Restriction and Removable Singularities for Fully Nonlinear Partial Differential Equations

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Reese Harvey

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They apply, for example, to calibrated and symplectic geometry

Nonlinear PDE's – The Classical "Operator" Viewpoint

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F is call a **subequation**.

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Definition. A function $u \in C^2(X)$ is F-**subharmonic** (a subsolution) if

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We want to extend the notion of F-subharmonicity to upper semi-continuous functions.

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F(X) $\equiv$ the set of these.

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and $J^1(X) = \mathbf{R} \oplus T^*X$.

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The key to defining differential **equations** on X

is to use subequations and **duality**.

Duality and F-Harmonics

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- Note that
$$F \cap -\widetilde{F} = \partial F$$

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$$u \in C^2(X) \text{ is } F\text{-harmonic} \qquad \iff \qquad J^2_x u \in \partial F \quad \forall\, x \in X.$$

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The homogeneous real Monge-Ampère Equation

$$D^2 u \geq 0 \quad \textbf{and} \quad \det(D^2 u) = 0.$$

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$$\widetilde{P}_k \;=\; P_{n-k+1}$$

Examples: Other Elementary Symmetric Functions

$$\mathcal{S}_k \equiv \{A : \sigma_1(A) \geq 0, ..., \sigma_k(A) \geq 0\}$$

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The equation has $(k-1)$ other branches.

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$\mathcal{P}_p$-harmonics are solutions of the polynomial equation

$$MA_p(A) = \prod_{i_1 < \cdots < i_p} \left(\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A) \right) = 0.$$

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is $\mathcal{P}_p$**-harmonic** in $\mathbf{R}^n - \{0\}$

and $\mathcal{P}_p$-subharmonic across 0.

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All the O(n)-invariant subequations given in terms of the $\lambda_k(A)$

have U(n)-invariant analogues given by the same conditions on the $\lambda_k^{\mathbf{C}}(A)$

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Example: The homogeneous complex Monge-Ampère equation

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Examples: Complex Analogues

$$\mathbf{C}^n = (\mathbf{R}^{2n}, J).$$

$$\mathrm{Sym}^2_{\mathbf{C}}(\mathbf{C}^n) \ \subset \ \mathrm{Sym}^2(\mathbf{R}^{2n})$$

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(Note: $\mathcal{P}^{\mathbf{C}}(X)$ = the plurisubharmonic functions on X)

Examples: Quaternionic Analogues

$$\mathbf{H}^n = (\mathbf{R}^{4n}, I, J, K).$$

$$A_{\mathbf{H}} \equiv \frac{1}{4}(A - IAI - JAJ - KAK)$$

$A_{\mathbf{C}}$ has quaternionic eigenspaces and ordered eigenvalues

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Riemannian manifolds and the decomposition of $J^2(X)$

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where $\mathrm{Hess}\, u \in \Gamma(\mathrm{Sym}^2(T^*X))$ is the **Riemannian hessian** defined by

$$(\mathrm{Hess}\, u)(V, W) \;=\; VWu - (\nabla_V W)u$$

for vector fields V, W.

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Example. $\mathbf{F} \equiv \mathbf{R} \times \mathbf{R}^n \times \{\mathrm{tr} A \geq 0\}$ gives

$$\mathrm{tr}(\mathrm{Hess}\, u) = \Delta u \geq 0.$$

Universal Hermitian subequations

Let $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$. If

$$\mathbf{F} \ \subset \ \mathbf{J} \ \equiv \ \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^{2n})$$

- is closed
- **$\mathbf{U}(n)$-invariant**
- and satisfies (P), (N) and (T)

Then **F** **canonically determines a subequation**

$$F_X \ \subset \ J^2(X)$$

on any **almost complex, hermitian** manifold X.

Example. $\mathbf{F} \ \equiv \ \mathbf{R} \times \mathbf{R}^n \times \{A_{\mathbf{C}} \geq 0\}$ gives the homogeneous **complex Monge-Ampère** subequation.

Manifolds with Topological Structure Group G

Let

$$G \subset \mathrm{O}(n)$$

be a closed subgroup. If

$$\mathbf{F} \;\subset\; \mathbf{J} \;\equiv\; \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^{2n})$$

- is closed
- ***G*-invariant**
- and satisfies (P), (N) and (T)

Then **F** **canonically determines a subequation** F_X on any riemannian manifold with **topological** structure group G.

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is a smooth bundle map.

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such that

$$\Psi(F_1) = F_2.$$

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THEOREM B. Suppose E is a closed set with no interior, which is locally M-polar. Then for $u \in C(X)$,

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THEOREM. *Pluripolar sets (i.e. $\mathcal{P}^{\mathbf{C}}$-polar sets) in $\mathbf{C}^n$ are removable for* **all branches** *of the homogeneous Monge-Ampère equation.*

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The proof uses Riesz potentials

$$\mu * K_p \qquad \text{where} \qquad K_p(x) = \frac{-1}{|x|^{p-2}}.$$

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Suppose $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is a subequation with monotonicity cone M.

Riesz Characteristics

Let $M \subset \mathrm{Sym}^2(\mathbf{R}^n)$ be a convex cone subequation and $1 \le p \le n$

Lemma.

$$\mathcal{P}_p \ \subset \ M \qquad \iff \qquad I - p\,\pi_e \in M \quad \text{for all unit vectors } e \in \mathbf{R}^n.$$

Definition. The **Riesz characteristic** of M is

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Suppose $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is a subequation with monotonicity cone M.
Then any closed set E of locally finite Hausdorff $(p_M - 2)$-measure
is removable for F-subharmonics and F-harmonics.

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Labutin, Amensola-Galise-Vitolo, Caffarelli-Li-Nirenberg

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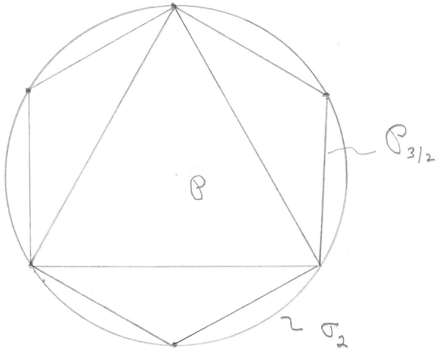

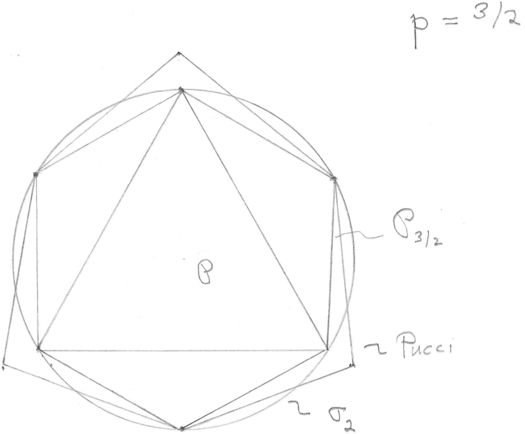
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For linear equations there is a simple and useful linear restriction hypothesis.

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Theorem. *Let $Y \subset X$ be a $\mathbb{G}$-submanifold which is minimal. Then restriction to Y holds for $F_{\mathbb{G}}$.*

In other words, the restriction of any $F_{\mathbb{G}}$-plurisubharmonic function to Y is subharmonic in the induced riemannian metric on Y.

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Then restriction holds for F on any totally geodesic submanifold $Y \subset X$.

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One now applies the **viscosity** definition as before.

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THEOREM. These three definitions are equivalent.

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Pali conjectured (3) $\Rightarrow$ (1) and proved it under certain assumptions on u.

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Let $\Omega \subset\subset \mathbf{R}^n$ be a domain with a smooth boundary which is both F and $\widetilde{F}$ strictly convex.

Theorem. *Suppose $0 \in \Omega$ and let $h \in C(B_\epsilon - \{0\})$ be an F-harmonic function with $\lim_{x\to 0} h(x) = -\infty$. Fix $\varphi \in C(\partial\Omega)$.*

Existence. *There exists $H \in C(\overline{\Omega} - \{0\})$ such that:*

(1) H *is F-harmonic on* $\Omega - \{0\}$,

(2) $H\big|_{\partial\Omega} = \varphi$,

(3) $h(x) + c \ \leq \ H(x) \ \leq \ h(x) + C$ *on a neighborhood of 0 for some constants* c, C.

Uniqueness. *There is at most one function $h \in C(\overline{\Omega} - \{0\})$ satisfying (1), (2), and (3).*