section $\mu : X \to V$ they can be pulled back to equations on X. The atomic hypothesis ensures that the $L^1_{\mathrm{loc}}(V)$ form σ pulls back to an $L^1_{\mathrm{loc}}(X)$ form on X, as well as ensuring that $\mathrm{Div}(\mu)$ is the appropriate replacement for $[X]$.

In order to describe the global L^1_{loc} forms τ_s, σ_s, and σ explicitely the following notation is useful. Let $e_1, \ldots, e_n$ denote a local frame for V and let e denote the column with i^{th} entry e_i. Then a section μ can be written as $\mu = ue$ with $u = (u_1, \ldots, u_n)$. The equation $D_V e = \omega_V e$ defines the local gauge ω_V as an $n \times n$ matrix of one forms. The curvature operator $R_V = D_V^2$ has matrix form $\Omega_V = d\omega_V - \omega_V \wedge \omega_V$. Since $D_V\mu = (du + u\omega_V)e$, it is convenient to let $Du \equiv du + u\omega_V$, so that $D_V\mu = (Du)e$. Let $h_V \equiv (\langle e_i, e_j\rangle_V)$ denote the metric matrix, and let $u^* \equiv h_V \bar{u}^t$ so that $|\mu|^2 = uh_V\bar{u}^t = uu^*$. (We will also find it convenient to let $|u|^2$ denote $|\mu|^2 = uu^*$ rather than $u\bar{u}^t$.) Also, let

$$Du^* = du^* - \omega_V u^*.$$

Just as Du is the matrix form of $D_V\mu$, this formula is the matrix form of $D\mu^*$ where μ^* is the adjoint of μ thought of as a bundle map from the trivial bundle $\underline{\mathbf{C}}$ to V.

In the real case we only consider oriented orthonormal frames e and hence $h_V = 1$ (the identity matrix) so that $u^* = u^t$, $|\mu|^2 = uu^t = |u|^2$, and $Du^t = du^t - \omega_V u^t$. As before $\chi(t) : [0, \infty] \to [0, 1]$ determines the approximation mode. For brevity let

$$\chi_s \;\equiv\; \chi\left(\tfrac{|u|^2}{s^2}\right).$$

In the complex case, consider the dual frame $e^* = (e_1^*, \ldots, e_n^*)$, together with the frame e, as elements of the grassmann algebra $\wedge(V^* \oplus V)$. Let $\lambda(e) \equiv e_1^* \wedge e_1 \wedge \cdots \wedge e_n^* \wedge e_n$ denote the **volume form**. Then for any matrix A, the determinant can be computed from

$$\frac{1}{n!}(e^* A e) \;=\; (\det A)\lambda(e).$$

Consequently, the equation

$$\det(A\;;\;B)\lambda(e) \;=\; \frac{1}{(n-1)!}(e^* A e)(e^* A e)^{n-1}$$

can be used to compute

$$\det(A\;;\;B) \;\equiv\; \frac{d}{dt}\det(B + tA)\,\Big|_{t=0}\,.$$

Complex Case:

$$\boldsymbol{\tau_s} \;=\; \left(\tfrac{i}{2\pi}\right)^n (1-\chi_s)\det\left(\Omega_V - \chi_s \frac{Du^* Du}{|u|^2}\right)$$
$$+\left(\tfrac{i}{2\pi}\right)^n \left(\chi_s(1-\chi_s) - \chi_s' \tfrac{|u|^2}{s^2}\right) \tfrac{d|u|^2}{|u|^2} \det\left(\frac{u^* Du}{|u|^2}\;;\;\Omega_V - \chi_s \frac{Du^* Du}{|u|^2}\right).$$

$$\boldsymbol{\sigma_s}\lambda(e) \;=\; -\tfrac{1}{n!}\left(\tfrac{i}{2\pi}\right)^n \frac{e^* u^* Due}{|u|^2}\;\;\frac{(e^* \boldsymbol{\Omega_V} e - \chi_s e^* \frac{Du^* Du}{|u|^2} e)^n - (e^* \boldsymbol{\Omega_F} e)^n}{e^* \frac{Du^* Du}{|u|^2} e}$$

$$\boldsymbol{\sigma}\lambda \;=\; -\tfrac{1}{n!}\left(\tfrac{i}{2\pi}\right)^n \frac{e^* u^* Due}{|u|^2}\;\;\frac{(e^* \boldsymbol{\Omega_V} e - e^* \frac{Du^* Du}{|u|^2} e)^n - (e^* \boldsymbol{\Omega_V} e)^n}{e^* \frac{Du^* Du}{|u|^2} e}$$

The choice $\chi(t) \equiv t/1+t$ is referred to as the **algebraic approximation** mode. (See [HL$_2$] and [Z]) for motivation for this choice). Note that σ is independent of the choice of approximation mode χ.

Complex Case with Algebraic Approximation Mode:

$$\boldsymbol{\tau_s} \;=\; \left(\tfrac{i}{2\pi}\right)^n \frac{s^2}{|u|^2 + s^2}\det\left(\boldsymbol{\Omega_V} - \frac{Du^* Du}{|u|^2 + s^2}\right),$$

or equivalently,

$$\boldsymbol{\tau_s}\lambda \;=\; \tfrac{1}{n!}\left(\tfrac{i}{2\pi}\right)^n \tfrac{s^2}{|u|^2+s^2}\left(e^* \boldsymbol{\Omega_V} e - \frac{e^* Du^* Due}{|u|^2 + s^2}\right)^n.$$

$$\boldsymbol{\sigma_s}\lambda \;=\; -\tfrac{1}{n!}\left(\tfrac{i}{2\pi}\right)^n e^* u^* Due \frac{\left(e^* \boldsymbol{\Omega_V} e - \frac{e^* Du^* Due}{|u|^2+s^2}\right)^n - (e^* \boldsymbol{\Omega_V} e)^n}{e^* Du^* Due}.$$

Now we consider the real case. The real rank of V is assumed to be even $(= 2n)$.

Real Case: Suppose A is a skew $2n \times 2n$ matrix. Let $e_1, \ldots, e_{2n}$ denote an **oriented** orthonormal (local) frame for V and let $\lambda \equiv e_1 \wedge \cdots \wedge e_{2n} \in \Lambda^{2n} V$ denote the unit volume element. Recall the definition of the **pfaffian** of A

$$\mathrm{Pf}(A)\lambda \;=\; \frac{1}{n!}\left(\tfrac{1}{2} e^t A e\right)^n.$$

Since V is oriented the unit volume form λ for V is globally defined, and hence $\mathrm{Pf}(A)$ is globally defined independently of the choice of oriented orthonormal frame e.

$$\boldsymbol{\tau}_{\mathbf{s}}\lambda \;=\; \tfrac{1}{n!}\left(\tfrac{-1}{4\pi}\right)^n (1-\chi_s)\left(e^t\Omega_V e - 2\chi_s\left(1-\tfrac{\chi_s}{2}\right)\frac{(Due)^2}{|u|^2}\right)^n \quad +$$

$$\tfrac{2}{(n-1)!}\left(\tfrac{-1}{4\pi}\right)^n\left(\chi_s(1-\chi_s)\left(1-\tfrac{\chi_s}{2}\right)-\chi_s'\tfrac{|u|^2}{s^2}\right)\tfrac{d|u|^2}{|u|^2}\,\tfrac{(ue)(Due)}{|u|^2}\left(e^t\Omega_V e-2\chi_s\left(1-\tfrac{\chi_s}{2}\right)\tfrac{(Due)^2}{|u|^2}\right)^{n-1}.$$

$$\boldsymbol{\sigma}_{\mathbf{s}}\lambda \;=\; \tfrac{2}{(n-1)!}\left(\tfrac{-1}{4\pi}\right)^n \tfrac{(ue)(Due)}{|u|^2}\int_0^{\chi_s}\left(e^t\Omega_V e - 2x\left(1-\tfrac{x}{2}\right)\frac{(Due)^2}{|u|^2}\right)^{n-1} dx.$$

The choice $\chi(t) = 1 - \frac{1}{\sqrt{1+t}}$ is referred to as the **real algebraic approximation** mode (See [HL$_2$]).

Real Case with Algebraic Approximation Mode:

$$\boldsymbol{\tau}_{\mathbf{s}}\lambda \;=\; \tfrac{1}{n!}\left(\tfrac{-1}{4\pi}\right)^n \frac{s}{\sqrt{|u|^2+s^2}}\left(e^t\Omega_V e - \frac{(Due)^2}{|u|^2+s^2}\right)^n.$$

$$\boldsymbol{\sigma}\lambda \;=\; \frac{1}{\pi^n}\sum_{p=0}^{n-1}(-1)^{n-p}\frac{p!}{(n-p-1)!(2p+1)!2^{2n-2p-1}}\;\frac{(ue)(Due)^{2p+1}}{|u|^{2p+2}}\;(e^t\Omega_V e)^{n-p-1}.$$

Thus the part of $\boldsymbol{\sigma}$ of top degree $2n-1$ in the 1-forms $du_1, \ldots, du_{2n}$ is

$$\boldsymbol{\sigma}_{2n-1} \;=\; \mathrm{vol}(S^{2n-1})^{-1}\theta(u)$$

where

$$\theta(u) \;\equiv\; \sum_{k=1}^{2n}(-1)^{k-1}\;\frac{u_k du_1 \wedge \cdots \wedge \widehat{du_k} \wedge \cdots \wedge du_{2n}}{|u|^{2n}}$$

denotes the **solid angle kernel** on $\mathbf{R}^{2n}$.

Remark. These explicit formula have two different interpretations. First they define forms on the total space of the bundle V. In this case $u \equiv u_1, \ldots, u_n$ is the fiber variable. Second they define forms on X where $u \equiv (u_1, \ldots, u_n)$ is the n-tuple of C^∞ functions on X representing the given atomic section μ in the frame e.

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