# Duality Relating Spaces of Algebraic Cocycles and Cycles

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**Abstract.** In this paper a fundamental duality is established between algebraic cocycles and algebraic cycles on a smooth projective variety. A map is constructed between these spaces and shown to be a weak homotopy equivalence. The proof makes use of a new Chow moving lemma for families. If $X$ is a smooth projective variety of ndimension $n$, the duality map induces isomorphisms $L^sH^k(X) \to L_{n-s}H_{2n-k}(X)$ for $2s \leq k$, which carry over via natural transformations to the Poincaré duality isomorphism $H^k(X;\mathbf{Z}) \to H_{2n-k}(X;\mathbf{Z})$. The most general duality result asserts that for smooth projective varieties $X$ and $Y$ the natural graphing homomorphism sending algebraic cocycles on $X$ with values in $Y$ to algebraic cycles on the product $X \times Y$ is a weak homotopy equivalence. The main results have a wide variety of applications. Among these is the determination of the homotopy type of certain algebraic mapping complexes. It also includes a determination of the group of algebraic $s$-cocycles modulo algebraic equivalence on a smooth projective variety.

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**Contents**



# Introduction

In [$FL_1$], the authors introduced the notion of an effective algebraic cocycle on an algebraic variety $X$ with values in a variety $Y$, and developed a "bivariant morphic cohomology theory" based on such objects. The theory was shown to have a number of intriguing properties, including Chern classes for algebraic bundles, operations, ring structure, and natural transformations to singular cohomology over $\mathbf{Z}$. The fundamental objects of the theory are simply families of algebraic cycles on $Y$ parametrized by $X$. More precisely they are defined as morphisms from $X$ to the Chow varieties of $r$-cycles on $Y$ and can be represented as cycles on the product $X \times Y$ which are equidimensional over $X$. Such cocycles form a topological abelian monoid, denoted $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$, and the morphic cohomology groups are defined to be the homotopy groups of its group completion $\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$. This stands in analogy with (and, in fact, recovers by letting $X =$ a point) the homology groups introduced and studied in [$F_1$], [L], and elsewhere.

When $Y = \mathbf{A}^n$, the theory is of strict cohomology type. It has a natural cup product given by the pointwise join of cycles, and a natural transformation (of ring functors) to $H^*(X; \mathbf{Z})$.

The main point of this paper is to establish a duality theorem between algebraic cycles and algebraic cocycles. The fundamental result (Theorem 3.3) states that if $X$ and $Y$ are smooth and projective, then the graphing map

$$\mathfrak{Mor}(X, \mathcal{C}_r(Y)) \hookrightarrow \mathcal{Z}_{m+r}(X \times Y)$$

which sends $Y$-valued cocycles on $X$ to cycles on $X \times Y$ is a homotopy equivalence. (Here $m = \dim(X)$). Stated in terms of homotopy groups this theorem asserts that the morphic cohomology groups of $X$ with values in $Y$ are isomorphic to the $L$-homology groups of $X \times Y$. This duality theorem was not forseen when we first formulated the concept of an algebraic cocycle and it represents a non-trivial result for algebraic cycles. In particular, computations of cycle spaces provide computations of mapping spaces consisting of algebraic morphisms.

The duality theorem also holds when $Y = \mathbf{A}^n$ and thereby gives a duality isomorphism

$$\mathcal{D} : L^s H^k(X) \xrightarrow{\cong} L_{m-s} H_{2m-k}(X)$$

between the morphic cohomology and the $L$-homology of any smooth, m-dimensional projective variety $X$. It is shown that this map has a number of interesting properties. The most compelling property is the compatibility of $\mathcal{D}$ with the Poincaré duality map $\mathcal{PD}$. It is shown in §5 that there is a commutative diagram

$$\begin{array}{ccc} L^s H^k(X) & \xrightarrow{\mathcal{D}} & L_{m-s} H_{2m-k}(X) \\ \Phi \Big\downarrow & & \Big\downarrow \Phi \\ H^k(X; \mathbf{Z}) & \xrightarrow{\mathcal{PD}} & H_{2m-k}(X; \mathbf{Z}) \end{array}$$

where the maps $\Phi$ are the natural transformations. It is also shown that on smooth varieties $\mathcal{D}$ intertwines the cup product on $L^* H^*(X)$ with the intersection product on $L_* H_*(X)$ that was established in [FG].

The map $\mathcal{D}$ has certain basic properties. It is compatible with morphisms, and for smooth varieties it intertwines certain Gysin maps. It is also compatible with the **s**-operations of [FM] which act on both theories. This shows that for smooth varieties Poincaré duality preserves the filtrations induced by these operations on singular theory (with **Z**-coefficients) [FM], [$\mathrm{FL}_1$].

The basic results have a wide range of applications. For example it is shown that for generalized flag manifolds $X, Y$ (smooth varieties with cell decompositions) there is an isomorphism

$$\pi_* \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \;\cong\; H_{2(m+r)+*}(X \times Y;\, \mathbf{Z}).$$

where $m = \dim(X)$. Furthermore, for any smooth $m$- dimensional variety $X$ there are natural isomorphisms

$$\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{P}^s))/\{\text{algebraic equivalence}\} \;\cong\; \mathcal{A}_{m-s}(X) \times \mathcal{A}_{m-s-1}(X) \times \cdots \times \mathcal{A}_0(X)$$

where $\mathcal{A}_r(X)$ denotes the group of algebraic $r$-cycles modulo algebraic equivalence on $X$, and where $t \le m$. One also shows that for a smooth variety $X$, the space of parametrized rational curves on $\mathcal{Z}_r(X)$ is weakly homotopy equivalent to $\mathcal{Z}_r(X) \times \mathcal{Z}_{r+1}(X)$.

The proof of the main theorem (3.3) is based on a Moving Lemma for Families of Cycles of Bounded Degree established in [$\mathrm{FL}_2$].

In this paper we have introduced the notation $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$ for the topological monoid of $Y$-valued cocycles of relative dimension $r$ on $X$. This notation emphasizes the nature of cocycles as mappings and differs from the notation in [$\mathrm{FL}_1$]. Furthermore, in the theory of cocycles developed in [$\mathrm{FL}_1$] we used the homotopy-theoretic group completion $\mathcal{Z}_{m+r}(X;\, Y) \equiv \Omega B \mathfrak{Mor}(X, \mathcal{C}_r(Y))$ of the monoid to define the morphic cohomology groups. In this paper we shall use instead the newer "technology" of naïve topological group completions as introduced in [$\mathrm{Li}_2$] and formulated in [FG]. These naïve group completions have directly accessible geometric properties and work better in many circumstances.

The two group completions give equivalent theories, for there is a natural weak homotopy equivalence $\Omega B\mathfrak{Mor}(X, \mathcal{C}_r(Y)) \cong \mathfrak{Mor}(X, \mathcal{Z}_r(Y))$ established in Appendix T. There we revisit the theory of tractable monoids and tractable actions, introduced in [FG], in order to provide the topological formalities needed to work with naïve group completions of cocycle spaces. In particular, we isolate the special topological property of varieties which we use: any closed, constructible embedding is a cofibration.

In Appendix C we prove that for any normal quasi-projective variety $X$ and any projective variety $Y$, the topology induced on $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$ by the embedding $\mathfrak{Mor}(X, \mathcal{C}_r(Y)) \to \mathcal{C}_r(X \times Y)$ is exactly the topology of uniform convergence with bounded degree on compact subsets. If $X$ is projective, then it is exactly the compact-open topology.

The results of this paper basically concern projective varieties. The authors made efforts to extend the methods to quasi-projective varieties with mixed success. We finally realized that a more sophisticated approach is required to appropriately realize functoriality, duality, and other desired properties for the topological abelian groups of cocycles on quasi-projective varieties. Such an approach can be found in [$\mathrm{F}_2$]. A more abstract treatment of duality for varieties over more general fields is given in [FV].

It is assumed throughout the main body of this paper that $X$ and $Y$ are projective varieties over $\mathbf{C}$ and that $X$ is normal.

## §0. Conventions and terminology.

By a **projective algebraic variety** $X$ we shall mean a reduced, irreducible scheme over $\mathbf{C}$ which admits a Zariski closed embedding in some (complex) projective space $\mathbf{P}^N$. Thus, $X$ is the zero locus in $\mathbf{P}^N$ of a finite collection of homogeneous polynomials, and the irreducibility condition is the condition that $X$ is not a non-trivial union of two such zero loci. By a closed subvariety $W \subset X$ we shall mean a Zariski closed subset with its structure of a reduced $\mathbf{C}$-scheme (but which is not necessarily irreducible). Unless explicit mention to the contrary, we shall view locally closed algebraic subsets of projective spaces with their analytic topology.

Throughout this paper we retain the convention that $X, Y$ are projective algebraic varieties of dimensions $m, n$ respectively and that $X$ **is normal**. Also throughout, $r$ and $t$ shall denote non-negative integers with $r \leq n$ and $t \leq m$. We recall that an $r$-**cycle** on $Y$ is a formal integer combination $Z = \sum n_i W_i$, where each $W_i \subset Y$ is an irreducible subvariety of dimension $r$ in $Y$. Such an $r$-cycle is said to be **effective** if each $n_i$ is positive. We denote by $|Z|$ the **support** of $Z = \sum n_i W_i$, defined as the union $|Z| = \cup_i W_i \subset Y$.

Our study involves the consideration of Chow varieties (see, for example, [S]). For each set of non-negative integers $r \leq N$ and $d$, there is Zariski closed subset

$$C_{r,d}(\mathbf{P}^N) \subset \mathbf{P}^M$$

where $M = M_{r,d,N}$, whose points are in natural 1-1 correspondence with effective $r$-cycles of degree $d$ on $\mathbf{P}^N$. For any algebraic subset $Y \subset \mathbf{P}^N$, the set of those effective

$r$-cycles of degree $d$ on $\mathbf{P}^N$ with support on $Y$ correspond to a Zariski closed subset $C_{r,d}(Y) \subset C_{r,d}(\mathbf{P}^N)$. Although there is not a "universal cycle" above $C_{r,d}(Y)$, there does exists the incidence correspondence $\mathcal{I}_{r,d}(Y) \subset C_{r,d}(Y) \times Y$, the Zariski closed subset consisting of pairs $(Z, y)$ with $y \in |Z|$. With the analytic topology on each $C_{r,d}(Y)$ the **Chow monoid**

$$\mathcal{C}_r(Y) \quad \stackrel{\text{def}}{=} \quad \coprod_d C_{r,d}(Y)$$

of effective $r$-cycles on $Y$ is an abelian topological monoid whose algebraic (and, hence, topological) structure is independent of the choice of projective embedding $Y \subset \mathbf{P}^N$ [B].

In this paper, we frequently work with homomorphisms of topological abelian groups which are weak homotopy equivalences. At times, we have need to invert such maps: we define a **weak homotopy inverse** of a weak homotopy equivalence $f : S \to T$ to be a map $g : T' \to S'$ which is a homotopy inverse on the CW-approximation of $f$. At other times, we deal with diagrams of such maps which "weakly homotopy commute"; in other words, compositions of maps in the diagram with same source and target have the property that they induce homotopic maps on CW-approximations. The reader comfortable with derived categories will recognize that these somewhat clumsy conventions would be avoided if we were to replace these topological abelian groups by their associated chain complexes and replace homomorphisms which are weak homotopy equivalences by quasi- isomorphisms.

## §1. Cocycles on Projective Varieties.

In this section we rework the definition of the space of algebraic cocycles on a projective variety. We retain the definition from [$\mathrm{FL}_1$] of the monoid of effective cocycles, but replace the formal construction of the homtopy-theoretic group completion with the more accessible construction of the naïve group completion. One satisfying aspect of the latter is that the naïve group completion of the topological monoid of effective cocycles is a topological abelian group whose points are in one-to-one correspondence with formal differences of effective cocycles.

We recall that the **naïve group completion** $M^+$ of an abelian topological monoid $M$ with the cancellation property is the topological quotient of $M \times M$ by the equivalence relation: $(m_1, m_2) \sim (n_1, n_2)$ iff $m_1 + n_2 = m_2 + n_1$.

**Definition 1.1.** By the topological monoid of **effective algebraic cocyles** of relative dimension $r$ on $X$ with values in $Y$ we mean the abelian monoid

$$\mathfrak{Mor}(X, \mathcal{C}_r(Y)) \tag{1.1.1}$$

of morphisms from $X$ to the Chow monoid $\mathcal{C}_r(Y)$ provided with the compact open topology. We define the topological abelian group of all such cocycles to be the naïve group completion of $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$,

$$\mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \quad \stackrel{\text{def}}{=} \quad [\mathfrak{Mor}(X, \mathcal{C}_r(Y))]^+. \tag{1.1.2}$$

A case of fundamental importance is where $Y$ is essentially the quasi-projective variety $\mathbf{A}^n$. This is defined as follows. By the **monoid of effective algebraic cocycles of codimension-$t$ on** $X$ we mean the topological quotient monoid

$$\mathcal{C}^t(X) \quad \stackrel{\text{def}}{=} \quad \mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^t))/\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^{t-1})). \tag{1.1.3}$$

Its naïve group completion

$$\mathcal{Z}^t(X) \quad \stackrel{\text{def}}{=} \quad [\mathcal{C}^t(X)]^+ \quad \stackrel{\text{def}}{=} \quad \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^t)). \tag{1.1.4}$$

is the **topological group of all algebraic cocycles of codimension-$t$ on** $X$.

As observed in [FL$_1$], $\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$ enjoys various functoriality properties. Composition with a morphism $f : X' \to X$ determines a continuous homomorphism

$$f^* : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \;\longrightarrow\; \mathfrak{Mor}(X', \mathcal{Z}_r(Y)).$$

Push-forward of cycles via a morphism $g : Y \to Y'$ determines $g_* : \mathcal{C}_r(Y) \to \mathcal{C}_r(Y')$ and thus

$$g_* : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \;\longrightarrow\; \mathfrak{Mor}(X, \mathcal{Z}_r(Y')).$$

Similarly, if $g : \tilde{Y} \longrightarrow Y$ is flat of relative dimension $k$, then flat pull-back determines $g^* : \mathcal{C}_r(Y) \to \mathcal{C}_{r+k}(\tilde{Y})$ and thus

$$g^* : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \;\to\; \mathfrak{Mor}(X, \mathcal{Z}_{r+k}(\tilde{Y}))$$

(denoted by $g^!$ in [FL$_1$]).

Since $X$ is assumed to be normal, there is an alternative, equivalent definition of the space of algebraic cocycles. We denote by

$$\mathcal{C}_r(Y)(X) \subset \mathcal{C}_{r+m}(X \times Y)$$

**the topological submonoid of those effective cycles on $X \times Y$ which are equidimensional** of relative dimension $r$ over $X$. In [F$_1$] it was shown that any morphism $\psi : X \to \mathcal{C}_r(Y)$ has a naturally associated graph $\mathcal{G}(\psi) \in \mathcal{C}_r(Y)(X)$, and in [FL$_1$] we showed that for normal varieties $X$ this map $\mathcal{G} : \mathfrak{Mor}(X, \mathcal{C}_r(Y)) \to \mathcal{C}_r(Y)(X)$ is a bijection. In Appendix C we establish the following.

**Theorem 1.2.** *The graphing construction*

$$\mathcal{G} : \mathfrak{Mor}(X, \mathcal{C}_r(Y)) \;\longrightarrow\; \mathcal{C}_r(Y)(X)$$

*is a homeomorphism.*

This alternative formulation of $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$ becomes the definition of the monoid of effective cocycles in the more general context of [F$_2$].

Consideration of naïve group completions is rare in algebraic topology because spaces constructed as quotients with the quotient topology typically have inaccessible algebraic invariants. The usual method of "group completing" a topological monoid $M$ is to take the loop space of the classifying space of $M$, $\Omega B[M]$, whose algebraic invariants are closely related to those of $M$ (cf. [M-S]). In Appendix T, we demonstrate that $\mathcal{C}_r(Y)(X)$ is a **tractable monoid** in the sense of [FG] which by Theorem 1.2 implies the following.

**Proposition 1.3.** *There is a natural weak homotopy equivalence*

$$\Omega B[\mathfrak{Mor}(X, \mathcal{C}_r(Y))] \xrightarrow{\cong} \mathfrak{Mor}(X, \mathcal{Z}_r(Y))$$

*(i.e., natural up to weak homotopy). Moreover, if $Y_\infty \subset Y$ is a closed subvariety, then the following triple is a* **fibration sequence** *(i.e., it determines a long exact sequence in homotopy groups):*

$$\mathfrak{Mor}(X, \mathcal{Z}_r(Y_\infty)) \;\to\; \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \;\to\; [\mathfrak{Mor}(X, \mathcal{C}_r(Y))/\mathfrak{Mor}(X, \mathcal{C}_r(Y_\infty))]^+.$$

We recall that the monoid of effective $k$-cycles on the quasi-projective variety $X \times \mathbf{A}^t$ is defined to be the quotient monoid

$$\mathcal{C}_k(X \times \mathbf{A}^t) \;\stackrel{\text{def}}{=}\; \mathcal{C}_k(X \times \mathbf{P}^t)/\mathcal{C}_k(X \times \mathbf{P}^{t-1}),$$

where $\mathbf{P}^{t-1} \subset \mathbf{P}^t$ is the linear embedding of a "hyperplane at $\infty$". Because $\mathcal{C}_0(\mathbf{P}^{t-1})(X) \subset \mathcal{C}_m(X \times \mathbf{P}^{t-1})$ is obtained by intersection with the Zariski closed subset $\mathcal{C}_m(X \times \mathbf{P}^{t-1}) \subset \mathcal{C}_m(X \times \mathbf{P}^t)$, we conclude easily that

$$\mathcal{C}^t(X) \;\cong\; \mathcal{C}_0(\mathbf{P}^t)(X)/\mathcal{C}_0(\mathbf{P}^{t-1})(X) \;\hookrightarrow\; \mathcal{C}_m(X \times \mathbf{A}^t) \tag{1.3.3}$$

is a continuous injective mapping. (The reader is cautioned that (1.3.3) is not a topological embedding however. See Example C.7 in Appendix C.)

As an immediate corollary of the preceding results, we conclude that our definitions of spaces of cocycles agree up to homotopy with those of [FL$_1$].

**Corollary 1.4.** *There are natural weak homotopy equivalences:*

$$\Omega B\mathfrak{Mor}(X, \mathcal{C}_r(Y)) \;\xrightarrow{\cong}\; \mathfrak{Mor}(X, \mathcal{Z}_r(Y)),$$

$$\text{htyfib}\Big\{B\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^{t-1})) \to B\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^t))\Big\} \;\xrightarrow{\cong}\; \mathcal{Z}^t(X).$$

The fundamental result (cf. [L]) about spaces of algebraic cycles is the Algebraic Suspension Theorem which asserts that the algebraic suspension map

$$\Sigma\!\!\!/ : C_{r,d}(Y) \;\longrightarrow\; C_{r+1,d}(\Sigma\!\!\!/ Y)$$

induces a homotopy equivalence on homtopy-theoretic group completions of Chow monoids,

$$\Sigma\!\!\!/ : \Omega B[\mathcal{C}_r(Y)] \xrightarrow{\cong} \Omega B[\mathcal{C}_{r+1}(\Sigma\!\!\!/ Y)].$$

Since the natural map $\Omega B[\mathcal{C}_r(Y)] \to \mathcal{Z}_r(Y)$ is a weak equivalence (cf. [Li$_2$], [FG], or Corollary T.5 of Appendix T), this implies that $\Sigma\!\!\!/$ also induces a weak homotopy equivalence

$$\Sigma\!\!\!/ : \mathcal{Z}_r(Y) \longrightarrow \mathcal{Z}_{r+1}(\Sigma\!\!\!/ Y).$$

In [FL$_1$;3.3], the algebraic suspension theorem was extended to equidimensional cycles by replacing $\Sigma\!\!\!/$ with the relative analogue $\Sigma\!\!\!/_X$. Thus, the equivalence of naïve and homotopy-theoretic group completions provides as above the following suspension isomorphism for cocycle spaces.

**Proposition 1.5.** *Composition of cocycles with $\Sigma\!\!\!/ : \mathcal{C}_r(Y) \longrightarrow \mathcal{C}_{r+1}(\Sigma\!\!\!/ Y)$ induces a weak homotopy equivalence*

$$\Sigma\!\!\!/_X : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \longrightarrow \mathfrak{Mor}(X, \mathcal{Z}_{r+1}(\Sigma\!\!\!/ Y)).$$

## §2. Duality Map

This section introduces our duality map from spaces of cocycles to spaces of cycles and verifies that this map is compatible with various constructions. On effective cocycles, this map is merely the inclusion of (1.1.1). On the naïve group completions, the map is that induced by (1.1.1). We point out that although it is injective and continuous, our duality map is not a topological embedding. (See examples at the end of Appendix C.)

**Definition 2.1.** The **duality map**

$$\mathcal{D} : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \longrightarrow \mathcal{Z}_{r+m}(X \times Y) \tag{2.1.1}$$

is the continuous injective homomorphism of topological abelian groups induced by the graphing construction:

$$\mathfrak{Mor}(X, \mathcal{C}_r(Y)) \xrightarrow{\mathcal{G}} \mathcal{C}_r(Y)(X) \subset \mathcal{C}_{r+m}(X \times Y). \tag{2.1.2}$$

of Theorem 1.2. Similarly for $Y = \mathbf{A}^n$ the duality map

$$\mathcal{D} : \mathcal{Z}^t(X) \longrightarrow \mathcal{Z}_{m-t}(X) \tag{2.1.3}$$

is defined to be the composition of the map on naïve group completions induced by (1.3.3) and the inverse of the natural homotopy equivalence $\mathcal{Z}_{m-t}(X) \to \mathcal{Z}_m(X \times \mathbf{A}^t)$ (cf. [FG]).

In the following proposition, we verify that the duality map $\mathcal{D}$ of (2.1.1) is natural with respect to functorial constructions on cycles and cocycles.

**Proposition 2.2.** *If $f : Y \to Y'$ is a morphism of projective algebraic varieties, then $f_*$ fits in the following commutative square*

$$\begin{array}{ccc} \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m}(X \times Y) \\ f_* \downarrow & & \downarrow (1 \times f)_* \\ \mathfrak{Mor}(X, \mathcal{Z}_r(Y')) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m}(X \times Y'). \end{array} \tag{2.2.1}$$

*If $g : \tilde{Y} \to Y$ is a flat map of projective varieties of relative dimension $k$, then $g^*$ fits in the following commutative square*

$$\begin{array}{ccc} \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m}(X \times Y) \\ g^* \downarrow & & \downarrow (1 \times g)^* \\ \mathfrak{Mor}(X, \mathcal{Z}_{r+k}(\tilde{Y})) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m+k}(X \times \tilde{Y}). \end{array} \tag{2.2.2}$$

*If $h : \tilde{X} \to X$ is a flat morphism of relative dimension $e$, then $h^*$ fits in the following commutative square*

$$\begin{array}{ccc} \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m}(X \times Y) \\ h^* \downarrow & & \downarrow (h \times 1)^* \\ \mathfrak{Mor}(\tilde{X}, \mathcal{Z}_r(Y)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m+e}(\tilde{X} \times Y). \end{array} \tag{2.2.3}$$

*If $i : X_0 \to X$ is a regular closed embedding of codimension $c$, then $i^*$ fits in the following weakly homotopy commutative square*

$$\begin{array}{ccc} \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m}(X \times Y) \\ i^* \downarrow & & \downarrow (i \times 1)^! \\ \mathfrak{Mor}(X_0, \mathcal{Z}_r(Y)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m-c}(X_0 \times Y), \end{array} \tag{2.2.4}$$

*where $(1 \times i)^!$ is the Gysin map of [FG].*

**Proof.** To prove the commutativity of (2.2.1), it suffices to verify the following: if $G = \mathcal{G}(\psi) \subset X \times Y$ is the graph of $\psi : X \to \mathcal{C}_r(Y)$, then $(1 \times f)_*(G)$ equals the graph

$G' = \mathcal{G}(f \circ \psi)$ of $f \circ \psi$. This is verified by observing that $(1 \times f)_*(G)$ and $G'$ are equal when restricted to $\mathrm{Spec}(K) \times Y'$, where $\eta : \mathrm{Spec}(K) \to X$ is the generic point, and both are given as the closures in $X \times Y'$ of these restrictions.

The commutativity of (2.2.2) follows by observing that $g^*$ on $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$ equals the restriction to $\mathcal{C}_r(Y)(X) \subset \mathcal{C}_{r+m}(X \times Y)$ of $(1 \times g)^*$. This is verified as in the proof of the commutativity of (2.2.1) by observing that $g^* \circ \psi : X \to \mathcal{C}_r(Y) \to \mathcal{C}_{r+k}(\tilde{Y})$ has graph whose restriction to $\mathrm{Spec}(K) \times \tilde{Y}$ equals the restriction of $(1 \times g)^*(\mathcal{G}(\psi))$, where $\eta : \mathrm{Spec}(K) \to X$ is the generic point.

To verify the commutativity of (2.2.3), we must show that $(h \times 1)^*(\mathcal{G}(\psi)) = \mathcal{G}(\psi \circ h)$ for a morphism $\psi : X \to \mathcal{C}_r(Y)$. Once again, this is verified by observing that the restrictions of these cycles to $\mathrm{Spec}(\tilde{K}) \times Y$ are equal, where $\tilde{\eta} : \mathrm{Spec}(\tilde{K}) \to \tilde{X}$ is the generic point.

As verified in [FG;3.4], the Gysin map

$$(i \times 1)^! : \mathcal{C}_{r+m}(X \times Y) \longrightarrow \mathcal{C}_{r+m-c}(X_0 \times Y)$$

can be represented by intersection (in the sense of [Fu]) with $X_0 \times Y$ on the submonoid $\mathcal{C}_{r+m}(X \times Y; X_0 \times Y)$ of those cycles which meet $X_0 \times Y$ properly. Clearly, $\mathcal{C}_r(Y)(X) \subset \mathcal{C}_{r+m}(X \times Y; X_0 \times Y)$. On the other hand, the homomorphism $i^* : \mathcal{C}_r(Y)(X) \to \mathcal{C}_r(Y)(X_0)$ given by intersection with $X_0 \times Y$ is identified in [FM$_1$;3.2] with $i^* : \mathfrak{Mor}(X, \mathcal{C}_r(Y)) \to \mathfrak{Mor}(X_0, \mathcal{C}_r(Y))$ given by composition with $i$. $\square$

We state without proof the following analogue of Proposition 2.2 for the duality map $\mathcal{D}$ of (2.1.3). This analogue follows easily from the naturality of the constructions involved in (2.2.3) and (2.2.4).

**Proposition 2.3.** *If $h : \tilde{X} \to X$ is a flat morphism of relative dimension $e$, then the following square commutes*

$$\begin{array}{ccc} \mathcal{Z}^t(X) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{m-t}(X) \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle (h \times 1)^*} \\ \mathcal{Z}^t(\tilde{X}) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{m+e-t}(\tilde{X}). \end{array} \tag{2.3.1}$$

*If $i : X_0 \to X$ is a regular closed immersion of codimension $c$, then the following square is weakly homotopy commutative*

$$\begin{array}{ccc} \mathcal{Z}^t(X) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{m-t}(X) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle (i \times 1)^!} \\ \mathcal{Z}^t(X_0) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{m-t-c}(X_0). \end{array} \tag{2.3.2}$$

We next proceed to exhibit a Gysin morphism on cocycles with respect to a regular embedding $\epsilon : Y_0 \to Y$. Essentially, we show that the Gysin map constructed in [FG] on cycle spaces for a regular embedding restricts to a map on cocycle spaces. To carry out this argument, we use the formulation of effective cocycles as graphs from Theorem 1.2, and appeal to a variant of our duality theorem (namely, Corollary 3.6 of the next section). For this we introduce the submonoid

$$\mathcal{C}_r(Y;Y_0)(X) \quad\subset\quad \mathcal{C}_r(Y)(X) \tag{2.3.3}$$

of those effective cocycles which intersect $X \times Y_0$ properly, where we have assumed $r \geq \operatorname{codim} Y_0$. It can be verified that (2.3.3) is a constructible embedding by applying the upper semi-continuity of the fibres of the projection

$$\mathcal{I} \cap [\mathcal{C}_r(Y)(X) \times (X \times Y_0)] \quad\longrightarrow\quad \mathcal{C}_r(Y)(X)$$

where $\mathcal{I} \subset \mathcal{C}_{r+m}(X \times Y) \times (X \times Y)$ is the incidence correspondence. (However, this basic fact is not used in the proof below). We set

$$\mathcal{Z}_r(Y;Y_0)(X) \quad\stackrel{\text{def}}{=}\quad [\mathcal{C}_r(Y;Y_0)(X)]^+. \tag{2.3.4}$$

Our appeal to duality in proving Proposition 2.4 explains the smoothness hypotheses. The result may hold in greater generality. However, one should note that the map $\epsilon^!$ is not given by a simple composition of the Gysin map $\mathcal{Z}_r(Y) \to \mathcal{Z}_{r-e}(Y_0)$ with cocycles.

**Proposition 2.4.** *Assume that $X, Y$ are smooth and consider a regular (Zariski) closed embedding $\epsilon : Y_0 \to Y$ of codimension $e$ where $e \leq r$. Then there is a natural weak homotopy class of maps*

$$\epsilon^! : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \quad\longrightarrow\quad \mathfrak{Mor}(X, \mathcal{Z}_{r-e}(Y_0))$$

*which fits in the following weakly homotopy commutative diagram*

$$\begin{array}{ccc} \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m}(X \times Y) \\ \epsilon^! \downarrow & & \downarrow (1 \times \epsilon)^! \\ \mathfrak{Mor}(X, \mathcal{Z}_{r-e}(Y_0)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m-e}(X \times Y_0). \end{array} \tag{2.4.1}$$

**Proof.** By Theorem 1.2 we may replace $\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$ by the naïve group completion $\mathcal{Z}_r(Y)(X)$ of the topological monoid $\mathcal{C}_r(Y)(X)$. By Corollary 3.6 there is a weak homotopy equivalence

$$j : \mathcal{Z}_r(Y;Y_0)(X) \quad\stackrel{\cong}{\longrightarrow}\quad \mathcal{Z}_r(Y)(X). \tag{2.4.2}$$

We define the Gysin map $\epsilon^!$ to be the composition

$$\epsilon^! : \mathcal{Z}_r(Y)(X) \stackrel{\cong}{\longrightarrow} \mathcal{Z}_r(Y;Y_0)(X) \longrightarrow \mathcal{Z}_{r-e}(Y_0)(X)$$

where the first map is the weak homotopy inverse of (2.4.2), and the second is the naïve group completion of the map $\mathcal{C}_r(Y;Y_0)(X) \to \mathcal{C}_{r-e}(Y_0)(X)$ given by intersection with $X \times Y_0$. So defined $\epsilon^!$ fits in the weakly homotopy commutative square (2.4.1) by [FG;3.4]. $\square$

In [FM$_1$], a basic operation $\mathbf{s} : \Omega B[\mathcal{C}_r(X)] \wedge S^2 \to \Omega B[\mathcal{C}_{r-1}(X)]$ was introduced and studied. (Here $S^2$ denotes the 2-sphere, the underlying topological space of $\mathbf{P}^1$.) This $\mathbf{s}$-map was originally defined using the algebraic suspension theorem and the join mapping

$$\# : \mathcal{C}_r(X) \times \mathcal{C}_0(\mathbf{P}^1) \longrightarrow \mathcal{C}_{r+1}(X\#\mathbf{P}^1).$$

In [FG], this operation was extended to an operation $\mathbf{s} : \mathcal{Z}_r(U) \wedge S^2 \to \mathcal{Z}_{r-1}(U)$ for cycles on a quasi-projective variety $U$ and was shown to be independent of the projective embedding. (A fact not previously known even for $X$ projective). In the formulation of [FG;2.6], the join mapping is replaced by the product mapping

$$\times : \mathcal{C}_r(X) \times \mathcal{C}_0(\mathbf{P}^1) \quad \longrightarrow \quad \mathcal{C}_r(X \times \mathbf{P}^1) \tag{2.5.0}$$

sending a pair $(Z, p)$ to $Z \times \{p\}$, and the algebraic suspension theorem is replaced by the Gysin map $\mathcal{Z}_r(X \times \mathbf{P}^1) \to \mathcal{Z}_{r-1}(X)$.

In the following proposition, we verify that the duality map is natural with respect to this $\mathbf{s}$-map. Since our proof once again uses duality, we require that $Y$ and $X$ be smooth.

**Proposition 2.5.** *Assume that $X, Y$ be are smooth. The $\mathbf{s}$-map determines a weak homotopy class of maps*

$$\mathbf{s} : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \wedge S^2 \quad \longrightarrow \quad \mathfrak{Mor}(X, \mathcal{Z}_{r-1}(Y)) \tag{2.5.1}$$

*which fits in the following weakly homotopy commutative square:*

$$\begin{array}{ccc} \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \wedge S^2 & \xrightarrow{\mathcal{D}\wedge 1} & \mathcal{Z}_{r+m}(X \times Y) \wedge S^2 \\ \mathbf{s}\Big\downarrow & & \Big\downarrow\mathbf{s} \\ \mathfrak{Mor}(X, \mathcal{Z}_{r-1}(Y)) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m-1}(X \times Y). \end{array} \tag{2.5.2}$$

**Proof.** As in the proof of Proposition 2.4, we replace $\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$ by $\mathcal{Z}_r(Y)(X)$. We consider the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Z}_r(Y)(X) \times \mathcal{Z}_0(\mathbf{P}^1) & \xrightarrow{\times} & \mathcal{Z}_r(Y \times \mathbf{P}^1)(X) & \xleftarrow{j} & \mathcal{Z}_r(Y \times \mathbf{P}^1; Y \times \{\infty\})(X) \\ \mathcal{D}\times 1\Big\downarrow & & \mathcal{D}\Big\downarrow & & \mathcal{D}\Big\downarrow \\ \mathcal{Z}_{r+m}(X \times Y) \times \mathcal{Z}_0(\mathbf{P}^1) & \xrightarrow{\times} & \mathcal{Z}_{r+m}(X \times Y \times \mathbf{P}^1) & \xleftarrow{j} & \mathcal{Z}_{r+m}(X \times Y \times \mathbf{P}^1; X \times Y \times \{\infty\}) \end{array} \tag{2.5.3}$$

where by Corollary 3.6 the maps $j$ are weak homotopy equivalences. We now consider the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_r(Y \times \mathbf{P}^1; Y \times \{\infty\})(X) & \xrightarrow{\bullet} & \mathcal{Z}_{r-1}(Y)(X) \\ \mathcal{D} \big\downarrow & & \big\downarrow \mathcal{D} \\ \mathcal{Z}_{r+m}(X \times Y \times \mathbf{P}^1; X \times Y \times \{\infty\}) & \xrightarrow{\bullet} & \mathcal{Z}_{r+m-1}(X \times Y) \end{array} \tag{2.5.4}$$

where the map $\bullet$ of the upper row denotes intersection with $Y \times \{\infty\}$, and the map $\bullet$ of the lower row denotes intersection with $X \times Y \times \{\infty\}$. We define $\mathbf{s} : \mathcal{Z}_r(Y)(X) \wedge S^2 \longrightarrow \mathcal{Z}_{r-1}(Y)(X)$ to be the composition of the maps in the upper rows of (2.5.3) and (2.5.4) (with $j$ replaced by its weak homotopy inverse) restricted to $\mathcal{Z}_r(Y)(X) \wedge S^2$, where $S^2 \simeq \mathbf{P}^1 \to \mathcal{Z}_0(\mathbf{P}^1)$ is defined to be the pointed map sending $p \in \mathbf{P}^1$ to $p - \{\infty\} \in \mathcal{Z}_0(\mathbf{P}^1)$. The composition of the maps in the lower rows of the diagrams (again with $j$ replaced by its weak homotopy inverse), when restricted to $\mathcal{Z}_{r+m}(X \times Y) \wedge S^2$, gives the $\mathbf{s}$-map by [FG;2.6]. The weak homotopy commutativity of (2.5.2) now follows. $\square$

Proposition 2.5 admits the following analogue for cocycle spaces.

**Proposition 2.6.** *If $X$ is smooth, then there is a natural weak homotopy class of maps*

$$\mathbf{s} : \mathcal{Z}^t(X) \wedge S^2 \longrightarrow \mathcal{Z}^{t+1}(X) \tag{2.6.1}$$

*which fits in the following weakly homotopy commutative square*

$$\begin{array}{ccc} \mathcal{Z}^t(X) \wedge S^2 & \xrightarrow{\mathcal{D}\wedge 1} & \mathcal{Z}_{m-t}(X) \wedge S^2 \\ \mathbf{s} \big\downarrow & & \big\downarrow \mathbf{s} \\ \mathcal{Z}^{t+1}(X) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{m-t-1}(X). \end{array} \tag{2.6.2}$$

**Proof.** To define the map (2.6.1) we consider the upper rows of (2.5.3) and (2.5.4) with $r = 1$ and $Y = \mathbf{P}^t$. We restrict this row to effective cycles, and map it, via the linear inclusion $\mathbf{P}^t \subset \mathbf{P}^{t+1}$, to the analogous row with $r = 1$ and $Y = \mathbf{P}^{t+1}$. Applying algebraic suspension (cf. 1.5) and the taking quotients, we obtain the chain of maps

$$\mathcal{C}^t(X) \times \mathcal{C}_0(\mathbf{P}^1) \xrightarrow{\Sigma \times 1} \left[\mathcal{C}_1(\mathbf{P}^{t+1})(X)/\mathcal{C}_1(\mathbf{P}^t)(X)\right] \times \mathcal{C}_0(\mathbf{P}^1)$$

$$\xrightarrow{\times} \mathcal{C}_1(\mathbf{P}^{t+1} \times \mathbf{P}^1)(X)/\mathcal{C}_1(\mathbf{P}^t \times \mathbf{P}^1)(X) \xleftarrow{j}$$

$$\mathcal{C}_1(\mathbf{P}^{t+1} \times \mathbf{P}^1; \mathbf{P}^{t+1} \times \infty)(X)/\mathcal{C}_1(\mathbf{P}^t \times \mathbf{P}^1; \mathbf{P}^t \times \infty)(X) \xrightarrow{\bullet} \mathcal{Z}^{t+1}(X).$$

Taking naïve group completions and replacing $j$ by its weak homotopy inverse, we obtain a chain of maps from $\mathcal{Z}^t(X) \times \mathcal{Z}_0(\mathbf{P}^1)$ to $\mathcal{Z}^{t+1}(X)$ which determines $\mathbf{s}$.

The strict naturality of the commutative diagrams (2.5.3) and (2.5.4) with respect to the linear embedding of a hyperplane $\mathbf{P}^t \subset \mathbf{P}^{t+1}$ enables us to conclude the weak homotopy commuativity of (2.6.2) as in the proof of Proposition 2.5. $\square$

We conclude this section with a verification that the join product on cocycle spaces defined in [FL$_1$;6.2]:

$$\#_X : \mathcal{Z}_0(\mathbf{P}^t)(X) \times \mathcal{Z}_0(\mathbf{P}^u)(X) \quad \longrightarrow \mathcal{Z}_1(\mathbf{P}^{t+u+1})(X) \simeq \mathcal{Z}_0(\mathbf{P}^{t+u})(X) \tag{2.7.0}$$

and the intersection product on cycle spaces defined in [FG;3.5] intertwine with the duality map.

**Proposition 2.7.** *If $X$ is smooth and if $t$ and $u$ are non-negative integers with $t+u \leq m$, then the join pairing of (2.7.0) fits in a homotopy commutative diagram*

$$\begin{array}{ccccc}
\mathcal{Z}_0(\mathbf{P}^t)(X) \times \mathcal{Z}_0(\mathbf{P}^u)(X) & \longrightarrow & \mathcal{Z}^t(X) \times \mathcal{Z}^u(X) & \xrightarrow{\mathcal{D}\times\mathcal{D}} & \mathcal{Z}_{m-t}(X) \times \mathcal{Z}_{m-u}(X) \\
\#_X \Big\downarrow & & & & \Big\downarrow \bullet \\
\mathcal{Z}_0(\mathbf{P}^{t+u})(X) & \longrightarrow & \mathcal{Z}^{t+u}(X) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{m-t-u}(X)
\end{array} \tag{2.7.1}$$

*where the left horizontal arrows are the defining projections and where $(-) \bullet (-)$ denotes the intersection product on cycle spaces.*

**Proof.** Let $W \subset \mathbf{P}^t \times \mathbf{P}^u \times \mathbf{P}^{t+u+1}$ denote the subvariety consisting of those triples $(a, b, c)$ with the property that $c$ lies on the line from $a$ to $b$, where $\mathbf{P}^t, \mathbf{P}^u \subset \mathbf{P}^{t+u+1}$ are embedded linearly and disjointly. Thus, $\pi : W \to \mathbf{P}^t \times \mathbf{P}^u$ is the projective bundle of the 2-plane bundle $\mathrm{pr}_1^*(\mathcal{O}_{\mathbf{P}^t}(1)) \otimes \mathrm{pr}_2^*(\mathcal{O}_{\mathbf{P}^u}(1))$ over $\mathbf{P}^t \times \mathbf{P}^u$. Now $\#_X$ factors as the composition of the maps

$$\mathcal{Z}_0(\mathbf{P}^t)(X) \times \mathcal{Z}_0(\mathbf{P}^u)(X) \xrightarrow{\times} \mathcal{Z}_0(\mathbf{P}^t \times \mathbf{P}^u)(X \times X) \xrightarrow{\Delta_X^*} \mathcal{Z}_0(\mathbf{P}^t \times \mathbf{P}^u)(X) \tag{2.7.2}$$

with the maps

$$\mathcal{Z}_0(\mathbf{P}^t \times \mathbf{P}^u)(X) \xrightarrow{\pi^*} \mathcal{Z}_1(W)(X) \xrightarrow{p_*} \mathcal{Z}_1(\mathbf{P}^{t+u+1})(X) \xrightarrow{i^!} \mathcal{Z}_0(\mathbf{P}^{t+u})(X) \tag{2.7.3}$$

where $\times$ sends a pair of 0-cycles to their product, $i^!$ is the Gysin map of Proposition 2.4, $\Delta_X : X \to X \times X$ denotes the diagonal, $\pi : W \to \mathbf{P}^t \times \mathbf{P}^u$ and $p : W \to \mathbf{P}^{t+u+1}$ are the projections, and $i : \mathbf{P}^{t+u} \to \mathbf{P}^{t+u+1}$ is a linear embedding.

We set (2.7.2) as the top row of the following diagram (whose Gysin maps are well defined up to weak homotopy)

$$\begin{array}{ccccc}
\mathcal{Z}_0(\mathbf{P}^t)(X) \times \mathcal{Z}_0(\mathbf{P}^u)(X) & \xrightarrow{\times} & \mathcal{Z}_0(\mathbf{P}^t \times \mathbf{P}^u)(X^2) & \xrightarrow{\Delta_X^*} & \mathcal{Z}_0(\mathbf{P}^t \times \mathbf{P}^u)(X) \\
\mathcal{D} \times \mathcal{D} \downarrow & & \mathcal{D} \downarrow & & \downarrow \mathcal{D} \\
\mathcal{Z}_m(X \times \mathbf{P}^t) \times \mathcal{Z}_m(X \times \mathbf{P}^u) & \xrightarrow{\times} & \mathcal{Z}_{2m}(X^2 \times \mathbf{P}^t \times \mathbf{P}^u) & \xrightarrow{(\Delta_X \times 1)^!} & \mathcal{Z}_m(X \times \mathbf{P}^t \times \mathbf{P}^u) \\
i_t^! \times i_u^! \downarrow & & i_{t+u}^! \downarrow & & \downarrow i_{t+u}^! \\
\mathcal{Z}_{m-t}(X) \times \mathcal{Z}_{m-u}(X) & \xrightarrow{\times} & \mathcal{Z}_{2m-t-u}(X^2) & \xrightarrow{\Delta_X^!} & \mathcal{Z}_{m-t-u}(X).
\end{array} \tag{2.7.4}$$

The bottom row of this diagram defines $\bullet : \mathcal{Z}_{m-t}(X) \times \mathcal{Z}_{m-u}(X) \longrightarrow \mathcal{Z}_{m-t-u}(X)$. Now the upper left square of this diagram commutes by inspection; the upper right square is weakly homotopy commutative as in (2.2.4); the lower left square is weakly homotopy commutative since $i_t^! : \mathcal{Z}_m(X \times \mathbf{P}^t) \to \mathcal{Z}_{m-t}(X)$ is the right weak homotopy inverse of $\mathrm{pr}_1^*$ (cf. Proposition 5.5); and the lower right square is weakly homotopy commutative by the commutativity property of the Gysin map proved in [FG;3.4]. The square

$$\begin{array}{ccc}
\mathcal{Z}_0(\mathbf{P}^t)(X) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_m(X \times \mathbf{P}^t) \\
\mathrm{pr} \downarrow & & \downarrow i^! \\
\mathcal{Z}^t(X) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{m-t}(X)
\end{array}$$

weakly homotopy commutes since we can arrange for $X \subset X \times \mathbf{P}^t$ to miss $X \times \mathbf{P}^{t-1} \subset X \times \mathbf{P}^t$. We therefore conclude that the analogue of (2.7.1) with $\#_X$ replaced by $\Delta_X^! \circ \times$ of (2.7.2) is weakly homotopy commutative.

To complete the proof, we use the fact that $\#_X$ is obtained from $\Delta_X^! \circ \times$ of (2.7.2) by composing with the maps of (2.7.3). The chain of maps of (2.7.3) is mapped with the duality map $\mathcal{D}$ to the following chain:

$$\mathcal{Z}_m(X \times \mathbf{P}^t \times \mathbf{P}^u) \xrightarrow{\pi^*} \mathcal{Z}_{m+1}(X \times W) \xrightarrow{p_*} \mathcal{Z}_{m+1}(X \times \mathbf{P}^{t+u+1}) \xrightarrow{i^!} \mathcal{Z}_m(X \times \mathbf{P}^{t+u}). \tag{2.7.5}$$

Each of the terms of this chain maps to $\mathcal{Z}_{m-t-u}(X)$ via a Gysin map associated to the inclusion of $X$ into the product occurring in that term. Using [FG;3.4] (which establishes the naturality and commutativity of the Gysin maps) or Proposition 5.5 below, we easily verify the maps of (2.7.5) fit in weakly homotopy commutative triangles over $\mathcal{Z}_{m-t-u}(X)$. In other words, we can extend (2.7.4) to the right in such a way that the diagram remains weakly homotopy commutative, the upper row represents the join product, and the bottom row represents the intersection product. □

**Remark.** As shown in [FL$_1$], the join product induces a well defined pairing on homotopy groups (i.e., morphic cohomology groups) $\pi_i(Z^t(X))\otimes\pi_j(Z^u(X)) \to \pi_{i+j}(Z^{t+u}(X))$. This implies that (2.7.1) establishes that the duality map interwines the join product on morphic cohomology groups with the intersection product on L-homology groups. In Corollary 3.4, we make this intertwining explicit.

## §3. Duality Theorems

In this section, we present our duality theorem (Theorem 3.3) and some of its immediate consequences. We begin by defining a topological concept which will be applicable in our study of maps of naïve group completions of abelian topological monoids.

**Definition 3.1.** A filtration of a topological space $T$ by a sequence of subspaces $T_0 \subset T_1 \subset \ \cdots \subset T_j \subset \cdots$ is said to be a **good filtration** if whenever $f : K \to T$ is a continuous map from a compact space $K$, there exists some $e \geq 0$ such that $f(K) \subset T_e$. A filtration-preserving continuous map $f : T' \to T$ of spaces with good filtrations is said to be a **very weak deformation retract** provided that for each $e \geq 0$ there exist maps

$$\phi'_e : T'_e \times I \longrightarrow T' \ , \quad \phi_e : T_e \times I \longrightarrow T \ , \quad \lambda_e : T_e \longrightarrow T'$$

whose restrictions $\phi'_e\big|_{T'_e\times\{0\}}$ and $\phi_e\big|_{T_e\times\{0\}}$ are the natural inclusions, and which fit in the following commuative diagrams

$$\begin{tikzcd}
T'_e \times I \arrow[r, "\phi'_e"] \arrow[d, "f_e \times Id"'] & T' \arrow[d, "f"] \\
T_e \times I \arrow[r, "\phi_e"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T'_e \times \{1\} \arrow[d, "f_e"'] & \subset & T'_e \times I \arrow[r, "\phi'_e"] & T' \arrow[d, "f"] \\
T_e \times \{1\} \arrow[urrr, "\lambda_e"] & \subset & T_e \times I \arrow[r, "\phi_e"] & T.
\end{tikzcd}$$

The next lemma extends the elementary result that a deformation retract is a homotopy equivalence.

**Lemma 3.2.** *Let $f : T' \to T$ be a very weak deformation retract of spaces with good filtrations. Then $f$ is a weak homotopy equivalence.*

**Proof.** Let $\xi : S^m \to T$ be a continuous map. Then there exists $e > 0$ such that $\xi(S^m) \subset T_e$. Apply the homotopy $\xi_t(\cdot) \equiv \phi_e(\xi(\cdot), t)$ and note that $\phi_1$ lifts over $f$ (via $\lambda_e$) to a map $\xi'_1 : S^m \to T'_e$. This proves surjectivity.

Suppose $\xi' : S^k \to T'$ is a map such that $\xi \equiv f \circ \xi'$ extends to a map $\xi : D^{k+1} \to T$ with image in $T_e$ for some $e$. Consider the homotopy $\xi'_t(\cdot) = \phi'_e(\xi'(\cdot), t)$ for $0 \leq t \leq 1$. When $t = 1$, we have the commutative diagram

$$\begin{tikzcd}
S^k \times \{1\} \arrow[r, "\xi' \times \{1\}"] \arrow[d, phantom, "\cap"] & T'_e \times \{1\} \arrow[r] \arrow[d] & T' \arrow[d] \\
D^{k+1} \times \{1\} \arrow[r, "\xi \times \{1\}"] & T_e \times \{1\} \arrow[r] \arrow[ur, "\lambda"] & T
\end{tikzcd}$$

which shows that $\xi'$ is homotopic to zero. This proves injectivity. $\square$

We now apply Lemma 3.2 in conjunction with the "Moving Lemma for Cycles of Bounded Degree" [FL$_2$] to prove our main theorem which asserts the equivalence of spaces of cocycles and cycles on smooth varieties.

**Theorem 3.3. (The Duality Theorem)** *Consider smooth projective varieties $X, Y$ of dimensions $m, n$ respectively and let $r \leq n,\ t \leq m$ be non- negative integers. Then the duality maps (2.1.1) and (2.1.3)*

$$\mathcal{D} : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \quad \longrightarrow \quad \mathcal{Z}_{r+m}(X \times Y)$$

$$\mathcal{D} : \mathcal{Z}^t(X) \quad \longrightarrow \quad \mathcal{Z}_{m-t}(X)$$

*are weak homotopy equivalences.*

**Proof.** By Theorem 1.2 we may replace $\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$ by $\mathcal{Z}_r(Y)(X)$, and we recall that $\mathcal{D}$ is induced by the topological embedding $j : \mathcal{C}_r(Y)(X) \hookrightarrow \mathcal{C}_{r_m}(X \times Y)$. Let

$$\pi : \mathcal{C}_{r+m}(X \times Y) \times \mathcal{C}_{r+m}(X \times Y) \quad \longrightarrow \quad \mathcal{Z}_{r+m}(X \times Y)$$

$$\pi' : \mathcal{C}_r(Y)(X) \times \mathcal{C}_r(Y)(X) \quad \longrightarrow \quad \mathcal{Z}_r(Y)(X)$$

denote the canonical projection maps. Then the filtration $\{K_e\}_{e=0}^{\infty}$ of $\mathcal{Z}_{r+m}(X \times Y)$ given by setting

$$K_e \equiv \pi \left\{ \coprod_{d+d' \leq e} C_{r+m,d}(X \times Y) \times C_{r+m,d'}(X \times Y) \right\}$$

is a good filtration. (See [Li$_1$] for example.) Consider the induced filtration $\{K'_e\}_{e=0}^{\infty}$ of $\mathcal{Z}_r(Y)(X)$:

$$K'_e \equiv \pi' \left\{ \coprod_{d+d' \leq e} C_{r,d}(Y)(X) \times C_{r,d'}(Y)(X) \right\}$$

where $C_{r,d}(Y)(X) \stackrel{\text{def}}{=} C_{r+m,d}(X \times Y) \cap \mathcal{C}_r(Y)(X)$. We claim this is also a good filtration. Indeed, if $K$ is compact and $f : K \to \mathcal{Z}_r(Y)(X)$ is continuous, then since $\{K_e\}_{e=0}^{\infty}$ is good, there exists an $e$ such that $(\mathcal{D} \circ f)(K) \subset K_e$. Since $\mathcal{D}$ is injective, this implies that $f(K) \subset K'_e$. Hence, $\{K'_e\}_{e=0}^{\infty}$ is also a good filtration and $\mathcal{D}$ is a filtration-preserving map.

We now apply our Moving Lemma for Cycles of Bounded Degree, which is proved in [FL$_2$] and summarized in Appendix M, to show that for all $e$ sufficiently large we can move the family $K_e$ so that every cycle in $K_e$ meets every fibre $\{x\} \times Y \subset X \times Y$ of the projection in proper dimension, i.e., so that every cycle in $K_e$ becomes a cocycle over $X$. Indeed let $e$ be any integer $\geq$ the (common) degrees of the $\{x\} \times Y,\ x \in X$, for some projective embedding of $X \times Y$. Let

$$\widetilde{\Psi} : \mathcal{C}_{r+m}(X \times Y) \times \mathcal{O} \ \longrightarrow \ \mathcal{C}_{r+m}(X \times Y) \times \mathcal{C}_{r+m}(X \times Y)$$

be the map guarenteed by Theorem M.1. By property (d) of Theorem M.1 and the fact that $\deg(\{x\} \times Y) \le e$ for all $x \in X$, we see that we have

$$\widetilde{\Psi}(\mathcal{C}_{r+m}(X \times Y) \times \{\tau\}) \subset \mathcal{C}_r(Y)(X) \times \mathcal{C}_r(Y)(X)$$

for all $\tau \neq 0$ in $\mathcal{O}$. In particular, the restriction of $\widetilde{\Psi}$ to $\mathcal{C}_r(Y)(X) \times \mathcal{O}$ determines a continuous map

$$\widetilde{\Psi}' : \mathcal{C}_r(Y)(X) \times \mathcal{O} \longrightarrow \mathcal{C}_r(Y)(X) \times \mathcal{C}_r(Y)(X).$$

Let

$$\Psi : \mathcal{Z}_{r+m}(X \times Y) \times \mathcal{O} \longrightarrow \mathcal{Z}_{r+m}(X \times Y), \qquad \Psi' : \mathcal{Z}_r(Y)(X) \times \mathcal{O} \longrightarrow \mathcal{Z}_r(Y)(X)$$

be the maps defined by linear extension of the maps $\pi \circ \widetilde{\Psi}$ and $\pi' \circ \widetilde{\Psi}'$ respectively. Note that the fact that $\pi \circ \widetilde{\Psi}$ (and therefore also $\pi' \circ \widetilde{\Psi}'$ ) is a monoid homomorphism on each $\mathcal{C}_{r+m}(X \times Y) \times \{\tau\}$, and therefore extends by linearity to $\mathcal{Z}_{r+m}(X \times Y) \times \{\tau\}$, is part of the assertion of Theorem M.1.

We now choose a smooth embedding $I \subset \mathcal{O}$ with endpoints 0 and 1, and we define

$$\phi_e : K_e \times I \longrightarrow \mathcal{Z}_{r+m}(X \times Y) \quad , \quad \phi'_e : K'_e \times I \longrightarrow \mathcal{Z}_r(Y)(X)$$

by restriction of $\Psi$ and $\Psi'$ respectively. One checks immediately that these maps satisfy the conditions of Definition 4.1, namely: $\phi'_e$ covers $\phi_e$ with respect to $\mathcal{D}$; $(\phi_e)\big|_{K_e \times \{0\}}$ and $(\phi'_e)\big|_{K'_e \times \{0\}}$ are the natural inclusions; and $(\phi_e)\big|_{K_e \times \{\tau\}}$ lifts to $\mathcal{Z}_r(Y)(X)$ for any $\tau \neq 0$. Thus, Lemma 3.2 implies that $\mathcal{D} : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \to \mathcal{Z}_{r+m}(X \times Y)$ is a weak homtopy equivalence.

We observe that $\mathcal{D}$ determines a map of fibration sequences (cf. Proposition 1.3):

$$\begin{array}{ccccc}
\mathcal{Z}_0(\mathbf{P}^{t-1})(X) & \longrightarrow & \mathcal{Z}_0(\mathbf{P}^{t})(X) & \longrightarrow & \mathcal{Z}^t(X) \\
\mathcal{D} \big\downarrow & & \mathcal{D} \big\downarrow & & \mathcal{D} \big\downarrow \\
\mathcal{Z}_m(X \times \mathbf{P}^{t-1}) & \longrightarrow & \mathcal{Z}_m(X \times \mathbf{P}^{t}) & \longrightarrow & \mathcal{Z}_m(X \times \mathbf{A}^{t}).
\end{array} \tag{3.3.1}$$

The preceding argument together with the 5-Lemma implies that the right vertical arrow is a weak homotopy equivalence. Thus, $\mathcal{D} : \mathcal{Z}^t(X) \to \mathcal{Z}_{m-t}(X)$, defined as the composition of this map and the weak homotopy equivalence $\mathcal{Z}_{m-t}(X) \to \mathcal{Z}_m(X \times \mathbf{A}^t)$, is also a weak homotopy equivalence. $\square$

We recall that the homotopy groups of $\mathcal{Z}^t(X)$ and $\mathcal{Z}_r(X)$ are called "morphic cohomology groups" and "L-homology groups" respectively. These are indexed as follows:

$$L^tH^k(X) \stackrel{\text{def}}{=} \pi_{2t-k}(\mathcal{Z}^t(X)) \quad , \quad L_rH_k(X) \stackrel{\text{def}}{=} \pi_{k-2r}(\mathcal{Z}_r(X)).$$

Using this notation, we re-state the second assertion of Theorem 3.3 and the remark following Proposition 2.7.

**Corollary 3.4**. *The duality map $\mathcal{D} : Z^t(X) \to Z_{m-t}(X)$ of (2.1.2) induces isomorphisms*

$$\boxed{L^tH^k(X) \overset{\mathcal{D}}{\cong} L_{m-t}H_{2m-k}(X).}$$

*Moreover, these isomorphisms fit in the following commutative square*

$$\begin{tikzcd}
L^tH^k(X) \otimes L^uH^\ell(X) \arrow[r, "\mathcal{D}\otimes\mathcal{D}"] \arrow[d, "\#_X"'] & L_{m-t}H_{2m-k}(X) \otimes L_{m-u}H_{2m-\ell}(X) \arrow[d, "\bullet"] \\
L^{t+u}H^{k+\ell}(X) \arrow[r, "\mathcal{D}"] & L_{m-t-u}H_{2m-k-\ell}(X)
\end{tikzcd}$$

Using Theorem 3.3 we now define a Gysin map for cocycle spaces compatible with the duality map. We can view this as a supplement to Propositions 2.2 and 2.3.

**Proposition 3.5.** *Assume that $X$ and $Y$ are smooth. Consider a regular closed embedding $i : X_0 \subset X$ of codimension $c$. Then there exists a weak homotopy class of maps (a* **Gysin map***)*

$$i^! : \mathfrak{Mor}(X_0, \mathcal{Z}_{r+c}(Y)) \longrightarrow \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \tag{3.5.1}$$

*which fits in the following weakly homotopy commutative square*

$$\begin{tikzcd}
\mathfrak{Mor}(X_0, \mathcal{Z}_{r+c}(Y)) \arrow[r, "\mathrm{i}^!"] \arrow[d, "\mathcal{D}"'] & \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \arrow[d, "\mathcal{D}"] \\
\mathcal{Z}_{r+m}(X_0 \times Y) \arrow[r, "\mathrm{i}_*"] & \mathcal{Z}_{r+m}(X \times Y).
\end{tikzcd} \tag{3.5.2}$$

*Moreover, there exists a weak homotopy class of maps*

$$i^! : \mathcal{Z}^{t-c}(X_0) \longrightarrow \mathcal{Z}^t(X) \tag{3.5.3}$$

*which fits in the following weakly homotopy commutative square*

$$\begin{tikzcd}
\mathcal{Z}^{t-c}(X_0) \arrow[r, "\mathrm{i}^!"] \arrow[d, "\mathcal{D}"'] & \mathcal{Z}^t(X) \arrow[d, "\mathcal{D}"] \\
\mathcal{Z}_{m-t}(X_0) \arrow[r, "\mathrm{i}_*"] & \mathcal{Z}_{m-t}(X).
\end{tikzcd} \tag{3.5.4}$$

**Proof.** Using Theorem 3.3, we define $i^!$ as follows:

$$i^! \stackrel{\text{def}}{=} \mathcal{D}^{-1} \circ i_* \circ \mathcal{D}. \tag{3.5.5}$$

So defined, $i^!$ fits in weakly homotopy commutative diagrams (3.5.2) and (3.5.3). □

In constructing the Gysin map $\epsilon^!$ of Proposition 2.4, we required the following variant of Theorem 3.3.

**Corollary 3.6** *Assume that $X$ and $Y$ are smooth, and let $Y_0 \subset Y$ is a closed subvariety of codimension $\leq r$. Then the naïve group completion*

$$\mathcal{Z}_r(Y;Y_0)(X) \longrightarrow \mathcal{Z}_r(Y)(X)$$

*of the embedding $\mathcal{C}_r(Y;Y_0)(X) \subset \mathcal{C}_r(Y)(X)$ of (2.3.3) is a weak homotopy equivalence.*

**Proof.** The Moving Lemma (Theorem M.1) enables one to move all effective cycles of degree $\leq e$ in $\mathcal{C}_{r+m}(X \times Y)$ so that the resulting cycles properly intersect all effective cycles of degree $\leq e$ and of dimension $\geq n-r$. We apply this result to move effective cycles in $\mathcal{C}_{r+m}(X \times Y)$ with respect to the cycles $x \times Y; x \in X$ and the cycle $X \times Y_0$. Thus, the proof of Theorem 3.3 applies with only notational changes to prove that

$$\mathcal{Z}_r(Y;Y_0)(X) \longrightarrow \mathcal{Z}_{r+m}(X \times Y)$$

is a weak homotopy equivalence. Combining this fact with Theorem 3.3 implies the corollary. □

## §4. Compatibility with Poincaré Duality

The purpose of this section is to prove that the duality isomorphism $\mathcal{D}$ of Corollary 3.4 is compatible with Poincaré duality. This gives some justification for our view that $\mathcal{D}$ is a natural duality for cycles. It also leads to some interesting applications.

We begin by recalling the natural transformations $\Phi : L_rH_k(X) \to H_k(X;\mathbf{Z})$ and $\Phi : L^tH^k(X) \to H^k(X;\mathbf{Z})$ introduced in [FM$_1$] and [FL$_1$] respectively. Let $\mathbf{s} : \mathcal{Z}_r(X) \wedge S^2 \to \mathcal{Z}_{r-1}(X)$ be the $\mathbf{s}$- operation discussed prior to Proposition 2.5. This induces a map $\mathbf{s} : \pi_j\mathcal{Z}_r(X) \to \pi_{j+2}\mathcal{Z}_{r-1}(X)$. Beginning with $\pi_{k-2r}\mathcal{Z}_r(X) \equiv L_rH_k(X)$ we iterate this map $r$-times and then apply the Dold-Thom isomorphism $\tau : \pi_k\mathcal{Z}_0(X) \xrightarrow{\cong} H_k(X;\mathbf{Z})$. This gives the following.

**Definition 4.1** The natural transformation

$$\Phi : L_rH_k(X) \longrightarrow H_k(X;\mathbf{Z})$$

is defined by setting $\Phi = \tau \circ \mathbf{s}^r$.

We recall that $\mathcal{Z}^t(X) \stackrel{\text{def}}{=} \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^t))$ is the naïve group completion of the monoid $\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^t))/\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^{t-1}))$ which admits an evident continuous homomorphism $\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^t))/\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^{t-1})) \longrightarrow \mathrm{Map}(X, \mathcal{Z}_0(\mathbf{A}^t))$. (Here $\mathrm{Map}(A, B)$ denotes the continuous maps with the compact-open topology.) Group completing this homomorphism gives a continuous homomorphism

$$\Phi : \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^t)) \longrightarrow \mathrm{Map}(X, \mathcal{Z}_0(\mathbf{A}^t)). \tag{4.1.1}$$

By the Dold-Thom Theorem there is natural homotopy equivalence $\mathcal{Z}_0(\mathbf{A}^t) \cong K(\mathbf{Z}, 2t)$.

**Definition 4.2** The natural transformation

$$\Phi : L^t H^k(X) \longrightarrow H^k(X; \mathbf{Z})$$

is defined by applying the map (4.1.1) to the homotopy groups $\pi_{2t-k}$ and then using the natural isomorphism $\pi_{2t-k}\mathrm{Map}(X, K(\mathbf{Z}, 2t)) \cong H^k(X; \mathbf{Z})$.

For the proof of Theorem 4.4 below we shall need the following special case of the "Kronecker pairing" induced by the slant product construction

$$\mathfrak{Mor}(X, \mathcal{C}_r(Y)) \times \mathcal{C}_p(X) \longrightarrow \mathcal{C}_{r+p}(Y) \tag{4.2.1}$$

given in [FL$_1$;7.2].

**Proposition 4.3.** *When $r = p = 0$, the slant product (4.2.1) is given by sending the pair $(f, \sum x_i) \in \mathfrak{Mor}(X, SP^d(Y)) \times SP^e(X))$ to $\sum f(x_i) \in SP^{de}(Y)$. By specializing to $Y = P^t$, this induces a slant product (or "Kronecker") pairing*

$$\pi_i \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^t)) \otimes \pi_{2t-i}\mathcal{Z}_0(X) \xrightarrow{\ \backslash\ } \pi_{2t}\mathcal{Z}_0(\mathbf{A}^t) \cong \mathbf{Z}.$$

**Proof.** The asserted identification of the special case of the slant product defined in [FL$_1$;7.2] for effective cocycles and cycles is immediate from the definitions. This clearly defines a pairing

$$[\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^t))/\mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{P}^{t-1}))] \times \mathcal{C}_0(X) \longrightarrow \mathcal{C}_0(\mathbf{P}^t)/\mathcal{C}_0(\mathbf{P}^{t-1}) \stackrel{\text{def}}{=} \mathcal{C}_0(\mathbf{A}^t). \tag{4.3.1}$$

The map on homotopy groups of the naïve group completion of this latter map is the asserted slant product pairing. $\square$

The following theorem demonstrates that the natural transformations above intertwine the duality map $\mathcal{D}$ with the Poincaré Duality map.

**Theorem 4.4.** *If $X$ is smooth, then the duality isomorphism $\mathcal{D} : L^tH^k(X) \to L_{m-t}H_{2m-k}(X)$ of Corollary 3.4 fits in the following commutative square*

$$\begin{array}{ccc} L^tH^k(X) & \xrightarrow{\mathcal{D}} & L_{m-t}H_{2m-k}(X) \\ \Phi \big\downarrow & & \big\downarrow \tau \circ \mathbf{s}^{m-t} \\ H^k(X;\mathbf{Z}) & \xrightarrow{\mathcal{P}} & H_{2m-k}(X;\mathbf{Z}) \end{array} \tag{4.4.1}$$

*where $\mathcal{P}$ is the Poincaré Duality map sending $\alpha \in H^k(X;\mathbf{Z})$ to $\alpha \cap [X] \in H_{2m-k}(X;\mathbf{Z})$.*

**Proof.** In [FL$_1$;5.2], we showed that the composition

$$\Phi \circ \mathbf{s}^{m-t} : L^tH^k(X) \longrightarrow L^mH^k(X) \longrightarrow H^k(X;\mathbf{Z})$$

equals $\Phi$. Thus, Proposition 2.6 reduces us to the special case $t = m$. The commutativity of (4.4.1) for the special case $t = m$ is equivalent to the commutativity of the associated squares

$$\begin{array}{ccc} L^mH^k(X,F) & \xrightarrow{\mathcal{D}} & L_0H_{2m-k}(X,F) \\ \Phi \big\downarrow & & \big\downarrow \tau \\ H^k(X,F) & \xrightarrow{\mathcal{P}} & H_{2m-k}(X,F) \end{array} \tag*{$(4.4.1)_F$}$$

as $F$ ranges each of the prime fields $\mathbf{Q}$, $\mathbf{Z}/\ell\mathbf{Z}$. Since the evaluation and intersection products

$$H^k(X,F) \otimes H_k(X,F) \xrightarrow{\langle,\rangle} H_0(X,F) \quad , \quad H_{2m-k}(X,F) \otimes H_k(X,F) \xrightarrow{\odot} H_0(X,F)$$

are perfect pairings, to prove the commutativity of $(4.4.1)_F$ it suffices to prove that

$$\langle \Phi(\alpha), \gamma \rangle \quad = \quad \tau(\mathcal{D}(\alpha)) \odot \gamma$$

for all $\alpha \in L^mH^k(X,F)$, and all $\gamma \in H_k(X,F)$. (Recall that $\langle \phi, \gamma \rangle = (\phi \cap [X]) \odot \gamma$ for $\phi \in H^k(X,F)$.) To prove this equality, it suffices to prove the commutativity of the following diagram:

$$\begin{array}{ccc} H^k(X,F) \otimes H_k(X,F) & \xrightarrow{\langle,\rangle} & H_0(X,F) \\ \Phi \otimes \tau \big\uparrow & & \big\uparrow \epsilon^{-1} \\ \pi_{2m-k}(\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^m)), F) \otimes \pi_k(\mathcal{Z}_0(X), F) & \xrightarrow{\backslash} & \pi_{2m}(\mathcal{Z}_0(\mathbf{A}^m), F) \\ \mathcal{D} \otimes \mathrm{pr}_1^* \big\downarrow & & \big\downarrow = \\ \pi_{2m-k}(\mathcal{Z}_m(X \times \mathbf{A}^m), F) \otimes \pi_k(\mathcal{Z}_m(X \times \mathbf{A}^m), F)) & \xrightarrow{\mathrm{pr}_{2*} \circ \bullet} & \pi_{2m}(\mathcal{Z}_0(\mathbf{A}^m), F) \\ \mathrm{pr}_1^* \circ \tau^{-1} \otimes \mathrm{pr}_1^* \circ \tau^{-1} \big\uparrow & & \big\uparrow \epsilon \\ H_{2m-k}(X,F) \otimes H_k(X,F) & \xrightarrow{\odot} & H_0(X,F) \end{array} \tag{4.4.2}$$

where $\backslash$ is the map in homotopy with $F$-coefficients induced by the naïve group completion of (4.3.1) and $\epsilon$ is the natural isomorphism

$$\epsilon = \mathrm{pr}_{2*} \circ \mathrm{pr}_1^* : \pi_0(Z_0(X), F) \longrightarrow \pi_{2m}(\mathcal{Z}_m(X \times \mathbf{A}^m), F) \longrightarrow \pi_{2m}(\mathcal{Z}_0(\mathbf{A}^m), F).$$

The commutativity of the upper square of (4.4.2) follows immediately from the observation that the evaluation product $\langle , \rangle : H^k(X, F) \otimes H_k(X, F) \to H_0(X, F)$ can be represented as the pairing on homotopy groups induced by the map

$$\mathrm{Map}(X, \mathcal{Z}_0(\mathbf{A}^m)) \times \mathcal{Z}_0(X) \xrightarrow{\backslash} \mathcal{Z}_0(\mathbf{A}^m)$$

sending $(f, \sum x_i)$ to $\sum f(x_i)$. (See [FL$_1$, §8] for example.)

To verify the commutativity of the middle square of (4.4.2), it suffices to establish the homotopy commutativity of the square

$$\begin{array}{ccc}
\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^m)) \times \mathcal{Z}_0(X) & \xrightarrow{\backslash} & \mathcal{Z}_0(\mathbf{A}^m) \\
\mathcal{D} \times \mathrm{pr}_1^* \downarrow & & \downarrow = \\
\mathcal{Z}_m(X \times \mathbf{A}^m) \times \mathcal{Z}_m(X \times \mathbf{A}^m) & \xrightarrow{\mathrm{pr}_{2*} \circ \bullet} & \mathcal{Z}_0(\mathbf{A}^m)
\end{array} \cdot \qquad (4.4.3)$$

Observe that for any $f : X \to \mathcal{C}_0(\mathbf{P}^m)$ and $\sum x_i \in SP^d(X)$, the cycles $\mathcal{D}(f)$ and $\mathrm{pr}_1^*(\sum x_i)$ intersect properly in $X \times \mathbf{P}^m$ and

$$\mathrm{pr}_{2*} \left[ \mathcal{D}(f) \bullet \mathrm{pr}_1^*(\sum x_i) \right] = \sum f(x_i).$$

Thus, the homotopy commutativity of (4.4.3) follows from the result proved in [FG;3.5.a] asserting that the intersection product on cycle spaces for the smooth variety $X \times \mathbf{A}^m$ can be represented by the usual intersection product when restricted to the naïve group completions of pairs of cycle spaces consisting of cycles which intersect properly.

Finally, to prove the commutativity of the bottom square of (4.4.2), we claim that it suffices to prove the commutativity of the following diagram

$$\begin{array}{ccc}
H_{2m-k}(X, F) \otimes H_k(X, F) & \xrightarrow{(\mathrm{pr}_1^*)^{\otimes 2}} & H^{BM}_{4m-k}(X \times \mathbf{A}^m, F) \otimes H^{BM}_{2m+k}(X \times \mathbf{A}^m, F) \\
\times \downarrow & & \downarrow \times \\
H_{2m}(X \times X, F) & \xrightarrow{(\mathrm{pr}_1 \times \mathrm{pr}_1)^*} & H^{BM}_{6m}((X \times \mathbf{A}^m)^2, F) \\
\Delta^! \downarrow & & \downarrow \Delta^! \\
H_0(X, F) & \xrightarrow{\mathrm{pr}_1^*} & H^{BM}_{2m}(X \times \mathbf{A}^m, F) \\
= \downarrow & & \downarrow \mathrm{pr}_{2*} \\
H_0(X, F) & \xrightarrow{\epsilon} & H^{BM}_{2m}(\mathbf{A}^m, F)
\end{array} \qquad (4.4.4)$$

where $H_*^{BM}(V)$ of a quasi-projective variety $V$ is the Borel-Moore homology of $V$. To verify this reduction, first recall that the Dold-Thom isomorphism extends to quasi-projective varieties $\tau : \pi_* \mathcal{Z}_0(V) \xrightarrow{\cong} H_*^{BM}(V)$. Thus, the composition of the maps in the right column can be identified with the map $\mathrm{pr}_{2*} \circ \bullet$ of (4.4.2) using the naturality of $\tau$ and the homotopy invariance of Lawson homology. On the other hand, the composition of the maps in the left column of (4.4.4) is the cap product pairing, so that it does indeed suffice to prove the commutativity of (4.4.4).

The evident intertwining of the external product $\times$ and the flat pull-back $\mathrm{pr}_1^*$ implies the commutativity of the top square of (4.4.4). As shown for example in [FG;3.4.d], the Gysin maps and flat pull- backs also suitably intertwine, thereby implying the commutativity of the middle square of (4.4.4). The bottom square commutes by the definition of $\epsilon$. $\square$

**Remark 4.5.** In [FL$_1$] we introduced the groups

$$L^t H^k(X;\, Y) \ \stackrel{\mathrm{def}}{=} \ \pi_{2t-k}\mathfrak{Mor}(X, \mathcal{Z}_{n-t}Y)$$

for $0 \le k \le 2t$. When $X$ and $Y$ are smooth, our Duality Theorem 3.3 gives isomorphisms

$$\mathcal{D} : L^t H^k(X;\, Y) \ \xrightarrow{\cong} \ L_{(n+m)-t} H_{2(n+m)-k}(X \times Y).$$

Now for smooth $X$ and $Y$ there is a diagram

$$\begin{array}{ccc}
L^t H^k(X;\, Y) & \xrightarrow{\mathcal{D}} & L_{(n+m)-t} H_{2(n+m)-k}(X \times Y) \\
\Phi \Big\downarrow & & \Big\downarrow \Phi \\
\bigoplus\limits_{j-i=2n-k} H^i(X; H_j(Y; \mathbf{Z})) & \xrightarrow{\mathcal{P}} & H_{2(n+m)-k}(X \times Y; \mathbf{Z})
\end{array}$$

where the maps $\Phi$ are natural transformations (cf. [FL$_1$], [FM$_1$]), and where $\mathcal{P}$ is the Poincaré duality map

$$\bigoplus_{j-i=2n-k} H^i(X; H_j(Y; \mathbf{Z})) \ \xrightarrow{\mathcal{P}} \ \bigoplus_{j+i=2(n+m)-k} H_i(X; H_j(Y; \mathbf{Z})) \ \cong \ H_{2(n+m)-k}(X \times Y; \mathbf{Z}).$$

It is natural to suppose that this diagram commutes.

## §5. Applications

The duality theorems have a variety of consequences, some of which we now present. Throughout this section $X$ and $Y$ will be smooth projective varieties of dimensions $m$ and $n$ respectively. We fix a positive integer $r \le m$ and set $q = m-r$. We recall for emphasis that $\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$ is simply the Grothendieck group of the monoid of morphisms $X \longrightarrow \mathcal{C}_r(Y)$ furnished with the compact-open topology. We also recall the definition of the morphic cohomology groups

$$L^qH^k(X;\, Y) = \pi_{2q-k}\mathfrak{Mor}(X, \mathcal{Z}_r(Y)).$$

**5.A Algebraic Cocycles modulo Algebraic Equivalence.** We recall the basic isomorphism (cf. [$F_1$;1.8])

$$L_pH_{2p}(X) \;=\; \mathcal{A}_p(X) \;\stackrel{\text{def}}{=}\; \{\text{algebraic } p\text{−cycles on } X\}/\{\text{algebraic equivalence}\}.$$

where $\mathcal{A}_p(X)$ is the Chow group of algebraic $p$-cycles on $X$ modulo algebraic equivalence. There is an analogous interpretation of the group $L^qH^{2q}(X;\, Y) = \pi_0\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$. Each element of $\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$ can be written as a difference $f-g$ where $f, g : X \to \mathcal{C}_r(Y)$ are algebraic families of effective cycles on $Y$ parameterized by $X$. Two such pairs $(f, g)$, $(f', g')$ determine the same element if there are morphisms $h, h'$ with $(f+h, g+h) = (f'+h', g'+h')$. The pairs $(f, g)$ , $(f', g')$ are **algebraically equivalent** in $\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$ if there exist $h, h'$ and an algebraic curve joining $(f+h, g+h)$ to $(f'+h', g'+h')$ in $\mathfrak{Mor}(X, \mathcal{C}_r(Y)) \times \mathfrak{Mor}(X, \mathcal{C}_r(Y))$. We have a similar description of $\pi_0\mathfrak{Mor}(X, \mathcal{Z}_r(Y))$, except that the condition is that two pairs can be connected (after translation) by a real curve. Since $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$ is a countable disjoint union of quasi-projective varieties (see [$FL_1$]), two points in $\mathfrak{Mor}(X, \mathcal{C}_r(Y))^2$ can be joined by a real curve if and only if they can be joined by an algebraic curve. This shows that

$$\begin{aligned} L^qH^{2q}(X;\, Y) \;&\cong\; \{\text{algebraic } q\text{−cocycles on } X \text{ with values in } Y\}/\{\text{ algebraic equivalence}\} \\ &=\; \mathfrak{Mor}(X, \mathcal{Z}_r(Y))/\{\text{algebraic equivalence}\} \end{aligned}$$

The Duality Theorem 3.3 now provides the following non- obvious isomorphism.

**Theorem 5.1.** *Let $X$ be a smooth projective variety of dimension $m$ and $q$ a non-negative integer with $q \le m$. Then there are natural isomorphisms*

$$\frac{\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{P}^q))}{\{\text{algebraic equivalence}\}} \;\cong\; \mathcal{A}_{m-q}(X) \times \mathcal{A}_{m-q-1}(X) \times \cdots \times \mathcal{A}_0(X)$$

*and*

$$\pi_0\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^q)) \;\cong\; \mathcal{A}_{m-q}(X).$$

**Proof.** The second isomorphism is obtained immediately by applying $\pi_0$ to the second homotopy equivalence of Theorem 3.3. The first isomorphism follows similarly by using the projective bundle theorem in L-homology proved in [FG;2.5] together with the first homotopy equivalence of Theorem 3.3. $\square$

**5.B Flag manifolds** Recall that a variety $Y$ is said to have a **cell decompositon** if there is a filtration of $Y$ by subvarieties $\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_k = Y$ such that $Y_j - Y_{j-1} = \coprod_i \mathbf{A}^{N_j}$ for each $j$. Such varieties include all homogeneous varieties, and in particular all generalized flag manifolds such as projective spaces, Grassmannians, etc. We recall from the work of Lima-Filho [Li$_1$] that for any such $Y$ the natural inclusion

$$\mathcal{Z}_r(Y) \ \subset \ \mathcal{J}_{2r}(Y)$$

into the topological group of integral cycles of (real) dimension $2r$ on $Y$, is a homotopy equivalence. In particular, there is a homotopy equivalence

$$\mathcal{Z}_r(Y) \ \cong \ \prod_j K(H_{2r+j}(Y; \mathbf{Z}), j) \tag{5.2.0}$$

As a result of duality there is the following cohomological version of these results

**Theorem 5.2.** *If $X$ and $Y$ are smooth projective varieties with cell decompositions, then there is an isomorphism*

$$\pi_* \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \ \cong \ H_{2(m+r)+*}(X \times Y;\, \mathbf{Z}). \tag{5.2.1}$$

*Furthermore, the inclusion*

$$\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^q)) \ \longrightarrow \ \mathrm{Map}(X, \mathcal{Z}_0(\mathbf{A}^q)) \tag{5.2.2}$$

*is a weak homotopy equivalence.*

**Proof.** The first statement follows immediately from the Duality Theorem 3.3 and (5.2.0). The second statement is equivalent to the assertion that the natural transformation

$$\Phi : L^q H^k(X) \ \xrightarrow{\cong} H^k(X; \mathbf{Z})$$

is an isomorphism for all $q, k$ with $2q \geq k$. Now in [Li$_1$] it is proved that the natural transformation $\tau \circ \mathbf{s}^{m-q} : L_{m-q} H_{2m-k}(X) \to H_{2m-k}(X; \mathbf{Z})$ is an isomorphism for all $q, k$ with $2(m-q) \leq 2m - k$. Together with (4.4.1) this proves the result. $\square$

**Remark.** If the general compatibility with Poincaré duality conjectured in 4.5 above holds, then Theorem 5.2 extends to the assertion that when $X$ and $Y$ have cell decompositions, the inclusion

$$\mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \ \longrightarrow \ \mathrm{Map}(X, \mathcal{Z}_r(Y))$$

is a homotopy equivalence.

**5.C Theorems of Segal-type** Duality tells us something about rational families of cycles on a smooth variety.

**Theorem 5.3.** *Let $Y$ be a smooth projective variety of dimension $n$. Then for each pair of non-negative integers $r, k$ with $r + k \le n$ there is a weak homotopy equivalence*

$$\mathfrak{Mor}(\mathbf{P}^k, \mathcal{Z}_r(Y)) \;\cong\; \mathcal{Z}_r(Y) \times \mathcal{Z}_{r+1}(Y) \times \cdots \times \mathcal{Z}_{r+k}(Y)$$

**Proof.** By Theorem 3.3 we have $\mathfrak{Mor}(\mathbf{P}^k, \mathcal{Z}_r(Y)) \overset{\text{w.h.e.}}{\cong} \mathcal{Z}_{k+r}(\mathbf{P}^k \times Y)$, and from [FG;2.5] we know that there is a weak homotopy equivalence $\mathcal{Z}_{k+r}(\mathbf{P}^k \times Y) \cong \mathcal{Z}_r(Y) \times \mathcal{Z}_{r+1}(Y) \times \cdots \times \mathcal{Z}_{r+k}(Y)$ $\square$

Setting $k = 1$ gives the following.

**Corollary 5.4.** *The space of parameterized rational curves on $\mathcal{Z}_r(Y)$ is weakly homotopy equivalent to $\mathcal{Z}_r(Y) \times \mathcal{Z}_{r+1}(Y)$.*

A basic result of Graeme Segal [Se] states that the natural embedding

$$\mathrm{Mor}_d(\mathbf{P}^1, \mathbf{P}^1) \;\subset\; \mathrm{Map}_d(\mathbf{P}^1, \mathbf{P}^1)$$

of the space of morphisms of degree $d$ into the space of continuous maps of degree $d$ (with the compact-open topology) is $2d$- connected. In particular, Segal asserts that

$$\lim_{d\to\infty} \mathrm{Mor}_d(\mathbf{P}^1, \mathbf{P}^1) \;\subset\; \lim_{d\to\infty} \mathrm{Map}_d(\mathbf{P}^1, \mathbf{P}^1)$$

is a weak homotopy equivalence. Subsequent work [CCMM] has identified the stable homotopy type (stable in the sense of spectra, not in the sense of increasing degree) of $\mathrm{Mor}_d(\mathbf{P}^1, \mathbf{P}^n)$. Setting $X = \mathbf{P}^1$ and using the identification $SP^n(\mathbf{P}^1) \cong \mathbf{P}^n$, one concludes as a special case of Corollary 5.4 in conjunction with Theorem 4.4 that

$$\varinjlim_{n,d} \mathrm{Mor}_d(\mathbf{P}^1, \mathbf{P}^n)) \quad\subset\quad \varinjlim_{n,d} \mathrm{Map}_d(\mathbf{P}^1, \mathbf{P}^n).$$

is a weak homotopy equivalence. For this reason we call the results of this section "theorems of Segal-type".

**5.D Inverse to Gysin Maps** In this section we present a consequence of the Moving Lemma M.1 which underpins the Duality Theorem. This result has proved useful in our early discussion (e.g., in the proof of Proposition 2.7).

**Proposition 5.5.** *Let $p : P \to Y$ be a flat map of relative dimension $c$ between smooth projective varieties with section $s : Y \to P$. Then*

$$s^! \circ p^* : \mathcal{Z}_r(Y) \longrightarrow \mathcal{Z}_{r+c}(P) \longrightarrow \mathcal{Z}_r(Y)$$

*is a weak homotopy equivalence.*

**Proof.** Let $\mathcal{C}_{r+c}(Y;P) \subset \mathcal{C}_{r+c}(Y)$ denote the submonoid of those effective $r+c$-cycles on $P$ which meet $s(Y) \subset P$ properly. By Corollary 3.6 (or, more directly, by its proof), we conclude that the map on naïve group completions

$$\mathcal{Z}_{r+c}(Y;P) \quad \longrightarrow \quad \mathcal{Z}_{r+c}(P)$$

is a weak homotopy equivalence. On the other hand, [FG;3.4] asserts that the Gysin map $s^!$ when restricted to $\mathcal{Z}_{r+c}(Y;P)$ is represented by intersection with $s(Y)$. Clearly, intersection of a cycle $p^*(Z)$ with $s(Y)$ is merely the cycle $Z$ for any $r$-cycle $Z$ on $Y$. $\square$

## 5.E A Dold-Thom Theorem and a fundamental class for morphic cohomology

The classical Dold-Thom Theorem [DT] establishes an equivalence of functors which in our context can be written as

$$L_0H_k(X) \;\cong\; H_k(X;\, \mathbf{Z}) \qquad\qquad \text{for} \quad 0 \le k \le 2m.$$

By the Theorems 3.3 and 4.4 this implies the following result in cohomology.

**Theorem 5.6** *For any smooth projective variety $X$ of dimension $m$ there are natural ismorphisms*

$$L^mH^k(X) \;\cong\; H^k(X;\, \mathbf{Z}) \qquad\qquad \text{for} \quad 0 \le k \le 2m.$$

The classical Dold-Thom isomorphism is conventionally written as the isomorphism

$$\pi_* Z_0(X) \cong H_*(X;\, \mathbf{Z}),$$

In analogy, the "morphic cohomology version" above can be rewritten as

$$\pi_*\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^m)) \;\cong\; H^{2m-*}(X;\, \mathbf{Z}).$$

As a corollary of Theorem 5.6 in the special case $k = 2m$, we obtain a "Noether normalization of virtual degree 1" giving a well-defined **fundamental class** in the morphic cohomology of $X$.

**Corollary 5.7.** *Let $f, g : X \to SP^d(\mathbf{P}^m)$ determine $[f-g] \in \pi_0\mathfrak{Mor}(X, \mathcal{Z}(\mathbf{A}^m))$ corresponding to the fundamental class $\iota_X \in H^{2m}(X;\mathbf{Z})$ under the isomorphism of Theorem 5.6. Let $\iota_{2m} \in H^{2m}(\mathcal{Z}_0(\mathbf{A}^m);\mathbf{Z})$ denote the canonical class. Then*

$$f^*(\iota_{2m}) - g^*(\iota_{2m}) = \iota_X \in H^{2m}(X).$$

**Proof.** The identification of $\pi_0\mathrm{Map}(X, \mathcal{Z}_0(\mathbf{A}^m))$ with $H^{2m}(X;\mathbf{Z})$ is achieved by sending $\phi : X \to \mathcal{Z}_0(\mathbf{A}^m)$ to $\phi^*(\iota_{2m})$. $\square$

**5.F Filtrations in Cohomology** Recall that $L^qH^k(X) = \pi_{2q-k}\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^q))$ is defined for *all* $q > 0$, and there are commutative triangles

$$\begin{tikzcd} L^qH^*(X) \arrow[d, "\mathbf{s}"'] \arrow[dr, "\Phi"] & \\ L^{q+1}H^*(X) \arrow[r, "\Phi"'] & H^*(X;\mathbf{Z}) \end{tikzcd}$$

where $\Phi$ denotes the natural transformation and $\mathbf{s}$ is the operation from [FL$_1$] discussed in §2. It is natural to ask at what point the images of these maps stabilize. As we verify in the following theorem, a strong form of stabilization is valid when $X$ is smooth.

**Theorem 5.8.** *Let $X$ be a smooth projective variety. Then for any $q \geq m$,*

$$\mathbf{s} : L^qH^*(X) \longrightarrow L^{q+1}H^*(X)$$

*is an isomorphism.*

**Proof.** We interpret the $\mathbf{s}$-map via the following diagram

$$\begin{tikzcd} \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^q)) \wedge S^2 \arrow[r, "\mathbf{s}"] \arrow[d, "\mathcal{D} \times i"'] & \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^{q+1})) \arrow[d, "\mathcal{D}"] \\ \mathcal{Z}_m(X \times \mathbf{A}^q) \times \mathcal{Z}_0(\mathbf{A}^1) \arrow[r, "\times"] & \mathcal{Z}_m(X \times \mathbf{A}^{q+1}) \\ \mathcal{Z}_0(X \times \mathbf{A}^{q-m}) \times \mathcal{Z}_0(\mathbf{A}^1) \arrow[r, "\times"] \arrow[u, "\mathrm{pr}^* \times 1"] & \mathcal{Z}_0(X \times \mathbf{A}^{q+1-m}) \arrow[u, "\mathrm{pr}^*"'] \end{tikzcd}$$

where $i : S^2 \longrightarrow \mathcal{Z}_0(\mathbf{A}^m)$ is given by $S^2 \cong \mathbf{P}^1 \longrightarrow \mathcal{Z}_0(\mathbf{P}^1) \longrightarrow \mathcal{Z}_0(\mathbf{A}^1)$ sending $p \in S^2$ to $p - \{\infty\}$. Since the map on homotopy induced by the bottom horizontal arrow is simply the Thom isomorphism assoicated to the trivial rank-1 bundle over $X \times \mathbf{A}^{q-m}$, we conclude that $\mathbf{s}$ is a weak homotopy equivalence. $\square$

In [FM$_1$], [FM$_2$], a "topological filtration on homology"

$$T_rH_k(X;\mathbf{Z}) \stackrel{\text{def}}{=} \text{image}\,\{\tau \circ \mathbf{s}^r : L_rH_k(X) \longrightarrow H_k(X;\mathbf{Z})\}$$

(decreasing with respect to $r$) was introduced and shown to have a number of interesting properties. In [FL$_1$], we considered the cohomological analogue:

$$T^qH^k(X;\mathbf{Z}) \stackrel{\text{def}}{=} \text{image}\left\{\Phi : L^qH^k(X) \longrightarrow H^k(X;\mathbf{Z})\right\}.$$

Our final theorem asserts that for $X$ smooth this cohomological analogue is exactly the Poincaré dual of the topological filtration on homology. The "proof" of this theorem consists in observing that Theorem 5.9 is merely a restatement of Theorem 4.4.

**Theorem 5.9.** *If $X$ is smooth, then the Poincaré duality map $\mathcal{P} : H^k(X;\mathbf{Z}) \longrightarrow H_{2m-k}(X;\mathbf{Z})$ induces isomorphisms*

$$\mathcal{P} : T^q H^k(X;\mathbf{Z}) \xrightarrow{\cong} T_{m-q}H_{2m-q}(X;\mathbf{Z})$$

*of the topological filtrations in all degrees.*

**5.G. Adjointness within the Kronecker pairing.** We shall now prove that under the duality isomorphism the Kronecker pairing is equivalent to the intersection pairing. This establishes the degeneracy of the Kronecker pairing in some cases.

We recall that in [FG] an intersection product $\mathcal{Z}_p(X) \otimes \mathcal{Z}_q(X) \xrightarrow{\bullet} \mathcal{Z}_{p+q-n}(X)$ was defined for smooth $n$- dimensional varieties $X$, when $p+q-n \geq 0$. This product has the property that when restricted to pairs of cycles which meet in proper dimension, it is homotopic to the standard intersection of cycles (as in [Fu]). One checks that this pairing is compatible with the equivalences $\mathcal{Z}_p(X) \xrightarrow{\cong} \mathcal{Z}_{p+1}(X \times \mathbf{A}^1)$ and thereby induces a pairing

$$L_pH_k(X) \otimes L_qH_\ell(X) \xrightarrow{\cong} L_{p+q-n}H_{k+\ell-2n}(X)$$

for $k+\ell \geq 2n$, where for negative integers $-r$ one defines $L_{-r}H_m(X) = L_0H_{m+2r}(X \times \mathbf{A}^r) \cong H_m(X;\,\mathbf{Z})$.

**Theorem 5.10.** *Let $X$ be a smooth projective variety of dimension $m$, and fix integers $p, k, q$ with $2p \leq k \leq 2q$. Consider on $X$ the Kronecker pairing*

$$\kappa \,:\, L^qH^k(X) \otimes L_pH_k(X) \longrightarrow \mathbf{Z}$$

*introduced in [FL$_1$], and the intersection pairing*

$$\bullet \,:\, L_{m-q}H_{2m-k}(X) \otimes L_pH_k(X) \longrightarrow \mathbf{Z}$$

*established in [FG]. Then*

$$\kappa(\varphi,\, c) \;=\; \mathcal{D}\varphi \bullet c$$

*for all $\varphi \in L^qH^k(X)$ and $c \in L_pH_k(X)$.*

**Proof.** Consider the diagram

$$\begin{array}{ccc}
\mathfrak{Mor}(X, \mathcal{Z}_0(\mathbf{A}^q)) \wedge \mathcal{Z}_p(X) & \xrightarrow{\xi} & \mathcal{Z}_p(\mathbf{A}^q) \\
{\scriptstyle \mathcal{D} \wedge \Sigma^q}\downarrow & & \\
\mathcal{Z}_m(X \times \mathbf{A}^q) \wedge \mathcal{Z}_{p+q}(X \times \mathbf{A}^q) & & \| \\
\bullet \downarrow & & \\
\mathcal{Z}_p(X \times \mathbf{A}^q) & \xrightarrow{(\mathrm{pr}_2)_*} & \mathcal{Z}_p(\mathbf{A}^q).
\end{array} \tag{5.10.1}$$

where the map $\xi$ is given by graphing the cocycle over the cycle and then pushing into $\mathbf{A}^q$. One verifies directly that $\xi$ coincides with the Kronecker map defined in [FL$_1$, §7]. If we assume for the moment that $\bullet$ is the standard intersection product defined only on the subset of pairs of cycles which meet properly, then the diagram (5.10.1) commutes on the nose. It therefore follows from [FG;3.5] that this diagram commutes up to homotopy. Taking homotopy groups "$\pi_{2q-k} \otimes \pi_{k-2p}$" yields a commutative diagram

$$\begin{array}{ccc}
L^qH^k(X) \otimes L_pH_k(X) & \xrightarrow{\kappa} & \mathbf{Z} \\
\mathcal{D} \otimes \mathrm{Id} \downarrow & & \\
L_{m-q}H_{2m-k}(X) \otimes L_pH_k(X) & & \| \\
\bullet \downarrow & & \\
L_{q-p}H_0(X) & \xrightarrow{\cong} & \mathbf{Z}.
\end{array} \qquad \square$$

Now the intersection pairing $\bullet$ above factors through the topological intersection product $\bullet_{\mathrm{top}}$, i.e., there is a factoring of $\bullet$ of the form

$$L_{m-q}H_{2m-k}(X) \otimes L_pH_k(X) \xrightarrow{\Phi \otimes \Phi} H_{2m-2p}(X;\mathbf{Z}) \otimes H_{2p}(X;\mathbf{Z}) \xrightarrow{\bullet_{\mathrm{top}}} \mathbf{Z}$$

where $\Phi$ is the natural transformation as above. For example when $2p = k = 2q$ we have

$$\mathcal{A}_{m-p} \otimes \mathcal{A}_p \xrightarrow{\Phi \otimes \Phi} H_{2m-k}(X;\mathbf{Z}) \otimes H_k(X;\mathbf{Z}) \xrightarrow{\bullet_{\mathrm{top}}} \mathbf{Z},$$

and there are many well-known cases where the kernel of $\Phi$ has large rank, i.e., where homological equivalence does not imply algebraic equivalence. It follows that the pairing

$$\mathcal{A}_{m-p} \otimes \mathcal{A}_p \xrightarrow{\bullet} \mathbf{Z}$$

is degenerate in such cases (numerical equivalence does not imply algebraic eqivalence). By Theroem 5.10 the Kronecker pairing is also degenerate over $\mathbf{Q}$ in such cases.

## Appendix C. Cocycles and the compact- open topology.

In this appendix, we correct the mistaken discussion given in [$\mathrm{FL}_1$;1.5] of the relationship between the compact-open topology on the monoid $\mathfrak{Mor}(U, \mathcal{C}_r(Y))$ for a quasi-projective variety $U$ and the topology induced by the embedding

$$\mathcal{G} : \mathfrak{Mor}(U, \mathcal{C}_r(Y)) \quad \longrightarrow \quad \mathcal{C}_{r+m}(U \times Y) \stackrel{\text{def}}{=} \mathcal{C}_{r+m}(X \times Y)/\mathcal{C}_{r+m}(X_\infty \times Y) \qquad (C.1.0)$$

via the graphing map. When $U$ is projective, these topologies coincide. However, if $U$ is not projective, then the topology induced by $\mathcal{G}$ is equivalent to the topology of uniform convergence on compacta **with uniformly bounded degree**. In [$\mathrm{FL}_1$] the added condition of uniformly bounded degree was overlooked. We present here a thorough discussion of these topologies.

Throughout this appendix, $U$ shall denote a quasi-projective variety of dimension $m$, $X \supset U$ will denote a projective closure with Zariski closed complement $X_\infty \subset X$, $Y$ will denote a projective variety of dimension $n$, and $r$ will be a non-negative integer $\leq n$.

We recall that the graphing construction (C.1.0) is always injective. We define $\mathcal{C}_r(Y)(U)$ to be the topological submonoid of $\mathcal{C}_{r+m}(U \times Y)$ given as the image of $\mathcal{G}$.

**Proposition C.1.** *The inverse of the graphing construction*

$$\mathcal{G}^{-1} : \mathcal{C}_r(Y)(U) \quad \longrightarrow \quad \mathfrak{Mor}(U, \mathcal{C}_r(Y))$$

*is continuous, where $\mathfrak{Mor}(U, \mathcal{C}_r(Y))$, considered as a space of continuous maps from $U$ to $\mathcal{C}_r(Y)$, is given the compact-open topology.*

**Proof.** It will suffice to construct an evaluation mapping

$$\epsilon : \mathcal{C}_r(Y)(U) \times U \longrightarrow \mathcal{C}_r(Y) \qquad (C.1.1)$$

and establish that it is continuous (since, by a standard lemma, the continuity of (C.1.1) implies that the adjoint mapping is continuous into the compact-open topology.) When $Y$ and $U$ are smooth, we define the evaluation map $\epsilon$ using intersection of cycles: $\epsilon(f, u) = \Gamma(f) \bullet (\{u\} \times Y)$. The continuity of $\epsilon$ in this case follows from the continuity of the intersection product proved in [Fu] for families of cycles which meet in proper dimension. When $U$ is smooth but $Y$ is not necessarily smooth, we consider a projective embedding $Y \subset \mathbf{P}^N$ and replace $Y$ by $\mathbf{P}^N$ (i.e., we now write $f(u) = \Gamma(f) \bullet (\{u\} \times \mathbf{P}^N)$ where the intersection takes place in $U \times \mathbf{P}^N$).

If $U$ is not smooth, we consider a resolution of singularities $\pi : \widetilde{U} \to U$ and let $\widetilde{\epsilon} : \mathcal{C}_r(Y)(\widetilde{U}) \times \widetilde{U} \longrightarrow \mathcal{C}_r(Y)$ be the evaluation map (C.1.1). Note that composition of morphisms $\phi : U \to \mathcal{C}_r(Y)$ with $\pi$ induces an injective map

$$\pi^* : \mathcal{C}_r(Y)(U) \hookrightarrow \mathcal{C}_r(Y)(\widetilde{U}).$$

The composition $\widetilde{\epsilon} \circ (\pi^* \times 1) : \mathcal{C}_r(Y)(U) \times \widetilde{U} \to \mathcal{C}_r(Y)$ descends to $\epsilon$ as in (C.1.1), since $\widetilde{\epsilon}(\mathcal{G}(\phi \circ \pi), \widetilde{u}) = \{u\} \times (\phi \circ \pi)(\widetilde{u})$.

We now prove the continuity of $\epsilon$, assuming for the moment that we have shown that $\pi^*$ is continuous. Let $a_n \to a$ be a convergent sequence in $\mathcal{C}_r(Y)(U) \times U$. Choose $\{\widetilde{a}_n\}$ in $\mathcal{C}_r(Y)(U) \times \widetilde{U}$ with $(\mathrm{Id} \times \pi)(\widetilde{a}_n) = a_n$. Since $\mathrm{Id} \times \pi$ is proper, for every subsequence of $\{a_n\}$ there is a sub-subsequence of $\{\widetilde{a}_n\}$ such that $\widetilde{a}_n \to \widetilde{a}$ upstairs, and so $\epsilon(a_n) = \widetilde{\epsilon}((\mathrm{Id} \times \pi)(\widetilde{a}_n)) \to \widetilde{\epsilon}((\mathrm{Id} \times \pi)(\widetilde{a})) = \epsilon(a)$. This proves the continuity of $\epsilon$.

It remains to prove that $\pi^*$ is continuous. Let $X$ be a compactification of $U$, and set $X_\infty = X - U$. Let $\widetilde{X} \supset \widetilde{U}$ be a compactification of $\widetilde{U}$ such that $\pi : \widetilde{U} \to U$ admits an extension $\pi : \widetilde{X} \to X$. Consider a subset $A \subset \mathcal{C}_{m+r}(U \times Y) = \mathcal{C}_{m+r}(X \times Y)/\mathcal{C}_{m+r}(X_\infty \times Y)$ which lies in the image of $\mathcal{C}_{m+r,\leq d}(X \times Y)$, the compact space of effective $(m+r)$-cycles of degree $\leq d$ on $X \times Y$ (for some projective embedding of $X \times Y$). Let $B \subset \mathcal{C}_{r+m,\leq d}(X \times Y)$ denote the constructible subset of those effective $r+m$-cycles each component of which when restricted to $U \times Y$ is equidimensional over $U$. As argued in [FG;1-6] using noetherian induction and generic flatness of families, $B \subset (\pi \times \mathrm{Id})_*(\mathcal{C}_{r_m,\leq e}(\widetilde{X} \times Y))$ for some sufficiently large $e$. (Any algebraic family of subschemes on $X \times Y$ parametrized by some variety $C$ is generically flat over $C$, and the closure in $\widetilde{X} \times Y$ of the intersection of the family with $\mathrm{Reg}(U) \times Y$ is also generically flat over $C$.)

The continuity of $\pi^*$ is now proved as follows. Consider a convergent sequence $\Gamma(f_n) \to \Gamma(f)$ in $\mathcal{C}_r(Y)(U)$. By the paragraph above, the cycles $\pi^*(\Gamma(f_n)) = \Gamma(\widetilde{f}_n)$ and $\pi^*(\Gamma(f)) = \Gamma(\widetilde{f})$ (where $\widetilde{f}_n = f_n \circ \pi$ and $\widetilde{f}_n = f \circ \pi$) have closures of uniformly bounded degree in $\widetilde{X} \times Y$. Furthermore, we see that $\Gamma(\widetilde{f}_n) \to \Gamma(\widetilde{f})$ over $\mathrm{Reg}(U) \times Y$ since $\pi \times \mathrm{Id}$ is an isomorphism there. By compactness, every subsequence of $\Gamma(\widetilde{f}_n)$ has a sub-subsequence which converges to an effective cycle, say $\Lambda$, on $\widetilde{U} \times Y$. From the convergence on $\mathrm{Reg}(U) \times Y$ we see that $\Lambda = \Gamma(\widetilde{f}) + c_0$ for some effective cycle $c_0$ supported on $\pi^{-1}\mathrm{Sing}(U) \times Y$. We shall show that $|c_0| \subset |\Gamma(\widetilde{f})|$ and therefore $c_0 = 0$ because $\dim\{|\Gamma(\widetilde{f})| \cap \pi^{-1}\mathrm{Sing}(U) \times Y\} \leq m + r - 1$. If $x \in |c_0|$, then there exist $x_n \in |\Gamma(\widetilde{f}_n)|$ such that $x_n \to x$ in $\widetilde{U} \times Y$. Then $(\pi \times \mathrm{Id})(x_n) \to (\pi \times \mathrm{Id})(x)$ and so $(\pi \times \mathrm{Id})(x) \in |\Gamma(f)|$ by Proposition C.2 below. This implies that $x \in |\Gamma(\widetilde{f})|$ as claimed. □

**Proposition C.2.** *Let $\{W_n\}$ be a sequence of effective $r$-cycles on $Y$ which converge in $\mathcal{C}_r(Y)$ to some $r$-cycle $W$. Consider a sequence of points $\{y_n\}$ with $y_n \in |W_n|$ and assume that this sequence converges to some point $y \in Y$. Then $y \in |W|$.*

**Proof.** We assume that $Y$ is provided with the metric induced from an embedding into some projective space. A basic result in the theory of positive currents (cf. [H]) states that

$$\mathrm{Mass}\left(W_n\big|_{B_\epsilon(z')}\right) \geq c_{2r}\epsilon^{2r}$$

for all $w \in W_n$ and all $\epsilon > 0$, where $c_{2r} > 0$ is a constant depending only upon $r$ and where $B_\epsilon(z')$ is the open $\epsilon$-ball centered at $z'$.

Fix some $\epsilon > 0$. Set $q = \chi\omega^r/r!$ where $\omega$ is the restriction of the Kahler form of projective space to $Y$ and where $\chi$ denotes the characteristic function of the ball $B_{2\epsilon}(z)$. Then for all $n$ sufficiently large that $y_n \in B_\epsilon(y)$ we have

$$c_{2r}\epsilon^{2r} \leq \int_{W_n} q \longrightarrow \int_W q \leq \mathrm{Mass}(W \cap B_{2\epsilon}(y)).$$

Hence, $y \in |W|$ as asserted. $\square$

We denote by $\mathrm{Map}(U, \mathcal{C}_r(Y))$ the space of continuous maps $f : U \to \mathcal{C}_r(Y)$ equipped with the compact-open topology, i.e., the topology of uniform convergence on compact subsets.

**Theorem C.3** *The topology on $\mathcal{C}_r(Y)(U)$ is characterized by the following property: a sequence $\{f_i : i \in \mathbf{N}\} \subset \mathcal{C}_r(Y)(U)$ converges for this topology if and only if*

*(i) $\{f_i : i \in \mathbf{N}\}$ converges when viewed in* $\mathrm{Map}(U, \mathcal{C}_r(Y))$.

*(ii) The associated sequence $\{Z_i : i \in \mathbf{N}\} \subset \mathcal{C}_r(Y)(U)$ of graphs has the property that for some Zariski locally closed embedding $U \times Y \subset \mathbf{P}^N$, there is a positive integer $E$ such that each $Z_i$ has closure $\bar{Z}_i \subset \mathbf{P}^N$ of degree $\leq E$.*

*We call this topology on $\mathcal{C}_r(Y)(U)$ inherited from that of $\mathcal{C}_r(Y)(U)$ the* **topology of convergence with bounded degree**.

**Proof.** Proposition C.1 implies that convergence of $\{f_i : i \in \mathbf{N}\} \subset \mathcal{C}_r(Y)(U)$ implies (i). Recall from [Li$_1$] that $\mathcal{C}_{r+m}(U \times Y)$ has the compactly generated topology associated to the increasing filtration

$$\ldots \subset \mathcal{C}_{r+m,\leq d}(U \times Y) \subset \ldots \subset \mathcal{C}_{r+m,\leq d+1}(U \times Y) \subset \ldots \mathcal{C}_{r+m}(U \times Y),$$

where

$$\mathcal{C}_{r+m,\leq d}(U \times Y) \quad \stackrel{\text{def}}{=} \quad \mathrm{image}\Big\{ \coprod_{e \leq d} \mathcal{C}_{r+m,e}(X \times Y) \longrightarrow \mathcal{C}_{r+m}(U \times Y) \Big\}.$$

If the sequence of graphs $\{Z_i : i \in \mathbf{N}\}$ associated to $\{f_i : i \in \mathbf{N}\}$ converges to $Z$, then the subset $\{Z_i : i \in \mathbf{N}\} \cup \{Z\}$ is compact and therefore lies in some $\mathcal{C}_{r+m,\leq d}(U \times Y)$. This immediately implies convergent condition (ii).

Conversely, assume $\{f_i : i \in \mathbf{N}\}$ satisfies condition (i) and that its associated sequence of graphs $\{Z_i : i \in \mathbf{N}\}$ satisfies conditions (ii). Let $g : U \to \mathcal{C}_r(Y)$ with graph $Z_g$ denote the limit in $\mathrm{Map}(U, \mathcal{C}_r(Y))$ of $\{f_i : i \in \mathbf{N}\}$. Let $\bar{Z}_i$ denote the closure of $Z_i$ in $X \times Y$. By hypothesis (ii), any subsequence of $\{\bar{Z}_i : i \in \mathbf{N}\}$ admits a convergent subsequence $\{\bar{Z}_j : j \in \mathbf{M} \subset \mathbf{N}\} \subset \mathcal{C}_{r+m}(X \times Y)$. Let $\bar{Z} \in \mathcal{C}_{r+m}(X \times Y)$ denote the limit of such a convergent sequence. It suffices to prove that the restriction $Z$ of $\bar{Z}$ to $U \times Y$ has support contained in the support of $Z_g$, for then Proposition C.1 implies that $Z = Z_g$.

Consider an arbitrary point $(x, y) \in |Z| \subset U \times Y$. The convergence of $\{\bar{Z}_j\}$ to $Z$ implies that for every $\epsilon > 0$ there exists some $N_\epsilon$ such that $|Z_j| \cap B_\epsilon(x, y) \neq \emptyset$ whenever $j \geq N_\epsilon$. Hence, there exists a sequence of points $(x_j, y_j) \in |Z_j|$ converging to $(x, y)$. In particular, the sequence $\{y_j \in |f_j(x_j)|\}$ converges to $y$. By choosing $j$ sufficiently large, we may assume that the $x_j$'s lie in some compact ball centered about $x$ in $U$. Because $\{f_j\}$ converges to $g$ on compact subsets of $U$, we conclude that $\{f_j(x_j)\}$ converges to $g(x)$. Proposition C.2 thus implies that $y \in |g(x)|$ as required. $\square$

As we make explicit in Theorem C.4, the condition of bounded degree is redundant if the domain $U$ equals $X$ (i.e., is projective).

**Theorem C.4.** *The graphing construction*

$$\mathcal{G} : \mathfrak{Mor}(X, \mathcal{C}_r(Y)) \quad \longrightarrow \quad \mathcal{C}_r(Y)(U)$$

*is a homeomorphism.*

**Proof.** By Theorem C.3, it suffices to show the following: for any convergent seqence $\{f_n\}$ in $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$, the associated sequence of graphs $\{Z_n\}$ has bounded degree. For this, it suffices to prove that homology classes $[Z_n] \in H_{2m+2r}(X \times Y)$ are independent of $n$ for $n$ sufficiently large. (Namely, the homology class $[Z]$ determines the multi-degree of $Z_n$ which in turn determines the degree of $Z_n$.) This is implied by the assertion that the maps $f_n$ are all homotopic for $n$ sufficiently large, for a homotopy between $f_n, f_m$ determines an integral current on $X \times Y \times I$ with restrictions to $Z_n, Z_m$.

The fact that $f_n$'s are homotopic for all sufficiently large $n$ is a consequence of the following elementary lemma.

**Lemma C.5** *Let $P$ be a polyhedron. Then there exists an $\epsilon > 0$ such that whenever $f, g : A \to P$ are continuous and satisfy*

$$\|f - g\|_\infty \quad < \quad \epsilon$$

*then $f$ is homotopic to $g$.*

**Proof.** We embed $P \subset \mathbf{R}^N$ and assume without loss of generality that the metric on $P$ is the one induced from this embedding. Let $W \supset P$ be a neighborhood of $P$ in $\mathbf{R}^N$ with a retraction $r : W \to P$. Then there exists $\epsilon > 0$ such that for all $x, y \in P$ with $d(x, y) < \epsilon$, the line segment $\overline{xy} \subset W$. Hence, if $\|f - g\|_\infty < \epsilon$ then $f_t(x) = (1 - t)f(x) + tg(x)$ is a homotopy from $f$ to $g$ in $W$ and $r \circ f_t$ gives the desired homotopy in $P$. $\square$

We conclude with two important examples which illustrate the subtleties of the topology on cocycle spaces.

**Example C.6** *The injective continuous map $\mathcal{D} : \mathfrak{Mor}(X, \mathcal{Z}_r(Y)) \to \mathcal{Z}_{m+r}(X \times Y)$ given by group completion of the graphing map, is not a topological embedding.* To see this, let $X = Y = \mathbf{P}^1$, and for each integer $n > 0$ let $\Gamma_n \in \mathfrak{Mor}(\mathbf{P}^1, \mathcal{C}_0(\mathbf{P}^1))$ defined by the mapping

$$y = x/n$$

in affine coordinates. Then the sequence of cocycles $\Sigma_n \stackrel{\text{def}}{=} \mathcal{D}(\Gamma_{n+1} - \Gamma_n)$ converges to 0 in $\mathcal{Z}_1(\mathbf{P}^1 \times \mathbf{P}^1)$. This is clear since $\mathcal{D}(\Gamma_n)$ converges to the effective cycle $\mathbf{P}^1 \times \{0\} + \{\infty\} \times \mathbf{P}^1$. However, **the sequence $\Sigma_n$ does not converge to 0 in $\mathcal{Z}_1(\mathbf{P}^1 \times \mathbf{P}^1)$**. This is seen as follows. Let $M = \mathfrak{Mor}(\mathbf{P}^1, \mathcal{C}_0(\mathbf{P}^1))$ and let $\Delta \subset M \times M$ denote the diagonal. Then by definition $\Sigma_n \longrightarrow 0$ in $\mathfrak{Mor}(\mathbf{P}^1, \mathcal{Z}_0(\mathbf{P}^1))$ iff for every $M$-saturated open neighborhood $\mathcal{U}$ of $\Delta$ there is an $N$ to that $(\Gamma_{n+1}, \Gamma_n) \in \mathcal{U}$ for all $n \geq N$. Now let $\mathcal{U} = M \times M - \{(\Gamma_{n+1}, \Gamma_n)\}_{n=1}^{\infty}$ and observe that $\mathcal{U}$ is a saturated open neighborhood of $\Delta$. (It is open because the limiting cycle $(c, c)$ with $c = \mathbf{P}^1 \times \{0\} + \{\infty\} \times \mathbf{P}^1$ is not in $\Delta$.)

**Example C.7** *The injective continuous map $\mathcal{G} : \mathfrak{Mor}(X, \mathcal{C}_0(\mathbf{A}^q)) \to \mathcal{C}_m(X \times \mathbf{A}^q)$ given by the graphing map, is not a topological embedding.* To see this, consider the sequence of mappings $\psi_n \in \mathfrak{Mor}(\mathbf{P}^1, \mathbf{P}^2) \subset \mathfrak{Mor}(\mathbf{P}^1, \mathcal{C}_0(\mathbf{P}^2))$ given in homogeneous coordinates by $\psi_n([x : y]) = [x : \frac{1}{n}y : \frac{1}{n^2}y]$, with graphs

$$\Gamma_n = \mathcal{G}(\psi_n) = \{([x : y], [x : \tfrac{1}{n}y : \tfrac{1}{n^2}y]) \in \mathbf{P}^1 \times \mathbf{P}^2 \ : \ x, y \in \mathbf{C}\}.$$

We take the distinguished line $\mathbf{P}^1 \subset \mathbf{P}^2$ to be $\mathbf{P}^1 = \{[z : w : 0]\}$. Then one verifies that $\Gamma_n \longrightarrow \mathbf{P}^1 \times [1 : 0 : 0] + [0 : 1] \times \mathbf{P}^1$ as cycles in $\mathbf{P}^1 \times \mathbf{P}^2$. Hence, the sequence $\Gamma_n$ converges to "0" in $\mathcal{C}_1(\mathbf{P}^1 \times \mathbf{A}^2) \equiv \mathcal{C}_1(\mathbf{P}^1 \times \mathbf{P}^2)/\mathcal{C}_1(\mathbf{P}^1 \times \mathbf{P}^1)$, however, it does not converge at all in $\mathfrak{Mor}(\mathbf{P}^1, \mathcal{C}_0(\mathbf{A}^2)) \equiv \mathfrak{Mor}(\mathbf{P}^1, \mathcal{C}_0(\mathbf{P}^2))/\mathfrak{Mor}(\mathbf{P}^1, \mathcal{C}_0(\mathbf{P}^1))$.

## Appendix M. The Moving Lemma for Families.

For the convenience of the reader we present here the statement of the Moving Lemma for Families of Cycles of Bounded Degree proved in [FL$_2$]. The main result there is substantially more general. We quote here the form of the theorem which is needed for the duality theorems of §3. It can be found in [FL$_2$; 3.2 and 3.3]

**Theorem M.1.** *Let $X \subset P^n_{\mathbf{C}}$ be a complex projective variety of dimension $m$. Let $r, s, e$ be non-negative integers with $r + s \geq m$. Then there exists a Zariski open neighborhood $\mathcal{O}$ of $\{0, 1\}$ in $\mathbf{C}$, and a continuous algebraic map*

$$\widetilde{\Psi} : \mathcal{C}_s(X) \times \mathcal{O} \longrightarrow \mathcal{C}_s(X)^{\times 2}$$

*which has the property that $\pi \circ \widetilde{\Psi}$ induces by linearity a continuous map*

$$\Psi : \mathcal{Z}_s(X) \times \mathcal{O} \longrightarrow \mathcal{Z}_s(X)$$

*satisfying the following. Set $\psi_\tau| = \Psi\big|_{\mathcal{Z}_s(X)\times\{\tau\}}$ for $\tau \in \mathcal{O}$.*

*(a)* $\psi_0 = \mathrm{Id}$.
*(b)* *For any $Z \in \mathcal{Z}_s(X)$, the restriction*

$$\Psi\big|_{\{Z\}\times\mathcal{O}} : \{Z\} \times \mathcal{O} \longrightarrow \mathcal{Z}_s(X),$$

*determines a rational equivalence between $Z$ and $\psi_1(Z)$.*
*(c)* *For any $\tau \in \mathcal{O}$, $\psi_\tau$ is a continuous group homomorphism.*
*(d)* *For any $Z \in \mathcal{Z}_{s,\leq e}(X)$, any $Y \in \mathcal{Z}_{r',\leq e}(X)$, $r' \geq r$ and any $\tau \neq 0$ in $\mathcal{O}$, each component of excess dimension (i.e., $> r' + s - m$) of the intersection*

$$|Y| \cap |\psi_\tau(Z)|$$

*is contained in the singular locus of $X$.*

# Appendix T. Tractable monoids.

We recall that a subset $C$ of a variety $V$ is said to be **constructible** if $C$ is a finite union of subsets $C_i \subset V$ each of which is locally closed in $V$ with respect to the Zariski topology. Since any locally closed subset $C_i \subset V$ has the property that its closure (with respect to the analytic topology) is Zariski closed in $V$ (cf. [Sh]), the closure of a constructible subset $C \subset V$ is also Zariski closed in $V$. If $T \subset V$ is constructible, then a subset $S \subset T$ is said to be a **constructible embedding** if $S \subset V$ is a constructible subset. If $S \subset T$ is a constructible embedding of constructible subsets, then the closure of $\overline{S}$ in $T$ is "Zariski closed" in the sense that $\overline{S}$ equals the intersection of $T$ with some Zariski closed subset of the ambient variety $V$.

The following lemma isolates the special property we use of the topology of algebraic varieties. The proof of this lemma relies on the existence of relative triangulations for semi-algebraic subsets (more general than constructible subsets) proved by Hironaka in [H].

**Lemma T.1** *Any closed, constructible embedding $S \subset T$ of constructible spaces is a cofibration.*

**Proof.** Let $\bar{S} \subset \bar{T}$ denote the closure of $S \subset T$ for some projective embedding of $T \subset \mathbf{P}^N$. Note that $S = \bar{S} \cap T$. By Hironaka's relative triangulation theorem [H], there is a (finite) semi-algebraic triangulation of $\bar{T}$ so that $\bar{S}$ and $\bar{T} - T$ are subcomplexes, and thus $T$ and $S$ are unions of some open simplices. We construct a deformation retraction $r_t,\ 0 \leq t \leq 1$ of a neighborhood $\widetilde{U}$ of $\bar{S}$ in $\bar{T}$ onto $\bar{S}$ with the property that $r_t$ maps each open simplex of the triangulation into itself for all $t$.

Namely, for some maximal (closed) simplex $\sigma$ of $\bar{T}$, consider $\mu \stackrel{\text{def}}{=} \sigma \cap \bar{S}$. Write $\mu$ as a union of (closed) faces, $\mu = \cup_i F_i$, and define $\mu^* = \cap_i F_i^*$, where $F_i^*$ is the open star in the first barycentric subdivision of $\sigma$ of the dual face $F_i^\vee$. Then $\sigma - \mu^*$ admits a linear retraction to $\mu$ which restricts naturally to a linear retraction of $(\sigma - \mu^*) \cap \tau$ to $\tau \cap \bar{S}$, for any face $\tau \subset \sigma$. We take $\widetilde{U} \cap \sigma \subset \sigma - \mu^*$ to be some $\epsilon$ -neighborhood of $\mu \subset \sigma$.

We thus obtain a deformation retraction of a neighborhood $U$ of $S$ in $T$ onto $S$ by taking the restriction of $r_t$ to $U \stackrel{\text{def}}{=} \widetilde{U} \cap T$. $\square$

We recall from [FG] the following useful notion of a tractable action of a monoid $M$ on a space $T$ and of a tractable monoid.

**Definition T.2.** The action of an abelian topological monoid on a topological space $T$ is said to be **tractable** if $T$ is the topological union of inclusions

$$\emptyset = T_{-1} \subset T_0 \subset T_1 \subset \dots$$

such that for each $n > 0$ $T_{n-1} \subset T_n$ fits into a push-out square of $M$-equivariant maps (with $R_0$ empty)

$$\begin{array}{ccc} R_n \times M & \longrightarrow & S_n \times M \\ \downarrow & & \downarrow \\ T_{n-1} & \longrightarrow & T_n \end{array} \qquad (T.1.1)$$

whose upper horizontal arrow is induced by a cofibration $R_n \subset S_n$ of Hausdorff spaces. The monoid $M$ itself is said to be **tractable** if the diagonal action of $M$ on $M \times M$ is tractable.

If $S = \coprod_d S_d$ is a countable disjoint union of constructible spaces, we shall call $S$ a **generalized constructible space**. If $i : S \subset T$ is a countable disjoint union of constructible embeddings $i_d : S_d \subset T_d$, we shall say that $i : S \subset T$ is a constructible embedding.

The following proposition is a simple modification of the proofs of tractability given in [FG;1.3].

**Proposition T.3.** Consider a submonoid $\mathcal{E}_r \subset \mathcal{C}_{r+m}(X \times Y) \stackrel{\text{def}}{=} \mathcal{C}_r$ whose embedding is a constructible embedding of generalized constructible spaces. Then $\mathcal{E}_r$ is a tractable monoid.

Let $\mathcal{F}_r \subset \mathcal{E}_r$ be a submonoid with the property that each $\mathcal{F}_{r,d} \stackrel{\text{def}}{=} \mathcal{F}_r \cap \mathcal{C}_{r,d}$ is a Zariski closed subset of $\mathcal{E}_{r,d} \stackrel{\text{def}}{=} \mathcal{E}_r \cap \mathcal{C}_{r,d}$, where $\mathcal{C}_{r,d} \stackrel{\text{def}}{=} \mathcal{C}_{r+m,d}(X \times Y)$. Then $\mathcal{E}_r$ is tractable as a $\mathcal{F}_r$-space and the quotient monoid $\mathcal{E}_r/\mathcal{F}_r$ is also a tractable monoid.

**Proof.** Set $M = \mathcal{C}_{r+m}$ and $M(d) = \mathcal{C}_{r+m,d}$. Let $T$ denote $M \times M$ and set

$$T_n = \left[ \bigcup_{\nu(a,b) \leq n} M(a) \times M(b) \right] \cdot M$$

where $\nu : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ is a suitable bijection. Set

$$S_n = M(a_n) \times M(b_n), \qquad \nu(a_n, b_n) = n$$

and define $R_n \subset S_n$ by

$$R_n = \text{image} \left\{ \bigcup_{c>0} M(a_n - c) \times M(b_n - c) \times M(c) \ \to \ M(a_n) \times M(b_n) \right\}.$$

These spaces fit into a push-out diagram

$$\begin{array}{ccc} R_n \times M & \longrightarrow & S_n \times M \\ \downarrow & & \downarrow \\ T_{n-1} & \longrightarrow & T_n \end{array} \tag{T.3.1}$$

This is precisely the set-up of [FG;1.3.i], establishing that $\mathcal{C}_{r+m}$ is tractable.

We now restrict the above picture to the submonoid $\mathcal{E}_r$. Set $M' = \mathcal{E}_r$ and $T' = M' \times M'$. For each $d$, let $M'(d) = M' \cap M(d)$ and define the filtration ${T'}_n$ of $T'$ by

$${T'}_n \stackrel{\text{def}}{=} \left[ \bigcup_{\nu(a,b) \leq n} M'(a) \times M'(b) \right] \cdot M'.$$

We set $S'_n = S_n \cap (M' \times M') = M'(a_n) \times M'(b_n)$ and $R'_n = R_n \cap (M' \times M')$ and observe that (1.2.2) restricts to a push-out diagram

$$\begin{array}{ccc} R'_n \times M' & \longrightarrow & S'_n \times M' \\ \big\downarrow & & \big\downarrow \\ T'_{n-1} & \longrightarrow & T'_n. \end{array}$$

Lemma T.1 implies that each $R'_n \subset S'_n$ is a cofibration, for $R'_n$ is a closed constructible subset of $S'_n$. Thus, $M' = \mathcal{E}_r$ is a tractable monoid.

To verify that $\mathcal{E}_r$ is tractable as an $\mathcal{F}_r$-space, we proceed exactly as in [FG;1.3.ii], replacing $\mathcal{C}_r(Y)$ and $\mathcal{C}_r(X)$ in that proof by $\mathcal{F}_r$ and $\mathcal{E}_r$ and appealing once again to Lemma T.1 to verify the cofibration condition. Namely, because each multiplication map $\mathcal{C}_{r,n-c} \times \mathcal{C}_{r,c} \to \mathcal{C}_{r,n}$ is proper, so is its restriction $\mathcal{E}_{r,n-c} \times \mathcal{E}_{r,c} \to \mathcal{C}_{r,n}$. Thus, the image of each $\mathcal{E}_{r,n-c} \times \mathcal{F}_{r,c}$ in $\mathcal{E}_{r,n}$ is closed and constructible, thereby implying that

$$image\,\{\cup_{c>0} \mathcal{E}_{r,n-c} \times \mathcal{F}_{r,c} \to \mathcal{E}_{r,n}\} \quad \subset \quad \mathcal{E}_{r,n}$$

is a cofibration by Lemma T.1. This is precisely the cofibration condition necessary for that proof.

The same changes, this time to [FG;1.3.iii], imply that $\mathcal{E}_r/\mathcal{F}_r$ is also a tractable monoid. $\square$

**Corollary T.4.** If $\mathcal{E}_r \subset \mathcal{C}_r(X \times Y)$ is a submonoid whose embedding is constructible, then the natural homotopy class of maps of H-spaces

$$[\mathcal{E}_r]^+ \quad \longrightarrow \Omega B[\mathcal{E}_r]$$

is a weak homotopy equivalence, where $[\mathcal{E}_r]^+$ is the naive group completion of the abelian topological monoid $\mathcal{E}_r$ and $B[\mathcal{E}_r]$ is its classifying space.

In particular, there is a natural weak homotopy equivalence

$$\mathcal{Z}_r(Y)(X) \quad \longrightarrow \quad \Omega B[\mathcal{C}_r(Y)(X)],$$

where $\mathcal{Z}_r(Y)(X) \overset{\text{def}}{=} [\mathcal{C}_r(Y)(X)]^+$.

**Proof.** In [FG;1.4], it is shown for any abelian tractable monoid $M$ with the canclellation property that

$$[\text{Sing.}(M)]^+ = \text{Sing.}(M \times M)/\text{Sing.}(M) \longrightarrow \text{Sing.}(M^+)$$

is a weak homotopy equivalence, where Sing.$(-)$ denotes the functor sending a space to its singular complex. On the other hand, by a theorem of D. Quillen (cf. [$\text{FM}_1$;AppQ]), $[\text{Sing.}(M)]^+$ is homotopy equivalent to the homotopy-theoretic group completion of the simplicial monoid Sing.$(M)$. $\square$

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