D-bar Sparks

by

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Abstract

A $\overline{\partial}$-analogue of differential characters for complex manifolds is introduced and studied using a new theory of homological spark complexes. Many essentially different spark complexes are shown to have isomorphic groups of spark classes. This has many consequences: It leads to an analytic representation of $\mathcal{O}^\times$-gerbes with connection, it yields a soft resolution of the sheaf $\mathcal{O}^\times$ by currents on the manifold, and more generally it gives a Dolbeault-Federer representation of Deligne cohomology as the cohomology of certain complexes of currents.

It is shown that the $\overline{\partial}$-spark classes $\widehat{\mathbf{H}}^*(X)$ carry a functorial ring structure. Holomorphic bundles have Chern classes in this theory which refine the integral classes and satisfy Whitney duality. A version of Bott vanishing for holomorphic foliations is proved in this context.

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Table of Contents

0. Introduction
1. Homological Spark Complexes
2. Quasi-Isomorphisms of Spark Complexes
3. Čech-Dolbeault Sparks
4. $\mathcal{O}^\times$-Gerbes with Connection
5. The Case of Degree 0
6. The Case of Degree 1
7. $\overline{\partial}$-Sparks
8. Čech-Dolbeault Hypersparks
9. A Current Resolution of the Sheaf $\mathcal{O}^\times$
10. Ring Structure
11. Functoriality
12. Refined Chern Classes for Holomorphic Bundles
13. Bott Vanishing for Holomorphic Foliations
14. Generalizations and the Relation to Deligne Cohomology
15. Theorems in the General Setting
Appendix A. Hypercohomology and Cone Complexes
Appendix B. Non-smoothness of Integrally Flat Currents

0. INTRODUCTION

In 1972 Jeff Cheeger and Jim Simons developed a theory of differential characters with applications to secondary characteristic invariants, conformal geometry, the theory of foliations, and much more [4], [23], [3], [5]. Over the intervening years the importance of this theory has steadily grown – it is now relevant to many areas of mathematics and mathematical physics. The purpose of this paper is to establish a "$\overline{\partial}$-analogue" of this theory with new applications to geometry and analysis. In fact for each integer $p \geq 0$ we establish a ring functor $\widehat{\mathbf{H}}^*(X, p)$ on the category of complex manifolds and holomorphic maps, with two natural transformations (graded ring homomorphisms)

$$\delta_1 : \widehat{\mathbf{H}}^{*-1}(X,p) \longrightarrow \mathcal{Z}^*_{\mathbb{Z}}(X,p) \qquad \text{and} \qquad \delta_2 : \widehat{\mathbf{H}}^{*-1}(X,p) \longrightarrow H^*(X;\mathbb{Z})$$

where $\mathcal{Z}^*_{\mathbb{Z}}(X,p)$ denote the closed differential forms, with certain integrality properties, in the truncated Dolbeault complex $\bigoplus_{j<p} \mathcal{E}^{j,*-j}(X)$. The kernel of δ_1 is Deligne cohomology ($\cong H^{*-1}(X, \mathcal{O}^\times)$ when $p = 1$). The kernel of δ_2 corresponds to classes with smooth representatives. These homomorphisms sit in an exact, functorial 3×3 grid.

For the sake of exposition we shall restrict attention primarily to the case $p = 1$, which is quite natural and already of considerable interest. The results and proofs for $p > 1$, which are strictly analogous, will be presented in §14.

We first introduce a number of distinct $\overline{\partial}$-spark complexes and examine them in detail in low degrees. One of these is related to $\overline{\partial}$-gerbes with connection, another is more analytical and concerns $\overline{\partial}$-equations relating smooth to rectifiable currents. All these complexes are shown to have isomorphic groups of spark classes $\widehat{\mathbf{H}}^*(X, 1)$. From this, one establishes the ring structure and functoriality. In certain presentations of the theory, there are quite explicit formulas for the product.

This theory is then applied to define refined Chern classes $\hat{d}_k(E) \in \widehat{\mathbf{H}}^{2k-1}(X, \mathcal{O}^\times)$ for holomorphic vector bundles E. These classes are constructed by using a hermitian metric and its associated canonical connection, but as in ordinary Chern-Weil theory, the result is independent of the choice. The associated *total class* $\hat{d}(E) = 1 + \hat{d}_1(E) + \hat{d}_2(E) + \dots$ satisfies the duality

$$\hat{d}(E \oplus F) \;=\; \hat{d}(E) * \hat{d}(F).$$

Furthermore, $\delta_2 \circ \hat{d}(E) = c(E) \in H^*(X;\mathbb{Z})$ is the ordinary total integral Chern class of E. These results, and their extension to $p > 1$, give a representation of Chern-Weil type for the Deligne characteristic classes of E.

In a subsequent section we consider holomorphic foliations and establish the following analogue of the Bott Vanishing Theorem.

Theorem. *Let N be a holomorphic bundle of rank q on a complex manifold X. If N is (isomorphic to) the normal bundle of a holomorphic foliation of X, then for every polynomial P of pure cohomology degree $k > 2q$, the associated refined Chern class satisfies*

$$P(\widehat{d}_1(N), ..., \widehat{d}_q(N)) \;\in\; H^{2k-1}(X;\mathbb{C}^\times) \subset H^{2k-1}(X;\mathcal{O}^\times)$$

A nice aspect of this theory is its natural presentation of Deligne cohomology in terms of forms and currents. In fact, we shall construct several different spark complexes, each yielding the same ring of spark classes containing Deligne cohomology. Hence, we also obtain several other geometric constructions of Deligne cohomology.

We note that the standard short exact sequence for the Deligne group $H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$ sits as the left column in our 3×3-grid. When X is compact Kaehler and $k = 2p$, this is the sequence: $0 \to \mathcal{J}_p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \mathrm{Hdg}^{p,p}(X) \to 0$ where $\mathcal{J}_p(X)$ is Griffiths' intermediate Jacobian. (See the diagrams following Proposition 14.4.) In our context it is trivial to see that every holomorphic cycle of codimension-p determines a class in $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$. Furthermore, one sees in a similar way that *every maximally complex cycle M of codimension $2p+1$ determines a class $[M] \in H^{2p+1}_{\mathcal{D}}(X, \mathbb{Z}(p))$, and if M bounds a holomorphic chain of (complex) codimension p, then $[M] = 0$* (cf. Prop. 14.6).

Recall that a powerful property of cohomology theory is its broad range of distinct formulations. Some presentations of the theory make it easy to compute, while others, such as de Rham theory, current theory or harmonic theory, lead to non-trivial assertions in analysis. Differential characters are similar in nature. There are many distinct formulations: some relatively simple and computable, and others rather more complicated, involving Čech-deRham complexes [19] or complexes of currents [20], [10], [14]. These latter approaches relate differential characters for example to refined characteristic classes for singular connections [16], [17], [22], to Morse Theory [18] and to harmonic theory.

In [19] the homological apparatus of *spark complexes* was introduced to establish the equivalence of the many approaches to differential characters. In §1 below, that apparatus is generalized to treat a wide range of situations. In particular, on a complex manifold one can replace the deRham component of differential character theory by the $\overline{\partial}$-complex of $(0,q)$-forms, or more generally, the deRham complex truncated at level p. The machine developed in §1 is of independent interest and applies to a broad range of interesting situations.

Using this machine we show that a variety of $\overline{\partial}$-spark complexes are equivalent and therefore lead to isomorphic groups of spark classes. Thus we are able to relate the classes of $\overline{\partial}$-gerbes with connection, which are defined within the Čech-Dolbeault double complex, to classes of currents satisfying a $\overline{\partial}$-spark equation of the form

$$\overline{\partial} a \;=\; \phi - \Psi(R)$$

where ϕ is a smooth $\overline{\partial}$-closed $(0,q)$-form and $\Psi(R)$ is the $(0,q)$-component of a rectifiable cycle R of codimension q. These two are related by a larger enveloping complex which contains them both.

Similar remarks apply to the case where one truncates at level $p > 1$.

Other geometrically motivated spark complexes will be studied in forthcoming papers.

1. HOMOLOGICAL SPARK COMPLEXES

We begin with a generalization of the homological algebra introduced in [19].

Definition 1.1. A **homological spark complex** is a triple of cochain complexes (F^*, E^*, I^*) together with morphisms

$$I^* \xrightarrow{\Psi} F^* \supset E^*$$

such that:

(i) $\Psi(I^k) \cap E^k = \{0\}$ for $k > 0$,

(ii) $H^*(E) \cong H^*(F)$, and

(iii) $\Psi : I^o \to F^0$ is injective.

Note that

$$\Psi(I^0) \cap E^0 \subset Z^0(E) = H^0(E) = H^0(F)$$

since for any $a \in \Psi(I^0) \cap E^0$, we have $da \in \Psi(I^1) \cap E^1 = \{0\}$

Definition 1.2. In a given spark complex (F^*, E^*, I^*) a **spark of degree** k is a pair

$$(a, r) \in F^k \oplus I^{k+1}$$

which satisfies the **spark equation**

(i) $da = e - \Psi(r)$ for some $e \in E^{k+1}$, and

(ii) $dr = 0$.

The group of sparks of degree k is denoted by $\mathcal{S}^k = \mathcal{S}^k(F^*, E^*, I^*)$. Note that by 1.1 (i)

(iii) $de = 0$.

Definition 1.3. Two sparks $(a, r), (a'r') \in \mathcal{S}^k(F^*, E^*, I^*)$ are **equivalent** if there exists a pair

$$(b, s) \in F^{k-1} \oplus I^k$$

(i) $a - a' = db + \Psi(s)$

(ii) $r - r' = -ds$.

The set of equivalence classes is called the **group of spark classes of degree** k associated to the given spark complex and will be denoted by $\widehat{\mathbf{H}}^k(F^*, E^*, I^*)$ or simply $\widehat{\mathbf{H}}^k$ when the complex in question is evident. Note that $\widehat{\mathbf{H}}^{-1} = H^0(I)$.

We now derive the fundamental exact sequences associated to a homological spark complex (F^*, E^*, I^*). Let $Z^k(E) = \{e \in E^k : de = 0\}$ and set

$$Z^k_I(E) \equiv \{e \in Z^k(E) \ : \ [e] = \Psi_*(\rho) \quad \text{for some} \quad \rho \in H^k(I)\} \tag{1.1}$$

where $[e]$ denotes the class of e in $H^k(E) \cong H^k(F)$.

Lemma 1.4. *There exist well-defined surjective homomorphisms:*

$$\widehat{\mathbf{H}}^k \xrightarrow{\delta_1} Z_I^{k+1}(E) \qquad \text{and} \qquad \widehat{\mathbf{H}}^k \xrightarrow{\delta_2} H^{k+1}(I) \tag{1.2}$$

given on any representing spark $(a,r) \in \mathcal{S}^k$ *by*

$$\delta_1(a,r) = e \qquad \text{and} \qquad \delta_2(a,r) = [r]$$

where $da = e - \Psi(r)$ *as in 1.2 (i).*

Proof. It is straightforward to see that these maps are well-defined. To see that δ_1 is surjective, consider $e \in Z_I^{k+1}(E)$. By definition there exists $r \in I^{k+1}$ with $dr = 0$ such that $[e] = \Psi_*[r] \in H^{k+1}(F) \cong H^{k+1}(E)$. Hence, there exists $a \in F^k$ with $da = e - \Psi(r)$ and $\delta_1(a,r) = e$ as desired. The map δ_2 is surjective because if $[r] \in H^{k+1}(I)$, then $\Psi(r)$ represents a class in $H^{k+1}(F) \cong H^{k+1}(E)$. Picking a representative $e \in E^{k+1}$ of this class yields a spark (a,r) with $da = e - \Psi(r)$. □

Lemma 1.5. *Let* $\widehat{\mathbf{H}}_E^k = \ker \delta_2$. *Then*

$$\widehat{\mathbf{H}}_E^k \;=\; E^k / Z_I^k(E)$$

Proof. Suppose $\alpha \in \widehat{\mathbf{H}}^k$ is represented by the spark (a,r). Then $da = e - \Psi(r)$ with $e \in E^{k+1}$, and $dr = 0$. Now $\delta_2 \alpha = 0$ means that $[r] = 0 \in H^{k+1}(I)$, i.e., $r = -ds$ for some $s \in I^k$. The equivalent spark $(a - \Psi(s), 0)$ satisfies $d(a - \Psi(s)) = e$. Since $H^*(E) \cong H^*(F)$, we know by [19, Lemma 1.5] that we can find $b \in F^{k-1}$ so that $a - \Psi(s) + db \in E^k$. This proves that each $\alpha \in \widehat{\mathbf{H}}_E^k$ has a representative of the form $(a,0)$ with $a \in E^k$. If $(a,0)$ is equivalent to 0, then $a = db + \Psi(s)$ for some $b \in F^{k-1}$ and some $s \in I^k$ with $ds = 0$. That is, $a \in Z_I^k(E)$. □

Definition 1.6. Associated to any spark complex (F^*, E^*, I^*) is the **cone complex** (G^*, D) defined by setting

$$\begin{aligned} G^k &\equiv F^k \oplus I^{k+1} \qquad\qquad k \geq -1 \\ D(a,r) &= (da + \Psi(r), -dr) \end{aligned} \tag{1.3}$$

Note that there is a short exact sequence of complexes

$$0 \;\to\; F^* \;\to\; G^* \;\to\; I^*(1) \;\to\; 0 \tag{1.4}$$

where $I^k(1) \equiv I^{k+1}$. The morphism Ψ defines a chain map of degree 1:

$$F^* \xleftarrow{\Psi} I^*(1)$$

which induces the connecting homomorphisms in the associated long exact sequence in cohomology.

Proposition 1.7. *There are two fundamental short exact sequences:*

$$
\begin{aligned}
0 \longrightarrow\ & H^k(G) \longrightarrow \widehat{\mathbf{H}}^k \xrightarrow{\delta_1} Z_I^{k+1}(E) \longrightarrow 0 \\
0 \longrightarrow\ & \widehat{\mathbf{H}}_E^k \longrightarrow \widehat{\mathbf{H}}^k \xrightarrow{\delta_2} H^{k+1}(I) \longrightarrow 0
\end{aligned}
\tag{1.5}
$$

Proof. This follows immediately from Lemmas 1.4, 1.5 and Definition 1.6. □

Consider the homomorphism $\Psi_* : H^k(I) \to H^k(F) \cong H^k(E)$, and define

$$H_I^k(E) \equiv \text{Image}\{\Psi_*\} \qquad \text{and} \qquad Ker^k(I) \equiv \ker\{\Psi_*\}$$

The exact sequences above fit into the following 3×3 commutative grid.

Proposition 1.8. *Associated to any spark complex (F^*, E^*, I^*) is the commutative diagram*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(E)}{H_I^k(E)} & \longrightarrow & \widehat{\mathbf{H}}_E^k & \longrightarrow & dE^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k(G) & \longrightarrow & \widehat{\mathbf{H}}^k & \xrightarrow{\delta_1} & Z_I^{k+1}(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_2} & & \downarrow & & \\
0 & \longrightarrow & Ker^{k+1}(I) & \longrightarrow & H^{k+1}(I) & \xrightarrow{\Psi_*} & H_I^{k+1}(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{1.6}
$$

whose rows and columns are exact.

2. QUASI-ISOMORPHISMS OF SPARK COMPLEXES

In this section we introduce a useful criterion for showing that two spark complexes have isomorphic groups of spark classes.

Definition 2.1. Two spark complexes (F^*, E^*, I^*) and $(\overline{F}^*, \overline{E}^*, \overline{I}^*)$ are **quasi-isomorphic** if there exists a commutative diagram of morphisms

$$
\begin{array}{ccccc}
\overline{I}^* & \xrightarrow{\overline{\Psi}} & \overline{F}^* & \supset & \overline{E}^* \\
{\scriptstyle \psi}\uparrow & & \cup & & \| \\
I^* & \xrightarrow{\Psi} & F^* & \supset & E^*
\end{array}
$$

inducing an isomorphism

$$\psi^* : H^*(I) \xrightarrow{\cong} H^*(\overline{I}) \tag{2.1}$$

Proposition 2.2. *A quasi-isomorphism of spark complexes (F^*, E^*, I^*) and $(\overline{F}^*, \overline{E}^*, \overline{I}^*)$ induces an isomorphism*

$$\widehat{\mathbf{H}}^*(F^*, E^*, I^*) \;\cong\; \widehat{\mathbf{H}}^*(\overline{F}^*, \overline{E}^*, \overline{I}^*)$$

of the associated groups of spark classes. In fact, it induces an isomorphism of the grids (1.6) associated to the two complexes.

Proof. There is evidently a mapping $\widehat{\mathbf{H}}^*(F^*, E^*, I^*) \to \widehat{\mathbf{H}}^*(\overline{F}^*, \overline{E}^*, \overline{I}^*)$. To see that it is onto, consider a spark $(\overline{a}, \overline{r}) \in \overline{\mathcal{S}}^k$ with $d\overline{r} = 0$ and $d\overline{a} = e - \overline{\Psi}(\overline{r})$ where $e \in \overline{E}^{k+1} = E^{k+1}$. By the RHS of (2.1) there exists an element $\overline{s} \in \overline{I}^k$ such that $\overline{r} = \psi(r) - d\overline{s}$ for some $r \in I^{k+1}$ with $dr = 0$. Hence, $d\overline{a} = e - \overline{\Psi}(\overline{r}) = e - \overline{\Psi}(\psi(r)) + \overline{\Psi}(d\overline{s}) = e - \Psi(r) + d\overline{\Psi}(\overline{s})$, and we have

$$d\{\overline{a} - \overline{\Psi}(\overline{s})\} \;=\; e - \Psi(r) \;\in\; F^{k+1}.$$

It follows (see [19, Lemma 1.5]) that there exists $\overline{b} \in \overline{F}^{k-1}$ such that

$$a \;\equiv\; \overline{a} - \overline{\Psi}(\overline{s}) + d\overline{b} \;\in\; F^k.$$

Consequently, $(\overline{a}, \overline{r})$ is equivalent to $(\overline{a} - \overline{\Psi}(\overline{s}) + d\overline{b}, \overline{r} + d\overline{s}) = (a, \psi(r))$ which is the image of the spark $(a, r) \in \mathcal{S}^k = \mathcal{S}^k(F^*, E^*, I^*)$, (note that $da = e - \Psi(r)$). Hence, the map is onto as claimed.

We now prove that the mapping is injective. Suppose that the image of $(a, r) \in \mathcal{S}^k$ is equivalent to 0 in $\overline{\mathcal{S}}^k$. This means that there exists a pair $(\overline{b}, \overline{s}) \in \overline{F}^{k-1} \oplus \overline{I}^k$ such that

$$a \;=\; d\overline{b} - \overline{\Psi}(\overline{s}) \qquad \text{and} \qquad \psi(r) \;=\; d\overline{s}. \tag{2.2}$$

Since $\psi_* : H^*(I) \to H^*(\overline{I})$ is an isomorphism , the RHS of (2.2) implies that there exists $s \in I^k$ such that $r = ds$ Hence (a, r) is equivalent in $\mathcal{S}^k$ to $(a', 0)$ where $a' = a + \Psi(s)$. The triviality condition (2.2) now becomes

$$a' \;=\; d\overline{b} - \overline{\Psi}(\overline{s}') \qquad \text{and} \qquad 0 \;=\; d\overline{s}'.$$

Again since $\psi_* : H^*(I) \to H^*(\overline{I})$ is an isomorphism, there exists $s' \in I^k$ with $ds' = 0$ such that $\psi(s') = \overline{s}' + d\overline{t}$ where $\overline{t} \in \overline{I}^{k-1}$. Hence, $\overline{\Psi}(\overline{s}') = \overline{\Psi}(\psi(s') - d\overline{t}) = \Psi(s') - d\overline{\Psi}(\overline{t})$, and we conclude that

$$a' + \Psi(s') \;=\; d(\overline{b}')$$

where $\overline{b}' = \overline{b} + \overline{\Psi}(\overline{t}) \in \overline{F}^{k-1}$. Since $H^*(\overline{F}) \cong H^*(F)$, there exists an element $b \in F^{k-1}$ with

$$a' + \Psi(s') \;=\; db.$$

Hence, $(a', 0)$ is equivalent in $\mathcal{S}^k$ to $(0, 0) = (a' + \Psi(s') - db, ds')$. □

Remark 2.3. Proposition 2.2 can be strengthened by replacing the inclusion $F^* \subset \overline{F}^*$ with a strong homological equivalence $F^* \to \overline{F}^*$ which induces an isomorphism $E^* \to \overline{E}^*$. A *strong homological equivalence* is a chain map whose kernel and cokernel are acyclic.

Proposition 2.2 unifies the diverse theories of sparks, gerbes, differential characters and holonomy maps (see [19]). In what follows we examine a "$\overline{\partial}$- analogue" of these objects.

3. ČECH-DOLBEAULT SPARKS

Suppose X is a complex manifold of dimension n, and let

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,2} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathcal{E}^{0,n}$$

denote the Dolbeault resolution of the sheaf $\mathcal{O} = \mathcal{O}_X$ of holomorphic functions on X by smooth $(0,q)$-forms. Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is a covering of X by Stein open sets such that all finite intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_N}$ are contractible, and consider the double Čech-Dolbeault complex

$$\begin{array}{ccccccc}
C^0(\mathcal{U}, \mathcal{E}^{0,n}) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{E}^{0,n}) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{E}^{0,n}) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & C^n(\mathcal{U}, \mathcal{E}^{0,n}) \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow \\
C^0(\mathcal{U}, \mathcal{E}^{0,2}) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{E}^{0,2}) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{E}^{0,2}) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & C^n(\mathcal{U}, \mathcal{E}^{0,2}) \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow \\
C^0(\mathcal{U}, \mathcal{E}^{0,1}) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{E}^{0,1}) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{E}^{0,1}) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & C^n(\mathcal{U}, \mathcal{E}^{0,1}) \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow \\
C^0(\mathcal{U}, \mathcal{E}^{0,0}) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{E}^{0,0}) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{E}^{0,0}) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & C^n(\mathcal{U}, \mathcal{E}^{0,0})
\end{array}$$

There are two edge complexes:

$$\{\ker(\delta) \text{ on the left column}\} \cong Z^0(\mathcal{U}, \mathcal{E}^{0,*}) \cong H^0(\mathcal{U}, \mathcal{E}^{0,*}) \cong \mathcal{E}^{0,*}(X) \tag{3.1}$$

the standard *smooth Dolbeault complex* with differential $\overline{\partial}$, and

$$\{\ker(\overline{\partial}) \text{ on the bottom row}\} \cong C^*(\mathcal{U}, \mathcal{O}) \tag{3.2}$$

the standard *Čech complex* with coefficients in the sheaf $\mathcal{O}$. There is also the *total complex* (F^*, D) where

$$F^k \equiv \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{E}^{0,q}) \qquad \text{and} \qquad D \equiv (-1)^p \delta + \overline{\partial} \tag{3.3}$$

Lemma 3.1. *The inclusions of the edge complexes*

$$\mathcal{E}^{0,*}(X) \subset F^* \qquad \text{and} \qquad C^*(\mathcal{U},\mathcal{O}) \subset F^*$$

each induce an isomorphism in cohomology.

Proof. There are two spectral sequences converging to $H^*(F)$ cf. [12, 4.8]. For the first one, ${}'E_1^{*,*} =$ the δ-cohomology of $C^*(\mathcal{U},\mathcal{E}^{0,*})$, which, by the fineness of the sheaves $\mathcal{E}^{0,*}$, reduces to the edge complex (3.1) in the far left column and zero elsewhere.

For the second sequence, ${}''E_1^{*,*} =$ the $\overline{\partial}$-cohomology of $C^*(\mathcal{U},\mathcal{E}^{0,*})$, which, since each $\mathcal{E}^{0,q}$ is fine and each $U_{\alpha_1}\cap\cdots\cap U_{\alpha_1}$ is Stein, reduces to the edge complex (3.2) in the bottom row and zero elsewhere. □

Associated to this double complex are many interesting spark complexes. The most basic is the following.

Definition 3.2. The *Čech-Dolbeault spark complex* is the homological spark complex (F^*,E^*,I^*) where F^* is defined by (3.3),

$$E^* = \mathcal{E}^{0,*}(X) \qquad \text{and} \qquad I^* \equiv C^*(\mathcal{U},\mathbb{Z}) \xrightarrow{\Psi} C^*(\mathcal{U},\mathcal{O})$$

and where Ψ is the chain map associated to the inclusion of the sheaf of locally constant integer-valued functions into $\mathcal{O}$. The groups of spark classes associated to this complex will be denoted by $\widehat{\mathbf{H}}^{0,*}(X)$.

Proposition 3.3. *The diagram (1.6) for the groups $\widehat{\mathbf{H}}^{0,*}(X)$ can be written as*

(3.4)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(X,\mathcal{O})}{H^k_{\mathbb{Z}}(X,\mathcal{O})} & \longrightarrow & \widehat{\mathbf{H}}^{0,k}_{\infty}(X) & \longrightarrow & \overline{\partial}\mathcal{E}^{0,k}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k(X,\mathcal{O}^\times) & \longrightarrow & \widehat{\mathbf{H}}^{0,k}(X) & \xrightarrow{\delta_1} & \mathcal{Z}^{0,k+1}_{\mathbb{Z}}(X) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle\delta_2}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \delta\{H^k(X,\mathcal{O}^\times)\} & \longrightarrow & H^{k+1}(X;\mathbb{Z}) & \longrightarrow & H^{k+1}_{\mathbb{Z}}(X,\mathcal{O}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $H^k_{\mathbb{Z}}(X,\mathcal{O}) = \text{Image}\{H^k(X;\mathbb{Z}) \to H^k(X,\mathcal{O})\}$ and $\delta : H^k(X,\mathcal{O}^\times) \to H^{k+1}(X;\mathbb{Z})$ is the coboundary map coming from the exponential sequence $0\to\mathbb{Z}\to\mathcal{O}\to\mathcal{O}^\times\to 0$ and where $\mathcal{Z}^{0,k}_{\mathbb{Z}}(X)$ denotes $\overline{\partial}$-closed $(0,k)$-forms representing classes in $H^k_{\mathbb{Z}}(X,\mathcal{O}) \cong H^{0,k}_{\mathbb{Z}}(X)$. The group $\widehat{\mathbf{H}}^{0,k}_{\infty}(X) \cong \mathcal{E}^{0,k}(X)/\mathcal{Z}^{0,k}_{\mathbb{Z}}(X)$ consists exactly of the spark classes representable by smooth $(0,k)$-forms.

Proof. This is essentially straightforward. The only real point is the identification of $H^*(G)$ in (1.6) with $H^*(X, \mathcal{O}^\times)$. This is done explicitly in Theorem 9.1.

Remark 3.4. If $\mathcal{U}'$ is another open cover having the same properties as $\mathcal{U}$, then the associated groups of Čech-Dolbeault spark classes and 3×3 diagrams are isomorphic. This follows from the fact that if $\mathcal{U}'$ is also a refinement of $\mathcal{U}$, the associated Čech-Dolbeault spark complexes are quasi-isomorphic.

Remark 3.5. In the above discussion one can replace $\mathbb{Z}$ with any subring $\Lambda \subset \mathbb{C}$. In this case the sheaf $\mathcal{O}^\times$ is replaced by $\mathcal{O}/\Lambda$

4. $\mathcal{O}^\times$-GERBES WITH CONNECTION

To each Čech-Dolbeault spark one can associate a slightly more geometric object called an "$\mathcal{O}^\times$-grundle"or a (generalized) "$\mathcal{O}^\times$-gerbe with connection". Under the correspondence, spark equivalence becomes a certain "gauge equivalence", so the groups of spark classes discussed in §3 become gauge equivalence classes of $\mathcal{O}^\times$-gerbes with connection. These objects have very nice interpretations in low dimensions.

An "$\mathcal{O}^\times$-grundle" of degree k is obtained from a Čech-Dolbeault spark A of degree k by replacing the bottom component $A^{k,0}$ with its exponential

$$g_{\alpha_1 \dots \alpha_{k+1}} \;=\; e^{2\pi i A_{\alpha_1 \dots \alpha_{k+1}}}$$

Therefore, $g \in C^k(\mathcal{U}, \mathcal{E}^\times)$ where $\mathcal{E}^\times$ denotes the sheaf of smooth $\mathbb{C}^*$-valued functions. The bottom component of the spark equation:

$$\delta A^{k,0} \;=\; r$$

where $r_{\alpha_1 \dots \alpha_{k+1}} \in \mathbb{Z}$, implies that g is a cocycle, that is

$$\delta g \;=\; 0.$$

Definition 4.1. An **$\mathcal{O}^\times$-grundle of degree** k is a pair (A, g) where $g \in C^k(\mathcal{U}, \mathcal{E}^\times)$ satisfies $\delta g = 0$ and $A \in \bigoplus_{p+q=k, q>0} C^p(\mathcal{U}, \mathcal{E}^{0,q})$ satisfies

$$\begin{aligned}
\overline{\partial} A^{0,k} &= \varphi \in \mathcal{E}^{0,k+1}(X) \\
\overline{\partial} A^{1,k-1} + (-1)^k \delta A^{0,k} &= 0 \\
\overline{\partial} A^{2,k-2} + (-1)^{k-1} \delta A^{1,k-1} &= 0 \\
&\vdots \\
\overline{\partial} A^{k-1,1} + \delta A^{k-2,2} &= 0 \\
\frac{1}{2\pi i} \frac{\overline{\partial} g}{g} - \delta A^{k-1,1} &= 0
\end{aligned}$$

Definition 4.2. Two $\mathcal{O}^\times$-grundles $(A, g), (\widetilde{A}, \widetilde{g})$ of degree k are **gauge equivalent** if there exists a pair (B, h) where $h \in C^{k-1}(\mathcal{U}, \mathcal{E}^\times)$ satisfies

$$\delta h = g\widetilde{g}^{-1}$$

and $B \in \bigoplus_{p+q=k-1, q>0} C^p(\mathcal{U}, \mathcal{E}^{0,q})$ satisfies

$$\begin{aligned} A^{0,k} - \widetilde{A}^{0,k} &= \overline{\partial} B^{0,k-1} \\ A^{1,k-1} - \widetilde{A}^{1,k-1} &= \overline{\partial} B^{1,k-2} + (-1)^{k-1} \delta B^{0,k-1} \\ A^{2,k-2} - \widetilde{A}^{2,k-2} &= \overline{\partial} B^{2,k-3} + (-1)^{k-2} \delta B^{1,k-2} \\ &\vdots \\ A^{k-1,1} - \widetilde{A}^{k-1,1} &= \frac{1}{2\pi i} \frac{\overline{\partial} h}{h} - \delta B^{k-2,1} \end{aligned}$$

The group of gauge equivalence classes of $\mathcal{O}^\times$-grundles is denoted by $\widehat{\mathbf{H}}^{0,k}_{\text{grundle}}(X)$. The following proposition is straightforward to check.

Proposition 4.3. *The correspondence above induces an isomorphism*

$$\widehat{\mathbf{H}}^{0,k}(X) \;\cong\; \widehat{\mathbf{H}}^{0,k}_{\text{grundle}}(X)$$

In the next two sections we examine these groups $\widehat{\mathbf{H}}^{0,k}(X)$ in low degrees where they have interesting interpretations analogous to those of differential characters [4], [20].

5. THE CASE OF DEGREE 0

From Proposition 4.3 one immediately deduces the isomorphism

$$\widehat{\mathbf{H}}^{0,0}(X) \;\cong\; \mathrm{Map}(X, \mathbb{C}^\times)$$

with the space of C^∞-maps to $\mathbb{C}^\times$. The two fundamental exact sequences (1.5) become

$$\begin{aligned} 0 \;\to\; \mathcal{O}^\times(X) \;\to\; \mathrm{Map}(X, \mathbb{C}^\times) \;\xrightarrow{\overline{\partial}}\; \mathcal{Z}^{0,1}_{\mathbb{Z}}(X) \;\to\; 0 \\ 0 \;\to\; \mathrm{Map}(X, \mathbb{C})/\mathbb{Z} \;\xrightarrow{\exp}\; \mathrm{Map}(X, \mathbb{C}^\times) \;\to\; H^1(X; \mathbb{Z}) \;\to\; 0 \end{aligned}$$

6. THE CASE OF DEGREE 1

From Proposition 4.3 one sees that an element of $\widehat{\mathbf{H}}^{0,1}(X)$ is represented by a pair (A, g) where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ is a Čech 1-cocycle, and $A_\alpha \in \mathcal{E}^{0,1}(U_\alpha)$ are (0,1)-forms satisfying

$$A_\alpha - A_\beta \;=\; \frac{1}{2\pi i} \frac{\overline{\partial} g_{\alpha\beta}}{g_{\alpha\beta}} \qquad \text{in } U_\alpha \cap U_\beta.$$

Thus g gives the data for a complex line bundle L on X and A represents the (0,1)-part of a connection on L. These objects have intrinsic interest.

Definition 6.1. Let $L \to X$ be a smooth complex line bundle on a complex manifold X. Then a **$\overline{\partial}$-connection** on L is a linear mapping

$$\overline{\partial}_A : \mathcal{E}^{0,0}(X, L) \longrightarrow \mathcal{E}^{0,1}(X, L)$$

from smooth sections of L to smooth (0,1)-forms with values in L such that

$$\overline{\partial}_A(f\sigma) = (\overline{\partial} f) \otimes \sigma + f\overline{\partial}_A(\sigma)$$

for all $f \in C^\infty(X)$ and $\sigma \in \mathcal{E}^{0,0}(X, L)$. Two $\overline{\partial}$-connections $\overline{\partial}_A$, $\overline{\partial}_B$ are said to be **gauge equivalent** if $\overline{\partial}_B = g \circ \overline{\partial}_A \circ g^{-1}$ for some bundle isomorphism $g : L \to L$.

A $\overline{\partial}$-connection extends naturally to define a Dolbeault sequence

$$\mathcal{E}^{0,0}(X, L) \xrightarrow{\overline{\partial}_A} \mathcal{E}^{0,1}(X, L) \xrightarrow{\overline{\partial}_A} \mathcal{E}^{0,2}(X, L) \xrightarrow{\overline{\partial}_A} \ldots \xrightarrow{\overline{\partial}_A} \mathcal{E}^{0,n}(X, L) \tag{6.1}$$

for non-holomorphic bundles L. In general $\overline{\partial}_A^2$ is not zero. However, on a section $\sigma \in \mathcal{E}^{0,0}(X, L)$ one has

$$\overline{\partial}_A^2 \sigma = \varphi_A \otimes \sigma$$

for $\varphi_A \in \mathcal{E}^{0,2}(X)$. The form φ_A is called the **$\overline{\partial}$-curvature** of the connection. It has the following properties.

Proposition 6.2.

(1) *φ_A depends only on the gauge equivalence class of the $\overline{\partial}$-connection.*
(2) *L admits a $\overline{\partial}$-connection with curvature $\varphi_A \equiv 0$ if and only if L is (smoothly) equivalent to a holomorphic line bundle. In fact each $\overline{\partial}_A$ with $\varphi_A \equiv 0$ determines a unique holomorphic structure on L.*

Proof. The first assertion is a calculation. The second is a standard consequence of the Newlander-Nirenberg Theorem.

Proposition 6.3. *Let X be a complex manifold. Then there is a natural isomorphism:*

$$\widehat{\mathbf{H}}^{0,1}(X) \cong \frac{\{\text{ smooth complex line bundles with } \overline{\partial}\text{-connection on } X\}}{\text{gauge equivalence}}$$

Proof. Exercise.

Note that the $\overline{\partial}$-spark equation in this case can be written

$$\overline{\partial}(A, g) = \varphi_A - C_1(L) \tag{6.2}$$

where φ_A is the curvature of the $\overline{\partial}$-connection and $C_1(L)$ is a Čech representative of the first Chern class of the line bundle L. In this case the two fundamental exact sequences (1.5) become:

$$0 \to H^1(X, \mathcal{O}^\times) \to \widehat{\mathbf{H}}^{0,1}(X) \xrightarrow{\varphi_A} \mathcal{Z}^{0,2}_{\mathbb{Z}}(X) \to 0$$

$$0 \to \widehat{\mathbf{H}}^{0,1}_{\text{triv}}(X) \to \widehat{\mathbf{H}}^{0,1}(X) \xrightarrow{c_1} H^2(X; \mathbb{Z}) \to 0$$

where $\widehat{\mathbf{H}}^{0,1}_{\text{triv}}(X) \subset \widehat{\mathbf{H}}^{0,1}(X)$ is the subgroup where the line bundles are topologically trivial.

7. $\overline{\partial}$-SPARKS

Suppose X is a complex manifold of dimension n. Let $\mathcal{E}^{p,q}(X)$ denote the smooth forms of bidegree (p,q) on X and $\mathcal{D}'^{p,q}(X) \supset \mathcal{E}^{p,q}(X)$ the *generalized forms* or *currents* of bidegree (p,q) on X. (Recall that by definition $\mathcal{D}'^{p,q}(X)$ is the topological dual space to $\mathcal{D}^{n-p,n-q}(X)$ with the C^∞-topology.) The standard Dolbeault decomposition induces a decomposition

$$\mathcal{D}'^{k}(X) \;=\; \bigoplus_{p+q=k} \mathcal{D}'^{p,q}(X) \tag{7.1}$$

of the currents of degree k on X.

Let $\mathcal{R}^k(X) \subset \mathcal{D}'^{k}(X)$ denote the group of locally rectifiable currents of degree k (and dimension $2n-k$) on X (cf. [8]). A current $T \in \mathcal{D}'^{k}(X)$ is called locally **integral** if $T \in \mathcal{R}^k(X)$ and $dT \in \mathcal{R}^{k+1}(X)$. The group of locally integral currents of degree k on X is denoted $\mathcal{I}^k(X)$. The complex of sheaves $0 \to \mathbb{Z} \to \mathcal{I}^*$ is a fine resolution of $\mathbb{Z}$ and one has a natural isomorphism

$$H^*(\mathcal{I}) \;\cong\; H^*(X;\mathbb{Z}) \tag{7.2}$$

For compact manifolds X this is a basic result of Federer and Fleming [9], [8, 4.4.5].

Note 7.1. In all that follows, one could replace the complex $\mathcal{I}^*(X)$ with the subcomplex $\mathcal{I}^*_{\text{deR}}(X)$ of de Rham *integral chain currents*, which are defined by integration over (locally finite) C^∞ singular chains with $\mathbb{Z}$-coefficients (cf. [7]). This follows from Proposition 2.2 and the fact that $H^*(\mathcal{I}^*_{\text{deR}}(X)) \;\cong\; H^*(X;\mathbb{Z})$. Readers unfamiliar with integral currents may want to replace them with integral chain currents.

Definition 7.2. By the **$\overline{\partial}$-spark complex** on X we mean the triple (F^*, E^*, I^*) where

$$\begin{aligned} F^* &\equiv \mathcal{D}'^{0,*}(X) \\ E^* &\equiv \mathcal{E}^{0,*}(X) \\ I^* &\equiv \mathcal{I}^*(X) \end{aligned}$$

with

$$\mathcal{E}^{0,*}(X) \;\subset\; \mathcal{D}'^{0,*}(X) \qquad \text{and} \qquad \Psi : \mathcal{I}^*(X) \;\to\; \mathcal{D}'^{0,*}(X)$$

where the inclusion that of smooth forms into the space of forms with distribution coefficients, and where for $r \in \mathcal{I}^k(X)$, we have $\Psi(r) \equiv r^{0,k}$, the $(0,k)^{\text{th}}$ component of r in the decomposition (7.1). The isomorphism $H^*(E) \cong H^*(F)$ is standard. The fact that $E^k \cap \Psi(I^k) = \{0\}$ for $k > 0$ is proved in Appendix B.

Thus a **$\overline{\partial}$-spark of degree** k is a pair $(a,r) \in \mathcal{D}'^{0,k}(X) \oplus \mathcal{I}^{k+1}(X)$ such that $dr = 0$ and a satisfies the **$\overline{\partial}$-spark equation:**

$$\overline{\partial} a \;=\; \varphi - r^{0,k+1} \tag{7.3}$$

The set of these will be denoted by $\mathcal{S}^{0,k}(X)$. A $\overline{\partial}$-spark $(a,r) \in \mathcal{S}^{0,k}(X)$ is equivalent to zero if there exits $(b,s) \in \mathcal{D}'^{0,k-1}(X) \oplus \mathcal{I}^k(X)$ such that

$$a \;=\; \overline{\partial} b + s^{0,k+1} \qquad \text{and} \qquad r \;=\; -ds \tag{7.4}$$

The set of spark classes is denoted by

$$\widehat{\mathbf{H}}^{0,k}_{\overline{\partial}-\text{spark}}(X) \;\equiv\; \mathcal{S}^{0,k}(X)/\sim .$$

8. ČECH-DOLBEAULT HYPERSPARKS

Suppose X is a complex manifold of dimension n, and let

$$0 \;\to\; \mathcal{O} \;\to\; \mathcal{D}'^{0,0} \;\xrightarrow{\overline{\partial}}\; \mathcal{D}'^{0,1} \;\xrightarrow{\overline{\partial}}\; \mathcal{D}'^{0,2} \;\xrightarrow{\overline{\partial}}\; \dots \;\xrightarrow{\overline{\partial}}\; \mathcal{D}'^{0,n}$$

denote the Serre resolution of the sheaf $\mathcal{O} = \mathcal{O}_X$ by generalized $(0,q)$-forms. Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ is a covering of X as in §3, and consider the double Čech-Serre complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
C^0(\mathcal{U},\mathcal{D}'^{0,2}) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}'^{0,2}) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}'^{0,2}) & \xrightarrow{\delta} & \dots \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
C^0(\mathcal{U},\mathcal{D}'^{0,1}) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}'^{0,1}) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}'^{0,1}) & \xrightarrow{\delta} & \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
C^0(\mathcal{U},\mathcal{D}'^{0,0}) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}'^{0,0}) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}'^{0,0}) & \xrightarrow{\delta} & \dots
\end{array}$$

There are two edge complexes:

$$\begin{gathered}
\{\ker(\delta) \text{ on the left column}\} \;\cong\; Z^0(\mathcal{U},\mathcal{D}'^{0,*}) \;\cong\; \mathcal{D}'^{0,*}(X) \;\supset\; \mathcal{E}^{0,*}(X), \\
\{\ker(\overline{\partial}) \text{ on the bottom row}\} \;\cong\; C^*(\mathcal{U},\mathcal{O})
\end{gathered}$$

where in the second line we are using the standard regularity results for $\overline{\partial}$. Consider the *total complex* $(\overline{F}^*, D)$ where

$$\overline{F}^k \;\equiv\; \bigoplus_{p+q=k} C^p(\mathcal{U},\mathcal{D}'^{0,q}) \qquad \text{and} \qquad D \;\equiv\; (-1)^p\delta + \overline{\partial} \tag{8.1}$$

Lemma 8.1. *The inclusions of the edge complexes*

$$\mathcal{E}^{0,*}(X) \subset \overline{F}^* \qquad \text{and} \qquad C^*(\mathcal{U}, \mathcal{O}) \subset \overline{F}^*$$

each induce an isomorphism in cohomology.

Proof. The argument is essentially the same as the one given for Lemma 3.1. □

Let $\mathcal{I}^q$ denote the sheaf of germs of locally integrally flat currents of degree q on X. The standard boundary operator gives a resolution

$$0 \to \mathbb{Z} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \mathcal{I}^3 \to \dots$$

of the constant sheaf $\mathbb{Z}$ (cf. [21, Appendix A]). Consider the double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
{\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\
C^0(\mathcal{U}, \mathcal{I}^2) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{I}^2) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{I}^2) & \xrightarrow{\delta} & \dots \\
{\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\
C^0(\mathcal{U}, \mathcal{I}^1) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{I}^1) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{I}^1) & \xrightarrow{\delta} & \\
{\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\
C^0(\mathcal{U}, \mathcal{I}^0) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{I}^0) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{I}^0) & \xrightarrow{\delta} & \dots
\end{array}$$

There are two edge complexes:

$$\begin{aligned}
\{\ker(\delta) \text{ on the left column}\} &\cong Z^0(\mathcal{U}, \mathcal{I}^*) \cong \mathcal{I}^*(X), \\
\{\ker(d) \text{ on the bottom row}\} &\cong C^*(\mathcal{U}, \mathbb{Z})
\end{aligned}$$

Consider the *total complex* $(\overline{I}^*, D)$ where

$$\overline{I}^k \equiv \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{I}^q) \qquad \text{and} \qquad D \equiv (-1)^p \delta + \overline{\partial} \tag{8.2}$$

Lemma 8.2. *The inclusions of the edge complexes* $\mathcal{I}^*(X) \subset \overline{I}^*$ *and* $C^*(\mathcal{U}, \mathbb{Z}) \subset \overline{I}^*$ *induce isomorphisms*

$$H^*(\mathcal{I}^*(X)) \cong H^*(C^*(\mathcal{U}, \mathbb{Z})) \cong H^*(\overline{I}^*) \cong H^*(X; \mathbb{Z}).$$

Proof. This follows as before since the sheaves $\mathcal{I}^q$ are acyclic. (See [21, Appendix A].) □

Inclusion $\mathcal{I}^q \subset \mathcal{D}'^q$ followed by projection $\mathcal{D}'^q \to \mathcal{D}'^{0,q}$ gives a morphism of sheaves $\mathcal{I}^q \to \mathcal{D}'^{0,q}$ which induces a mapping of double complexes

$$\Psi : C^p(\mathcal{U}, \mathcal{I}^q) \to C^p(\mathcal{U}, \mathcal{D}'^{0,q}) \tag{8.3}$$

Definition 8.3. By the *Čech-Dolbeault hyperspark complex* on X we mean the homological spark complex $(\overline{F}^*, \overline{E}^*, \overline{I}^*)$ where $\overline{E}^* \equiv \mathcal{E}^{0,*}(X) \subset \overline{F}^*$ is given as in Lemma 8.1 and $\Psi : \overline{I}^* \to \overline{F}^*$ is defined by (8.3). The associated group of spark classes will be denoted by $\widehat{\mathbf{H}}^{0,k}_{\text{hyperspark}}(X)$

Theorem 8.4. *The Čech-Dolbeault hyperspark complex is quasi-isomorphic to the (smooth) Čech-Dolbeault spark complex and also to the (analytic) $\overline{\partial}$-spark complex of §8. Hence there are natural isomorphisms*

$$\widehat{\mathbf{H}}^{0,k}(X) \;\cong\; \widehat{\mathbf{H}}^{0,k}_{\overline{\partial}-\text{spark}}(X) \;\cong\; \widehat{\mathbf{H}}^{0,k}_{\text{hyperspark}}(X)$$

Proof. Let (F^*, E^*, I^*) denote the Čech-Dolbeault spark complex defined in 3.2. The natural inclusion $\mathcal{E}^{0,*} \subset \mathcal{D}'^{0,*}$ of the smooth forms into the generalized forms gives an inclusion $F^* \to \overline{F}^*$ which is the identity on E^*. We define a mapping $I^* \to \overline{I}^*$ by the inclusions

$$I^* \equiv C^*(\mathcal{U}, \mathbb{Z}) \;\subset\; C^*(\mathcal{U}, \mathcal{I}^0) \;\subset\; C^*(\mathcal{U}, \mathcal{I}^*) \equiv \overline{I}^*.$$

This commutes with the maps Ψ, and by Lemma 8.2 it induces an isomorphism $H^*(I) \cong H^*(\overline{I})$. This establishes the first assertion.

Now let (F^*, E^*, I^*) denote the $\overline{\partial}$-spark complex defined in 7.2, and consider the inclusion

$$F^k = \mathcal{D}'^{0,k}(X) \;=\; Z^0(\mathcal{U}, \mathcal{D}'^{0,k}) \;\subset\; \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{D}'^{0,q}) \;=\; \overline{F}^k$$

which is essentially the identity on E^*. One sees that under this inclusion $I^* = F^* \cap \overline{I}^*$ is the vertical edge complex of the double complex $\overline{I}^*$. Hence, by Lemma 8.2 one has that $H^*(I^*) \cong H^*(\overline{I}^*)$, and so this inclusion is a quasi-isomorphism as claimed. □

9. A CURRENT RESOLUTION OF THE SHEAF $\mathcal{O}^\times$

One of the first consequences of Theorem 8.4 is the following theorem of deRham-Federer type, giving an isomorphism between $H^*(X, \mathcal{O}^\times)$ and the cohomology of a certain complex of currents on a complex manifold X.

Consider the complex of sheaves on X:

$$0 \to \mathcal{G}^{-1} \to \mathcal{G}^0 \xrightarrow{\overline{D}} \mathcal{G}^1 \xrightarrow{\overline{D}} \mathcal{G}^2 \xrightarrow{\overline{D}} \dots \tag{9.1}$$

with

$$\mathcal{G}^q \;\equiv\; \mathcal{D}'^{0,q} \oplus \mathcal{I}^{q+1} \qquad \text{and} \qquad \overline{D}(a, r) \;\equiv\; (\overline{\partial} a + r^{0,q+1}, -dr).$$

where $\mathcal{D}'^{0,*}$ and $\mathcal{I}^*$ are as in §8. Let

$$G^q(X) \equiv \mathcal{D}'^{0,q}(X) \oplus \mathcal{I}^{q+1}(X)$$

be the group of global sections and consider the associated complex

$$G^{-1}(X) \xrightarrow{\overline{D}} G^0(X) \xrightarrow{\overline{D}} G^1(X) \xrightarrow{\overline{D}} G^2(X) \xrightarrow{\overline{D}} \dots \tag{9.2}$$

Note that $\mathcal{G}^{-1} = \mathcal{I}^0$ is the sheaf of locally integrable $\mathbb{Z}$-valued functions, and $\overline{D} : \mathcal{G}^{-1} \to \mathcal{G}^0 = \mathcal{D}'^0 \oplus \mathcal{I}^1$ is given by $\overline{D}(f) = (f, -df)$. We define

$$\widetilde{\mathcal{G}}^0 \;\equiv\; \mathcal{G}^0/\overline{D}\mathcal{G}^{-1}$$

to be the quotient sheaf and claim that the kernel of $\overline{D}$ on $\widetilde{\mathcal{G}}^0$ is the sheaf $\mathcal{O}^\times$. To see this fix a contractible open set U and consider an element $(a, r) \in G^0(U)$ with $\overline{D}(a, r) = (\overline{\partial}a + r^{0,1}, -dr) = (0, 0)$. Since $H^1(\mathcal{I}^*(U)) = H^1(U, \mathbb{Z}) = 0$, we have $r = df$ for $f : U \to \mathbb{Z}$ locally integrable. Hence $(a, r) \sim (a+f, r-df) = (\widetilde{a}, 0)$ and $\overline{\partial}\widetilde{a} = \bar{a}+[df]^{0,1} = \overline{\partial}a+r^{0,1} = 0$. Two such holomorphic functions $(\widetilde{a}, 0)$, $(\widetilde{a}', 0)$ are equivalent if they differ by a integer constant. Thus we obtain a complex of sheaves

$$0 \to \mathcal{O}^\times \to \widetilde{\mathcal{G}}^0 \xrightarrow{\overline{D}} \mathcal{G}^1 \xrightarrow{\overline{D}} \mathcal{G}^2 \xrightarrow{\overline{D}} \dots \tag{9.3}$$

Theorem 9.1. *The complex (9.3) is a soft resolution of the sheaf $\mathcal{O}^\times$, and there are natural isomorphisms*

$$H^q(G(X)) \;\cong\; H^q(X; \mathcal{O}^\times) \tag{9.4}$$

for all $q \geq 0$.

Proof. Fix $x \in X$ and consider the cofinal family of contractible Stein neighborhoods of x. For any such neighborhood U we have $H^q(G^*(U)) = 0$ for all $q > 0$. This results from the long exact sequence in cohomology associated to the short exact sequence of complexes: $0 \to \mathcal{D}'^{0,*}(U) \to G^*(U) \to I^{*+1}(U) \to 0$. Exactness at $q = 0$ was proved above. Since the sheaves $\mathcal{D}'^{0,*}$ and $\mathcal{I}$ are soft, so are the sheaves $\mathcal{G}^*$ and $\widetilde{\mathcal{G}}^0$. This easily implies (9.4), which can also be deduced from the quasi-equivalence of the complex (9.1) with the trivial complex $\mathcal{O}^\times$ (in degree 0). □

10. RING STRUCTURE

In this section we show that on any complex manifold X, the group $\widehat{\mathbf{H}}^{0,*}(X)$ carries the structure of a graded commutative ring with the property that the homomorphisms δ_1 and δ_2 in the fundamental exact sequences (3.4) are ring homomorphisms. To do this we shall use the $\overline{\partial}$-spark representation of $\widehat{\mathbf{H}}^{0,*}(X)$ and its analogue in the real case.

Recall from [20] (cf. [10] and [14]) that on any C^∞-manifold X the *differential characters* of degree k can be defined as the spark classes $\widehat{\mathbf{H}}^k(X)$ of the spark complex (F^*, E^*, I^*) where

$$F^* \equiv \mathcal{D}'^*(X), \qquad E^* \equiv \mathcal{E}^*(X), \qquad I^* \equiv \mathcal{I}^*(X). \tag{10.1}$$

There is a ring structure on $\widehat{\mathbf{H}}^*(X)$ defined at the spark level as follows. Fix $\alpha \in \widehat{\mathbf{H}}^k(X)$ and $\beta \in \widehat{\mathbf{H}}^\ell(X)$ and choose representatives $(a, r) \in \alpha$ and $(b, s) \in \beta$ with

$$da \;=\; \phi - r \qquad \text{and} \qquad db \;=\; \psi - s$$

where $a \wedge s$, $r \wedge b$ and $r \wedge s$ are all well defined and $r \wedge s \in \mathcal{I}^{k+\ell+2}(X)$. It is shown in [20] that this can always be done and that

$$\alpha * \beta \stackrel{\text{def}}{=} [a \wedge \psi + (-1)^{k+1} r \wedge b] = [a \wedge s + (-1)^{k+1} \phi \wedge b] \in \widehat{\mathbf{H}}^{k+\ell+1}(X) \tag{10.2}$$

defines a graded commutative ring structure on $\widehat{\mathbf{H}}^*(X)$ which coincides with the one defined by Cheeger in [3].

Note. Given a spark a $\in \mathcal{D}'^k(X)$ with da=ϕ-r where ϕ is smooth and r is integrally flat, the elements ϕ and r are uniquely determined by a (cf. [20]). For this reason we refer to the spark (a,r) simply by a.

Remark 10.1. In discussing differential characters and sparks the standard references only consider *real* differential forms and currents. However, if one replaces these with *complex* differential forms and currents and if one replaces $\mathbb{R}/\mathbb{Z}$ with $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$, the standard discussion remains valid. We shall assume that the forms and currents in (10.1) are complex.

Let us denote the $\overline{\partial}$-spark complex of §7 by $(F^{0,*}, E^{0,*}, I^*)$ where

$$F^{0,*} \equiv \mathcal{D}'^{0,*}(X), \qquad E^{0,*} \equiv \mathcal{E}^{0,*}(X), \qquad I^* \equiv \mathcal{I}^*(X)$$

Proposition 10.2. *The projection $\pi : \mathcal{D}'^*(X) \to \mathcal{D}'^{0,*}(X)$ determines a morphism of spark complexes $(F^*, E^*, I^*) \to (F^{0,*}, E^{0,*}, I^*)$ which induces a surjective additive homomorphism*

$$\Pi : \widehat{\mathbf{H}}^*(X) \to \widehat{\mathbf{H}}^{0,*}(X) \tag{10.3}$$

whose kernel is an ideal.

Proof. Note that for any $a \in \mathcal{D}'^k(X)$ one has

$$[da]^{0,k+1} = \overline{\partial}(a^{0,k}).$$

From this it is straightforward to see that

$$F^* \oplus E^* \oplus I^* \xrightarrow{\pi \oplus \pi \oplus \mathrm{Id}} F^{0,*} \oplus E^{0,*} \oplus I^*$$

is a map of complexes which commutes with structure maps: $I^* \to F^*$, $I^* \to F^{0,*}$, etc. in the definition of a spark complex. Consequently, the the induced map $(a, r) \mapsto (a^{0,q}, r)$ on sparks descends to a well-defined homomorphism $\Pi : \widehat{\mathbf{H}}^k(X) \to \widehat{\mathbf{H}}^{0,k}(X)$ as claimed.

To see that Π is surjective, consider a $\overline{\partial}$-spark $(A, r) \in F^{0,k} \oplus I^{k+1}$ with $\overline{\partial} A = \Phi - r^{0,k+1}$. By theorems of de Rham [7] there exist $\phi_0 \in E^{k+1}$ and $a_0 \in F^k$ such that $da_0 = \phi_0 - r$. (That is, there exists a smooth representative of the cohomology class of r in $H^*(F^*)$.) Let $A_0 = [a_0]^{0,k}$ and $\Phi_0 = [\phi_0]^{0,k+1}$ and note that $\overline{\partial} A_0 = \Phi_0 - r^{0,k+1}$. Hence, $\overline{\partial}(A - A_0) = \Phi - \Phi_0$ is a smooth form. It follows that there exist $b \in F^{0,k-1}$ and $\psi \in E^{0,k}$ with $A - A_0 = \psi + \overline{\partial} b$ (cf. [19, Lemma 1.5]). Set $a = a_0 + \psi + db$ and note that $da = (\phi_0 + d\psi) - r$. Hence, a is a spark of degree k and $a^{0,k} = A_0 + \psi + \overline{\partial} b = A$. Hence, the mapping Π is surjective as claimed.

We now invoke the following.

Lemma 10.3. *On* $\widehat{\mathbf{H}}^k(X)$, *one has that*

$$\ker(\Pi) \;=\; \{\alpha \in \widehat{\mathbf{H}}^k(X) : \exists (a,0) \in \alpha \text{ where } a \text{ is smooth and } a^{0,k} = 0\}$$

Proof. Clearly if $\alpha = [(a,0)]$ where $a^{0,k} = 0$, then $\Pi(\alpha) = 0$. Conversely, suppose $\alpha \in \ker(\Pi)$ and choose any spark $(a,r) \in \alpha$. Then $\Pi(\alpha) = 0$ means that there exist $b \in \mathcal{D}'^{0,k-1}(X)$ and $s \in \mathcal{I}^k(X)$ with $r = -ds$ and $a^{0,k} = -\overline{\partial} b + s^{0,k} = (-db+s)^{0,k}$. Replace (a,r) by the equivalent element $(\widetilde{a},0) \equiv (a+db-s, r+ds)$ and note that $\widetilde{a}^{0,k} = 0$. Repeated application of [19, Lemma 1.5] now shows that after modification by some $d\widetilde{b}$ we may assume that $\widetilde{a}$ is smooth. Specifically, since $\overline{\partial}\widetilde{a}^{1,k-1}$ is smooth, there exists $\widetilde{b}^{1,k-2}$ so that $\widetilde{a}^{1,k-1} + \overline{\partial}\widetilde{b}^{1,k-2}$ is smooth. Replacing $\widetilde{a}$ by $\widetilde{a} + d\widetilde{b}$ we can assume that $\widetilde{a}^{1,k-1}$ is smooth and $d\widetilde{a} = 0$. Thus $\overline{\partial}\widetilde{a}^{2,k-2} = -\partial\widetilde{a}^{1,k-1}$ is smooth, and by [19, Lemma 1.5] there exists $\widetilde{b}^{1,k-2}$ with $\widetilde{a}^{2,k-2} + \overline{\partial}\widetilde{b}^{1,k-2}$ smooth. Continuing inductively completes the proof. □

It remains to show that $\ker(\Pi)$ is an ideal. Fix $\alpha \in \ker(\Pi)$ and choose $(a,0) \in \alpha$ with $a^{0,k} = 0$ by Lemma 10.3. Let $\beta \in \widehat{\mathbf{H}}^\ell(X)$ be any class, and $(b,s) \in \beta$ any representative with spark equation $db = \psi - s$. Then by formula (10.2) the class $\alpha * \beta$ is represented by $(a \wedge \psi, 0)$ and $(a \wedge \psi)^{0,k+\ell+1} = a^{0,k} \wedge \psi^{0,\ell+1} = 0$. Hence, $\alpha * \beta \in \ker(\Pi)$, and so $\ker(\Pi)$ is a ideal as claimed. □

The projection Π is well behaved with respect to the fundamental exact sequences.

Proposition 10.4. *There are commutative diagrams*

$$\begin{array}{ccc} \widehat{\mathbf{H}}^k(X) & \xrightarrow{\delta_1} & \mathcal{Z}^{k+1}_{\mathbb{Z}}(X) \\ {\scriptstyle\Pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widehat{\mathbf{H}}^{0,k}(X) & \xrightarrow{\delta_1} & \mathcal{Z}^{0,k+1}_{\mathbb{Z}}(X) \end{array} \qquad\qquad \begin{array}{ccc} \widehat{\mathbf{H}}^k(X) & \xrightarrow{\delta_2} & H^{k+1}(X;\mathbb{Z}) \\ {\scriptstyle\Pi}\downarrow & & \downarrow{=} \\ \widehat{\mathbf{H}}^{0,k}(X) & \xrightarrow{\delta_2} & H^{k+1}(X;\mathbb{Z}) \end{array}$$

where π *denotes the projection. In fact the surjective homomorphism* Π *expands to a morphism of the fundamental diagram for* $\widehat{\mathbf{H}}^k(X)$ *([19]):*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{H^k(X,\mathbb{C})}{H^k_{\text{free}}(X;\mathbb{Z})} & \longrightarrow & \widehat{\mathbf{H}}^k_\infty(X) & \longrightarrow & d\mathcal{E}^k(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^k(X,\mathbb{C}^\times) & \longrightarrow & \widehat{\mathbf{H}}^k(X) & \xrightarrow{\delta_1} & \mathcal{Z}^{k+1}_{\mathbb{Z}}(X) & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle\delta_2}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^k_{\text{tor}}(X,\mathbb{Z}) & \longrightarrow & H^{k+1}(X;\mathbb{Z}) & \longrightarrow & H^{k+1}_{\text{free}}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{10.4}$$

to the diagram (3.4).

Proof. Fix $\alpha \in \widehat{\mathbf{H}}^{0,k}(X)$ and $\alpha' \in \widehat{\mathbf{H}}^{k}(X)$ with $\Pi(\alpha') = \alpha$. Choose a representative $(a, r) \in \alpha'$ with $da = \phi - r$ where $\phi \in \mathcal{Z}^{k+1}(X)$. Then $(\pi(a), r) = (a^{0,k}, r)$ represents α and we see that $\overline{\partial} a^{0,k} = [da]^{k+1} = \phi^{k+1} - r^{k+1}$. Thus $\delta_1 \alpha = \phi^{k+1} = \pi(\phi) = \pi(\delta_1 \alpha')$. The argument for δ_2 is similar. The assertion concerning the diagrams follows directly. □

This brings us to the main result.

Theorem 10.5. *For any complex manifold X the $\overline{\partial}$-spark classes $\widehat{\mathbf{H}}^{0,*}(X)$ carry a biadditive product*

$$* : \widehat{\mathbf{H}}^{0,k}(X) \times \widehat{\mathbf{H}}^{0,\ell}(X) \longrightarrow \widehat{\mathbf{H}}^{0,k+\ell+1}(X)$$

which makes $\widehat{\mathbf{H}}^{0,*}(X)$ into a graded commutative ring (commutative in the sense that

$$\alpha * \beta = (-1)^{k+\ell+1} \beta * \alpha$$

for $\alpha \in \widehat{\mathbf{H}}^{0,k}(X)$ and $\beta \in \widehat{\mathbf{H}}^{0,\ell}(X)$.) With respect to this ring structure the fundamental maps δ_1 and δ_2 are ring homomorphisms.

Proof. The first assertions follow immediately from Proposition 10.2. For the last assertion we apply Proposition 10.4 as follows. Suppose $\alpha \in \widehat{\mathbf{H}}^{0,k}(X)$ and $\beta \in \widehat{\mathbf{H}}^{0,\ell}(X)$ are given and choose $\alpha' \in \widehat{\mathbf{H}}^{k}(X)$ and $\beta' \in \widehat{\mathbf{H}}^{\ell}(X)$ with $\Pi(\alpha') = \alpha$ and $\Pi(\beta') = \beta$. If $\delta_1 \alpha' = \phi$ and $\delta_1 \beta' = \psi$, then $\delta_1(\alpha' * \beta') = \phi \wedge \psi$ and so $\delta_1(\alpha * \beta) = \delta_1 \Pi(\alpha' * \beta') = \pi \delta_1(\alpha' * \beta') = \pi(\phi \wedge \psi) = \pi(\phi) \wedge \pi(\psi) = \delta_1(\alpha) \wedge \delta_1(\beta)$. The argument for δ_2 is similar. □

Remark 10.5. The product on $\widehat{\mathbf{H}}^{0,*}(X)$ can be defined at the spark level as follows. Let $(a, r) \in \alpha \in \widehat{\mathbf{H}}^{0,k}(X)$ and $(b, s) \in \beta \in \widehat{\mathbf{H}}^{0,\ell}(X)$ be sparks with $\overline{\partial} a = \phi - r$ and $\overline{\partial} b = \psi - s$. Assume that the currents $a \wedge s^{0,\ell+1}$, $r^{0,k+1} \wedge b$ and $r \wedge s$ are well defined and that $r \wedge s$ is integrally flat. (The intersection theory for currents in [20] shows that such representatives exist,) Then the product $\alpha * \beta$ is represented by each of the following:

$$(a \wedge \psi + (-1)^{k+1} r^{0,k+1} \wedge b,\ r \wedge s) \qquad \text{and} \qquad (a \wedge s^{0,\ell+1} + (-1)^{k+1} \phi \wedge b,\ r \wedge s).$$

Corollary 10.6. The subgroup $H^*(X, \mathcal{O}^\times) = \ker \delta_1$ is an ideal in $\widehat{\mathbf{H}}^{0,*}(X)$. In particular, $H^*(X, \mathcal{O}^\times)$ has a ring structure compatible with the spark product. In this product the coboundary map

$$\delta : H^*(X, \mathcal{O}^\times) \longrightarrow H^*(X; \mathbb{Z})$$

from the sheaf sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$ is a ring homomorphism.

11. FUNCTORIALITY

The main result of this section is the following.

Theorem 11.1. *Any holomorphic map $f : Y \to X$ between complex manifolds induces a graded ring homomorphism*

$$f^* : \widehat{\mathbf{H}}^{0,*}(X) \to \widehat{\mathbf{H}}^{0,*}(Y)$$

with the property that if $g : Z \to Y$ is holomorphic, then $(f \circ g)^* = g^* \circ f^*$. In other words, $\widehat{\mathbf{H}}^{0,*}(\bullet)$ is a graded ring functor on the category of complex manifolds and holomorphic maps.

Proof. We know that f induces a ring homomorphism $f^* : \widehat{\mathbf{H}}^*(X) \to \widehat{\mathbf{H}}^*(Y)$ with the asserted property. We need only show that $f^* \ker(\Pi) \subset \ker(\Pi)$. This follows directly from the Lemma 10.3. □

12. REFINED CHERN CLASSES FOR HOLOMORPHIC BUNDLES

In this section we construct refined Chern classes $\widehat{d}_k(E) \in H^{2k-1}(X, \mathcal{O}^\times)$ for holomorphic bundles E which extend the tautological case $k = 1$. These classes possess the usual properties and map to the integral Chern classes under the coboundary map $\delta : H^{2k-1}(X, \mathcal{O}^\times) \to H^{2k}(X, \mathbb{Z})$. In fact they descend to holomorphic K-theory in the sense of Grothendieck.

These classes can also be accessed through Deligne cohomology, and our construction could be considered *a Chern-Weil approach to Deligne cohomology* in lowest degree. In [13] this is extended to the full Deligne theory.

Our point of departure is the fundamental work of Cheeger and Simons [4] who showed that for a smooth complex vector bundle $E \to X$ with unitary connection ∇ there exist refined Chern classes $\widehat{c}_k(E, \nabla) \in \widehat{\mathbf{H}}^{2k-1}(X)$ with

$$\delta_1(\widehat{c}_k(E,\nabla)) \;=\; c_k(\Omega^\nabla) \qquad \text{and} \qquad \delta_2(\widehat{c}_k(E,\nabla)) \;=\; c_k(E) \tag{12.1}$$

where $c_k(E)$ is the integral k^{th} Chern class and $c_k(\Omega^\nabla)$ is the Chern-Weil form representing $c_k(E) \otimes \mathbb{R}$ in the curvature of ∇ (cf. (10.4)). Setting $\widehat{c}(E) = 1 + \widehat{c}_1 + \widehat{c}_2 + \dots$ they show

$$\widehat{c}(E \oplus E', \nabla \oplus \nabla') \;=\; \widehat{c}(E,\nabla) * \widehat{c}(E',\nabla') \tag{12.2}$$

Whenever X is a complex manifold we can take the projections

$$\widehat{d}_k(E,\nabla) \;\equiv\; \Pi\{\widehat{c}_k(E,\nabla)\} \in \widehat{\mathbf{H}}^{0,2k-1}(X)$$

and note that by (12.1), (12.2) and Proposition 10.4

$$\delta_1(\widehat{d}_k(E,\nabla)) \;=\; c_k(\Omega^\nabla)^{0,2k} \qquad \text{and} \qquad \delta_2(\widehat{d}_k(E,\nabla)) \;=\; c_k(E) \tag{12.3}$$

and with $\widehat{d}(E,\nabla) \equiv 1 + \widehat{d}_1(E,\nabla) + \widehat{d}_2(E,\nabla) + \dots,$

$$\widehat{d}(E \oplus E', \nabla \oplus \nabla') \;=\; \widehat{d}(E,\nabla) * \widehat{d}(E',\nabla'). \tag{12.4}$$

Suppose now that E is holomorphic and is provided with a hermitian metric h. Let ∇ be the associated canonical hermitian connection. Then $\widehat{c}_k(E,\nabla)$ is of type (k,k) and so by (12.3) we have

$$\widehat{d}_k(E,\nabla) \;\in\; \ker(\delta_1) \;=\; H^{2k-1}(X, \mathcal{O}^\times) \tag{12.5}$$

Proposition 12.1. *The class in (12.5) is independent of the choice of hermitian metric.*

Proof. Let h_0, h_1 be hermitian metrics on E with canonical connections ∇^0, ∇^1 respectively. Then (see [4])

$$\widehat{c}_k(E, \nabla^1) - \widehat{c}_k(E, \nabla^0) \;=\; [T]$$

where $[T]$ is the differential character represented by the smooth transgression form

$$T \;=\; T(\nabla^1, \nabla^0) \;\equiv\; k \int_0^1 C_k(\nabla^1 - \nabla^0, \Omega_t, ..., \Omega_t)\, dt$$

where $C_k(X_1, ..., X_k)$ is the polarization of the k^{th} elementary symmetric function and where Ω_t is the curvature of the connection $\nabla^t \equiv t\nabla^1 - (1-t)\nabla^0$. Fix a local holomorphic frame field for E and let H_j be the hermitian matrix representing the metric h_j with respect to this trivialization. Then

$$\nabla^1 - \nabla^0 = \omega_1 - \omega_0 \qquad\qquad \text{where} \qquad \omega_j \equiv H_j^{-1} \partial H_j = \partial \log H_j.$$

In this framing, $\nabla^t = d + \omega_t$ where $\omega_t = t\omega_1 - (1-t)\omega_0$ and so its curvature $\Omega_t = d\omega_t - \omega_t \wedge \omega_t$ only has Hodge components of type $(1,1)$ and $(2,0)$. It follows that the Hodge components

$$T^{p,q} \;=\; 0 \qquad \text{for} \quad p < q. \tag{12.6}$$

In particular, $T^{0,2k-1} = 0$ and so $\Pi[T] = \widehat{d}_k(E, \nabla^1) - \widehat{d}_k(E, \nabla^0) = 0$. □

By Proposition 12.1 each holomorphic vector bundle E of rank k has a well defined total refined Chern class

$$\widehat{d}(E) \;=\; 1 + \widehat{d}_1(E) + \cdots + \widehat{d}_k(E) \;\in\; H^*(X, \mathcal{O}^\times)$$

Denote by $\mathcal{V}^k(X)$ the set of isomorphism classes of holomorphic vector bundles of rank k on X, and note that $\mathcal{V}(X) = \coprod_{k \geq 0} \mathcal{V}^k(X)$ is an additive monoid under Whitney sum.

Theorem 12.3. *On any complex manifold there is a natural transformation of functors*

$$\widehat{d} : \mathcal{V}(X) \;\longrightarrow\; H^*(X, \mathcal{O}^\times)$$

with the property that:

(i) $\widehat{d}(E \oplus F) \;=\; \widehat{d}(E) * \widehat{d}(F)$,

(ii) $\widehat{d} : \mathcal{V}^1(X) \xrightarrow{\approx} 1 + H^1(X, \mathcal{O}^\times)$ *is an isomorphism, and*

(iii) under the coboundary map $\delta : H^\ell(X, \mathcal{O}^\times) \to H^{\ell+1}(X, \mathbb{Z})$,

$$\delta \circ \widehat{d} \;=\; c \quad \textit{(the total integral Chern class).}$$

Proof. Property (i) follows from (12.4), Property (ii) is classical, and Property (iii) follows from (12.3). □

Property (i) implies that $\widehat{d}$ extends to the additive group completion

$$\mathcal{V}^+(X) \equiv \mathcal{V}(X) \times \mathcal{V}(X)/\sim \tag{12.7}$$

where $(E,F) \sim (E',F')$ iff there exists $G \in \mathcal{V}(X)$ with $E \oplus F' \oplus G \cong E' \oplus F \oplus G$. It is natural to ask is whether $\widehat{d}$ then descends to the Grothendieck quotient. It does.

Theorem 12.4. *For any short exact sequence of holomorphic vector bundles on X*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

one has

$$\widehat{d}(E) \;=\; \widehat{d}(E') * \widehat{d}(E'')$$

Proof. Choose hermitian metrics h' and h'' for E' and E'' respectively, and choose a C^∞-splitting

$$0 \longrightarrow E' \overset{i}{\longrightarrow} E \underset{\sigma}{\overset{\pi}{\rightleftarrows}} E'' \longrightarrow 0.$$

Define a hermitian metric $h = h' \oplus h''$ on E via the smooth isomorphism

$$(i,\sigma) : E' \oplus E'' \longrightarrow E.$$

We must show that the $\overline{\partial}$-spark classes for the direct sum of the canonical hermitian connections on $E' \oplus E''$ agree with the spark classes for the canonical hermitian connection on E. For this it will suffice to work over an open set $U \subset X$ on which there exist holomorphic framings

$$(e'_1,...,e'_n) \quad \text{for} \quad E' \qquad \text{and} \qquad (e''_1,...,e''_m) \quad \text{for} \quad E''$$

We now choose two framings form E over U.
Framing 1 (holomorphic):

$$(e_1,...,e_{n+m}) \;=\; (e'_1,...,e'_n,\widehat{e}''_1,...,\widehat{e}''_m)$$

where each $\widehat{e}''_k$ is a holomorphic lift of e''_k.
Framing 2 (smooth and direct-sum compatible):

$$(\widetilde{e}_1,...,\widetilde{e}_{n+m}) \;=\; (e'_1,...,e'_n,\sigma e''_1,...,\sigma e''_m).$$

Let H' and H'' be the smooth hermitian-matrix-valued functions representing the metric h' and h'' in their respective holomorphic frames over U. Then the connection 1-form for the direct sum connection on E in framing 2 is

$$\omega \;=\; \omega' \oplus \omega'' \qquad \text{where} \quad \omega' \;=\; \partial H' \cdot (H')^{-1} \;\text{ and }\; \omega'' \;=\; \partial H'' \cdot (H'')^{-1}.$$

In particular in framing 2 this connection form is of type (1,0).

Now let H be the smooth hermitian-matrix-valued function representing the metric h in the holomorphic framing 1 of E over U. In this framing the connection 1-form of the canonical hermitian connection on E is

$$\theta \;=\; \partial H \cdot H^{-1}.$$

We want to compute the connection 1-form $\widetilde{\theta}$ for this connection in the second framing $(\widetilde{e}_1, ..., \widetilde{e}_{n+m})$. For this we consider the change of framing $\widetilde{e}_k = \sum_{\ell=1}^{n+m} g_{k\ell} e_\ell$ and recall (cf. [11, p. 72]) that

$$\widetilde{\theta} \;=\; dg \cdot g^{-1} + g \cdot \theta \cdot g^{-1} \tag{12.8}$$

where $g = (g_{k\ell})$. We now observe that $g_{k\ell} = \delta_{k\ell}$ for $1 \le k \le n$ and so g has the form

$$g \;=\; \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \qquad \text{and} \qquad g^{-1} \;=\; \begin{pmatrix} I & 0 \\ -A & I \end{pmatrix}.$$

In particular we have that

$$dg \cdot g^{-1} \;=\; dg \cdot g \;=\; g \cdot dg \;=\; dg \;=\; \begin{pmatrix} 0 & 0 \\ dA & 0 \end{pmatrix} \tag{12.9}$$

Suppose now that Φ is any Ad-invariant symmetric k-multilinear function on the Lie algebra of $GL_{n+m}(\mathbb{C})$. Then the two given connections on E give rise to two Cheeger-Simons differential characters, say $\widehat{\Phi}_0$ and $\widehat{\Phi}_1$, and the difference

$$\widehat{\Phi}_1 - \widehat{\Phi}_0 \;=\; [T]$$

where $[T]$ is the character associated to the smooth differential form

$$T \;=\; k \int_0^1 \Phi(\widetilde{\theta} - \omega, \Omega_t, ..., \Omega_t)\, dt.$$

where Ω_t is the curvature 2-form of the connection $\omega_t \equiv (1-t)\widetilde{\theta} + t\omega$. To compute the corresponding difference in $\overline{\partial}$-characters we take the projection onto $(0, 2k-1)$-forms

$$\begin{aligned} T^{0,2k-1} \;&=\; k \int_0^1 \Phi(\widetilde{\theta} - \omega, \Omega_t, ..., \Omega_t)^{0,2k-1}\, dt \\ &= k \int_0^1 \Phi(dg \cdot g^{-1}, \Omega_t, ..., \Omega_t)^{0,2k-1}\, dt \\ &= k \int_0^1 \Phi(dg, \Omega_t, ..., \Omega_t)^{0,2k-1}\, dt \end{aligned} \tag{12.10}$$

where the second line follows from the fact that ω and θ are of type (1,0) and the last line follows from (12.9).

Observe now that

$$\Phi(dg, \Omega_t, ..., \Omega_t)^{0,2k-1} \;=\; \Phi(\overline{\partial} g, \Omega_t^{0,2}, ..., \Omega_t^{0,2})$$

and from type considerations one computes that

$$\Omega_t^{0,2} = \{d\omega_t - \omega_t \wedge \omega_t\}^{0,2} = (1-t)^2 \overline{\partial} g \wedge \overline{\partial} g = (1-t)^2 [\overline{\partial} g, \overline{\partial} g].$$

From the Adjoint-invariance of Φ we conclude that

$$\Phi(\overline{\partial} g, [\overline{\partial} g, \overline{\partial} g], ..., [\overline{\partial} g, \overline{\partial} g]) \;=\; 0.$$

Indeed for any matrix-valued 1-form w, invariance implies that

$$\begin{aligned} \Phi([w,w], w, [w,w], ...,[w,w]) + \Phi(w, [w,w], ..., [w,w]) & \\ + \Phi(w, w, [w, [w,w]], [w,w]..., [w,w]) + \dots & \\ = \; 2\Phi(w, [w,w], ..., [w,w]) \;=\; 0 & \end{aligned}$$

since $[w, [w,w]] = 0$ by the Jacobi Identity. We conclude that $T^{0,2k-1} = 0$ and the proof is complete. □

Consider the natural transformation

$$\widehat{d} : \mathcal{V}(X)^+ \;\longrightarrow H^*(X, \mathcal{O}^\times)$$

where $\mathcal{V}(X)^+$ is the group completion of $\mathcal{V}(X)$ (cf. (12.7)). Following Grothendieck we define the holomorphic K-theory of X to be the quotient

$$K_{\mathrm{hol}}(X) \;\equiv\; \mathcal{V}(X)^+ / \sim$$

where $\sim$ is the equivalence relation generated by setting $[E] \sim [E' \oplus E'']$ whenever there is a short exact sequence of holomorphic bundles $0 \to E' \to E \to E'' \to 0$ on X.

Corollary 12.5. *The natural transformation $\widehat{d}$ defined above descends to a natural transformation*

$$\widehat{d} : K_{\mathrm{hol}}(X) \;\longrightarrow\; H^*(X, \mathcal{O}^\times)$$

such that properties (i), (ii) and (iii) of Theorem 12.3 continue to hold. In particular for algebraic manifolds this gives a total Chern class map

$$\widehat{d} : CH(X) \;\longrightarrow\; H^*(X, \mathcal{O}^\times)$$

from the group of algebraic cycles modulo rational equivalence.

13. BOTT VANISHING FOR HOLOMORPHIC FOLIATIONS

In 1969 R. Bott constructed a family of connections on the normal bundle of any smooth foliation of a manifold. Using these classes he established the vanishing of characteristic classes of the normal bundle in sufficiently high degrees [1]. Cheeger and Simons then showed that for these classes which vanish, the corresponding differential characters are well defined (independent of the choice of Bott connection) and represent secondary invariants of the foliation [4, §7]. These invariants are highly non-trivial and can vary continuously as $\mathbb{R}^k$-valued objects.

Suppose now that N is the normal bundle to a holomorphic foliation of codimension-q on a complex manifold X. Then there are two natural families of connections to consider on N: the family of Bott connections and the family of canonical hermitian connections.

Proposition 13.1. *Let $P(c_1, ..., c_q)$ be a polynomial in Chern classes which is of pure cohomology degree $2k$ with $k > 2q$. Then the $\overline{\partial}$-character $P(\widehat{c}_1, ..., \widehat{c}_q)^{0,2k-1}$ for the Bott connections agrees with the $\overline{\partial}$-character $P(\widehat{d}_1, ..., \widehat{d}_q)$ for the canonical hermitian connections.*

Note. The polynomial P has *pure cohomology degree* $2k$ if it satisfies the weighted homogeneity condition: $P(tc_1, t^2c_2, ..., t^qc_q) = t^kP(c_1, ..., c_q)$ for all $t \in \mathbb{R}$.

Theorem 13.2. *Let N be a holomorphic bundle of rank q on a complex manifold X. If N is (isomorphic to) the normal bundle of a holomorphic foliation of X, then for every polynomial P of pure cohomology degree $k > 2q$, the associated refined Chern class satisfies*

$$P(\widehat{d}_1(N), ..., \widehat{d}_q(N)) \ \in \ H^{2k-1}(X; \mathbb{C}^\times) \subset H^{2k-1}(X; \mathcal{O}^\times)$$

Proof. This is an immediate consequence of Proposition 13.1 which we shall now prove.

Suppose N is the normal bundle to a holomorphic foliation $\mathcal{F}$ of codimension-q. Then X has a distinguished atlas of coordinate charts $(z_\alpha, w_\alpha) : U_\alpha \to \mathbb{C}^{n-q} \times \mathbb{C}^q$ such that in the open subset $U_\alpha \subset X$, $\mathcal{F}$ is defined by the equation $w_\alpha =$ constant. Under the change of such coordinates one has

$$\frac{\partial w_\alpha}{\partial z_\beta} \ = \ 0, \qquad \text{that is} \qquad w_\alpha \ = \ w_\alpha(w_\beta) \tag{13.1}$$

depends on w_β alone. Note that therefore the operator

$$\partial_w \ \equiv \ \sum dw^j \wedge \frac{\partial}{\partial w^j}$$

is independent of the choice of distinguished coordinates (z, w) and therefore globally defined on X.

Suppose now that a hermitian metric h is given for the normal bundle $N = \text{span}\{dw\}$ and let H_α be the hermitian matrix representing h in the holomorphic frame $dw_\alpha^1, ..., dw_\alpha^q$ We define a connection 1-form θ_α for N in this frame by setting

$$\theta_\alpha \ \equiv \ \partial_w H_\alpha \cdot H_\alpha^{-1}$$

The transition functions for the $N =$ are given by the jacobian matrix

$$g \;=\; g_{\alpha,\beta} \;=\; \frac{\partial(w_\alpha)}{\partial(w_\beta)}.$$

A straightforward calculation shows that

$$\theta_\alpha \;=\; g\cdot\theta_\beta\cdot g^{-1} + \partial_w g\cdot g^{-1} \;=\; g\cdot\theta_\beta\cdot g^{-1} + dg\cdot g^{-1},$$

and so these 1-forms assemble to give a well-defined Bott connection ∇ on N.

Recall that in the local frame $dw_\alpha^1, ..., dw_\alpha^q$ for N the canonical hermitian connection $\widetilde{\nabla}$is given by the 1-form

$$\widetilde{\theta}_\alpha \;\equiv\; \partial H_\alpha\cdot H_\alpha^{-1}.$$

We can now explicitly compute the transgression term in any distinguished coordinate system (z, w). (We shall drop the α's for convenience.) Given P, let $\Phi(X_1, ..., X_k)$ be the Ad-invariant k-multilinear symmetric function on the Lie algebra $\mathfrak{gl}_q(\mathbb{C})$ such that $P(\sigma_1(X), ..., \sigma_q(X)) = \Phi(X, ..., X)$ where $\sigma_j(X)$ is the j^{th} elementary symmetric function of the eigenvalues of X. Then the difference between the $\overline{\partial}$-differential characters associated to P for the two connections $\widetilde{\nabla}$ and ∇ is the character associated to the smooth form

$$T^{0,2k-1} \;=\; k\int_0^1 \Phi(\widetilde{\theta}-\theta, \Omega_t, ..., \Omega_t)^{0,2k-1}\, dt.$$

where

$$\widetilde{\theta}-\theta \;=\; \partial_z H\cdot H^{-1}$$

and $\Omega_t \;=\; d\theta_t - \theta_t\wedge\theta_t$ with

$$\theta_t \;\equiv\; (1-t)\widetilde{\theta} + t\theta \;=\; \partial_w H\cdot H^{-1} - t(\partial_z H\cdot H^{-1}).$$

Since θ_t is of type $1,0$ we have that Ω_t is of type $1,1$ plus $2,0$. Hence, $T^{0,2k-1} = 0$. $\square$

Note 13.3. The above calculation shows that

$$T^{p,q} \;=\; k\int_0^1 \Phi(\widetilde{\theta}-\theta, \Omega_t, ..., \Omega_t)^{p,q}\, dt \;=\; 0 \qquad \text{for all } p<q.$$

This will allow us to generalize Theorem 13.2 to all Deligne cohomology.

14. GENERALIZATIONS AND THE RELATION TO DELIGNE COHOMOLOGY

Most of the discussion above can be easily generalized to other truncations of the de Rham complex. This leads to spark complexes with multiplicative structure on the associated spark classes. The role of $H^*(X,\mathcal{O}^\times)$ in the above is now played by more general Deligne cohomology groups.

Fix an integer $p>0$ and consider the truncated de Rham complex $(\mathcal{D}'^*(X,p),\overline{d})$ with

$$\mathcal{D}'^k(X,p) \;\equiv\; \bigoplus_{\substack{r+s=k\\ r<p}} \mathcal{D}'^{r,s}(X) \qquad \text{and} \qquad \overline{d}\equiv\Psi\circ d$$

where

$$\Psi:\mathcal{D}'^k(X) \;\longrightarrow\; \mathcal{D}'^k(X,p)$$

is the projection $\Psi(a)=a^{0,k}+a^{1,k-1}+\cdots+a^{p-1,k-p+1}$. Note the subcomplex

$$\mathcal{E}^k(X,p) \;\equiv\; \bigoplus_{\substack{r+s=k\\ r<p}} \mathcal{E}^{r,s}(X)$$

of smooth forms with projection $\Psi:\mathcal{E}^*(X)\to\mathcal{E}^*(X,p)$ whose kernel is a **d-closed ideal.**

Definition 14.1. By the **$\overline{d}$-spark complex of level** p we mean the triple (F^*,E^*,I^*) where

$$\begin{aligned} F^k &\equiv \mathcal{D}'^k(X,p)\\ E^k &\equiv \mathcal{E}^k(X,p)\\ I^k &\equiv \mathcal{I}^k(X)\end{aligned}$$

with maps

$$E^* \;\subset F^* \qquad \text{and} \qquad \Psi:I^* \;\longrightarrow\; F^*$$

given by the inclusion and projection above. The group of associated spark classes in degree k will be denoted by $\widehat{\mathbf{H}}^k(X,p)$.

Note that a spark in this complex is a pair $(a,r)\in\mathcal{D}'^k(X,p)\times\mathcal{I}^{k+1}(X)$ such that $dr=0$, and a satisfies the **$\overline{d}$-spark equation**

$$\overline{d}a \;=\; \phi-\Psi(r) \tag{14.1}$$

for some smooth $(k+1)$-form ϕ on X. Note that $\phi^{\ell,k+1-\ell}=0$ for all $\ell\geq p$ and that $\overline{d}\phi=0$, i.e., $d\phi\equiv 0 \pmod{\ker\Psi}$.

The $\overline{\partial}$-spark complexes of level 1 are exactly the $\overline{\partial}$-sparks discussed in §7.

Of course we have not yet established that the triple (F^*,E^*,I^*) in Definition 14.1 is a spark complex. The fact that $E^k\cap I^k=\{0\}$ for $k>0$ is proved in Proposition B.1. To show that the inclusion $E^*\subset F^*$ induces an isomorphism in cohomology

$$H^*(E) \;\cong\; H^*(F) \;\equiv\; H^*(X,p),$$

consider E^* and F^* as double complexes and note that the inclusion induces an isomorphism on vertical $\overline{\partial}$-cohomology (and hence in total cohomology by a standard spectral sequence argument (cf. [2, Lemma 1.2.5]). Note that $H^*(X,p)$ is the hypercohomology of the complex of sheaves: $0\to\Omega^0_X\to\Omega^1_X\to\Omega^2_X\to\cdots\to\Omega^{p-1}_X\to 0$.

To analyze the groups of spark classes $\widehat{\mathbf{H}}^k(X,p)$ we recall the following (cf. [2] or [24]).

Definition 14.2. By the **Deligne complex of level** p we mean the complex of sheaves

$$\underline{\mathbb{Z}}_{\mathcal{D}}(p) : \qquad 0 \to \mathbb{Z} \to \Omega^0_X \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{p-1}_X \to 0$$

where Ω^k_X, the sheaf of holomorphic k-forms on X, is considered to live in degree $k+1$. (Typically the constant sheaf $\mathbb{Z}$ is embedded into $\mathcal{O}_X = \Omega^0_X$ by $m \mapsto (2\pi i)^p m$. We will not adopt this convention here.) By the **Deligne cohomology** of X in level p we mean the hypercohomology of this complex:

$$H^*_{\mathcal{D}}(X, \mathbb{Z}(p)) \equiv H^*(X, \underline{\mathbb{Z}}_{\mathcal{D}}(p))$$

Note that when $p = 1$ the complex of sheaves $\underline{\mathbb{Z}}_{\mathcal{D}}(1)$ is quasi-equivalent to $\mathcal{O}^\times_X[-1]$, the sheaf $\mathcal{O}^\times_X$ shifted to the right by 1 so its formal degree is 1 and not 0. Thus we have that

$$H^k_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^{k-1}(X, \mathcal{O}^\times_X),$$

and so in the fundamental exact grid (3.4), the left-middle term could be replaced by $H^k_{\mathcal{D}}(X, \mathbb{Z}(1))$. In this form the picture generalizes to all levels.

Proposition 14.3. *The first fundamental exact sequence (1.5) for the group* $\widehat{\mathbf{H}}^k(X, p)$ *is*

$$0 \longrightarrow H^{k+1}_{\mathcal{D}}(X, \mathbb{Z}(p)) \longrightarrow \widehat{\mathbf{H}}^k(X, p) \xrightarrow{\delta_1} \mathcal{Z}^{k+1}_{\mathbb{Z}}(X, p) \longrightarrow 0$$

where $\mathcal{Z}^{k+1}_{\mathbb{Z}}(X, p)$ *is the set of* $\overline{d}$*-closed forms in* $\mathcal{E}^{k+1}(X, p)$ *which represent classes in* $H^{k+1}_{\mathbb{Z}}(X, p) \equiv \text{Image}\{\Psi_* : H^{k+1}(X; \mathbb{Z}) \longrightarrow H^{k+1}(X, p)\}$.

Proof. The identification of Image(δ_1) is straightforward. To identify the kernel consider the acyclic resolution

$$\begin{array}{ccc} \mathcal{I}^*_X & \xrightarrow{\Psi} & \mathcal{D}'^{*,*}_X \\ \uparrow & & \uparrow \\ \mathbb{Z} & \xrightarrow{\psi} & \Omega^*_X \end{array}$$

of the Deligne complex. Proposition A.3 then gives the following.

Proposition 14.4. *There is a natural isomorphism*

$$H^*_{\mathcal{D}}(X, \mathbb{Z}(p)) \cong H^*(\text{Cone}\,(\mathcal{I}^*(X) \xrightarrow{\Psi} \mathcal{D}'^*(X, p)))$$

This cone complex is exactly the cone complex G^* associated to our spark complex as in (1.5). □

The diagram (1.6) for the groups $\widehat{\mathbf{H}}^{k-1}(X,p)$ can be written as

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^{k-1}(X,p)}{H^{k-1}_{\mathbb{Z}}(X,p)} & \longrightarrow & \widehat{\mathbf{H}}^{k-1}_{\infty}(X,p) & \longrightarrow & \overline{d}\mathcal{E}^{k-1}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & \widehat{\mathbf{H}}^{k-1}(X,p) & \xrightarrow{\delta_1} & \mathcal{Z}^k_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \delta_2\downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\Psi_*) & \longrightarrow & H^k(X;\mathbb{Z}) & \xrightarrow{\Psi_*} & H^k_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where as usual $\widehat{\mathbf{H}}^{k-1}_{\infty}(X,p)$ denotes the spark classes representable by smooth forms.

Note that when X is Kaehler, we have

$$\begin{aligned}
\ker(\Psi_*) &= H^k(X;\mathbb{Z}) \cap \bigoplus_{j\geq p} H^{j,k-j}(X) \\
&= H^k(X;\mathbb{Z}) \cap \bigoplus_{\substack{|r-s|\leq k-2p \\ r+s=k}} H^{r,s}(X)
\end{aligned}$$

where the second line is deduced from the reality of $H^k(X;\,\mathbb{Z})$. In particular when $k=2p$ we deduce

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_p(X) & \longrightarrow & \widehat{\mathbf{H}}^{2p-1}_{\infty}(X,p) & \longrightarrow & \overline{d}\mathcal{E}^{2p-1}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & \widehat{\mathbf{H}}^{2p-1}(X,p) & \xrightarrow{\delta_1} & \mathcal{Z}^{2p}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \delta_2\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hdg}^{p,p}(X) & \longrightarrow & H^{2p}(X;\mathbb{Z}) & \xrightarrow{\Psi_*} & H^{2p}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{J}_p(X)$ denotes the Griffiths' p^{th} intermediate Jacobian and $\mathrm{Hdg}^{p,p}(X) \subset H^{2p}(X;\mathbb{Z})$ are the *Hodge classes*, i.e., the integral classes representable over $\mathbb{R}$ by closed (p,p)-forms.

The left vertical sequence, which is classical for Deligne cohomology, is deduced directly from this theory.

Remark 14.5. (The Deligne class of a holomorphic chain.) A *holomorphic k-chain* on X is an integral current which can be written as a locally finite sum $Z = \sum_j n_j V_j$ where for each j, $n_j \in \mathbb{Z}$ and V_j is an irreducible complex subvariety of dimension-k. Any such chain determines a spark $(0, Z) \in \mathcal{S}^{2p-1}(X)$ where $p = n - k$ (since $d0 = 0 - \Psi(Z) = 0$) and therefore a class in $\widehat{\mathbf{H}}^{2p-1}(X)$. This is clearly in the kernel of δ_1 and we obtain a class

$$[(0, Z)] \in H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

This retrieves Griffiths' generalized Abel-Jacobi mapping when X is compact Kaehler and Z is homologous to zero. The existence of this class goes back to Deligne [6].

Remark 14.6. (The Deligne class of a maximally complex cycle.) The above construction clearly generalizes to any integral cycle of restricted Hodge type. An interesting case is the following. An integral cycle $M \in \mathcal{I}^{2p+1}(X)$ with Dolbeault decomposition $M = M^{p,p+1} + M^{p+1,p}$ is called *maximally complex* (cf. [15]). As above, any such a cycle determines a class in Deligne cohomology

$$[(0, M)] \in H^{2p+1}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

As in [15] we shall say that M is the *boundary of a holomorphic chain* if $M = dZ$ where Z is a current defined by a holomorphic $(n-p)$-chain in $X - M$.

Proposition 14.6. *If M is a maximally complex cycle which bounds a holomorphic chain, then its Deligne class in $H^{2p+1}_{\mathcal{D}}(X, \mathbb{Z}(p))$ is zero.*

Proof. If $M = dZ$ where $Z \in \mathcal{I}^{2p}(X) \cap \mathcal{D}'^{p,p}(X)$, then $[(0, M)] = [(\Psi(Z), M - dZ)] = [(0, 0)] = 0$ in $H^{2p+1}_{\mathcal{D}}(X, \mathbb{Z}(p))$. □

The above remarks also apply to cycles with compact support with classes in *compactly supported Deligne cohomology*. In this case one retrieves the moment conditions characterizing boundaries of holomorphic chains in [15].

15. THEOREMS IN THE GENERAL SETTING

Most of the results proved for $\overline{\partial}$-sparks carry over to the general $\overline{d}$-sparks of §14.

Theorem 15.1. *The projection Ψ induces a morphism from the de Rham-Federer spark complex to the $\overline{d}$-spark complex in 14.1. This induces a surjective homomorphism*

$$\Pi : \widehat{\mathbf{H}}^*(X) \longrightarrow \widehat{\mathbf{H}}^*(X, p)$$

of abelian groups whose kernel is an ideal. Hence, $\widehat{\mathbf{H}}^*(X, p)$ carries a ring structure given at the spark level by explicit formulas.

Theorem 15.2. *Holomorphic vector bundles E have Chern classes $d_p(E) \in H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ defined as the image of the Cheeger-Simons classes $\widehat{c}_p(E, \nabla) \in \widehat{\mathbf{H}}^{2p}(X)$ where ∇ is any canonical hermitian connection on E.*

This constitutes a form of Chern-Weil Theory for Deligne characteristic classes.

Proofs of these theorems appear in [13] where Ning Hao also establishes the following.

- Analytic formulas for the full product in Deligne cohomology.
- A Čech-deRham spark complex equivalent to the one given above.
- A strengthened Bott vanishing theorem and the construction of secondary classes in Deligne cohomology for holomorphic foliations.

APPENDIX A. HYPERCOHOMOLOGY AND CONE COMPLEXES

In this section we present some elementary homological algebra relevant to Deligne cohomology and the following cone construction.

Definition A.1. Let $\Psi : \mathcal{I}^* \to \mathcal{J}^*$ be a degree-0 mapping of cochain complexes. The associated **cone complex** $C^* \equiv \mathrm{Cone}\,(\mathcal{I}^* \xrightarrow{\Psi} \mathcal{J}^*)$ is defined by

$$C^k \equiv \mathcal{I}^{k+1} \oplus \mathcal{J}^k$$

with differential $D : C^k \to C^{k+1}$ given by

$$D(a, b) \equiv (-da, db + \Psi(a)).$$

Consider now a two-step complex of sheaves

$$\mathcal{A} \xrightarrow{\psi} \mathcal{B}$$

on a manifold X (with $\mathcal{A}$ in degree 0), and an acyclic resolution

$$\begin{array}{ccc} I^* & \xrightarrow{\Psi} & J^* \\ \uparrow & & \uparrow \\ \mathcal{A} & \xrightarrow{\psi} & \mathcal{B} \end{array}$$

Proposition A.2. *There is a natural isomorphism*

$$H^*(X, \mathcal{A} \to \mathcal{B}) \;\cong\; H^*(Cone\,(\Gamma I^* \to \Gamma J^*))$$

where Γ denotes the sections functor.

Note. This is the a relative version of the classical isomorphisms:

$$H^*(X, \mathcal{A}) \;\cong\; H^*(\Gamma I^*) \qquad \text{and} \qquad H^*(X, \mathcal{B}) \;\cong\; H^*(\Gamma J^*).$$

Proof. By definition the hypercohomology $H^*(X, \mathcal{A} \to \mathcal{B})$ is the total cohomology of the double complex

$$\begin{array}{ccc}
\vdots & & \vdots \\
{\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\
\Gamma I^2 & \xrightarrow{\Psi} & \Gamma J^2 \\
{\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\
\Gamma I^1 & \xrightarrow{\Psi} & \Gamma J^1 \\
{\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\
\Gamma I^0 & \xrightarrow{\Psi} & \Gamma J^0
\end{array}$$

whose total differential

$$\Gamma I^p \oplus \Gamma J^{p-1} \;\xrightarrow{D}\; \Gamma I^{p+1} \oplus \Gamma J^p$$

is given by $D(a, b) \equiv (-da, \Psi(a) + db)$. This is exactly the cone complex as asserted. □

The simple fact asserted in Proposition A.2 has a useful generalization. Consider a complex of sheaves on X:

$$\mathcal{A} \;\xrightarrow{\psi}\; \mathcal{B}^0 \to \mathcal{B}^1 \to \mathcal{B}^2 \to \cdots \to \mathcal{B}^N$$

where $\mathcal{B}^k$ has formal degree $k+1$, and suppose we are given an acyclic resolution

$$\begin{array}{ccc}
I^* & \xrightarrow{\Psi} & J^{**} \\
\uparrow & & \uparrow \\
\mathcal{A} & \xrightarrow{\psi} & \mathcal{B}^*
\end{array}$$

so that $H^*(X, \mathcal{A}) \cong H^*(\Gamma I^*)$ and $H^*(X, \mathcal{B}^*) \cong H^*(\Gamma J^{**})$. Arguing exactly as above proves the following.

Proposition A.3. *There is a natural isomorphism*

$$H^*(X, \mathcal{A} \to \mathcal{B}^*) \;\cong\; H^*(\mathit{Cone}(\Gamma I^* \to \Gamma J^{**}))$$

APPENDIX B. NON-SMOOTHNESS OF INTEGRAL CURRENTS

Recall that a current T on a manifold is called *locally integral* if both T and dT are locally rectifiable. The main result of this section is the following.

Proposition B.1. *Let X be a complex manifold and for fixed integers $k, p > 0$ consider the projection*

$$\Psi : \mathcal{D}'^k(X) \longrightarrow \mathcal{D}'^k(X, p)$$

defined in §14. Then for any locally rectifiable current $T \in \mathcal{I}^k(X)$,

$$\Psi(T) \text{ is smooth} \qquad \Rightarrow \qquad \Psi(T) = 0.$$

Proof. Consider an open subset $U \subset \mathbb{R}^N$. For any current $S \in \mathcal{D}'^k(U)$ and any $x \in U$ we have the *upper m-density*:

$$\overline{\Theta}^m(x, \|S\|) \equiv \overline{\lim_{r \to 0}} \frac{\|S\|(B(x,r))}{\alpha_m r^m}$$

where $\|S\|$ is the *total variation measure* of S (cf. [8]), $m = N - k$ is the dimension of S, $B(x,r)$ is the ball of radius r centered at x and α_m is the volume of the unit ball in $\mathbb{R}^m$.

We now observe that

$$S \text{ is smooth} \qquad \Rightarrow \qquad \overline{\Theta}^m(x, \|S\|) = 0 \tag{B.1}$$

for all $x \in U$ and all $m < N$. To see this, suppose S is smooth and fix $\delta > 0$ with $B(x, \delta) \subset\subset U$. Let $c = \sup_{|y-x|<\delta} \|S_y\|$. Then $\|S\|(B(x,r))) \leq cr^N$ for $r < \delta$, and so $\overline{\Theta}^m(x, \|S\|) \leq cr^{N-m}$ for $r < \delta$, which proves (B.1).

Suppose now that T is a locally rectifiable current of dimension $m < N$ (i.e., of degree $k = N - m > 0$). Let $\mathcal{H}^m$ denote Hausdorff measure in dimension m. Then $T = \eta \mathcal{H}^m\big|_B$ for some $(\mathcal{H}^m, m)$ locally rectifiable subset B and some $\bigwedge^m \mathbb{R}^N$-valued function η which is L^1_{loc} with respect to $\mathcal{H}^m\big|_B$. In fact η is a simple m-vector of with $|\eta| \in \mathbb{Z}^+$, $\mathcal{H}^m\big|_B$-a.e. The corresponding total variation measure is just $\|T\| = |\eta| \mathcal{H}^m\big|_B$.

Suppose now that X is an open subset of $U \subset \mathbb{C}^n$ and that T is a locally rectifiable current on X with $\|T\| = \eta \mathcal{H}^m\big|_B$ as above. Then

$$S \equiv \Psi(T) = \Psi(\eta) \mathcal{H}^m\big|_B$$

where $\Psi(\eta)$ denotes the pointwise projection of η onto $\bigoplus_{\ell<p} \bigwedge^{\ell, k-\ell}$.

Now assume that S is smooth. Then by (B.1) we have that

$$\overline{\Theta}^m(x, \|S\|) = \overline{\Theta}^m(x, |\Psi(\eta)| \mathcal{H}^m\big|_B) = 0$$

for all $x \in U$. However, this implies that $\Psi(\eta) = 0$, $\mathcal{H}^m\big|_B$-a.e., and so $S = 0$ as claimed.□

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