AN A_1 WEIGHT NOT COMPARABLE WITH ANY QUASICONFORMAL JACOBIAN

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ABSTRACT. We construct an A_1 weight on the plane which is not comparable to the Jacobian of any quasiconformal mapping. As a consequence we construct a locally linearly connected, Ahlfors 2-regular surface in $\mathbb{R}^3$ which is not bi-Lipschitz equivalent to $\mathbb{R}^2$.

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1. Introduction

A locally integrable function w is called an A_1 weight if there is a $C < \infty$ so that

$$\frac{1}{\text{area}(B)} \int_B w \leq C \text{essinf}_{\text{B}} \text{w},$$

for every ball B in the plane. We say that w is comparable to a quasiconformal Jacobian if there is quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ and an $M < \infty$ so that $\frac{w}{M} \leq J_f \leq Mw$ holds almost everywhere. In this paper we prove:

Theorem 1.1. *There is an A_1 weight w on $\mathbb{R}^2$ which is not comparable to any quasiconformal Jacobian.*

The problem originates in the "quasiconformal Jacobian problem" (formulated by David and Semmes in their 1990 paper [6]) which asks how to characterize quasiconformal Jacobians up to comparability, and (so far as I know) was first explicitly stated by Semmes in [13] (following Conjecture 5.3 of that paper). The question also appeared as the second of 33 questions posed in [10] and in various other sources such as Section 4.6 of [9], Problem 3 of [8] and Question 4.5 of [4]. Also see [3], [12], [14] for the history of this problem and various partial results that have been obtained.

The basic idea of the proof of Theorem 1.1 is to construct a Sierpinski carpet E and a weight $w \geq 1$ which blows up on E. The set E is an intersection of sets $E_1 \supset E_2 \supset \dots$ so that each E_n contains thick annuli around each of its "holes" and w is chosen so that any corresponding $f(E_n)$ would have the same property. Using this one can show that a polygonal path in $f(E_n)$ can be modified to also remain inside $f(E_{n+1})$ with only a small addition of length. This implies $f(E)$ contains a rectifiable curve γ. Since w blows up on E, the Jacobian of f^{-1} vanishes on $f(E)$ and so $f^{-1}(\gamma)$ would have zero length, i.e., be a point, which is impossible since f is a homeomorphism. Thus w can't be comparable to any quasiconformal Jacobian.

If w is bounded and bounded away from zero, then it is comparable to the Jacobian of the identity map. However, the proof of Theorem 1.1 will show that given any positive function $\varphi(t)$ on $[0, \infty)$ so that $\varphi(t) \to 0$ as $t \to \infty$, we can take the weight w to satisfy $\text{area}(\{x : w(x) > \lambda\}) \leq \varphi(\lambda)$. Thus no additional assumption on the distribution function of an A_1 weight (weaker than boundedness) will force it to be comparable to a quasiconformal Jacobian.

As pointed out by Bonk, Heinonen and Saksman in [4], the quasiconformal Jacobian problem is closely related to the problem of determining which metric spaces are biLipschitz equivalent to the plane. Obviously, such a space must be homeomorphic to $\mathbb{R}^2$. Two additional necessary conditions are that the surface be linearly locally connected (LLC) and Ahlfors 2-regular. A metric space is LLC if there is a $C < \infty$ so that $B(x, r)$ can always be contracted to a point in $B(x, Cr)$. Being Ahlfors 2-regular means that Hausdorff 2-dimensional measure, $\mathcal{H}_2$, satisfies $r^2/C \leq \mathcal{H}_2(B(x, r)) \leq Cr^2$ for some $C < \infty$ which is independent of x and r.

A result of Bonk and Kleiner [5] states these these conditions imply the space is quasisymmetrically equivalent to $\mathbb{R}^2$, but Laakso [12] gave an example of such a space which is not bi-Lipschitz equivalent to the plane. However, his surface is not embeddable in any finite dimensional Euclidean space (or even any uniformly convex Banach space) and in [8] Heinonen asked if such an example could be embedded in $\mathbb{R}^n$. He also pointed out that the example in Theorem 1.1 implies the answer is yes, i.e., there is a LLC, Ahlfors 2-regular surface in $\mathbb{R}^n$, for some $n < \infty$, which is not biLipschitz equivalent to $\mathbb{R}^2$. This is because a theorem of Semmes (see Theorem 5.2 of [13] or Theorem 4.5 of [9]) implies that any A_1 weight is the Jacobian of an embedding of $\mathbb{R}^2$ into some finite dimensional $\mathbb{R}^n$ and the image is LLC and 2-regular. The surface corresponding to the weight w in Theorem 1.1 can't be biLipschitz equivalent to $\mathbb{R}^2$, for this would give a quasisymmetric map of the plane to itself (and hence a quasiconformal map) with Jacobian comparable to w. Some extra work shows we can take $n = 3$:

Corollary 1.2. *There is a surface $S \subset \mathbb{R}^3$ that is Ahlfors 2-regular and locally linearly connected, but is not bi-Lipschitz equivalent to $\mathbb{R}^2$.*

Theorem 1.1 was obtained while I attending the Ahlfors-Bers colloquium in Ann Arbor, May 2005. I thank the organizers for a stimulating conference and Mario Bonk and Juha Heinonen for very helpful comments concerning the problem addressed here. Thanks to Leonid Kovalev for some corrections to an earlier version of the paper. Also many heartfelt thanks to the referee, especially for finding an error in the original proof of the "good path lemma" (Lemma 2.2), for pointing out the specific statement of the problem in [13] and for various other helpful comments on two different versions of the manuscript. I greatly appreciate the effort the referee put into carefully reading and reporting on my work.

2. The construction

2.1. **The definition of E.** Our weight w will tend to ∞ on a set E which we describe now. E will be a type of Sierpinski carpet, but with a construction that varies at different generations. For odd, positive integers $L \geq 3$, $N \geq 3$ and any positive integer M we define a (L, M, N)-piece as follows. Take a square Q and divide it into L^2 equal subsquares. These are called the type 1 subsquares of Q. Let Q_0 denote the center square of this collection and divide Q_0 into M^2 equal subsquares (the type 2 subsquares of Q). Then divide each of type 2 squares into N^2 equal subsquares (called type 3). See Figure 1. Remove the center type 3 square from each type 2 squares. These are called the omitted squares associated to Q. This leaves a set $\Omega = \Omega(L, M, N) \subset Q$ so that

1. Ω consists of Q with M^2 squares of side length $\ell(Q)/LMN$ removed.
2. $\text{area}(\Omega) = \text{area}(Q)(1 - (LN)^{-2})$.
3. Ω contains disjoint annuli around each omitted square with radii ratio $\sim N$.

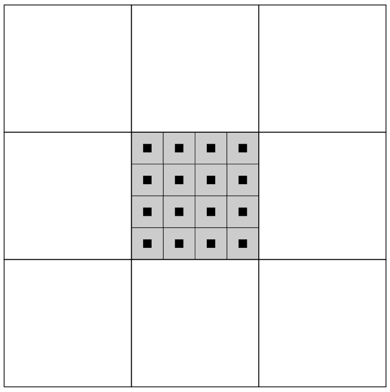

FIGURE 1. This shows an (L, M, N)-piece for $L = 3$, $M = 4$, $N = 3$. The type 1 squares are white, the type 2 are gray and the omitted squares are black.

Given sequences $\{L_n\}, \{M_n\}, \{N_n\}$, $n = 1, 2, \ldots$, we define a compact set as follows. Let $t_n = (L_n M_n N_n)^{-1}$ and $s_n = \prod_{k=1}^n t_n$. E_0 is the unit square. $E_1 \subset E_0$ is the associated (L_1, M_1, N_1)-piece. Divide E_1 into disjoint squares of size s_1 (these are called the first generation squares) and replace each of these with the associated (L_2, M_2, N_2)-piece. This gives $E_2 \subset E_1$. In general, given E_{n-1} we divide it into $(n-1)$st generation squares of size s_{n-1} and replace each of these by the associated (L_n, M_n, N_n)-piece to get E_n. Let $E = \cap_n E_n$.

For our construction to work it suffices to take $\{L_n\}$ constant, and for $\{M_n\}$ and $\{N_n\}$ to satisfy:

$$\sum_n N_n^{-2} = \infty, \tag{2.1}$$

$$\sum_n N_n^{-3} < \infty, \quad \sum_n M_n^{-2} < \infty. \tag{2.2}$$

To be specific, we can take $L_n = 7$ for all n, $M_n = n$, and $N_n = 3 + \frac{1}{8}\lfloor\sqrt{n}\rfloor$. Then $L_n N_n \leq \sqrt{n}$ for n large (say $n \geq n_0 = 168$) and hence

$$\text{area}(E_n) \leq \prod_{k=n_0}^{n} (1 - (L_k N_k)^{-2}) \leq \prod_{k=n_0}^{n} (1 - \frac{1}{k}) = O(n^{-1}). \tag{2.3}$$

In particular, (2.1) implies area$(E) = 0$. Similarly, if S is an nth generation square and $k > n$, then

$$\frac{\text{area}(E_k \cap S)}{\text{area}(S)} \leq C\frac{k}{n}.$$

Suppose we are given a nth generation square Q. Because $L_n = 7$ we can find a collection of type 1 subsquares of size $\ell(Q)/L_n$ whose union separates the central type 1 square from the boundary of Q and such that the distance of any of these squares from the central square or from the boundary of Q is $\geq \ell(Q)/7$. We will call these the "ring squares" associated to Q (since they form a topological annulus which separates the center of Q from its boundary). See Figure 2. We denote the union of the ring squares by W (or by W_Q to be more precise). Note that by the compactness of K-quasiconformal maps, there is a constant $0 < C_K < \infty$ so that

$$\text{dist}(f(W), \partial f(Q) \cup f(Q')) \geq C_K \text{diam}(f(Q)), \tag{2.4}$$

for any K-quasiconformal f which fixes 0 and ∞, where Q' is the central type 1 square in Q.

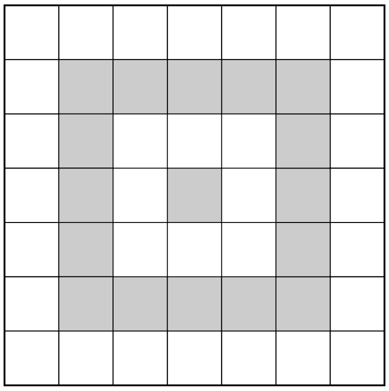

FIGURE 2. The ring W of type 1 squares which separates the central type 1 square Q' from the boundary of Q.

2.2. **The definition of the weight w.** Let F_n denote a s_n-neighborhood of E_n (this fills in the nth generation omitted squares and adds a "collar" of width s_n to the boundary of earlier omitted squares). Note that $\text{area}(F_n) \leq 2\text{area}(E_n)$ and $\text{dist}(F_n, F_{n-1}^c) \geq \frac{1}{2}s_{n-1}$. Given a sequence $A_n \nearrow \infty$ with $A_0 = 1$, we define the weight w by $w(x) = 1$ for $x \notin F_1$ and $w(x) = A_n$ for $x \in F_n \setminus F_{n+1}$. Let $a_n = A_n - A_{n-1}$, for $n = 1, 2, \ldots$. Then on F_1, $w(x) = 1 + \sum_{k=1}^{\infty} a_k \chi_{F_k}(x)$, so w is locally integrable as long as

$$\sum_{k=1}^{\infty} a_k \text{area}(F_k) \leq 2 \sum_{k=1}^{\infty} a_k \text{area}(E_k) = O(\sum_{k=1}^{\infty} \frac{a_k}{k}) < \infty. \tag{2.5}$$

Finally, assume there is a $\lambda < \sqrt{2}$ so that

$$A_{2n} \leq \lambda A_n, \quad n = 1, 2, 3, \ldots. \tag{2.6}$$

This is a "slow growth" condition on A_n which clearly implies

$$A_n = o(\sqrt{n}) = o(n), \tag{2.7}$$

(to see this, note that if $2^m < n \leq 2^{m+1}$, then $A_n \leq A_{2^{m+1}} \leq \lambda^k = o(2^{m/2}) = o(\sqrt{n})$). Moreover, (2.6) implies

$$\begin{aligned} n\sum_{k=n}^{N}\frac{a_k}{k} &= n[-\frac{A_{n-1}}{n} + \sum_{k=n}^{N-1} A_k(\frac{1}{k} - \frac{1}{k+1}) + \frac{A_N}{N}] \\ &= O(A_n) + O(n\sum_{k=n}^{N-1} A_k \frac{1}{k^2}) + n\frac{A_N}{N} \\ &\leq O(A_n) + O(n\sum_{j=0}^{\infty} n2^j \cdot A_{2^{j+1}n} \cdot n^{-2}2^{-2j}) + n\frac{A_N}{N} \\ &\leq O(A_n) + O(A_n\sum_{j=0}^{\infty} \lambda^j 2^{-j}) + n\frac{A_N}{N} \\ &\leq O(A_n) + n\frac{A_N}{N}. \end{aligned}$$

For a fixed n the last term tends to zero as $N \to \infty$, so

$$n\sum_{k=n}^{\infty}\frac{a_k}{k} = O(A_n) \tag{2.8}$$

Thus for any nth generation square S,

$$\frac{1}{\text{area}(S)}\int_S w \leq A_n + O(n\sum_{k=n}^{\infty}\frac{a_k}{k}) \leq O(A_n) \tag{2.9}$$

Since $A_n = \text{essinf}_{E_n} w$, this is similar to the A_1 condition (but only for generation squares; we will verify it for all balls below). Also note that we can take $A_n \to \infty$ as slowly as we wish, which means that we can make area$(\{x : w(x) > \lambda\}) \to 0$ as quickly as we wish, as claimed in the introduction.

Next we record an observation that we will need in the proof of Corollary 1.2. If $x \notin E_n$ then $\text{dist}(x,E) = \text{dist}(x,E_n)$ and if $x \in E_n \setminus E$ then $\text{dist}(x,E) \leq s_{n+1}/2$ (since x must be in one of the omitted squares of E_k for some $k \geq n+1$). Thus

$$\{x : s_{n+1} \leq \text{dist}(x,E) < s_n\} = \{x : s_{n+1} \leq \text{dist}(x,E_n) < s_n\} = F_n \setminus F_{n+1}.$$

Thus $w(x)$ only depends on the distance of x to E. Hence we can write

$$\sqrt{w(x)} = f(\text{dist}(x,E)), \tag{2.10}$$

for some function f on $(0,\infty)$. Moreover, we can take f to satisfy

$$f(r) = 1 \quad \text{if } r \geq r_0, \tag{2.11}$$

$$f \text{ is non-increasing,} \tag{2.12}$$

$$1 \leq \frac{f(r)}{f(2r)} \leq 1+\tau, \tag{2.13}$$

and we can take τ as small as we like by choosing the $\{A_n\}$ appropriately.

2.3. **Proof that w is an A_1 weight.** We have seen that w satisfies the A_1 condition with respect to nth generation squares, so all we need to do is check that it also satisfies it with respect to any ball. Suppose $B = B(x,r)$ is a ball of radius r and choose n so that $s_{n+1} \leq 4r < s_n$. The infimum of w on B is finite and must equal A_k for some k, i.e., $B \subset F_k$, and some point of B lies outside F_{k+1}.

If $k \leq n$, then $4r < s_n \leq s_k$ so this implies $2B$ lies outside E_{k+1} and hence B lies outside F_{k+2}. Thus w is bounded above by A_{k+2} on B. Since $A_k \geq 1$ and $a_k \leq 1/2$ for all k, we have $A_{k+2} = A_k + a_{k+1} + a_{k+2} \leq 2A_k$.

If $k > n$, then $r \geq \frac{1}{4}s_{n+1} \geq \frac{1}{4}s_k$. Then B can be covered by disjoint squares of size s_{k+1} and which are all contained in $2B$. Thus the average of w over B can be no worse than twice the average over any of these squares, which by (2.9) is at most $2A_{k+1} \leq 4A_k$. Thus w is an A_1 weight.

Later we will also need the following observation. Suppose Q is an nth generation square, $Q' \subset Q$ is a type 2 subsquare and $Q'' \subset Q'$ is the corresponding omitted square. Suppose B_1, V_2 are balls, such that $Q'' \subset B_1 \subset B_2 \subset Q'$. Then the average of w over B_1 and B_2 are comparable with a constant which is independent of n. This holds since w is an A_1 weight and the essential infimum over both balls is the same (since both are in F_n and neither is in F_{n+1}).

2.4. **$f(E_n)$ has big annuli.** Suppose f is a K-quasiconformal map whose Jacobian J_f is comparable with constant M to the weight w defined above. Eventually, we will prove that $f(E)$ contains a rectifiable arc γ. If we can do this, then f^{-1} has a Jacobian that vanishes on γ which means that $f^{-1}(\gamma)$ has length zero, i.e., is a single point. This is impossible since f is a homeomorphism, so we deduce that there is no such f.

The first step is to show that if we consider nested squares in adjacent generations, then their images under f have about the same size ratio as the original squares. This uses the slow growth of our weight near E and will follow from:

Lemma 2.1. *Suppose B_1 is a unit ball centered at the origin and $B_2 \subset B_1$ is the concentric ball with radius ϵ. Suppose f is a K-quasiconformal map of the plane which fixes 0 and ∞ and also suppose that the averages of the Jacobian, J_f, over the two balls are comparable with constant M. Then there is an $r > 0$ such that $f(B_1 \setminus B_2)$ contains a round Euclidean annulus with outer radius r and inner radius at most $\sqrt{M}\epsilon C_K^2 r$, where C_K is a constant that depends only on K.*

Proof. Let $A = \text{area}(f(B_1))$ and $a = \text{area}(f(B_2))$. By the compactness of K-quasiconformal maps there is a $C_K < \infty$ (depending only on K) and that there are radii $r > s > 0$ so that

$$B(0, r) \subset f(B_1) \subset B(0, C_K r),$$

$$B(0, s/C_K) \subset f(B_2) \subset B(0, s)$$

Thus $\pi r^2 \leq A \leq \pi C_K^2 r^2$ and $\pi s^2/C_K^2 \leq a \leq \pi s^2$. Since the average of the Jacobian over B_2 is comparable to the average over B_1 with constant M, we have $a \leq M\epsilon^2 A$ and hence $\pi s^2/C_K^2 \leq M\epsilon^2 \pi C_K^2 r^2$, or

$$s \leq \sqrt{M} C_K^2 \epsilon.$$

Therefore $\{z : \sqrt{M}\epsilon C_K^2 r < |z| < r\} \subset f(B_1) \setminus f(B_2)$, as desired □

Note that if we used merely that f is K-quasiconformal, we would have gotten an estimate involving $\epsilon^{1/K}$ by the Hölder continuity of quasiconformal maps. Since we are assuming something strong about the Jacobian of the map, we get a stronger conclusion.

We will make several uses of the lemma in the remainder of the proof. First, recall that if Q is an omitted square of E_n then Q has side length s_n and is contained in a ball radius $s_n\sqrt{2}$. Moreover, it is contained in a ball of radius $N_n s_n/\sqrt{2}$ which hits no other omitted squares. By the remark at the end of subsection 2.3, the averages of w over these to balls are comparable and hence the Jacobian of f has averages over these two balls that differ by at most a factor of CM^2. Hence, by the lemma, we see that $f(E_n)$ contains a round Euclidean annulus around $f(Q)$ where the outer radius exceeds the inner radius by at least a factor of $N_n/M\sqrt{2C}C_K^2$.

Second, if Q' is a type two subsquare of Q then we can deduce $\text{diam}(f(Q')) \leq C\text{diam}(f(Q))\sqrt{M}C_K^2/M_n$. Combined with (2.4) this means that if M_n is large enough (depending on K) then there is a $\delta = O(1/M_n)$ so that

$$2\mathrm{diam}(f(Q')) \leq \delta \mathrm{diam}(f(Q)) \leq \frac{1}{2}\mathrm{dist}(f(R), \partial f(Q)). \tag{2.14}$$

2.5. **Statement of the "good path" lemma.** A "good path" for $f(E_n)$ is a polygonal path which is contained in $f(E_n)$ and all of whose vertices lie in sets of the form $f(\partial Q)$ where Q is a nth generation square. The main point of the whole construction is contained in the following lemma.

Lemma 2.2. *Suppose f is K-quasiconformal and has Jacobian comparable to w with constant M. Then there are n_0, C_1, C_2 depending on K and M so that if $n \geq n_0$ the following holds. Suppose γ is a good path for $f(E_{n-1})$. Then there is a good path γ' for $f(E_n)$ such that*

1. *every vertex of γ is also a vertex of γ',*
2. *the length of γ' satisfies $\ell(\gamma') \leq \ell(\gamma)(1 + C_1/M_n^2 + C_2/N_n^3)$.*

It is obvious that $f(E_{n_0})$ contains a good path γ_{n_0} (just take a line segment). Thus the lemma and induction produce a sequence of good paths $\gamma_n \subset f(E_n)$ which connects the endpoints of γ_{n_0} and whose lengths are bounded by $\prod_{n=1}^{\infty}(1 + O(\frac{1}{M_n^2}) + O(\frac{1}{N_n^3}))$. This is finite if the conditions in (2.2) hold. Standard arguments then imply that $f(E)$ contains a rectifiable path connecting the endpoints of γ_1 (which were in $f(E)$ by construction). Thus to prove Theorem 1.1 it only remains to prove Lemma 2.2

The key point of the lemma is the power of 3 in the N_n term. We need $\sum_n N_n^{-2} = \infty$ to make E have zero area, and we need $\sum_n N_n^{-3} < \infty$ to produce a rectifiable arc in $f(E)$, so the gap between 2 and 3 is what allows our to example work.

2.6. **Proof of the good path lemma.** Let Y_n denote the f image of the boundaries of all nth generation squares. Since $Y_{n-1} \subset Y_n$, and since the endpoints of a good path γ for E_{n-1} lie in Y_{n-1}, they also lie in Y_n. If γ happens to lie inside $f(E_n)$ just let $\gamma' = \gamma$. Clearly the lemma is satisfied.

Otherwise, γ hits some omitted image squares for $f(E_n)$ and hence hits the central type 1 image squares for the corresponding $(n-1)$st generation squares. Enumerate these as $\{f(Q_k)\}$. If γ hits $f(Q_k)$ then it hits it along a line segment in γ (since γ has no vertices in $f(Q_k)$). Indeed, γ can have no vertices until it reaches the boundary of the corresponding $(n-1)$st generation image square, thus the line segment that hits $f(Q_k)$ also hits $f(W_k)$, where W_k is the ring that surrounds Q_k (see Figure 2). If J is an edge of γ which hits $f(Q_k)$ then add two vertices to γ at the two points of $f(W_k)$ first hit by J after leaving $f(Q_k)$ in either direction. This adds a finite set of new vertices to γ. Let $\{I_j\}$ be an enumeration of edges of the new γ which intersect some central image square.

Let Q be a $(n-1)$st generation square and Q' its central type 1 square. For convenience we drop the "j" and consider a single interval I hitting the central image square $f(Q')$ and with its endpoints on the boundary of corresponding image ring squares. By (2.4), the length of I and the distance of its endpoints to the corresponding image $(n-1)$st generation square, $f(Q)$, are both comparable to $\mathrm{diam}(f(Q))$. To simplify we renormalize so that $I=[0,1]$ and use terms like "left", "right", "top", "bottom" accordingly.

Suppose I is divided into five, disjoint pieces, $I_1,\dots I_5$, so that all five pieces have length comparable to the length of I (with constant depending on K), but such that I_1, I_2, I_4 and I_5 have length less that $\frac{1}{2}\mathrm{dist}(f(W), \partial f(Q) \cup f(Q'))$. Thicken the interval I to make a rectangle $R = I \times [-\frac{1}{2}\delta, -\frac{1}{2}\delta]$ of width δ where δ is as in (2.14).

Let R' be the rectangle with the same axis as R, but three times as thick. The rectangle R is divided into five subrectangles R_1, R_2, R_3, R_4, R_5 corresponding to the thickenings of the intervals $I_1,\dots,I_5$. Let $E_j = \partial R_j \cap \partial R_{j+1}$ for $j=1,2,3,4$. See Figure 3. Note that these are short segments perpendicular to I. Also note that R_1, R_2, R_4, R_5 are contained in $f(Q)$ and disjoint from $f(Q')$ (by our condition on the lengths of the intervals $I_1,\dots,I_5$). Let $X_2 \subset Y_n \cap R_2$ be a curve which connects the top and bottom edges of R_2 (such an X_2 clearly exists because the images of nth generation squares cover R_1 and have diameter $\le \frac{1}{2}\delta$ which is much smaller than the length of R_2). Similarly define $X_4 \subset R_4$.

Next we define a family of paths Γ_1 in R which all start at 0 and end at 1. Inside R_3 they run parallel to I and continue parallel to I through R_2 until they first hit X_2 and then run in a straight line to 0. On the other side of R_3, they continue straight into R_4 until they first hit X_4 and then run in a straight line to 1. See Figure 3. Thus each path in Γ_1 begins and ends at the same points and by the Pythagorean theorem, they all have length bounded by $|I|(1+O(\delta^2)) = |I|(1+O(M^{-2})$. Moreover, all the vertices of these polygonal paths lie in Y_n, as required.

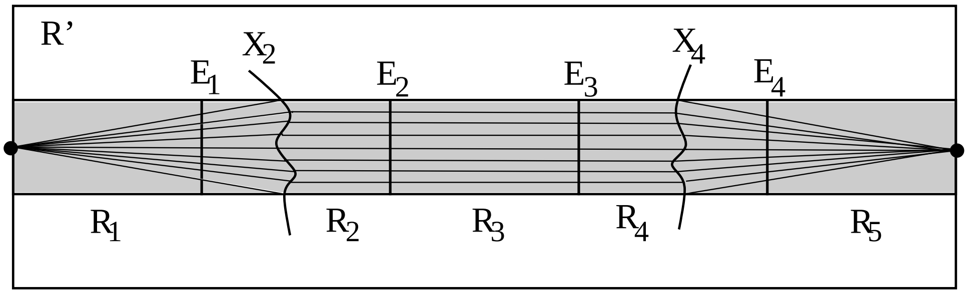

FIGURE 3. The construction of Γ_1.

Each omitted image square $f(Q_k)$ in $f(Q')$ hit by R can be put it into a Euclidean square S_k with sides parallel to those of R so that expanding S_k by a factor of $\lambda^{-1} \geq N_n/CMC_K^2$ (again by Lemma 2.1) we obtain a collection of disjoint squares, denoted $\lambda^{-1}S_k$, which are all contained in R'. By the disjointness of the expanded squares we get

$$\sum_j (\lambda^{-1}\ell(S_k))^2 \leq \text{area}(R') \leq 3\delta. \tag{2.15}$$

The orthogonal projection of S_k on E_2 will be denoted J_k (this corresponds to a horizontal projection in Figure 3). Fix a k and let L be a line parallel to I which contains one side of S_k. Choose a closed path $X_0 \subset Y_n$ which surrounds S_k and has comparable diameter and hence hits L at two points $\{a, b\}$, separated on L by ∂S_k. Let W be the rectangle in $\lambda^{-1}S_k$ which is the union of segments parallel to L which hit S_k (see Figure 4). We also define curves $X_1, X_2 \subset Y_n$ which each connect the two long sides of W, are separated in W by S_k and each with distance $\geq \frac{1}{4}\lambda^{-1}\ell(S_k)$ from X_0 (this is possible since $(n+1)$st generation image squares cover $W \setminus S_k$ and have much smaller diameter than W).

Now for each omitted image square S_j hit by R_3 we change the definition of the paths in Γ_1 which correspond to parameters in J_j. Outside $\lambda^{-1}S_j$ we leave them alone. Inside $\lambda^{-1}S_j$, we replace each path by one that passes around S_j, being careful to place the vertices in Y_n, as shown in Figure 4. More precisely, since the path hits S_k it enters $\lambda^{-1}S_k$ in a "short" side of W and defines a line segment L' which is parallel to L. We let the path follow L' until it hits X_1. From this point we let it run in a straight line to the closer of the points a, b, then along L to the other point, and then in a straight line to a point of $X_2 \cap L'$. The path then continues along L' until it leaves W. Since W has width $\sim \ell(S_k)$ and $\mathrm{dist}(X_0, X_i) \sim \lambda^{-1}\ell(S_k)$ for $i = 1, 2$, the new path is at most $O(\lambda^2 \ell(L')) \sim \lambda \ell(S_k)$ longer than L' by the Pythagorean theorem. Doing this for every square gives the path family Γ_2.

It may be difficult to see what happens to any one path, but we can easily compute the length of an average path crossing R. If we sum up over all the squares S_j then the additional length of the path crossing R_2 and starting at $x \in E_1$ is at most

$$L(x) = \sum_j C\lambda \ell(S_j) \chi_{J_j}(x).$$

Integrating over E_1 and dividing by the length of E_1 gives the average additional length as

$$\frac{1}{|S_1|}\int_{E_1} L(x) \leq \delta\lambda \sum_j \ell(S_j)^2 \leq \frac{C}{\delta}\lambda^3\delta = O(\lambda^3),$$

by (2.15). Thus there is at least one path through R' which connects E_1 and E_2 and which has length $\leq |I|(1 + C\lambda^3)$.

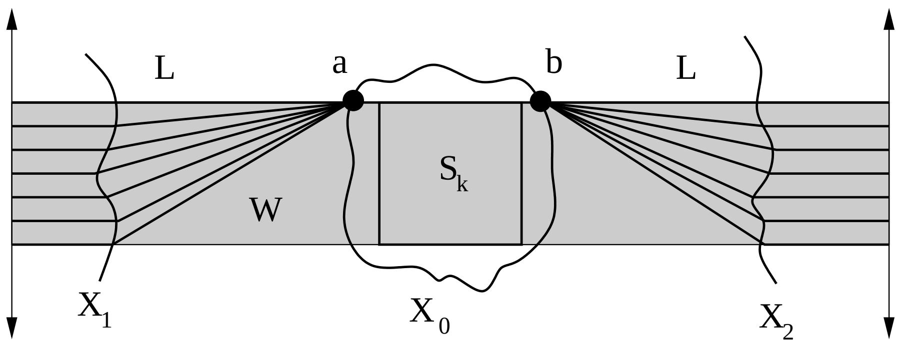

FIGURE 4. The construction of Γ_2.

Thus there is a polygonal path with the same endpoints as I, which stays inside $f(E_n)$ and which has length $\leq |I|(1 + O(\delta^2))(1 + O(\lambda^3))$. Since $\delta \leq O(1/M_n)$ and $\lambda \leq O(1/N_n)$ we see that (2.2) implies Lemma 2.2 and hence Theorem 1.1.

3. A SURFACE IN $\mathbb{R}^3$ NOT EQUIVALENT TO THE PLANE

Throughout this section w and E will refer to the weight and set constructed in Section 2. Our goal is to prove Corollary 1.2.

Our surface S will be the image of $\mathbb{R}^2$ under a quasisymmetric map Ψ which will be the composition of two maps. The first map P is given by the vertical projection from $\mathbb{R}^2$ to the graph of a Lipschitz function on $\mathbb{R}^2$, namely,

$$P(x) = (x, \min(1, \mathrm{dist}(x, E))).$$

Clearly P is biLipschitz. The second map is the quasiconformal map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ given by the following result from [1]

Theorem 3.1. *There is a non-increasing function f on $(0, \infty)$ which satisfies*

$$\lim_{t\to 0^+} f(t) = \infty,$$

and a a quasiconformal mapping $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that each of the following conditions hold.

1. *f can be taken to grow as slowly as we wish, e.g., given a sequence $s_n \searrow 0$ and any $\lambda > 1$ we may assume $f(s_{2n}) \leq \lambda f(s_n)$.*
2. *There is a $C < \infty$ so that for all $z, w \in \mathbb{R}^2 \times \{0\}$,*
$$\frac{1}{C} \leq \frac{|\Phi(z) - \Phi(w)|}{|x-y| f(|z-w|)} \leq C.$$
3. *The map Φ is smooth on $\mathbb{R}^2 \times (0, \infty)$ and its Jacobian satisfies $J_\Phi(z, t) \sim f^3(t)$.*

This is closely related to a result of David and Toro. If a function f satisfies conditions (2.11), (2.12) and (2.13), then second claim of Theorem 3.1 is Theorem 2.10 of [7]. The third claim is not explicitly stated in [7], but it probably follows from the estimates found there (some details about bounds for derivatives in the construction would have to be checked).

So assume we have f and Φ as given by Theorem 3.1 and let $\Psi = \Phi \circ P$. Clearly Ψ is quasisymmetric so $S = \Psi(\mathbb{R}^2) \subset \mathbb{R}^3$ is locally linearly connected. Define $A_n = f^2(s_n)$ and let w be the corresponding weight. We may assume A_n satisfies (2.6) by taking f to grow slowly enough (e.g., $\lambda < 2^{1/4}$ in part 1 of Theorem 3.1).

Let $\mathcal{H}_2$ denote 2-dimensional Hausdorff measure and let dm denote area measure in the plane. To see that S is Ahlfors 2-regular, we first want to check that $\Psi(E)$ (which also equals $\Phi(E)$) has zero 2-dimensional measure. Note that E can be covered by $N = \mathrm{area}(E_n) s_n^{-2}$ squares of side length s_n and recall that $\mathrm{area}(E_n) = O(1/n)$ by (2.3). Thus $\Psi(E)$ can be covered by the images of these squares, each of which has diameter at most $C s_n f(s_n) \sim s_n A_n \sim s_n o(\sqrt{n})$. Thus the 2-dimensional Hausdorff measure of $\Psi(E)$ is at most $N s_n^2 A_n^2 = (1/n) o(n)$. Since this tends to zero as $n \to \infty$ we deduce $\Psi(E)$ has zero measure.

Next suppose $x \in S$ and $r > 0$ and let $B_0 = B(x,r) \cap S$. Let $y = \Psi^{-1}(x)$. By quasisymmetry there are balls B_1, B_2 with

$$B_1 = B(y,s) \subset \Psi^{-1}(B_0) \subset B_2 = B(y, Cs) \subset \mathbb{R}^2,$$

where $r \sim sf(s)$ and C depends only on f. By our previous remarks, $|\Psi(E \cap B_1)$ has zero measure. Also, Ψ is locally Lipschitz on $B_1 \setminus E$ with Jacobian almost everywhere comparable to w. Thus the 2-dimensional measure of $\Psi(B_1 \setminus E)$ is comparable to

$$\int_{B_1 \setminus E} w(x)dm(x).$$

Since E has zero area and w is an A_1 weight, this is comparable to $s^2 \text{essinf}_{B_1} w(x)$.

By the construction of E, if $2s_n < s \leq 2s_{n-1}$, then for any ball B_1 of radius s, $B_1 \setminus E$ contains a square of side length s_{n+1}. Thus by the definition of A_1 weight and condition (2.10) we have

$$\mathcal{H}_2(\Psi(B_1)) \sim \int_{B_1} wdm \sim s^2 \text{essinf}_{B_1} w \sim s^2 f^2(s_{n+1}) \sim s^2 f^2(s_{n-1}).$$

Since $2s_n < s \leq 2s_{n-1}$ we have $f(s_n) \geq f(s) \geq f(s_{n-1})/\lambda$. Thus

$$\mathcal{H}_2(B_0) \geq \mathcal{H}_2(\Psi(B_1)) \sim s^2 f(s)^2 \sim r^2.$$

By a similar argument,

$$\mathcal{H}_2(B_0) \leq \mathcal{H}_2(\Psi(B_2)) \sim (Cs)^2 f(Cs)^2 \sim r^2.$$

Thus $S = \Psi(\mathbb{R}^2)$ is Ahlfors 2-regular.

As previously noted, if there were a biLipschitz map of S to $\mathbb{R}^2$, then the composition of this with Ψ would give a quasiconformal map of the plane to itself with Jacobian comparable to w. We proved before that this is impossible, so S can't be biLipschitz equivalent to $\mathbb{R}^2$. This proves Corollary 1.2.

4. Concluding remarks

Here are a few questions raised by our example:

Question 4.1. *Is there a compact set E of measure zero so that no A_1 weight w that blows up on E (i.e., so that $w(z) \to \infty$ as $z \to E$) can be comparable to a quasiconformal Jacobian?*

Question 4.2. *Is there a compact set E of measure zero so that no locally integrable w which blows up on E can be comparable to a quasiconformal Jacobian?*

Question 4.3. *Is there a compact set E of measure zero so that every quasiconformal image of E contains a rectifiable curve?*

The questions above are arranged so that an example for one is also an example for the previous ones. It is important that we only consider maps into the plane: by Theorem 3.1 the whole plane can be quasisymmetrically mapped into $\mathbb{R}^3$ so that the image has no rectifiable curves and the Jacobian is ∞ everywhere on $\mathbb{R}^2$ (see [1], [7]).

Our construction heavily uses that E_n contains big annuli around its complementary components. This forces E to have dimension 2.

Question 4.4. *Can the example in this paper have Hausdorff dimension < 2? What are the answers to Questions 4.1 - 4.3 if we require that* $\dim(E) < 2$*?*

In [11], Kovalev and Maldonado introduce the following definition: E is a quasiconformal ∞-set if there is a quasiconformal Jacobian with essential limit ∞ everywhere on E (essential limit means the limit after changing values, if necessary, on a set of measure zero). They show (Corollary 4.3 of [11]) that every set $E \subset \mathbb{R}^n$ of Hausdorff dimension < 1 is a quasiconformal ∞-set. Thus the set in Question 4.2 must have dimension at least 1.

Do the answers to the questions above for K-quasiconformal maps depend on the size of K, i.e., might the answers be different for large or small K? For example, in [2] I show that there a compact set E of measure zero and a $K_0 > 1$ so that every K-quasiconformal image of E contains a rectifiable curve for every $K < K_0$, but there is a K-quasiconformal image without rectifiable curves for some $K > K_0$. The proof that small deformations of the set contain rectifiable curves uses the following lemma, which follows from the proof of Lemma 2.2.

Lemma 4.5. *Suppose that $E_0 \supset E_1 \supset E_2 \supset E_3 \dots$ are compact sets and $E = \cap E_n$. Suppose there is a $K < \infty$, $C > 0$ and sequences of positive numbers $\{P_n\}$, $\{Q_n\}$ so that the following holds. For $n = 1, 2, \dots,$ suppose $E_n = \cup_k \Omega_{k,n} = E_{n-1} \setminus \cup_{k,j} W_{j,k,n}$ (disjoint except for boundaries) where $\Omega_{k,n}$ consists of a closed K-quasidisk $W_{k,n}$ with a finite number of open K-quasidisks $W_{j,k,n}$ removed (called the omitted* *regions). Moreover, for each j, k assume there is a collection of disjoint disks $\{D_n\} = \{D(x_n, r_n)\}$ so that with the properties that*

1. $\mathrm{dist}(D_n, \partial W_{k,n}) \geq C\mathrm{diam}(W_{k,n})$.
2. $W_{j,k,n} \subset D(x_n, r_n/Q_n) \subset D(x_n, P_n r_n) \subset W_{j,k}$.

If $\sum_n P_n^{-2} < \infty$ and $\sum_n Q_n^{-3} < \infty$, then E contains rectifiable curves.

Finally, how does one characterize biLipschitz images of the plane? Is there a natural condition on a surface that (in addition to LLC and Ahlfors 2-regular) that is not satisfied by the example in this paper?

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