FINITE VOLUME FLOWS and MORSE THEORY

by

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§**0. Introduction.** In this paper we present a new approach to Morse theory based on the de Rham - Federer theory of currents. The full classical theory is derived in a transparent way. The methods carry over uniformly to the equivariant and the holomorphic settings. Moreover, the methods are substantially stronger than the classical ones and have interesting applications to geometry. They lead, for example, to formulas relating characteristic forms and singularities of bundle maps.

The ideas came from addressing the following.

Question. Given a smooth flow $\varphi_t : X \to X$ on a manifold X, when does the limit

$$\mathbf{P}(\alpha) \equiv \lim_{t\to\infty} \varphi_t^* \alpha$$

exist for a given smooth differential form α?

We do not demand that this limit be smooth. As an example consider the gradient flow of a linear function restricted to the unit sphere $S^n \subset \mathbb{R}^{n+1}$. For any n-form α, $\mathbf{P}(\alpha) = c[p]$ where $c = \int_{S^n} \alpha$ and $[p]$ is the point measure of the bottom critical point p.**

Our key to the problem is the concept of a *finite volume flow* – a flow for which the graph T of the relation $x \prec y$, defined by the forward motion of the flow, has finite volume in $X \times X$. Such flows are abundant. They include generic gradient flows, flows for which the graph is analytic (e.g., algebraic flows), and any flow with fixed points on S^1.

We show that for any flow of finite volume, the limit $\mathbf{P}$ exists for all α and defines a continuous linear operator

$$\mathbf{P} : \mathcal{E}^*(X) \longrightarrow \mathcal{D}'^*(X)$$

from smooth forms to generalized forms , i.e., currents. Furthermore, this operator is chain homotopic to the inclusion $\mathbf{I} : \mathcal{E}^*(X) \hookrightarrow \mathcal{D}'^*(X)$. More precisely using the graph T we shall construct a continuous operator $\mathbf{T} : \mathcal{E}^*(X) \longrightarrow \mathcal{D}'^*(X)$ of degree -1 such that

$$\text{(FME)} \qquad d \circ \mathbf{T} + \mathbf{T} \circ d \;=\; \mathbf{I} - \mathbf{P}.$$

By de Rham [deR], $\mathbf{I}$ induces an isomorphism in cohomology. Hence so does $\mathbf{P}$.

Now let $f : X \to \mathbb{R}$ be a Morse function with critical set $Cr(f)$, and suppose there is a riemannian metric on X for which the gradient flow φ_t of f has the following properties:

(i) φ_t is of finite volume.

(ii) The stable and unstable manifolds, S_p, U_p for $p \in Cr(f)$, are of finite volume in X.

(iii) $p \prec q \;\Rightarrow\; \lambda_p < \lambda_q$ for all $p, q \in Cr(f)$, where λ_p denotes the index of p and where $p \prec q$ means there is a piecewise flow line connecting p in forward time to q.

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**When α has compact support in $\mathbb{R}^n = S^n - \{\infty\}$ this simple example is the useful "approximate identity" argument widespread in analysis.

Metrics yielding such gradient flows always exist. In fact Morse-Smale gradient systems have these properties [HL$_4$]. Under the hypotheses (1)–(3) we prove that the operator $\mathbf{P}$ has the following simple form

$$\mathbf{P}(\alpha) \;=\; \sum_{p\in Cr(f)} \left(\int_{U_p} \alpha\right)[S_p]$$

where $\int_{U_p} \alpha = 0$ if $\deg \alpha \neq \lambda_p$. Thus $\mathbf{P}$ gives a retraction

$$\mathbf{P} : \mathcal{E}^*(X) \;\longrightarrow\; \mathcal{S}_f \;\stackrel{\text{def}}{=}\; \text{span}_{\mathbb{R}}\Big\{[S_p]\Big\}_{p\in Cr(f)}$$

onto the finite dimensional subspace of currents spanned by the stable manifolds of the flow. Using (FME) we see that $\mathcal{S}_f$ is d-invariant and show that $H^*(\mathcal{S}_f) \;\cong\; H^*_{\text{de Rham}}(X)$. This immediately yields the strong Morse inequalities.

The restriction of d to $\mathcal{S}_f$ has the form

$$d[S_p] \;=\; \sum_{\lambda_q=\lambda_p-1} n_{pq}[S_q]$$

where, by a basic result of Federer, the constants n_{pq} are integers. One concludes that $\mathcal{S}_f^{\mathbf{Z}} \equiv \text{span}_{\mathbf{Z}}\left\{[S_p]\right\}_{p\in Cr(f)}$ is a finite-rank subcomplex of the integral currents $\mathcal{I}_*(X)$ whose inclusion into $\mathcal{I}_*(X)$ induces an isomorphism $H(\mathcal{S}_f^{\mathbf{Z}}) \;\cong\; H_*(X;\mathbb{Z})$. In the Morse-Smale case the constants n_{pq} can be computed explicitly by counting flow lines from p to q. This fact follows directly from Stokes' Theorem [La].

The operator $\mathbf{T}$ above can be thought of as the *fundamental solution* for the de Rham complex provided by the Morse flow. It is the Morse-theoretic analogue of the Hodge operator $d^*\circ G$ where G is the fundamental solution for the Laplacian (the Greens operator). The equation (FME) will be referred to as the *fundamental Morse equation*. As noted above it naturally implies the two homology isomorphisms (over $\mathbb{R}$ and $\mathbb{Z}$) which together encapsule Morse Theory. The proof of this equation employs the kernel calculus of [HP] to convert current equations on $X \times X$ to operator equations.

The method introduced here has many applications. It was used in [HL$_2$] to derive a local version of a formula of MacPherson [Mac$_{1,2}$] which relates the singularities of a generic bundle map $A : E \to F$ to characteristic forms of E and F. However, the techniques apply as well to a significantly larger class of bundle maps called *geometrically atomic*. These include every real analytic bundle map as well as generic direct sums and tensor products of bundle maps. New explicit formulas relating curvature and singularities are derived in each case [HL$_3$]. This extends and simplifies previous work on singular connections and characteristic currents [HL$_1$].

The approach also works for holomorphic $\mathbb{C}^*$-actions with fixed-points on Kähler manifolds. One finds a complex analogue of the current T. General results of Sommese imply that all the stable and unstable manifolds of the flow are analytic subvarieties. One retrieves classical results of Bialynicki-Birula [BB] and Carrell-Lieberman-Sommese

[CL],[CS]. The approach also fits into MacPherson's Grassmann graph construction and the construction of transgression classes in the refined Riemann-Roch Theorem [GS].

The method has many other extensions. It applies to the multiplication and comultiplication operators in cohomology whose kernel is the triple diagonal in $X \times X \times X$ (§11). This idea can be further elaborated as in [BC]. Similarly, the method fits into constructions of invariants of knots and 3-manifolds from certain "Feynman graphs" (cf. [K], [BT]).

The cell decomposition of a manifold by the stable manifolds of a gradient flow is due to Thom [T], [S]. That these stable manifolds embed into the de Rham complex of currents was first observed by Laudenbach [La].

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Notation. For an n-manifold X, $\mathcal{E}^k(X)$ will denote the space of smooth k-forms on X, and $\mathcal{D}'^k(X)$ the space of k-forms with distribution coefficients (called the *currents* of dimension $n-k$ on X). Under the obvious inclusion $\mathcal{E}^k(X) \subset \mathcal{D}'^k(X)$, exterior differentiation d has an extension to all of $\mathcal{D}'^k(X)$, and $d = (-1)^{k+1}\partial$ where ∂ denotes the current boundary. (See [deR] for full definitions.) We assume here that X is oriented, but this assumption is easily dropped and all results go through.

§**1. Finite Volume Flows.** Let X be a compact oriented manifold of dimension n, and let $\varphi_t : X \to X$ be the flow generated by a smooth vector field V on X. For each $t > 0$ consider the compact submanifold with boundary

$$\mathcal{T}_t \;\equiv\; \{(s, \varphi_s(x), x) \,:\, 0 \le s \le t \text{ and } x \in X\} \;\subset\; \mathbb{R} \times X \times X.$$

oriented so that

$$\partial \mathcal{T}_t = \{0\} \times \Delta - \{t\} \times P_t$$

where $\Delta \subset X \times X$ is the diagonal and $P_t = [\text{graph}(\varphi_t)] = \{(\varphi_t(x), x) \,:\, x \in X\}$ is the graph of the diffeomorphism φ_t. Let $\text{pr} : \mathbb{R} \times X \times X \to X \times X$ denote projection and consider the push-forward

$$T_t \equiv (\text{pr})_*(\mathcal{T}_t).$$

of $\mathcal{T}_t$ as a de Rham current. Since ∂ commutes with $(\text{pr})_*$, we have that

$$\partial T_t = [\Delta] - P_t$$

Note that the current T_t can be equivalently defined by

$$T_t \;=\; \Phi_*([0,t] \times X)$$

where $\Phi : \mathbb{R} \times X \to X \times X$ is the smooth mapping defined by $\Phi(s,x) = (\varphi_s(x), x)$. This mapping is an immersion exactly on the subset $\mathbb{R} \times (X - Z(V))$ where $Z(V) = \{x \in X : V(x) = 0\}$. Thus if we fix a riemannian metric g on X, then $\Phi^*(g \times g)$ is a non-negative symmetric tensor which is > 0 exactly on the subset $\mathbb{R} \times (X - Z(V))$.

Definition 1.1. A flow φ_t on X is called a **finite volume flow** if $\mathbb{R}^+ \times (X - Z(V))$ has finite volume with respect to the metric induced by the immersion Φ. (This concept is independent of the choice of riemannian metric on X.)

Theorem 1.2. *Let φ_t be a finite volume flow on a compact manifold X, and let T_t be the family of currents defined above with*

$$\text{(A)} \qquad \partial T_t = [\Delta] - P_t \qquad \text{on } \ X \times X.$$

Then both the limits

$$\text{(B)} \qquad P \ \equiv \ \lim_{t\to\infty} P_t \ = \ \lim_{t\to\infty} [\text{graph}\, \varphi_t] \qquad \text{and} \qquad T \ \equiv \ \lim_{t\to\infty} T_t$$

exist as currents, and taking the boundary of T gives the equation of currents

$$\text{(C)} \qquad \partial T = [\Delta] - P \qquad \text{on } \ X \times X.$$

Proof. Since φ_t is a finite-volume flow, the current $T \equiv \Phi_*((0,\infty)\times X)$ is the limit in the mass norm of the currents $T_t = \Phi_*((0,t)\times X)$ as $t \to \infty$. The continuity of the boundary operator and equation (A) imply the existence of $\lim_{t\to\infty} P_t$ and also establish equation (C). ∎

We shall see in the next section that Theorem 1.2 has an important reinterpretation in terms of operators on the differential forms on X.

Remark 1.3. If we define a relation on $X \times X$ by setting $x \overset{o}{\prec} y$ if $y = \varphi_t(x)$ for some $0 \le t < \infty$, then T is just the (reversed) graph of this relation. This relation is always transitive and reflexive, and it is antisymmetric if and only if φ_t has no periodic orbits (i.e., $\overset{o}{\prec}$ is a partial ordering precisely when φ_t has no periodic orbits).

Remark 1.4. The immersion $\Phi : \mathbb{R}\times (X - Z(V)) \to X \times X$ is an embedding outside the subset $\mathbb{R}\times Per(V)$ where

$$Per(V) = \{x \in X \ : \ \varphi_t(x) = x \ \ \text{for some} \ \ t > 0\}$$

are the non-trivial periodic points of the flow. Thus, if $Per(V)$ has measure zero, then T_t is given by integration over the embedded finite-volume submanifold $\Phi(R_t)$, where $R_t = (0,t)\times \widetilde{X}$ and $\widetilde{X} = X - Z(V) \cup Per(V)$. If furthermore the flow has finite volume, then T is given by integration over the embedded, finite-volume submanifold $\Phi(R_\infty)$.

There is evidence that any flow with periodic points cannot have finite volume. However, a gradient flow never has periodic points, and many such flows are of finite volume.

Note that any flow with fixed points on S^1 has finite volume.

Remark 1.5. A standard method for showing that a given flow is finite volume can be outlined as follows. Pick a coordinate change $t \mapsto \rho$ which sends $+\infty$ to 0 and $[t_0, \infty]$ to $[0, \rho_0]$. Then show that

$$\widehat{\mathcal{T}} \ \equiv \ \{(\rho, \varphi_{t(\rho)}(x), x) \ : \ 0 < \rho < \rho_0\} \ \text{ has finite volume in } \ \mathbb{R}\times X \times X.$$

Pushing forward to $X \times X$ then yields the current T with finite mass. Perhaps the most natural such coordinate change is $r = 1/t$. Another natural choice (if the flow is considered multiplicatively) is $s = e^{-t}$. Of course finite volume in the r coordinate insures finite volume in the s coordinate since $r \mapsto s = e^{-1/r}$ is a C^∞-map.

Many interesting flows can be seen to be finite volume as follows.

Lemma 1.6. *If X is analytic and $\widehat{\mathcal{T}} \subset \mathbb{R} \times X \times X$ is contained in a real analytic subvariety of dimension $n+1$, then φ_t is a finite volume flow.*

Proof. The manifold points of a real analytic subvariety have (locally) finite volume. ∎

Example 1.7. Consider S^n with two coordinate charts $\mathbb{R}^n$ and the coordinate change $x = y/|y|^2$. In the y-coordinates consider the translational flow $\varphi_t(y) = y + tu$ where $u \in \mathbb{R}^n$ is a fixed unit vector. In the x-coordinates this becomes

$$\varphi_t(x) \;=\; \frac{|x|^2}{|x + t|x|^2 u|^2}\left(x + t|x|^2 u\right)$$

To see that φ_t is finite volume flow let $r = 1/t$ and note that $\widehat{\mathcal{T}}$ is defined by algebraic equations so that Lemma 1.6 is applicable. The vector field V has only one zero and is therefore not a gradient vector field. It is however a limit of gradient vector fields.

§2. The Operator Equations.

The current equations (A), (B), (C) of Proposition 1.2 can be translated into operator equations by a kernel calculus which was introduced in [HP] and which we shall now briefly review. For clarity of exposition we assume our manifolds to be orientable. However, the discussion here extends straightforwardly to the non-orientable case and to forms and currents with values in a flat bundle.

Let X and Y be compact oriented manifolds, and let π_Y and π_X denote projection of $Y \times X$ onto Y and X respectively. Then each current (or *kernel*) $K \in \mathcal{D}'^*(Y \times X)$, determines an operator $\mathbf{K} : \mathcal{E}^*(Y) \to \mathcal{D}'^*(X)$ by the formula

$$\mathbf{K}(\alpha) = (\pi_X)_*(K \wedge \pi_Y^* \alpha),$$

i.e., for a smooth form β of appropriate degree on X

$$\mathbf{K}(\alpha)(\beta) = K(\pi_Y^* \alpha \wedge \pi_X^* \beta).$$

Example 2.1. Suppose $\varphi : X \to Y$ is a smooth map, and let

$$P_\varphi = \{(\varphi(x), x) : x \in X\} \subset Y \times X.$$

be the graph of φ with orientation induced from X. Then $\mathbf{P}_\varphi$ is just the pull-back operator $\mathbf{P}_\varphi(\alpha) \equiv \varphi^*(\alpha)$ on differential forms α. In particular if $X = Y$ and $K = [\Delta] \subset X \times X$ is the diagonal (the graph of the identity), then $\mathbf{K} = \mathbf{I} : \mathcal{E}^*(X) \to \mathcal{D}'^*(X)$ is the standard inlcusion of the smooth forms into the currents on X.

Lemma 2.2. *Suppose an operator $\mathbf{T} : \mathcal{E}^*(Y) \to \mathcal{D}'^*(X)$ has kernel $T \in \mathcal{D}'^*(Y \times X)$. Suppose that $\mathbf{T}$ lowers degree by one, or equivalently, that $\deg(T) = \dim Y - 1$. Then*

$$\text{(2.1)} \qquad \textit{The operator } \ d \circ \mathbf{T} + \mathbf{T} \circ d \ \textit{ has kernel } \ \partial T.$$

Proof. The boundary operator ∂ is the dual of exterior differentiation. That is ∂T is defined by $(\partial T)(\pi_Y^* \alpha \wedge \pi_X^* \beta) = T(d(\pi_Y^* \alpha \wedge \pi_X^* \beta))$. Also, by definition, $\mathbf{T}(d\alpha)(\beta) \equiv T(\pi_Y^*(d\alpha) \wedge \pi_X^* \beta)$, and

$$d(\mathbf{T}(\alpha))(\beta) \equiv (-1)^{\deg \alpha} \mathbf{T}(\alpha)(d\beta) = (-1)^{\deg \alpha} T(\pi_Y^* \alpha \wedge \pi_X^* d\beta),$$

since $\mathbf{T}(\alpha)$ has degree equal to $\deg \alpha - 1$. ∎

The results that we need are summarized in the following table.

Operators	Kernels
$\mathbf{I}$	$[\Delta]$
$\mathbf{P}_t \equiv \varphi_t^*$	$P_t \equiv [\mathrm{graph}\, \varphi_t]$
$\mathbf{T}$	T
$d \circ \mathbf{T} + \mathbf{T} \circ d$	∂T

From this table Theorem 1.2 can be reformulated as follows.

Theorem 2.3. *Let φ_t be a finite volume flow on a compact manifold X, and let $\mathbf{P}_t : \mathcal{E}^*(X) \to \mathcal{E}^*(X)$ be the operator given by pull-back*

$$\mathbf{P}_t(\alpha) \equiv \varphi_t^*(\alpha).$$

Let $\mathbf{T}_t : \mathcal{E}^k(X) \to \mathcal{E}^{k-1}(X)$ be the family of operators associated to the currents T_t defined in §1. Then for all t one has that

$$\textbf{(A)} \qquad d \circ \mathbf{T}_t + \mathbf{T}_t \circ d = \mathbf{I} - \mathbf{P}_t.$$

Furthermore, the limits

$$\textbf{(B)} \qquad \mathbf{T} = \lim_{t \to \infty} \mathbf{T}_t \quad \textit{and} \quad \mathbf{P} = \lim_{t \to \infty} \mathbf{P}_t$$

exist as operators from forms to currents, and they satisfy the equation

$$\textbf{(C)} \qquad d \circ \mathbf{T} + \mathbf{T} \circ d = \mathbf{I} - \mathbf{P}.$$

§3. Morse-Stokes Gradients; Axioms.

Let $f \in C^\infty(X)$ be a Morse function on a compact n-manifold X, and let $\mathrm{Cr}(f)$ denote the (finite) set of critical points of f. Recall that f is a **Morse function** if its Hessian at each critical point is non-degenerate. The standard Morse Lemma asserts that in a neighborhood of each $p \in \mathrm{Cr}(f)$ of index λ, there exist **canonical local coordinates** $(u_1, ..., u_\lambda, v_1, ..., v_{n-\lambda})$ for $|u| < r$, $|v| < r$ with $(u(p), v(p)) = (0, 0)$ such that

$$f(u, v) = f(p) - |u|^2 + |v|^2. \tag{3.1}$$

Fix a riemannian metric g on X and let φ_t denote the flow associated to ∇f. Suppose that at $p \in \mathrm{Cr}(f)$ there exists a canonical coordinate system (u, v) in which $g = |du|^2 + |dv|^2$. Then in these coordinates the gradient flow is given by

$$\varphi_t(u, v) = (e^{-t}u, e^t v) \tag{3.2}$$

Metrics g with this property at every $p \in \mathrm{Cr}(f)$ will be called ***f*-tame**.

Now to each $p \in \mathrm{Cr}(f)$ are associated the **stable** and **unstable manifolds** of the flow, defined respectively by

$$S_p = \{x \in X : \lim_{t\to\infty} \varphi_t(x) = p\} \qquad \text{and} \qquad U_p = \{x \in X : \lim_{t\to-\infty} \varphi_t(x) = p\}.$$

If g is f-tame, then for coordinates (u, v) at p, chosen as above, we consider the disks

$$S_p(\epsilon) = \{(u, 0) : |u| < \epsilon\} \qquad \text{and} \qquad U_p(\epsilon) = \{(0, v) : |v| < \epsilon\}$$

and observe that

$$S_p = \bigcup_{-\infty<t<0} \varphi_t\left(S_p(\epsilon)\right) \qquad \text{and} \qquad U_p = \bigcup_{0<t<+\infty} \varphi_t\left(U_p(\epsilon)\right). \tag{3.3}$$

Hence, S_p and U_p are contractible submanifolds (but not closed subsets) of X with

$$\dim S_p = \lambda_p \qquad \text{and} \qquad \dim U_p = n - \lambda_p \tag{3.4}$$

where λ_p is the index of the critical point p. For each p we choose an orientation on U_p. This gives an orientation on S_p via the splitting $T_pX = T_pU_p \oplus T_pS_p$.

The flow φ_t induces a partial ordering on X by setting $x \prec y$ if there is a continuous path consisting of a finite number of foward-time orbits, which begins with x and ends with y. We shall see in the proof of 3.4 that this relation is simply the closure of the partial ordering of Remark 1.3.

Definition 3.1. The gradient flow of a Morse function f on a riemannian manifold X is called **Morse-Stokes** if the metric is f-tame and if:

(i) The flow is a finite-volume flow.
(ii) Each of the stable and unstable manifolds S_p and U_p for $p \in \mathrm{Cr}(f)$ has finite volume.
(iii) If $p \prec q$ and $p \neq q$ then $\lambda_p < \lambda_q$, for all p, $q \in \mathrm{Cr}(f)$.

Remark 3.2. If the gradient flow of f is Morse-Smale, then it is Morse-Stokes. Furthermore, for any Morse function f on a compact manifold X there exist riemannnian metrics on X for which the gradient flow of f is Morse-Stokes. In fact these metric are dense in the space of all f-tame metrics on X. See [HL$_4$].

Theorem 3.3. *Let $f \in C^\infty(X)$ be a Morse function on a compact riemannian manifold whose gradient flow is Morse-Stokes. Then the current P in the fundamental equation*

$$\partial T \;=\; [\Delta] - P \tag{3.5}$$

from Theorem 1.2 is given by

$$P \;=\; \sum_{p \in Cr(f)} [U_p] \times [S_p] \tag{3.6}$$

where U_p and S_p are oriented so that $U_p \times S_p$ agrees in orientation with X at p.

Proof. For each critical point $p \in \mathrm{Cr}(f)$ we define

$$\widetilde{U}_p \stackrel{\mathrm{def}}{=} \bigcup_{p \prec q} U_q = \{x \in X : p \prec x\}.$$

Lemma 3.4. *Let $f \in C^\infty(X)$ be any Morse function whose gradient flow is of finite volume, and let $\mathrm{spt}\, P \subset X \times X$ denote the support of the current P defined in Theorem 1.2. Then*

$$\mathrm{spt}\, P \subset \bigcup_{p \in \mathrm{Cr}(f)} \widetilde{U}_p \times S_p.$$

Proof. Since $P = \lim_{t\to\infty} P_t$ exists and $P_t = \{(\varphi_t(x), x) \;:\; x \in X\}$, it is clear that $(y, x) \in \mathrm{spt}\, P$ only if there exist sequences $x_i \to x$ in X and $s_i \to \infty$ in $\mathbb{R}$ such that $y_i \equiv \varphi_{s_i}(x_i) \to y$. Let $L(x_i, y_i)$ denote the oriented flow line from x_i to y_i. Since the lengths of these lines are bounded, an elementary compactness argument implies that a subsequence converges to a piecewise flow line $L(x, y)$ from x to y. By the continuity of the boundary operator on currents, $\partial L(x, y) = [y] - [x]$. Finally, since $s_i \to \infty$ there must be at least one critical point on $L(x, y)$, and we define $p = \lim_{s\to\infty} \varphi_s(x)$. ∎

Consider now the subset

$$\Sigma \;\equiv\; \bigcup_{\substack{p \prec q \\ p \neq q}} U_q \times S_p \;\subset\; X \times X.$$

and define $\Sigma' = \Sigma \cup \{(p, p) \,:\, p \in Cr(f)\}$. Note that from (3.4) and Axiom (iii) in 3.1 that

$$\Sigma' \text{ is a finite disjoint union of submanifolds of dimension } \leq n - 1. \tag{3.7}$$

Lemma 3.5. *Consider the embedded submanifold*

$$T = \{(y,x) : x \notin Cr(f), \text{ and } y = \varphi_t(x) \text{ for some } 0 < t < \infty\}.$$

of $X \times X - \Sigma'$. The closure $\overline{T}$ of T in $X \times X - \Sigma'$ is a proper C^∞-submanfold with boundary

$$\partial\overline{T} = \Delta - \sum_{p \in \mathrm{Cr}(f)} U_p \times S_p.$$

Proof. We first show that it will suffice to prove the assertion in a neighborhood of $(p,p) \in X \times X$ for $p \in Cr(f)$. Consider $(\bar{y},\bar{x}) \in \overline{T} - T$. If $(\bar{y},\bar{x}) \notin \Sigma'$, then the proof of Lemma 3.4 shows that either $(\bar{y},\bar{x}) \in \Delta$ or $(\bar{y},\bar{x}) \in U_p \times S_p$ for some $p \in Cr(f)$. Near points $(\bar{x},\bar{x}) \in \Delta$, $\bar{x} \notin Cr(f)$, one easily checks that $\overline{T}$ is a submanifold with boundary Δ. If $(\bar{y},\bar{x}) \in U_p \times S_p$, then for sufficiently large $s > 0$, the diffeomorphism $\psi_s(y,x) \equiv (\varphi_{-s}(y), \varphi_s(x))$ will map $(\bar{y},\bar{x})$ into any given neighborhood of (p,p). Note that ψ_s leaves the subset $U_p \times S_p$ invariant, and that ψ_s^{-1} maps T into T. Hence, if $\partial\overline{T} = \Delta - U_p \times S_p$, in a neighborhood of (p,p), then $\partial\overline{T} = -U_p \times S_p$ near $(\bar{y},\bar{x})$.

Now in a neighborhood $\mathcal{O}$ of (p,p) we may choose coordinates as in (3.2) so that T consists of points $(y,x) = (\bar{u},\bar{v},u,v)$ with $\bar{u} = e^{-t}u$ and $\bar{v} = e^t v$ for some $0 < t < \infty$. Consequently, in $\mathcal{O}$ the set $\overline{T}$ is given by the equations

$$\bar{u} = su \quad \text{and} \quad v = s\bar{v} \qquad \text{for some } \ 0 \le s \le 1.$$

This obviously defines a submanifold in $\mathcal{O} - \{(p,p)\}$ with boundary consisting of Δ and the set $\{\bar{u} = 0, \ v = 0\} \cong (U_p \times S_p) \cap \mathcal{O}$. ∎

Lemma 3.5 has the following immediate consequence

$$\mathrm{spt}\left\{P - \sum_{p \in \mathrm{Cr}(f)} [U_p] \times [S_p]\right\} \subset \Sigma' \tag{3.8}$$

We now apply the following elementary but key result of Federer.

Proposition 3.6. ([F, 4.1.15]) *Let $[W]$ be a current in $\mathbb{R}^n$ defined by integration over a k-dimensional oriented submanifold W of locally finite volume. Suppose* $\mathrm{supp}\,(d[W]) \subset \mathbb{R}^\ell$, *a linear subspace of dimension $\ell < k-1$. Then $d[W] = 0$.*

Combining this with (3.7) and (3.8) proves (3.6) and completes the proof of Theorem 3.3. ∎

§**4. Morse Theory.** Combining Theorems 2.3 and 3.3 gives the following.

Theorem 4.1. *Let $f \in C^\infty(X)$ be a Morse function on a compact riemannian manifold X whose gradient flow φ_t is Morse-Stokes. Then for every differential form $\alpha \in \mathcal{E}^k(X)$, $0 \le k \le n$, one has*

$$\mathbf{P}(\alpha) = \lim_{t \to \infty} \varphi_t^* \alpha = \sum_{p \in \mathrm{Cr}(f)} r_p(\alpha)[S_p]$$

where the "residue" $r_p(\alpha)$ of α at p is defined by $r_p(\alpha) = \int_{U_p} \alpha$ if $k = n-\lambda$ and 0 otherwise. Furthermore, there is an operator $\mathbf{T}$ of degree -1 on $\mathcal{E}^*(X)$ with values in flat currents, such that

$$d \circ \mathbf{T} + \mathbf{T} \circ d \;=\; \mathbf{I} - \mathbf{P} \tag{4.1}$$

Note that $\mathbf{P} : \mathcal{E}^*(X) \longrightarrow \mathcal{D}'^*(X)$ maps onto the finite-dimensional subspace

$$\mathcal{S}_f^* \;\stackrel{\text{def}}{=}\; \text{span}_{\mathbb{R}} \left\{[S_p]\right\}_{p \in \text{Cr}(f)}, \tag{4.2}$$

and that from (4.1) we have

$$\mathbf{P} \circ d \;=\; d \circ \mathbf{P}.$$

Theorem 4.2. *The subspace $\mathcal{S}_f^*$ is d-invariant and is therefore a subcomplex of $\mathcal{D}'^*(X)$. The surjective linear map of cochain complexes*

$$\widehat{\mathbf{P}} : \mathcal{E}^*(X) \longrightarrow \mathcal{S}_f^*$$

defined by$\mathbf{P}$ *induces an isomorphism*

$$\widehat{\mathbf{P}}_* : H_{\text{deR}}(X) \;\xrightarrow{\cong}\; H(\mathcal{S}_f^*).$$

Proof. It follows immediately from Theorem 4.1 that $\mathbf{P}_* = \mathbf{I}_* : H(\mathcal{E}^*(X)) \to H(\mathcal{D}'^*(X))$, and by [deR], $\mathbf{I}_*$ is an isomorphism. Since $i_* \circ \widehat{\mathbf{P}}_* = \mathbf{P}_*$, where $i : \mathcal{S}_f^* \to \mathcal{D}'^*(X)$ denotes the inclusion, we see that $\widehat{\mathbf{P}}_*$ is injective.

However, it does not follow formally that $\widehat{\mathbf{P}}_*$ is surjective (as examples show). We must prove that every $S \in \mathcal{S}_f^*$ with $dS = 0$ is of the form $S = \mathbf{P}(\gamma)$ where $d\gamma = 0$.

To do this we note the following consequence of 3.1(iii). For all $p, q \in Cr(f)$ of the same index, one has that $\overline{S}_p \cap \partial U_q = \emptyset$, $\overline{S}_p \cap \overline{U}_q = \emptyset$ if $p \neq q$, and $\overline{S}_p \cap \overline{U}_p = \{p\}$. Now let $S = \sum r_p S_p$ be a k-cycle in $\mathcal{S}_f$. Then by the above there is an open neighborhood N of spt$\,(S)$ such that $N \cap \partial U_q = \emptyset$ for all q of index k. Since by [deR] cohomology with compact supports can be computed by either smooth forms or all currents, there exists a current σ with compact support in N such that $d\sigma = \gamma - S$ where γ is a smooth form. Now for each q of index k, U_q is a closed submanifold of N, and we can choose a family U_ϵ of smooth closed forms in N such that $\lim_{\epsilon\to 0} U_\epsilon = U_q$ in $\mathcal{D}'(N)$ and $\lim_{\epsilon\to 0}(S_p, U_\epsilon) = (S_p, U_q) = \delta_{pq}[p]$ for all $p \in Cr(f)$ of index k. This is accomplished via standard Thom form constructions for the normal bundle using canonical coordinates at q (e.g. [HL$_1$]). It follows that $(\gamma - S, U_q) = \lim_{\epsilon\to 0}(\gamma - S, U_\epsilon) = \lim_{\epsilon\to 0}(d\sigma, U_\epsilon) = \lim_{\epsilon\to 0}(\sigma, dU_\epsilon) = 0$. We conclude that $\int_{U_q} \gamma = (S, U_q) = r_q$ for all q and so $\mathbf{P}(\gamma) = S$ as claimed. ∎

The vector space $\mathcal{S}_f^*$ has a distinguished lattice

$$\mathcal{S}_f^{\mathbb{Z}} \;\stackrel{\text{def}}{=}\; \text{span}_{\mathbb{Z}} \left\{[S_p]\right\}_{p \in Cr(f)} \;\subset\; \mathcal{S}_f$$

generated by the stable manifolds. Note that $\mathcal{S}_f^{\mathbb{Z}}$ is a subgroup of the integral currents $\mathcal{I}(X)$ on X.

Theorem 4.3. *The lattice $\mathcal{S}_f^{\mathbb{Z}}$ is preserved by d, i.e., $(\mathcal{S}_f^{\mathbb{Z}}, d)$ is a subcomplex of $(\mathcal{S}_f, d)$. Furthermore, the inclusion of complexes $(\mathcal{S}_f^{\mathbb{Z}}, d) \subset (\mathcal{I}(X), d)$ induces an isomorphism*

$$H(\mathcal{S}_f^{\mathbb{Z}}) \;\cong\; H_*(X;\mathbb{Z})$$

Proof. Theorem 4.2 implies that for any $p \in Cr(f)$ we have

$$d[S_p] = \sum_{\lambda_q=\lambda_p-1} n_{p,q}\,[S_q] \tag{4.3}$$

for real numbers $n_{p,q}$. In particular, $d[S_p]$ has finite mass, and so by [F; 4.2.16(2)] it is a *rectifiable* current. This implies that $n_{p,q} \in \mathbb{Z}$ for all p, q, and the first assertion is proved.

Now the domain of the operator $\mathbf{P}$ extends to include any C^1 k-chain c which is transversal to the submanifolds $U_p,\ p \in Cr_k(f)$, while the domain of $\mathbf{T}$ extends to any C^1 chain c for which $c \times X$ is transversal to T. Standard transversality arguments show that such chain groups (over $\mathbb{Z}$) compute $H_*(X;\mathbb{Z})$. Furthermore they include $\mathcal{S}_f^{\mathbb{Z}}$. The result then follows from (4.1). ■

Corollary 4.4. *Let G be a finitely generated abelian group. Then there are natural isomorphisms*

$$H(\mathcal{S}_f^{\mathbb{Z}} \otimes_{\mathbb{Z}} G) \;\cong\; H_*(X;G).$$

The first assertion of Theorem 4.3 has a completely elementary proof whenever the flow is **Morse-Smale**, i.e., when S_p is transversal to U_q for all $p, q \in Cr(f)$. Suppose the flow is Morse-Smale and that $p, q \in Cr(f)$ are critical points with $\lambda_q = \lambda_p - 1$. Then $U_q \cap S_p$ is the union of a finite set of flow lines from q to p which we denote $\Gamma_{p,q}$. To each $\gamma \in \Gamma_{p,q}$ we assign an index n_γ as follows. Let $B_\epsilon \subset S_p$ be a small ball centered at p in a canonical coordinate system (cf. (3.1)), and let y be the point where γ meets ∂B_ϵ. The orientation of S_p induces an orientation on $T_y(\partial B_\epsilon)$, which is identified by flowing backward along γ with $T_q(S_q)$. If this identification preserves orientations we set $n_\gamma = 1$, and if not, $n_\gamma = -1$. We then define

$$N_{p,q} \;\stackrel{\text{def}}{=}\; \sum_{\gamma\in\Gamma_{p,q}} n_\gamma. \tag{4.4}$$

As in [La] Stokes' Theorem now directly gives us the following (cf. [W]).

Proposition 4.5. *When the gradient flow of f is Morse-Smale, the coefficients in (4.3) are given by*

$$n_{p,q} \;=\; (-1)^{\lambda_p}\, N_{p,q}$$

Proof. Given a form α of degree $\lambda_p - 1$, we have

$$(-1)^{\lambda_p} d[S_p](\alpha) \;=\; \int_{S_p} d\alpha \;=\; \lim_{r\to\infty} \int_{dS_p(r)} \alpha$$

where $S_p(r) = \varphi_{-r}(S_p(\epsilon))$ as in §3. It suffices to consider forms α with support near q where $\lambda_q = \lambda_p - 1$. Near such q, the set $S_p(r)$, for large r, consists of a finite number of manifolds with boundary, transversal to U_q. There is one for each $\gamma \in \Gamma_{p,q}$. As $r \to \infty$ along one such γ, $dS_p(r)$ converges to $\pm S_q$ where the sign is determined by the agreement (or not) of the orientation of $dS_p(r)$ with the chosen orientation of S_q. ∎

Remark 4.6. The integers $N_{p,q}$ have a simple definition in terms of currents. Set $S_p(r) = \varphi_{-r}(S_p(\epsilon))$ and $U_q(r) = \varphi_r(U_q(\epsilon))$ (cf. (3.3)). Then for all r sufficiently large

$$N_{p,q} \;=\; \int_X [U_q(r)] \wedge d\,[S_p(r)] \tag{4.5}$$

where the integral denotes evaluation on the fundamental class.

Next we observe that our Morse theory enables one to compute the full integral homology of X (including torsion) using smooth differential forms.

Definition 4.7. Let $\mathcal{E}_f^{\mathbb{Z}} = \mathbf{P}^{-1}(\mathcal{S}_f^{\mathbb{Z}})$ denote the group of **smooth forms with integral residues** and $\mathcal{E}_f^0 \equiv \ker(\mathbf{P})$ the subgroup of **forms with zero residues**. Note that $\alpha \in \mathcal{E}_f^{\mathbb{Z}}$ iff $\int_{U_p} \alpha \in \mathbb{Z}$ for all $p \in Cr(f)$. Since $\mathbf{P}$ commutes with d, both $\mathcal{E}_f^{\mathbb{Z}}$ and $\mathcal{E}_f^0$ are d-invariant, and we have a short exact sequence of chain complexes

$$0 \;\to \mathcal{E}_f^0 \to \mathcal{E}_f^{\mathbb{Z}} \xrightarrow{\mathbf{P}} \mathcal{S}_f^{\mathbb{Z}} \to 0 \tag{4.6}$$

showing that $\mathcal{E}_f^{\mathbb{Z}}$ is an extension of the lattice $\mathcal{S}_f^{\mathbb{Z}}$ by the vector space $\mathcal{E}_f^0$.

Proposition 4.8. *The operator* $\mathbf{P}$ *induces an isomorphism* $\mathbf{P}_* : H(\mathcal{E}_f^{\mathbb{Z}}) \xrightarrow{\cong} H(\mathcal{S}_f^{\mathbb{Z}})$.

Proof. Note that (4.6) embeds into the short exact sequence $0 \;\to \mathcal{E}_f^0 \to \mathcal{E} \xrightarrow{\mathbf{P}} \mathcal{S}_f \to 0$, and by 4.2 we know that $\mathbf{P}_* : H(\mathcal{E}) \to H(\mathcal{S}_f)$ is an isomorphism. Hence, $H(\mathcal{E}_f^0) = 0$ and the result follows. ∎

§**5. Poincaré Duality.** In this context there is a simple proof of Poincaré duality. Given two oriented submanifolds A and B of complementary dimensions in X which meet transversally in a finite number of points, let $A \bullet B = \int_X [A] \wedge [B]$ denote the algebraic number of intersections points. Then for any k we have

$$U_q \bullet S_p \;=\; \delta_{pq} \qquad\qquad \text{for all} \;\; p, q \in Cr_k(f) \tag{5.1}$$

where $Cr_k(f) \equiv \{p \in Cr(f) \;:\; \lambda_p = k\}$. This gives a formal identification

$$\mathcal{U}_f^{\mathbb{Z}} \;\stackrel{\text{def}}{=}\; \mathbb{Z} \cdot \left\{[U_p]\right\}_{p\in Cr(f)} \;\cong\; \mathrm{Hom}\left(\mathcal{S}_f^{\mathbb{Z}},\, \mathbb{Z}\right). \tag{5.2}$$

Therefore, taking the adjoint of d gives a differential δ on $\mathcal{U}_f^{\mathbb{Z}}$ with the property that $H_{n-*}(\mathcal{U}_f^{\mathbb{Z}}, \delta) \cong H^*(X;\mathbb{Z})$. On the other hand the arguments of §§1–4 (with f replaced by $-f$) show that $\mathcal{U}_f^{\mathbb{Z}}$ is d-invariant with $H_*(\mathcal{U}_f^{\mathbb{Z}}, d) \cong H_*(X;\mathbb{Z})$. However, these two differentials on $\mathcal{U}_f^{\mathbb{Z}}$ agree up to sign as we see in the next lemma.

Lemma 5.1. *One has*

$$(dU_q) \bullet S_p \ = \ (-1)^{n-k}\, U_q \bullet (dS_p) \tag{5.3}$$

for all $p \in Cr_k(f)$ and $q \in Cr_{k-1}(f)$, and for any k.

Proof. One can see directly from the definition that the integers $N_{p,q}$ are invariant (up to a global sign) under time-reversal in the flow. However, for a simple current-theoretic proof consider the 1-dimensional current $[U_q(r)] \wedge [S_p(r)]$ consisting of a finite sum of oriented line-segments in the flow lines of $\Gamma_{p,q}$ (cf. Remark 4.6). Note that

$$d\,[U_q(r)] \wedge [S_p(r)] \ = \ (d\,[U_q(r)]) \wedge [S_p(r)] \ + \ (-1)^{n-k+1}[U_q(r)] \wedge (d\,[S_p(r)])$$

and apply (4.5). ∎

Corollary 5.2. (Poincaré Duality)

$$H^{n-k}(X;\mathbb{Z}) \ \cong \ H_k(X;\mathbb{Z}) \qquad \textit{for all} \ \ k.$$

Note 5.3. The Poincaré duality isomorphism can be realized in our operator picture as follows. Consider the "total graph" of the flow: in $X \times X$

$$T_{\text{tot}} \ = \ T^* + T \ = \ \{(y, x) \, : \, y = \varphi_t(x) \ \text{ for some } t \in \mathbb{R}\}$$

with corresponding operator $\mathbf{T}_{\text{tot}}$. Then one has the operator equation

$$d \circ \mathbf{T}_{\text{tot}} + \mathbf{T}_{\text{tot}} \circ d \ = \ \mathbf{P} - \check{\mathbf{P}}$$

where

$$\mathbf{P} = \sum_{p \in Cr(f)} [U_p] \times [S_p] \qquad \text{and} \qquad \check{\mathbf{P}} = \sum_{p \in Cr(f)} [S_p] \times [U_p].$$

This chain homotopy induces an isomorphism $H_*(\mathcal{U}_f^{\mathbb{Z}}) \cong H_*(\mathcal{S}_f^{\mathbb{Z}})$, which after identifying $\mathcal{U}_f^{\mathbb{Z}}$ with the cochain complex via (5.1) and (5.3), gives the duality isomorphism 5.2. When X is not oriented, a parallel analysis yields Poincaré duality with mod 2 coefficients.

§**6. Generalizations.** As seen in §2, for any flow φ_t of finite volume the operator $\mathbf{P}(\alpha) = \lim_{t\to\infty} \varphi_t^*(\alpha)$ exists and is chain homotopic to the identity. This situation occurs often. Suppose for example that $f : X \to \mathbb{R}$ is a smooth function whose critical set is a finite disjoint union

$$Cr(f) \ = \ \coprod_{j=1}^{\nu} F_j$$

of compact submanifolds F_j in X and that Hess(f) is non-degenerate on the normal spaces to $Cr(f)$. Then for any f-tame gradient flow φ_t (cf. [L]) there are stable and unstable manifolds

$$S_j = \{x \in X \ : \ \lim_{t\to\infty} \varphi_t(x) \in F_j\} \qquad \text{and} \qquad U_j = \{x \in X \ : \ \lim_{t\to-\infty} \varphi_t(x) \in F_j\}$$

for each j, with projections

$$S_j \xrightarrow{\tau_j} F_j \xleftarrow{\sigma_j} U_j. \tag{6.1}$$

where

$$\tau_j(x) = \lim_{t\to\infty} \varphi_t(x) \qquad \text{and} \qquad \sigma_j(x) = \lim_{t\to-\infty} \varphi_t(x).$$

For each j, let $n_j = \dim(F_j)$ and set $\lambda_j = \dim(S_j) - n_j$. Then $\dim(U_j) = n - \lambda_j$. For $p \in F_j$ we define $\lambda_p \equiv \lambda_j$ and $n_p \equiv n_j$.

Definition 6.1. The gradient flow φ_t of a smooth function $f \in C^\infty(X)$ on a riemannian manifold X is called a **generalized Morse-Stokes flow** if if is f-tame and

(i) The critical set of f consists of a finite number of submanifolds $F_1, ..., F_\nu$ on the normals of which Hess(f) is non-degenerate.

(ii) The manifolds T, and T^*, and the stable and unstable manifolds S_j, U_j for $1 \le j \le \nu$ are submanifolds of finite volume. Furthermore, for each j, the fibres of the projections τ_j and σ_j are of uniformly bounded volume.

(iii) $p \prec q \quad \Rightarrow \quad \lambda_p + n_p < \lambda_q \qquad \forall p, q \in Cr(f)$.

Theorem 6.2. *Suppose φ_t is a gradient flow satisfying the generalized Morse-Stokes conditions 6.1 on a compact oriented manifold X. Then there is an equation of currents*

$$\partial T \ = \ [\Delta] - P$$

on $X \times X$, where T, Δ, and P are as in Theorem 1.2, and

$$P \ = \ \sum_{j=1}^{\nu} [U_j \times_{F_j} S_j] \tag{6.2}$$

where $U_j \times_{F_j} S_j \equiv \{(y, x) \in U_j \times S_j \subset X \times X \ : \ \sigma_j(y) = \tau_j(x)\}$ denotes the fibre product of the projections (6.1)

Proof. The argument follows closely the proof of Theorem 3.3. Details are omitted. ■

In [L] Janko Latschev has found a Smale-type condition which yields this result in cases where when 6.1 (iii) fails. In particular, his condition implies only that: $p \prec q \Rightarrow \lambda_p < \lambda_q$.

The result above can be translated into the following operator form.

Theorem 6.3. *Let φ_t be a gradient flow satisfying the generalized Morse-Stokes Conditions on a manifold X as above. Then for all smooth forms α on X, the limit*

$$\mathbf{P}(\alpha) = \lim_{t\to\infty} \varphi_t^*(\alpha)$$

exists and defines a continuous linear operator $\mathbf{P} : \mathcal{E}^*(X) \longrightarrow \mathcal{D}'^*(X)$ with values in flat currents on X. This operator fits into a chain homotopy

$$d \circ \mathbf{T} + \mathbf{T} \circ d \ = \mathbf{I} - \mathbf{P}. \tag{6.3}$$

Furthermore, $\mathbf{P}$ is given by the formula

$$\mathbf{P}(\alpha) \ = \ \sum_{j=1}^{\nu} Res_j(\alpha)\,[S_j] \tag{6.4}$$

where

$$Res_j(\alpha) \ \equiv \ \tau_j^* \left\{(\sigma_j)_* \left(\alpha\big|_{U_j}\right)\right\} \tag{6.5}$$

Proof. This is a direct consequence of Theorem 6.2 except for the formulae (6.4)-(6.5). To see this consider the pull-back square

$$\begin{array}{ccc} U_j \times_{F_j} S_j & \xrightarrow{t_j} & U_j \\ {\scriptstyle s_j}\downarrow & & \downarrow{\scriptstyle \sigma_j} \\ S_j & \xrightarrow{\tau_j} & F_j \end{array} \tag{6.6}$$

where t_j and s_j are the obvious projections. One sees from the definitions that

$$\mathbf{P}(\alpha) \ = \ \sum_{j=1}^{\nu} (s_j)_* \left\{(t_j)^* \left(\alpha\big|_{U_j}\right)\right\}.$$

The commutativity of the diagram (6.6) allows us to rewrite these terms as in (6.5). ∎

Corollary 6.4. *Suppose that $\lambda_p + n_p + 1 < \lambda_q$ for all critical points $p \prec q$. Then the homology of X is spanned by the images of the groups $H_{\lambda_j+\ell}(\overline{S_j})$ for $j = 1, ..., \nu$ and $\ell \geq 0$.*

Proof. Under this hypothesis $\partial(U_j \times_{F_j} S_j) = 0$ for all j, and so (6.2) yields a decomposition of $\mathbf{P}$ into operators that commute with d. ∎

Each map $\tau_j : S_j \to F_j$ can be given the structure of a vector bundle of rank λ_j. The closure $\overline{S_j} \subset X$ is a compactification of this bundle with a complicated structure at infinity (cf. [CJS]). There is nevertheless a homomorphism $\Theta_j : H_*(F_j) \longrightarrow H_{\lambda_j+*}(\overline{S_j})$ which after pushing forward to the one-point compactification of S_j, is the Thom isomorphism. This leads to the following (cf. [AB]).

Corollary 6.5. *Suppose that $\lambda_p + n_p + 1 < \lambda_q$ for all critical points $p \prec q$ and that X and all F_j and S_j are oriented. Then there is an isomorphism*

$$H_*(X) \;\cong\; \bigoplus_j H_{*-\lambda_j}(F_j)$$

One can drop the orientation assumptions by taking homology with appropriately twisted coefficients. Extensions of 6.3 and 6.5 to integral homology groups are found in [L]. Latschev also derives a spectral sequence, with geometrically computable differentials, associated to any *Bott-Smale* function satisfying a certain transversality hypothesis. Assuming that everything is oriented, the E^1-term is given by $E^1_{p,q} \;=\; \bigoplus_{\lambda_j=p} H_q(F_j;\mathbb{Z})$ and $E^k_{p,q} \Rightarrow H_*(X;\mathbb{Z})$

Remark 6.6. In standard Morse Theory one considers a proper exhaustion function $f : X \to \mathbb{R}$ and studies the change in topology as one passes from $\{x : f(x) \leq a\}$ to $\{x : f(x) \leq b\}$. Our operator approach is easily adapted to this case (See [HL_4]).

§**7. The local MacPherson formula and generalizations.** The ideas above can be applied to the study of curvature and singularities. Suppose

$$\alpha : E \longrightarrow F$$

is a map between smooth vector bundles with connection over a manifold X. Let $G = G_k(E \oplus F) \to X$ denote the Grassmann bundle of k-planes in $E \oplus F$ where $k = \mathrm{rank}(E)$. There is a flow φ_t on G induced by the flow $\psi_t : E \oplus F \to E \oplus F$ where $\psi_t(e, f) = (te, f)$. On the "affine chart" $\mathrm{Hom}(E, F)$, this flow has the form $\varphi_t(A) = \frac{1}{t}A$. (Note that here it is more natural to consider the flow multiplicatively ($\varphi_{ts} = \varphi_t \circ \varphi_s$) than additively.)

Definition 7.1. The section α is said to be **geometrically atomic** if its *radial span*

$$\mathcal{T}(\alpha) \overset{\mathrm{def}}{=} \left\{ \tfrac{1}{t}\alpha(x) \;:\; 0 < t \leq 1 \;\;\text{and}\;\; x \in X \right\}$$

has finite volume in G.

This hypothesis is sufficient to guarantee the existence of $\lim_{t\to 0} \alpha_t^* \Phi$ where $\alpha_t \equiv \frac{1}{t}\alpha$ and where Φ is any differential form on G. Choosing $\Phi = \Phi_0(\Omega_U)$ where Φ_0 is an Ad-invariant polynomial on $\mathfrak{gl}_k(\mathbb{R})$ and Ω_U is the curvature of the tautological k-plane bundle over G, one obtains formulas on X which relate the singularities of α to characteristic forms of E and F. For *normal maps*, maps which are transversal to the universal singularity sets, this process yields a local version of a classical formula of MacPherson [$\mathrm{Mac}_{1,2}$]. In other cases, such as generic direct sum or tensor products mappings, the process yields new formulas. The method also applies whenever α is real analytic. Details appear in [$\mathrm{HL}_{2,3}$].

§**8. Equivariant Morse Theory.** Our approach carries over to equivariant cohomology by using Cartan's equivariant de Rham theory (cf. [BGV]). Suppose G is a compact

Lie group with Lie algebra $\mathfrak{g}$ acting on a compact n-manifold X. By an **equivariant differential form** on X we mean a G-equivariant polynomial map $\alpha : \mathfrak{g} \longrightarrow \mathcal{E}^*(X)$. The set of equivariant forms is denoted by

$$\mathcal{E}_G^*(X) \;=\; \{S^*(\mathfrak{g}^*) \otimes \mathcal{E}^*(X)\}^G$$

and is graded by declaring elements of $S^p(\mathfrak{g}^*) \otimes \mathcal{E}^q(X)$ to have total degree $2p+q$. The equivariant differential $d_G : \mathcal{E}_G^*(X) \longrightarrow \mathcal{E}_G^{*+1}(X)$ is defined by setting

$$(d_G\alpha)\,(V) \;=\; d\alpha(V) - i_{\widetilde{V}}\alpha(V)$$

for $V \in \mathfrak{g}$ where $i_{\widetilde{V}}$ denotes contraction with the vector field $\widetilde{V}$ on X corresponding to V. The complex $\mathcal{D}'^*_G(X)$ of **equivariant currents** is similarly defined by replacing smooth forms $\mathcal{E}^*(X)$ by forms $\mathcal{D}'^*(X)$ with distribution coefficients.

Consider a G-invariant function $f \in C^\infty(X)$ and suppose X is given a G-invariant riemannian metric. Suppose the (G-invariant) gradient flow has finite volume and let

$$\partial T \;=\; \Delta - P \tag{8.1}$$

be the current equation derived in §3. Let G act on $X \times X$ by $g \cdot (x, y) = (gx, gy)$.

Lemma 8.1. *The current T satisfies:*

(i) $g_*T = T$ *for all* $g \in G$, *and*

(ii) $i_{\widetilde{V}}T = 0$ *for all* $V \in \mathfrak{g}$.

So also does the current P.

Proof. The current T corresponds to integration over the finite volume submanifold $\{(x, y) \in X \times X - \Delta \;:\; y = \varphi_t(x) \text{ for some } 0 < t < \infty\}$. Since $g\varphi_t(x) = \varphi_t(gx)$, assertion (i) is clear. The invariance of T implies the invariance of P by (8.1). For assertion (ii) note that for any n-form β on $X \times X$

$$\left(i_{\widetilde{V}}T\right)(\beta) \;=\; \int_T i_{\widetilde{V}}\beta \;=\; 0$$

since V is tangent to T. Similarly, $i_{\widetilde{V}}\Delta = 0$. Since $d \circ i_{\widetilde{V}} + i_{\widetilde{V}} \circ d = \mathcal{L}_{\widetilde{V}}$ (Lie derivative) and $\mathcal{L}_{\widetilde{V}}T = 0$, we conclude that $i_{\widetilde{V}}P = 0$. ∎

Corollary 8.2. *Consider $T \equiv 1 \otimes T \in 1 \otimes \mathcal{D}'^{n-1}(X \times X)^G$ as an equivariant current of total degree $(n-1)$ on $X \times X$. Consider Δ and P similarly as equivariant currents of degree n. Then*

$$\partial_G T \;= \Delta - P \qquad \textit{on} \;\; X \times X. \tag{8.2}$$

The correspondence between operators and kernels discussed in §2 carries over to the equivariant context. Currents in $\mathcal{D}'^{n-\ell}_G(X \times X)$ yield G-equivariant operators $\mathcal{E}^*(X) \longrightarrow$

$\mathcal{D}'^{*+\ell}(X)$, and hence operators $\mathcal{E}^*_G(X) \longrightarrow \mathcal{D}'^{*+\ell}_G(X)$. Equations of type (8.2) translate into operator equations

$$d_G \circ \mathbf{T} + \mathbf{T} \circ d_G \ = I - \mathbf{P}. \tag{8.3}$$

Applying the arguments of §4 proves the following.

Proposition 8.3. *Let φ_t be an invariant flow on a compact G-manifold X. If φ_t has finite volume, then the limit*

$$\mathbf{P}(\alpha) \ = \ \lim_{t\to\infty} \varphi_t^* \alpha \tag{8.4}$$

exists for all $\alpha \in \mathcal{E}^*_G(X)$ and defines a continuous linear operator $\mathbf{P} : \mathcal{E}^*_G(X) \longrightarrow \mathcal{D}'^*_G(X)$ of degree 0, which is equivariantly chain homotopic to the identity on $\mathcal{E}^*_G(X)$.

The fact that $S^*(\mathfrak{g}^*)^G \cong H^*(BG)$ leads as in §4 to the following result.

Theorem 8.4. *Let $f \in C^\infty(X)$ be an invariant Morse function on a compact riemnannian G-manifold whose gradient flow φ_t is Morse-Stokes. Then the continuous linear operator (8.4) defines a map of equivariant complexes*

$$\mathbf{P} : \mathcal{E}^*_G(X) \longrightarrow S^*(\mathfrak{g}^*)^G \otimes \mathcal{S}_f$$

where $\mathcal{S}_f = \mathrm{span}\{[S_p]\}_{p\in Cr(f)}$ as in (4.2) and where the differential on $S^*(\mathfrak{g}^*)^G \otimes \mathcal{S}_f$ is $1 \otimes \partial$. This map induces an isomorphism

$$H^*_G(X) \ \xrightarrow{\cong} H^*(BG) \otimes H^*(X)$$

Examples of this phenomenon arise in moment map constructions. For a simple example consider $G = (S^1)^{n+1}/\Delta$ acting on $\mathbb{P}^n_{\mathbb{C}}$ via the standard action on homogeneous coordinates $[z_0, ..., z_n]$, and set $f([z]) = \sum k|z_k|^2/\|z\|^2$. One sees immediately the well-known fact that $H^*_G(\mathbb{P}^n_{\mathbb{C}})$ is a free $H^*(BG)$-module with one generator in each dimension $2k$ for $k = 0, ..., n$. This extends to all generalized flag manifolds and to products.

J. Latschev has pointed out that there exists an invariant Morse function for which no choice of invariant metric gives a Morse-Stokes flow.

However, the method applies to quite general functions and yields results as in §§6-7. Suppose for example that f is an invariant function whose critical set consists of a finite number of non-degenerate critical orbits $\mathcal{O}_i = G/H_i$, $i = 1, ..., N$ (the generic case). There is a spectral sequence with (assuming for simplicity that everything is oriented)

$$E^1_{p,*} \ = \ \bigoplus_{\lambda_i = p} H^*_G(\mathcal{O}_i) \ = \ \bigoplus_{\lambda_i = p} H^*(BH_i)$$

and computable differentials such that $E^k_{*,*} \ \Rightarrow \ H^*_G(X)$ (cf. [L]).

§**9. Holomorphic Flows and the Carrell-Lieberman-Sommese Theorem.** This method also applies to the holomorphic case. For example, given a $\mathbb{C}^*$-action φ_t on a compact Kähler manifold X, there is a complex graph

$$\mathcal{T} \stackrel{\text{def}}{=} \{(t, \varphi_t(x), x) \in \mathbb{C}^* \times X \times X \,:\, t \in \mathbb{C}^* \text{ and } x \in X\} \subset \mathbb{P}^1(\mathbb{C}) \times X \times X$$

analogous to the graphs considered above. The following is a result of Sommese [So].

Theorem 9.1. *If φ_t has fixed-points, then $\mathcal{T}$ has finite volume and its closure $\overline{\mathcal{T}}$ in $\mathbb{P}^1(\mathbb{C}) \times X \times X$ is an analytic subvariety.*

The relation of $\mathbb{C}^*$-actions to Morse-Theory is classical. The action φ_t decomposes into an "angular" S^1-action and a radial flow. Averaging a Kähler metric over S^1 and applying an argument of Frankel [Fr], we find a function $f : X \to \mathbb{R}$ of Bott-Morse type whose gradient generates the radial action. Theorem 9.1 implies that the associated current T on $X \times X$, which is a real slice of $\mathcal{T}$, is real analytic and hence of finite volume. One can then apply the methods of this paper, in particular those of §6.

Thus when φ_t has fixed-points, $\overline{\mathcal{T}}$ gives a rational equivalence between the diagonal Δ in $X \times X$ and an analytic cycle P whose components consist of fibre products of stable and unstable manifolds over components of the fixed-point set of the action.

When the fixed-points are all isolated, P becomes a sum of analytic Künneth components $P = \sum \overline{S}_p \times \overline{\widetilde{U}}_p$, and we recover the well-known fact that the cohomology of X is freely generated by the stable subvarieties $\{\overline{S}_p\}_{p \in \mathrm{Zero}(\varphi)}$. It follows that X is algebraic and that all cohomology theories on X (eg. algebraic cycles modulo rational equivalence, algebraic cycles modulo algebraic equivalence, singular cohomology) are naturally isomorphic. (See [BB], [ES], [Fr]).

When the fixed-point set has positive dimension, one can recover results of Carrell-Lieberman-Sommese for $\mathbb{C}^*$-actions ([CL], [CS]), which assert among other things that if $\dim(X^{\mathbb{C}^*}) = k$, then $H^{p,q}(X) = 0$ for $|p - q| > k$.

§**10. Local coefficient systems.** Our method applies immediately to forms with coefficients in a flat bundle $E \to X$. In this case the kernels of §2 are currents on $X \times X$ with coefficients in $\mathrm{Hom}(\pi_1^* E, \pi_2^* E)$. Given a Morse-Stokes flow φ_t on X we consider the kernel $T_E = h \otimes T$ where T is defined as in §3 and $h : E_{\varphi_t(x)} \to E_x$ is parallel translation along the flow line. One obtains the equation $\partial T_E = \Delta_E - P_E$ where $\Delta_E = \mathrm{Id} \otimes \Delta$ and $P_E = \sum_p h_p \otimes ([U_p] \times [S_p])$ with $h_p : E_y \to E_x$ given by parallel translation along the broken flow line. Thus, h_p corresponds to $\mathrm{Id} : E_p \to E_p$ under the canonical trivializations $E\big|_{U_p} \cong U_p \times E_p$ and $E\big|_{S_p} \cong S_p \times E_p$. We obtain the operator equation

$$d \circ \mathbf{T}_E + \mathbf{T}_E \circ d \;=\; \mathbf{I} - \mathbf{P}_E \tag{10.1}$$

where $\mathbf{P}_E$ maps onto the finite complex

$$\mathcal{S}_E \stackrel{\text{def}}{=} \bigoplus_{p \in Cr(f)} E_p \otimes [S_p]$$

by integration of forms over the unstable manifolds. The restriction of d to $\mathcal{S}_E$ is given as in (4.3) by $d(e \otimes [S_p]) = \sum h_{p,q}(e)[S_q]$ where $h_{p,q} = (-1)^{\lambda_p} \sum_\gamma h_\gamma$ and $h_\gamma : E_p \to E_q$ is parallel translation along $\gamma \in \Gamma_{p,q}$. By (10.1) the complex $(\mathcal{S}_E, d)$ computes $H^*(X; E)$.

Reversing time in the flow shows that the complex $\mathcal{U}_E = \bigoplus_p E_p^* \otimes [U_p]$ with differential defined as above computes $H^*(X; E^*)$. As in §5 the dual pairing of these complexes establishes Poincaré duality. Furthermore, this extends to integral currents twisted by representations of $\pi_1(X)$ in $\mathrm{GL}_n(\mathbf{Z})$ or $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$ and gives duality with local coefficient systems.

§**11. Cohomology operations.** Our method has many extensions. For example, consider the triple diagonal $\Delta_3 \subset X \times X \times X$ as the kernel of the wedge-product operator $\wedge : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{E}^*(X)$. Let f and f' be functions with Morse-Stokes flows φ_t and φ_t' respectively. Assume that for all $(p, p') \in Cr(f) \times Cr(f')$ the stable manifolds S_p and $S'_{p'}$ intersect transversely in a manifold of finite volume, and similarly for the unstable manifolds U_p and $U'_{p'}$. Degenerating Δ_3 gives a kernel

$$T \;\equiv\; \{(\varphi_t(x), \varphi_t'(x), x) \in X \times X \times X \;:\; x \in X \quad \text{and} \quad 0 \le t < \infty\}$$

and a corresponding operator $\mathbf{T} : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{D}'^*(X)$ of degree -1. One calculates that $\partial T \;=\; \Delta_3 - M$ where $M = \sum_{(p,p')} [U_p] \times [U'_{p'}] \times [S_p \cap S'_{p'}]$. The corresponding operator $\mathbf{M} : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{D}'^*(X)$ is given by

$$\mathbf{M}(\alpha, \beta) \;=\; \sum_{(p,p') \in Cr(f) \times Cr(f')} \left(\int_{U_p} \alpha \right) \left(\int_{U'_{p'}} \beta \right) [S_p \cap S'_{p'}]. \tag{11.1}$$

The arguments of §§1-3 adapt to prove the following.

Theorem 11.1. *There is an equation of operators $\wedge - \mathbf{M} = d \circ \mathbf{T} + \mathbf{T} \circ d$ from $\mathcal{E}^*(X \times X)$ to $\mathcal{D}'^*(X)$ (where $\wedge$ denotes restriction to the diagonal). In particular for $\alpha, \beta \in \mathcal{E}^*(X)$ we have the chain homotopy*

$$\alpha \wedge \beta - \mathbf{M}(\alpha, \beta) \;=\; d\mathbf{T}(\alpha, \beta) + \mathbf{T}(d\alpha, \beta) + (-1)^{\deg \alpha} \mathbf{T}(\alpha, d\beta). \tag{11.2}$$

Note that the operator $\mathbf{M}$ has range in the finite dimensional vector space $\mathcal{M} \overset{\text{def}}{=} \mathrm{span}_{\mathbb{R}} \{ [S_p \cap S'_{p'}] \}_{(p,p')}$. It converts a pair of smooth forms α, β into a linear combination of the pairwise intersections of the stable manifolds $[S_p]$ and $[S'_{p'}]$. If $d\alpha = d\beta = 0$, then $\mathbf{M}(\alpha, \beta) \;=\; \alpha \wedge \beta - d\mathbf{T}(\alpha, \beta)$, and so $\mathbf{M}(\alpha, \beta)$ is a cycle homologous to the wedge product $\alpha \wedge \beta$. This operator maps onto the subspace $\mathcal{M}$, and for forms $\alpha, \beta \in \mathcal{E}^*(X)$, it satisfies the equation

$$d\mathbf{M}(\alpha, \beta) \;=\; \mathbf{M}(d\alpha, \beta) + (-1)^{\deg \alpha} \mathbf{M}(\alpha, d\beta) \tag{11.3}$$

It follows that $d(\mathcal{M}) \subset \mathcal{M}$. Furthermore, for $(p, p') \in Cr(f) \times Cr(f')$ one has

$$d\,[S_p \cap S'_{p'}] \;=\; \sum_{q \in Cr(f)} n_{pq} [S_q \cap S'_{p'}] + (-1)^{n - \lambda_p} \sum_{q' \in Cr(f')} n'_{p'q'} [S_p \cap S'_{q'}] \tag{11.4}$$

where the n_{pq} are defined as in §4. Thus we retrieve the cup product over the integers in the Morse complex.

A basic example of a pair satisfying these hypotheses is simply $f, -f$ where the gradient flow is Morse-Smale. Here $U'_p = S_p$ and $S'_p = U_p$ for all $p \in Cr(f)$, and we have:

Proposition 11.2. *Suppose the gradient flow of f is Morse-Smale and Y is a cycle in X which is transversal to $U_p \cap S_{p'}$ for all $p, p' \in Cr(f)$. Then for any closed forms α, β*

$$\int_Y \alpha \wedge \beta \;=\; \sum_{p,p' \in Cr(f)} \left(\int_{U_p} \alpha\right)\left(\int_{S_{p'}} \beta\right)[S_p \cap U_{p'} \cap Y].$$

Analogous formulas are derived by the same method for the comultiplication operator. Following ideas in [BC] one can extend these constructions to other cohomology operations.

§**12. Knot invariants.** From the finite volume flow 1.7 our methods construct an $(n+1)$-current T on $S^n \times S^n$ with the property that $\partial T \;=\; S^n \times \{*\} + \{*\} \times S^n$. This is a singular analogue of the form used by Bott and Taubes [BT], and can be used to develop a combinatorial version of their study of knot invariants.

§**13. Differential characters.** The ring of differential characters on a smooth manifold, introduced by Cheeger and Simons [CS] in 1973, has played an important role in geometry. In deRham-Federer formulations of the theory (see [HL_5]), differential characters are represented by *sparks*. These are currents T with the property that $dT = \alpha - R$ where α is smooth and R is integrally flat. Our Morse Theory systematically produces such creatures.

Consider the group $\mathcal{Z}_f^{\mathbb{Z}} = \mathcal{E}_f^{\mathbb{Z}} \cap \ker d$ of closed forms with integral residues (See §4). Then $\mathbf{T} : \mathcal{Z}_f^{\mathbb{Z}} \to \mathcal{D}'$ is a continuous map with the property that $d\mathbf{T}(\alpha) \;=\; \alpha - \mathbf{P}(\alpha)$ for all α. Thus $\mathbf{T}$ gives a continuous map of $\mathcal{Z}_f^{\mathbb{Z}}$ into the space of sparks, and therefore also into the ring of differential characters on X.

References

[AB] M. Atiyah and R. Bott, *The Yang-Mills equations over a Riemann surface*, Phil. Trans., R. Soc. London. A **308** (1982), 523-615.

[BC] M. Betz and R.L. Cohen, *Graph moduli spaces and cohomology operations* , Stanford Preprint, 1993.

[BB] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Annals of Math., **98** (1973), 480-497.

[BGV] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and the Dirac Operator*, Grundl. der Math. Wiss. Band 298, Springer, Berlin-Heidelberg-New York, 1992.

[BT] R. Bott and C. Taubes, *On the self-linking of knots*, J. Math. Phys., **35** (10) (1994), 5247-5287.

[CL] J. B. Carrell and D. I. Lieberman, *Holomorphic vector manifolds and compact Kähler manifolds*, Invent. Math., **21** (1973), 303-309.

[CS] J. B. Carrell and A. J. Sommese, *Some topological aspects of $\mathbb{C}^*$-actions on compact Kähler manifolds*, Comm. Math. Helv., **54** (1979), 583-800.

[CS] J. Cheeger and J. Simons, *Differential Characters and Geometric Invariants*, in Geometry and Topology, Lecture Notes in Math. **1167**, Springer–Verlag, New York, 1985, 50–80.

[CJS] R.L. Cohen, J. D. S. Jones and G. B. Segal, *Morse theory and classifying spaces* , Stanford Preprint, 1993.

[deR] G. de Rham, Variétés Différentiables, Hermann, Paris, 1973.

[ES] G. Ellingsrud and S. A. Stromme, *Toward the Chow ring of the Hilbert scheme of* $\mathbb{P}^2$, J. Reine Agnew. Math. **441** (1993), 33-44.

[F] H. Federer, Geometric Measure Theory, Springer Verlag, New York, 1969.

[Fr] T. Frankel, *Fixed points and torsion on Kähler manifolds*, Ann.of Math.,**70** (1959), 1-8.

[GS] H. Gillet and C. Soulé, *Characteristic classes for algebraic vector bundles with Hermitian metrics I, II*, Ann. of Math., **131** (1990), 163-203, 205-238.

[HL_1] F.R. Harvey and H.B. Lawson, Jr., A Theory of Characteristic Currents Associated with a Singular Connection, Astérisque **213**, Société Math. de France, Paris, 1993.

[HL_2] — , *Singularities and Chern-Weil theory, I – The local MacPherson formula*, Asian J. Math., **4**, No. 1 (2000), 71-96.

[HL_3] — , *Singularities and Chern-Weil theory, II – Geometric atomicity*, (to appear).

[HL_4] — , *Morse Theory and Stokes' Theorem*, in Surveys in Differential Geometry, VII, International Press, (to appear).

[HL_5] — , *Poincaré-Pontrjagin duality for differential characters*, (to appear).

[HP] R. Harvey and J. Polking, *Fundamental solutions in complex analysis, Part I*, Duke Math. J. **46** (1979), 253-300.

[K] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II, Birhäuser Verlag, Basel, 1994, pp. 97-122.

[L] J. Latschev, *Gradient flows of Morse-Bott functions*, Math. Annalen **318** no. 4 (2000), 731-759.

[La] F. Laudenbach, *On the Thom-Smale complex*, in An Extension of a Theorem of Cheeger and Müller, by J.-M. Bismut and W. Zhang, Astérisque, v. 205: S.M.F., 1992.

[Mac_1] R. MacPherson, *Singularities of vector bundle maps – Proceedings of Liverpool Singularities Symposium , I*, Springer Lecture Notes in Mathematics, **192** (1971), 316-318.

[Mac_2] — , *Generic vector bundle maps* in " Dynamical Systems, Proceedings of Symposium – University of Bahia, Salavador 1971", Academic Press, New York, 1973, pp. 165-175.

[S] S. Smale, *On gradient dynamical systems* , Annals of Math. **74** (1961), 199-206.

[So] A. J. Sommese, *Extension theorems for reductive groups actions on compact Kähler manifolds*, Math. Ann., **218** (1975), 107-116.

[T] R. Thom, *Sur une partition en cellules associées à une fonction sur une variété*, CRAS **228** (1949), 973-975.

[W] E. Witten, *Supersymmetry and Morse theory* , J. Diff. Geom., **17** , no. 4 (1982), 661-692.