BOUNDARIES OF POSITIVE HOLOMORPHIC CHAINS AND THE RELATIVE HODGE QUESTION

by

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Abstract

We characterize the boundaries of positive holomorphic chains in an arbitrary complex manifold.

We then consider a compact oriented real submanifold of dimension $2p-1$ in a compact Kähler manifold X and address the question of which relative homology classes in $H_{2p}(X, M; \mathbf{Z})$ are represented by positive holomorphic chains. Specifically, we define what it means for a class $\tau \in H_{2p}(X, M; \mathbf{Z})$ to be of type (p,p) and positive. It is then shown that τ has these properties if and only if $\tau = [T+S]$ where T is a positive holomorphic chain with $dT = \partial\tau$ and S is a positive (p,p)-current with $dS = 0$.

§**1. Introduction.** In the first part of this note we establish a general result concerning boundaries of positive holomorphic chains in a complex manifold X. In the second part we address the "Relative Hodge Question": *When is a homology class $\tau \in H_{2p}(X, M; \mathbf{Z})$ represented by a positive holomorphic chain?* Assuming M is a real $(2p-1)$-dimensional submanifold we are able to give a surprisingly full answer.

We begin our discussion of the first part by presenting some interesting special cases which are quite different in nature. The first main theorem is then formulated and proved in Section 2.

To start, suppose X compact and let Γ be a current of dimension $2p-1$ in X. By a positive holomorphic p-chain with boundary Γ we mean a finite sum $V = \sum_k m_k V_k$ with $m_k \in \mathbf{Z}^+$ and V_k an irreducible complex analytic variety of dimension p and finite volume in $X - \text{supp}\Gamma$, such that $dV = \Gamma$ as currents on X.

Equip X with a hermitian metric and let ω denote its associated $(1,1)$-form. A real $(2p-1)$-form α will be called a (p,p)**-positive linking form** if

$$d^{p,p}\alpha + \frac{1}{p!}\omega^p \geq 0 \qquad \text{(strongly positive)}$$

where $d^{p,p}\alpha$ denotes the (p,p)-component of $d\alpha$. (See [HK] or [H_2] for the definition of strongly and weakly positive currents.) The numbers $\int_\Gamma \alpha$ with α as above, will be called the (p,p)**-linking numbers** of Γ.

*Partially supported by the N.S.F.

THEOREM 1.1. *Let $\Gamma = \sum_{k=1}^{N} n_k \Gamma_k$ be an integer linear combination of compact, mutually disjoint, C^1-submanifolds of dimension $2p-1$ in X, each of which has a real analytic point. Then $\Gamma = dV$ where V is a positive holomorphic p-chain if and only if the (p,p)-linking numbers of Γ are bounded below.*

Note 1.2. The condition that the linking numbers of Γ are bounded below is easily seen to be independent of the choice of hermitian metric on X. However, for any given metric we have the precise statement that Γ bounds a positive holomorphic chain of mass $\leq \Lambda$ if and only if

$$\int_\Gamma \alpha \ \geq \ -\Lambda \qquad \text{for all } (p,p)-\text{positive linking forms } \ \alpha \tag{1.1}$$

Note 1.3. We shall actually prove the theorem in the more general situation where Γ is allowed to have a "scar" set and the real analyticity assumption is replaced by a weaker "push-out" hypothesis (see section 2). When $p > 1$, this hypothesis is satisfied at any point where the boundary is smooth and its Levi form has at least one negative eigenvalue. In all these cases, one has regularity at almost all points of Γ. This boundary regularity is discussed in [HL_1] and [H_2].

Remark 1.4. When X is a projective surface and $p = 1$, a much stronger result is conjectured: namely, Γ bounds a positive holomorphic 1-chain if and only if

$$\int_\Gamma d^c u \ \geq -\Lambda \qquad \text{for all } u \in C^\infty(X) \text{ with } dd^c u + \omega \geq 0. \tag{1.2}$$

Functions u with $dd^c u + \omega \geq 0$ are called *quasi-plurisubharmonic*. They were introduced by Demailly and play an important role in complex analysis [D], [GZ]. Condition (1.2) is equivalent to the condition that

$$\frac{1}{\ell}\text{Link}_{\mathbf{P}}(\Gamma, Z) \ \geq \ -\Lambda \qquad \text{for all positive divisors } Z \text{ in } X - \Gamma$$

of sections $\sigma \in H^0(X, \mathcal{O}(\ell))$, $\ell > 0$, where $\text{Link}_{\mathbf{P}}$ denotes the *projective linking number* introduced in [HL_5]. In this form the conjecture extends to all dimensions and codimensions (for X projective) and is a consequence of the above case: $p = 1$ in surfaces. All this is established in [HL_5] where the conjectures are also related to the projective hull introduced in [HL_4].

Although the hypothesis of Theorem 1.1 is conjecturally too strong for projective manifolds, it does give the "correct" result in the general case. For example, if X is a non-algebraic K3-surface, there appears to be no simpler condition characterizing the boundaries of positive holomorphic 1-chains.

Remark 1.5. The Linking Condition (1.1) forces the components of Γ to be maximally complex CR-manifolds. Maximal complexity is equivalent to the assertion that $\Gamma = \Gamma_{p-1,p} + \Gamma_{p,p-1}$ where $\Gamma_{r,s}$ denotes the Dolbeault component of Γ in bidimension (r,s). To see that this must hold, note that any $\alpha \in \mathcal{E}^{r,2p-1-r}(X)$ with $r \neq p-1, p$ satisfies $d^{p,p}\alpha + \omega \geq 0$ since $d^{p,p}\alpha = 0$.

Theorem 1.1 extends to characterize boundaries of compactly supported holomorphic chains in certain non-compact spaces. A complex n-manifold X is called *q-convex* if there exists a proper exhaustion function $f : X \to \mathbf{R}^+$ such that $dd^c f$ has at least $n - q + 1$ strictly positive eigenvalues outside some compact subset of X.

THEOREM 1.6. *Theorem 1.1 remains valid (for compactly supported holomorphic chains V) in any q-convex hermitian manifold with $q \leq p$.*

Remark 1.7. If X is 1-convex (i.e., strongly pseudoconvex), then Theorem 1.1 is valid for all p. If, further, X admits a proper exhaustion which is strictly plurisubharmonic everywhere (i.e., X is Stein), much stronger results are known. Condition (1.1) implies maximal complexity, and for $p > 1$ this condition alone implies that Γ bounds a holomorphic p-chain [HL_1]. Condition (1.1) also implies the *moment condition*: $\Gamma(\alpha) = 0$ for all $(p, p-1)$-forms α with $\overline{\partial}\alpha = 0$. When $p = 1$ this implies that Γ bounds a holomorphic 1-chain [HL_1].

Analogous remarks apply to results of [HL_2] in the q-convex spaces $\mathbf{P}^n - \mathbf{P}^{n-q}$.

Remark 1.8. Condition (1.1) implies that $\int_\Gamma \alpha \geq 0$ for all α with $d^{p,p}\alpha \geq 0$. If X is a Stein manifold embedded in some $\mathbf{C}^n$, this in turn implies that the linking number $\mathrm{Link}(\Gamma, Z) \geq 0$ for all algebraic subvarieties Z of codimension p in $\mathbf{C}^n - \Gamma$. By Alexander-Wermer [AW], [W_2] this last condition alone implies that Γ bounds a positive holomorphic p-chain in X.

Theorem 1.1 also holds "locally", that is, it extends to any non-compact hermitian manifold X where neither Γ nor V are assumed to have compact support.

THEOREM 1.9. *Suppose X is a non-compact hermitian manifold, and let $\Gamma = \sum_j n_j \Gamma_j$ be a locally finite integral combination of disjointly embedded C^1-submanifolds of dimension $2p-1$, each of which has a real analytic point. Then Γ is the boundary of a holomorphic p-chain V of mass $M(V) \leq \Lambda$ (whose support is a closed but not necessarily compact analytic subvariety of $X - \mathrm{supp}\Gamma$) if and only if $\int_\Gamma \alpha \geq -\Lambda$ for all (p,p)-positive linking forms α with compact support on X.*

In the last section of this paper we further weaken our hypotheses on Γ to an assumption that each component Γ_k be *residual* at some point. (See §3 for the definition.) The concept of residual submanifolds leads to questions of some independent interest.

In Section three we address a question related to the Characterization Theorems above. Let $j : M \subset X$ be a compact oriented real submanifold of dimension $2p - 1$ in a compact Kähler manifold X. Represent the relative homology group $H_{2p}(X, M; \mathbf{R})$ by $2p$-currents T on X with $dT = j_* S$ for some $(2p-1)$-current S on M. One can ask: *When does a given class $\tau \in H_{2p}(X, M; \mathbf{R})$ contain a positive holomorphic chain?*

As a first step we show that for every T as above and every d-closed form φ on X the pairing $T(\varphi)$ depends only on the relative class $\tau = [T]$. This allows us to introduce a real Hodge filtration on $H_{2p}(X, M; \mathbf{Z})_{\text{mod tor}}$ which extends the standard one on the subgroup $H_{2p}(X; \mathbf{Z})_{\text{mod tor}}$. It also allows us to formulate the following.

Definition 1.10. A class $\tau \in H_{2p}(X, M; \mathbf{R})$ is a **positive (p,p)-class** if $\tau(\varphi) \geq 0$ for all $2p$-forms φ with $d\varphi = 0$ and $\varphi^{p,p} \geq 0$.

THEOREM 1.11. *Let $M \subset X$ be as above and suppose each component of M has a real analytic point. Let $\tau \in H_{2p}(X, M; \mathbf{Z})_{\text{mod tor}}$ be a positive (p,p)-class. Then there exists a positive holomorphic p-chain T on X with $dT = \partial\tau$ and a positive (p,p)-current S with $dS = 0$ such that $\tau = [T + S]$.*

In particular, if the positive classes in $H_{p,p}(X; \mathbf{Q})$ are represented by positive holomorphic chains with rational coefficients, then so are all the positive classes in $H_{p,p}(X, M; \mathbf{Q})$.

Remark 1.12. This last result is a strengthening of the previous ones (in the Kähler case). Let τ be as in Theorem 1.11 and note that $\Gamma = \partial\tau = \sum_k n_k[M_k]$ where $M_1, ..., M_\ell$ are the connected components of M and the n_k's are integers. If τ is a positive (p,p)-class, then $\tau(d\alpha + \frac{1}{p!}\omega^p) \geq 0$ whenever $d^{p,p}\alpha + \frac{1}{p!}\omega^p \geq 0$. Therefore for any (p,p)-positive linking form α we have $\Gamma(\alpha) = (\partial\tau)(\alpha) = \tau(d\alpha) = \tau(d^{p,p}\alpha) = \tau(d^{p,p}\alpha + \frac{1}{p!}\omega^p) - \tau(\frac{1}{p!}\omega^p) \geq -\tau(\frac{1}{p!}\omega^p)$, and we conclude from Theorem 1.1 that Γ bounds a positive holomorphic p-chain T. Theorem 1.11 asserts that, moreover, the absolute class $\tau - [T]$ is represented by a positive (p,p)-current.

§2. The Characterization Theorem.

In this section we prove a general theorem which implies all of the results discussed in §1 except Theorem 1.11. We shall assume throughout that X is a hermitian manifold which is not necessarily compact.

Definition 2.1. Suppose there exists a closed subset Σ_Γ of Hausdorff $(2p-1)$-measure zero and an oriented, properly embedded, $(2p-1)$-dimensional C^1 submanifold of $X - \Sigma_\Gamma$ with connected components $\Gamma_1, \Gamma_2, ...$ If for given integers $n_1, n_2, ...,$

$$\Gamma = \sum_{k=1}^{\infty} n_k \Gamma_k$$

defines a current of locally finite mass in$X$ which is d-closed, then Γ will be called a **scarred** $(2p-1)$**–cycle of class** C^1 in X. By a unique choice of orientation on Γ_k we assume each $n_k > 0$.

Example . Any real analytic $(2p-1)$-cycle is automatically a scarred $(2p-1)$-cycle (see [F, p. 433]).

Definition 2.2. By a **positive holomorphic p-chain with boundary** Γ in X we mean a sum $V = \sum_k m_k V_k$ with $m_k \in \mathbf{Z}^+$ and V_k an irreducible p-dimensional complex analytic subvariety of $X - \text{supp}\Gamma$ such that V has locally finite mass in X and $dV = \Gamma$ as currents.

Definition 2.3. Suppose Γ is an embedded $(2p-1)$-dimensional oriented submanifold of a complex manifold. We say that Γ can be **pushed out at** $p \in \Gamma$ if there exists a complex p-dimensional submanifold-with-boundary $(V, -\Gamma)$ containing the point p (i.e., $\partial V = -\Gamma$ as oriented manifolds).

Our main result is the following.

THEOREM 2.4. *Let Γ be a scarred $(2p-1)$-cycle of class C^1 in X such that each component Γ_k can be pushed out at some point. Then $\Gamma = dV$ where V is a positive holomorphic p-chain with mass $M(V) \leq \Lambda$ if and only if the (p,p)-linking numbers of Γ are bounded below by $-\Lambda$.*

Remark 2.5. We say Γ is **two sided at** p if there exists a complex p-dimensional submanifold V near p with $\Gamma \subset V$ near p. Note that if Γ is real analytic and maximally complex at p, then Γ is two-sided at p. Note also that if Γ is two-sided at p, then Γ can be pushed out at p.

The proof of Theorem 2.4 has two parts. First the linking condition is shown to be equivalent to the existence of a weakly positive current T of bidimension p,p satisfying $dT = \Gamma$. In the second part it is shown that the existence of a positive T with $dT = \Gamma$ together with the pushout hypothesis on Γ implies the existence of a positive holomorphic chain with boundary Γ.

Solving $dT = \Gamma$ for T positive.

THEOREM 2.6. *Let $\Gamma \in \mathcal{D}'_{2p-1}(X)$ be an arbitrary current of dimension $2p-1$ on X. Then $dT = \Gamma$ for some weakly positive (p,p)-current with mass $M(T) \leq \Lambda$ if and only if the linking condition*

$$\int_\Gamma a \;\geq\; -\Lambda \tag{2.1}$$

is satisfied for all compactly supported, strongly positive (p,p)-linking forms α on X.

Proof. Let

$$S \;\equiv\; \{\alpha \in \mathcal{D}^{2p-1}(X) : d^{p,p}\alpha + \tfrac{1}{p!}\omega^p \geq 0 \;\; \text{(strongly positive)}\}$$

and let

$$C \;\equiv\; \{\Gamma \in \mathcal{D}'_{2p-1}(X) : \Gamma = dT \;\; \text{for some} \;\; T \geq 0 \;\; \text{(weakly positive) with } M(T) \leq 1\}$$

It suffices to prove the theorem for $\Lambda = 1$. In this case the theorem states that $\Gamma \in C$ if and only if $\Gamma \in S^0$, where $S^0 \equiv \{\Gamma \in \mathcal{D}'_{2p-1}(X) : \Gamma(\alpha) \geq -1 \text{ for all } \alpha \in S\}$ is the *polar* of S. So we must prove that

$$C \;=\; S^0.$$

Note that C is a closed convex set in $\mathcal{D}'_{2p-1}(X)$ since the set of weakly positive (p,p)-currents T with $M(T) \leq 1$ is compact in $\mathcal{D}'_{p,p}(X)$. Hence by the Bipolar Theorem [S] $C = (C^0)^0$, and it will suffice to prove that $C^0 = S$.

To see this first note that

$$T\left(d^{p,p}\alpha + \frac{1}{p!}\omega^p\right) \;=\; (dT)(\alpha) + M(T) \tag{2.2}$$

for all weakly positive (p,p)-currents T and all $\alpha \in \mathcal{D}^{2p-1}(X)$. If, in addition, $\alpha \in S$ and $\Gamma \in C$, then $0 \leq \Gamma(\alpha) + M(T) = \Gamma(\alpha) + 1$, so that $S \subseteq C^0$.

It remains to show that $C^0 \subseteq S$. Choose $\Gamma = dT$ with $T = \delta_x \xi$ where ξ is a weakly positive (p,p)-vector of mass norm one at $x \in X$. Note that $\Gamma \in C$. By (2.2) we have $(d^{p,p}\alpha + \frac{1}{p!}\omega^p)_x(\xi) = \Gamma(\alpha) + M(T) = \Gamma(\alpha) + 1$. If $\alpha \in C^0$, then $\Gamma(\alpha) \geq -1$ which proves that $\alpha \in S$. ∎

Replacing the Positive Solution by a Holomorphic Chain

THEOREM 2.7. *Suppose Γ is a scarred $(2p-1)$-cycle (of class C^1) in an arbitrary complex manifold X. Assume each component Γ_k of Γ can be pushed out at some point. If $\Gamma = dT$ for some weakly positive (p,p)-current T, then there exists a positive holomorphic p-chain V with $\Gamma = dV$ and $T - V \geq 0$, so in particular, $M(V) \leq M(T)$ and* $\mathrm{supp}(V) \subset \mathrm{supp}(T)$.

The proof depends on the following local result.

Lemma 2.8. *Suppose Γ is an oriented connected $(2p-1)$-dimensional submanifold near $0 \in \Gamma$ in $\mathbf{C}^n$.*

(1) If Γ can be pushed out at $0 \in \Gamma$ and $r\Gamma = dT$ for some $T \geq 0$ and $r > 0$, then Γ is two-sided near 0. That is, near 0 there exists a (unique) complex p-dimensional subvariety V containing Γ, so that $V = V^+ \cup \Gamma \cup V^-$ and $dV^{\pm} = \pm\Gamma$.

(2) If Γ is two-sided near 0 and $r\Gamma = dT$ for some $T \geq 0$ and $r > 0$, then

$$T = rV^+ + S \qquad \text{with} \;\; S \geq 0 \;\; \textit{and} \;\; dS = 0.$$

Proof. By the push-out hypothesis we have that $-\Gamma = dZ$ for some irreducible subvariety Z of $B(0,R) - \Gamma$. By taking a small piece V^- of Z we may assume that the positive current $T^+ \equiv T + rV^- \geq 0$ has boundary Γ^+ which does *not* contain the origin. (See Figure 1.) Consider the subset

$$E_r(T^+) \;=\; \{z : \Theta(T^+, z) \geq r\} \subset B(0,R) - \Gamma^+$$

where $\Theta(T^+, z)$ denotes the standard density, or Lelong number, of T^+ at z. Since $dT^+ = 0$ in $B(0,R) - \Gamma^+$ we know by a fundamental theorem of Siu [Siu] that

$$E_r(T^+) \textit{ is a complex subvariety of complex dimension} \leq p \textit{ and}$$
$$T^+ - rW \geq 0 \quad \textit{where } W \textit{ is the } p\textit{-dimensional part of } E_r(T^+).$$

Since $E_r(T^+)$ contains V^-, it must have an irreducible p-dimensional component $V \supset V^-$, defined in a neighborhood of the origin. This proves (1). Since $V \subset W$, we have $T^+ - rV \geq 0$. Note also that $d(T^+ - rV) = 0$ near the origin. This proves (2) since $S \equiv T - rV^+ = T^+ - rV$. ∎

Corollary 2.9. Under the hypotheses of Lemma 2.8 (1), $-\Gamma$ can also be pushed out at 0.

Proof of Theorem 2.7. As an easy consequence of Siu's Theorem (See, for example, Theorem 2.4, p. 638 in [H_1]), there exist irreducible p-dimensional subvarieties V_j of $X - \mathrm{supp}\Gamma$ and positive constants c_j so that

$$T \;=\; \sum_{j=1}^{\infty} c_j V_j + R \tag{2.3}$$

where $R \geq 0$ and, for each $c > 0$, the complex subvariety $E_c(R)$ is of dimension $\leq p-1$. This representation (2.3) is unique. (Note that $R \geq 0$ implies that the mass of T dominates the mass of $\sum_j c_j V_j$ on any set.)

Near the point where Γ_1 can be pushed out, Lemma 2.8 (with $r = n_1$) implies that

$$T \;=\; n_1 V^+ + S \qquad \text{with} \;\; S \geq 0 \;\; \text{and} \;\; dS = 0. \tag{2.4}$$

By uniqueness V^+ must be contained in one of the V_j, say V_1. Moreover, since $S \geq 0$ we have $c_1 \geq n_1$. This implies $\widetilde{T} \equiv T - n_1 V_1 \geq 0$.

Near the point where Γ_1 can be pushed out we have $dV_1 = \Gamma_1$. Hence, on X we have $dV_1 = \Gamma_1 + \sum_{k=2}^{\infty} m_k \Gamma_k$ with $m_k \in \mathbf{Z}$. Consequently,

$$d\widetilde{T} = \sum_{k=2}^{\infty} (n_k - n_1 m_k)\Gamma_k,$$

and so we have eliminated one of components of the boundary. Now the coefficients in this sum may not all be positive, and to make them all positive we may have to reverse the orientation of some of the Γ_k. However, by Corollary 2.9 these orientation-reversed components can also be pushed out at some point. Hence, $\widetilde{\Gamma} = d\widetilde{T} = \sum_{k=2}^{\infty} \widetilde{n}_k \widetilde{\Gamma_k}$ satisfies all the hypotheses of Theorem 2.7.

If Γ has only a finite number of components, then we are done by induction on the number of components. If not, then by continuing this process we obtain a sequence of positive currents $\widetilde{T}_k = T - (n_1 V_1 + n_2' V_2 + \cdots + n_k' V_k)$ where the n_j' are positive integers and

$$d\widetilde{T}_k \;=\; \sum_{j=k+1}^{\infty} n_{kj} \Gamma_j.$$

Since $T - \widetilde{T}_k \geq 0$, we may assume, by passing to a subsequence, that $\{\widetilde{T}_k\}_{k=1}^{\infty}$ converges in mass norm to a positive current $\widetilde{T}_\infty$, which must be flat since each $\widetilde{T}_k$ is a normal current. Note that $\mathrm{supp}(d\widetilde{T}_\infty) \subset \Sigma_\Gamma$ and recall that, by assumption, the scar set Σ_Γ has Hausdorff (2p-1)-measure zero. Hence, by [F, 4.1.20], we have $d\widetilde{T}_\infty = 0$. We conclude that $V = \sum n_j' V_j = T - \widetilde{T}_\infty$ is a positive holomorphic chain with $dV = \Gamma$. ∎

Proof of Theorems 1.1 and 1.9. Remarks 1.5 and 2.5 show that if $\Gamma = \sum_k m_k \Gamma_k$ satisfies the linking hypothesis, then Γ_k is two-sided at any real analytic point.

Proof of Theorem 1.6. It suffices to show that when X is q-convex for $q \leq p$, then Theorem 2.6 also holds with Γ and T having compact support. For this we change the definitions of S and C in the proof of Theorem 2.6 by permitting the α's in S to have arbitrary support and restricting the T's in C to have compact support. The argument will carry through as before once it is established that the cone C is closed in the weak topology. This follows from standard compactness theorems and the following fact. Suppose $f : X \to \mathbf{R}^+$ is the proper exhaustion with $n - q + 1$ positive eigenvalues on $\{x : f(x) \geq 1\}$. If $T \in \mathcal{P}_{p,p}(X)$ for $p \geq q$, then

$$\mathrm{supp}\, T \;\subset \left\{ x \in X : f(x) \leq \max\left\{ 1, \sup_{\mathrm{supp}\, dT} f \right\} \right\}.$$

3. Relative Hodge Classes and Representability.

In this chapter we address the question of when a relative homology class can be represented by a positive holomorphic chain. More specifically, let X be a compact Kähler manifold and $M \subset X$ a smooth orientable compact submanifold of real dimension $2p-1$. Then we have the two closely related questions:

Relative Hodge Question: *Which classes in $H_{2p}(X, M; \mathbf{Z})$ can be represented by holomorphic chains?*

Relative Hodge Question (Positive Version): *Which classes in $H_{2p}(X, M; \mathbf{Z})$ can be represented by positive holomorphic chains?*

We shall work in the space $\overline{H}_{2p}(X, M; \mathbf{Z}) = H_{2p}(X, M; \mathbf{Z})/\text{Tor}$ where Tor is the torsion subgroup and use the following Relative de Rham Theorem. Consider the short exact sequence of chain complexes of Fréchet spaces

$$0 \longrightarrow \mathcal{E}^*(X, M) \longrightarrow \mathcal{E}^*(X) \xrightarrow{j^*} \mathcal{E}^*(M) \longrightarrow 0, \tag{3.1}$$

where $j : M \to X$ denotes the inclusion, and the dual sequence of topological dual spaces

$$0 \longleftarrow \frac{\mathcal{E}'_*(X)}{j_*\mathcal{E}'_*(M)} \longleftarrow \mathcal{E}'_*(X) \xleftarrow{j_*} \mathcal{E}'_*(M) \longleftarrow 0. \tag{3.2}$$

The complex $\mathcal{E}^*(X, M)$, consisting of forms which vanish when restricted to M, computes the relative cohomology $H^*(X, M; \mathbf{R})$, and the complex $\mathcal{E}'_*(X, M) \equiv \mathcal{E}'_*(X)/j_*\mathcal{E}'_*(M)$ computes the relative homology $H_*(X, M; \mathbf{R})$.

The Relative de Rham Theorem states that:

$H^k(X, M; \mathbf{R})$ and $H_k(X, M; \mathbf{R})$ are dual to each other.

This can be proven as follows. Consider the dual triples

$$\begin{array}{ccccc} \mathcal{E}^{k-1}(X, M) & \xrightarrow{d} & \mathcal{E}^k(X, M) & \xrightarrow{d} & \mathcal{E}^{k+1}(X, M) \\ \mathcal{E}'_{k-1}(X, M) & \xleftarrow{d} & \mathcal{E}'_k(X, M) & \xleftarrow{d} & \mathcal{E}'_{k+1}(X, M) \end{array}$$

where $H^k(X, M; \mathbf{R}) = Z/B$ using the cycles Z and boundaries B in the first sequence, and $H_k(X, M; \mathbf{R}) = \widetilde{Z}/\widetilde{B}$ using the cycles $\widetilde{Z}$ and boundaries $\widetilde{B}$ in the second sequence. By the Hahn-Banach Theorem it suffices to show that B and $\widetilde{B}$ are closed. These spaces are images of continuous linear maps. If they are of finite codimension in Z and $\widetilde{Z}$ respectively, then they are closed by a standard result in functional analysis. Thus is remains to show that $H^k(X, M; \mathbf{R})$ and $H_k(X, M; \mathbf{R})$ are finite dimensional. That $H^k(X, M; \mathbf{R})$ is finite dimensional follows from the long exact sequence

$$\cdots \longrightarrow H^{k-1}(M; \mathbf{R}) \longrightarrow H^k(X, M; \mathbf{R}) \longrightarrow H^k(X; \mathbf{R}) \longrightarrow H^k(M; \mathbf{R}) \longrightarrow \cdots$$

derived from (3.1) and the fact that $H^{k-1}(M; \mathbf{R})$ and $H^k(X; \mathbf{R})$ are finite dimensional by the standard de Rham Theorem. That $H_k(X, M; \mathbf{R})$ is finite dimensional follows similarly from the long exact sequence derived from (3.2).

In the special case $k = 2p = \dim M + 1$ we have:

$$\begin{aligned} H^{2p}(X, M; \mathbf{R}) &= Z/B \qquad \text{where} \\ Z \;=\; \{\varphi \in \mathcal{E}^{2p}(X) : d\varphi = 0\} \quad &\text{and} \quad B \;=\; d\mathcal{E}^{2p-1}(X, M) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} H_{2p}(X, M; \mathbf{R}) &= \widetilde{Z}/\widetilde{B} \qquad \text{where} \\ \widetilde{Z} \;=\; \{T \in \mathcal{E}'_{2p}(X) : dT \in j_*\mathcal{E}'_{2p-1}(M)\} \quad &\text{and} \quad \widetilde{B} \;=\; d\mathcal{E}'_{2p+1}(X). \end{aligned} \tag{3.4}$$

It is an interesting fact, established in the next section, that the group $\overline{H}_{2p}(X, M; \mathbf{Z}) \subset H_{2p}(X, M; \mathbf{R})$ carries a "real Hodge filtration". A key point is the following lemma.

Lemma 3.1. *Fix $\tau \in H_{2p}(X, M; \mathbf{R})$. If $T, T' \in \mathcal{E}_{2p}(X, M)$ are relatively closed currents representing τ, then*

$$T(\varphi) \;=\; T'(\varphi) \quad \textit{for all} \;\; \varphi \in \mathcal{E}^{2p}(X) \;\; \textit{with} \;\; d\varphi = 0. \tag{3.5}$$

Hence, the notion of $\tau(\varphi)$ is well defined for such φ. Furthermore,

$$dT \;=\; dT' \;=\; \sum_{j=1}^{L} r_j [M_j] \tag{3.6}$$

where $M = M_1 \cup \cdots \cup M_L$ is the decomposition into connected components and the r_j are real numbers. Thus, $\partial\tau \;=\; \sum_{j=1}^{L} r_j [M_j]$ is well defined.

Proof. Since T and T' both represent $\tau \in \widetilde{Z}/\widetilde{B}$ and $\widetilde{B} = d\mathcal{E}'_{2p+1}(X)$ by (3.4), we have $T - T' = dR$ for $R \in \mathcal{E}'_{2p+1}(X)$. This proves (3.5) and that $dT = dT'$. Since $T \in \widetilde{Z}$, (3.4) says that $dT = j_*u$ where $u \in \mathcal{E}_{2p-1}(M)$. This implies that $du = 0$. Hence, u is a locally constant function on M. ∎

Definition 3.2. A class $\tau \in H_{2p}(X, M; \mathbf{R})$ is called **positive** if $\tau(\varphi) \geq 0$ for all closed, real $2p$-forms φ such that the component

$$\varphi^{p,p} \;\geq\; 0 \quad \text{(is weakly positive) on } X.$$

If τ is positive, then it is of type (p, p) as defined in 4.1 below.

Proposition 3.3. *A class $\tau \in H_{2p}(X, M; \mathbf{R})$ is positive if and only if it is represented (in the complex $\mathcal{E}_*(X, M)$) by a strongly positive current of type (p, p).*

This proposition will be proved below. We first observe that it leads to the following main result.

THEOREM 3.4. *Suppose $\tau \in \overline{H}_{2p}(X, M; \mathbf{Z})$ is positive. Suppose each component of M has a real analytic point (or, more generally, is two-sided at some point). Then there exists a positive holomorphic p-chain T on X with $dT = \partial\tau$. Furthermore, there exists a positive d-closed (p, p)-current S with $\tau = [T + S]$.*

In particular, if the positive classes in $H_{2p}(X;\mathbf{Q})$ are all represented by positive holomorphic chains with rational coefficients, then so are all the positive classes in $H_{2p}(X,M;\mathbf{Q})$.

Thus for example, given any real analytic M in a Grassmann manifold X, we conclude that every positive class in $H_{2p}(X,M;\mathbf{Z})$ carries a positive holomorphic chain. However there are projective manifolds X with positive (p,p)-classes in $\overline{H}_{2p}(X;\mathbf{Z})$ which do not carry positive holomorphic cycles. In fact, for every integer $k \geq 2$ there exists an abelian variety X of complex dimension $2k$ and a class $\tau \in H_{2k}(X;\mathbf{Z})$ which is represented by a positive (k,k)-current and also by an algebraic k-cycle, but τ is not represented by a positive algebraic k-cycle (see [L]).

Proof. By Proposition 3.3 and (3.6) in Lemma 3.1, the class τ is represented by a positive (p,p)-current T with $dT = \partial\tau = \sum_i n_i[M_i]$ for integers n_i (cf. the argument for (3.2) above.) Applying Theorem 2.7 with $\Gamma = dT$, we deduce the existence of a positive holomorphic chain T_0 with

$$dT_0 \;=\; dT \qquad \text{and} \qquad T - T_0 \geq 0. \qquad \blacksquare$$

Proof of Proposition 3.3. Consider the closed convex cones

$$\begin{aligned} P \;&\equiv\; \{\varphi \in \mathcal{E}^{2p}(X) : \varphi^{p,p} \text{ is weakly positive}\} \;\subset\; \mathcal{E}^{2p}(X) \\ \widetilde{P} \;&\equiv\; \{T \in \mathcal{E}'_{2p}(X) : T = T_{p,p} \text{ is strongly positive}\} \;\subset\; \mathcal{E}'_{2p}(X) \end{aligned}$$

These are polars of each other in the dual pair $\mathcal{E}^{2p}(X)$, $\mathcal{E}'_{2p}(X)$. Moreover, by the Relative de Rham Theorem and (3.3) and (3.4) we have:

(i) $\widetilde{B} \subset \mathcal{E}'_{2p}(X)$ is closed (in the weak topology).

(ii) Z and $\widetilde{B}$ are polars of each other in the dual pair $\mathcal{E}^{2p}(X)$, $\mathcal{E}'_{2p}(X)$.

(iii) $B \subset \mathcal{E}^{2p}(X)$ is closed.

(iv) B and $\widetilde{Z}$ are polars of each other in the dual pair $\mathcal{E}^{2p}(X)$, $\mathcal{E}'_{2p}(X)$.

Lemma 3.5. *The subset $\widetilde{P} + \widetilde{B}$ is closed in the standard topology on $\mathcal{E}'_{2p}(X)$.*

Proof. Let $\{T_i\} \subset \mathcal{P}_{p,p}$ and $\{dS_i\} \subset \widetilde{B}$ be sequences such that

$$T_i + dS_i \;\longrightarrow\; R \qquad \text{weakly in } \; \mathcal{E}'_{2p}(X)$$

Let ω denote the Kähler form on X. Then

$$p!M(T_i) \;=\; T_i\,(\omega^p) \;=\; (T_i + dS_i)\,(\omega^p) \;\longrightarrow\; R\,(\omega^p)\,,$$

and so the masses $M(T_i)$ are uniformly bounded. By the compactness theorem for positive currents there is a subsequence, again denoted by T_i, converging to a positive current T. Hence, $dS_i \longrightarrow R - T$ weakly, and since d has closed range, there exists $S \in \mathcal{E}'_{2p+1}(X)$ with $dS = R - T$. $\blacksquare$

Proposition 3.6.

$$[(P \cap Z) + B]^0 \;=\; (\widetilde{P} \cap \widetilde{Z}) + \widetilde{B}$$

Proof. By standard principles we have $[(P \cap Z) + B]^0 = (P \cap Z)^0 \cap B^0 = \overline{(P^0 + Z^0)} \cap B^0$. By (ii), (iv) and Lemma 3.5 we have $\overline{(P^0 + Z^0)} \cap B^0 = \overline{(\widetilde{P} + \widetilde{B})} \cap \widetilde{Z} = (\widetilde{P} + \widetilde{B}) \cap \widetilde{Z}$. Finally it is easy to see that $(\widetilde{P} + \widetilde{B}) \cap \widetilde{Z} = (\widetilde{P} \cap \widetilde{Z}) + \widetilde{B}$ since $\widetilde{B} \subset \widetilde{Z}$. ∎

To complete the proof of Proposition 3.3 choose a current $T \in \widetilde{Z}$ which represents the class τ. By hypothesis T is in the polar of $(P \cap Z) + B$. Therefore, by Proposition 3.6 and (3.4), $T = T_0 + dS$ with $T_0 \in \widetilde{P}$. ∎

§4. A Real Hodge Filtration on $H_{2p}(X, M; \mathbf{R})$.

Definition 4.1. A homology class $\tau \in H_{2p}(X, M; \mathbf{R})$ is of **filtration level** k if $\tau(\varphi) = 0$ for all closed complex valued forms φ of type (r, s) with $r > p + k$. Classes of filtration level 0 are called **type (p,p)**.

Note 4.2. This induces a real Hodge filtration $F^k H_{2p}(X, M; \mathbf{R})$ on $H_{2p}(X, M; \mathbf{R})$ which extends the basic one $F^k H_{2p}(X; \mathbf{R}) = \bigoplus_{r=0}^{k} \{H_{p-r,p+r}(X) \oplus H_{p+r,p-r}(X)\}_{\mathbf{R}}$ on $H_{2p}(X; \mathbf{R})$.

Proposition 4.3. *Suppose $\tau \in H_{2p}(X, M; \mathbf{R})$ has filtration level k. Then τ is represented by a current*

$$T \;\in\; \{\mathcal{E}'_{p-k,p+k}(X) \oplus \cdots \oplus \mathcal{E}'_{p+k,p-k}(X)\}_{\mathbf{R}}\,, \tag{4.1}$$

and therefore,

$$dT \;\in\; \{\mathcal{E}'_{p-k-1,p+k}(X) \oplus \cdots \oplus \mathcal{E}'_{p+k,p-k-1}(X)\}_{\mathbf{R}}\,. \tag{4.2}$$

In particular, if τ is of type (p, p), then $\tau = [T]$ for a some (p, p)-current T, and each non-zero boundary component of $\partial\tau$ is maximally complex (cf. [HL$_1$]).

Proof. We start by establishing (4.2). Write $\partial\tau = \sum_j r_j [M_j]$ as in Lemma 3.1. Choose any smooth form $\psi \in \mathcal{E}^{r,s}(X)$ with $r + s = 2p - 1$ and either $r > p + k$ or $s > p + k$. Then $0 = \tau(d\psi) = (\partial\tau)(\psi) = \sum_j r_j \int_{M_j} \psi$. Since ψ is arbitrary, we conclude that for each M_j with $r_j \neq 0$, the Dolbeault components

$$[M_j]_{r,s} = 0 \quad \text{if either} \;\; s > p + k \;\; \text{or} \;\; r > p + k.$$

This gives (4.2). When $k = 0$ this means M_j is maximally complex.

Consider the case where τ is of type (p, p) with $2p \leq n$. Choose a current T representing τ. Then by standard harmonic theory $T_{2p,0} = h_{2p,0} - \overline{\partial}\beta$ where h is harmonic (in particular, smooth) and $\beta \in \mathcal{E}'_{2p,1}(X)$. Then $[T - d\beta]_{2p,0} = T_{2p,0} - \overline{\partial}\beta = h_{2p,0} \equiv h$ and because $T(\bar{*}h) = \|h\|^2 = 0$ (since $\tau = [T]$ is type (p, p)), we have $h = 0$. Thus replacing T by $T - \overline{\partial}\beta - \partial\overline{\beta}$ we can assume $T_{2p,0} = T_{0,2p} = 0$.

If $p = 1$, we are done. If $p > 1$, we note that $\overline{\partial} T_{2p-1,1} = M_{2p-1,0} = 0$, and so $T_{2p-1,1} = h_{2p-1,1} + \overline{\partial}\beta$ where $h_{2p-1,1}$ is harmonic and $\beta \in \mathcal{E}'_{2p-1,2}(X)$. We conclude as above that $h_{2p-1,1} = 0$, and then replace T by $T - \overline{\partial}\beta - \partial\overline{\beta}$ so that $T_{2p-1,1} = T_{1,2p-1} = 0$. Continuing in this fashion gives the result. All other cases are entirely analogous and details are left to the reader. ∎

§5. Residual Currents.

Definition 5.1. Let R be a weakly positive, d-closed (p,p)-current. Then R is **residual** if for each $c > 0$ the complex dimension of the subvariety $E_c(R) = \{z : \Theta(R,z) \geq c\}$ is $\leq p-1$.

Suppose T is a a weakly positive, d-closed (p,p)-current defined in the complement of $\mathrm{supp}(\Gamma)$ where Γ is a scarred $2p-1$ cycle (of class C^1). By the main result of [H] (see Theorem 6, p. 71 and the note added in proof) T has locally finite mass across $\mathrm{supp}(\Gamma)$. That is, T has a unique "extension by zero" across $\mathrm{supp}(\Gamma)$. Let T also denote this extension. It follows easily that from two theorems of Federer that $dT = \sum_{k=1}^{\infty} r_k \Gamma_k$ with constants $r_k \in \mathbf{R}$.

Definition 5.2. The set $\mathrm{supp}(\Gamma)$ is **residual** if each residual current R on $X - \mathrm{supp}(\Gamma)$ satisfies $dR = 0$ on X.

Proposition 5.3. *If each component of* Γ *has a two-sided point, then* $\mathrm{supp}(\Gamma)$ *is residual.*

Proof. Suppose R is a residual current on $X - \mathrm{supp}(\Gamma)$ with $dR = \sum_k r_k \Gamma_k$, $r_k \in \mathbf{R}$. Near a two-sided point of one of the components, sayΓ_1, we have $dR = r_1 \Gamma_1$. By Lemma 2.8 we can write R locally as $R = r_1 V^+ + S$ with $S \geq 0$ and $dS = 0$ across Γ_1. This contradicts the hypothesis that R is residual unless $r_1 = 0$. ∎

Remark. This Proposition combined with the first half of Lemma 2.8 and the next result provides a second proof of Theorem 2.4.

THEOREM 5.4. *Suppose* Γ *is a scarred* $2p-1$ *cycle (of class* C^1*) in an arbitrary complex manifold* X*. Assume each component* Γ_k *of* Γ *is residual at some point. If* $\Gamma = dT$ *for some weakly positive* (p,p)*-current* T *on* X*, then there exists a positive holomorphic* p*-chain* V *with* $\Gamma = dV$ *and* $T - V \geq 0$.

Proof. Suppose $\Gamma = dT$ as in the theorem and consider the decomposition $T = S + R$ into a positive *real-coefficient holomorphic chain* $S = \sum_{j=1}^{\infty} c_j V_j$ plus a residual current R (on $X - \Gamma$), Now $dR = \sum_{k=1}^{\infty} r_k \Gamma_k$ for some $r_k \in \mathbf{R}$, but by the hypothesis each r_k must be zero. Hence, $\Gamma = dS$ bounds a positive real-coefficient holomorphic chain.

Proposition 5.5. *Let* Γ *be a scarred* $2p-1$ *cycle in an arbitrary complex manifold* X*. If* $\Gamma = dS$ *bounds a positive real-coefficient holomorphic chain* $S = \sum_{j=1}^{\infty} c_j V_j$*, then* $\Gamma = dV$ *bounds a positive (integer-coefficient) holomorphic chain* V *with* $S - V \geq 0$ *(and therefore also* $\mathrm{supp}V \subseteq \mathrm{supp}S$*).*

Proof. By hypothesis $d\left(\sum_{j=1}^{\infty} c_j V_j\right) = \sum_{k=1}^{\infty} n_k \Gamma_k$. Near a regular point x in Γ_1 each V_j satisfies $dV_j = \epsilon_j \Gamma_1$ with $\epsilon \in \{-1, 0, 1\}$. By uniqueness there is at most one of the subvarieties V_j with boundary Γ_1. Relabel so that $dV_1 = \Gamma_1$. Now there are two cases.

Case 1: $dV_j = 0$ for all $j \geq 2$. In this case we must have $c_1 = n_1$, and we can eliminate the component Γ_1 from Γ.

Case 2: $-\Gamma_1$ bounds exactly one of the subvarieties V_j, $j \geq 2$. Relabel so that $-\Gamma_1 = dV_2$. In this case $c_1 - c_2 = n_1$. Note that $V_1 + V_2$ is a subvariety without boundary near the point x on Γ_1. Set $\widetilde{S} = S - n_1 V_1 = c_2(V_1 + V_2) + \sum_{j=3}^{\infty} c_j V_j$. Then $\widetilde{S}$ is positive and $d\widetilde{S} = 0$

near the point x. Consequently, $d\widetilde{S} = \sum_{j=2}^{\infty} b_j V_j$. Finally, the b_j's must be integers. In fact $b_j = n_j - \epsilon_{1j} n_1$ where $dV_1 = \epsilon_{1j}\Gamma_j$ defines $\epsilon_{1j} \in \{-1, 0, 1\}$. Hence, we can eliminate the component Γ_1 from Γ in this case as well

The proof can now be completed exactly as in the last paragraph of the proof of Theorem 2.7. ∎

Question 5.6. Which (maximally complex) $(2p-1)$-dimensional submanifolds are residual? Note that if Γ is two-sided, then Γ is residual. Moreover, if Γ is one-sided, then Γ has a natural orientation so that $\Gamma = dW$ where W is complex, and in this case the residual property is equivalent to the following uniqueness property:

$$\text{If } T \geq 0 \text{ satisfies } dT = \Gamma, \text{ then } T = W + S \text{ with } S \geq 0 \text{ and } dS = 0.$$

If Γ is zero-sided, then Γ is residual if and only if

$$T \geq 0 \quad \text{and} \quad \text{supp}\{dT\} \subset \Gamma \qquad \Rightarrow \qquad dT = 0.$$

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