AN EXPLICIT CONSTANT FOR SULLIVAN'S CONVEX HULL THEOREM

CHRISTOPHER J. BISHOP

ABSTRACT. For any simply connected domain Ω in the plane, let S be the boundary of the convex hull of $\partial\Omega$ which faces Ω. We show that there is always a 7.82-quasiconformal map from S to Ω which extends to the identity on $\partial S = \partial\Omega$.

Date: September 27, 2000.

1991 *Mathematics Subject Classification.* Primary: 30C62 .

Key words and phrases. Brennan conjecture, Sullivan's theorem, quasiconformal mappings, convex hulls, hyperbolic manifolds, locally Lipschitz .

The author is partially supported by NSF Grant DMS 9800924.

1. Introduction

Let $\Omega \subsetneq \mathbb{R}^2$ be a simply connected domain and let $C(\partial\Omega) \subset \mathbb{H}^3 = \mathbb{R}^3_+$ be the hyperbolic convex hull of $\partial\Omega$ (this is the hyperbolic convex hull of the set of all hyperbolic geodesics with endpoints in $\partial\Omega$). Let S be the boundary component of $C(\partial\Omega)$ which separates Ω from $C(\partial\Omega)$. Equivalently, let $\tilde{\Omega} \subset \mathbb{H}^3$ be the union of all hemispheres whose bases are contained in Ω. Then $S = \partial\tilde{\Omega} \cap \mathbb{H}^3$. Let ρ_S denote the intrinsic path metric on S (using hyperbolic arclength) and let ρ denote the usual hyperbolic metric on the unit disk $\mathbb{D}$, the upper half space $\mathbb{H}^3$ or Ω. It is an observation of Thurston that there is an isometry ι from the metric space (S, ρ_S) to $(\mathbb{D}, \rho)$ (see [7] [4], [5]). Moreover, there is a K_0-biLipschitz map σ from (Ω, ρ) to (S, ρ_S) which extends continuously to the identity on $\partial\Omega$ (and K_0 is independent of Ω). Consequently, there is a universal $K < \infty$ so that σ is K-quasiconformal. The map σ is conformally natural in the sense that if Ω is invariant under a group of Möbius transformations then σ commutes with the group action. This appeared in Sullivan's paper [6] in the case when Ω is simply connected and invariant under a convex co-compact quasi-Fuchsian group. Epstein and Marden prove in [4] the more general statement above. Their proof gives biLipschitz constant $K_0 \approx 88.2$ and quasiconformal constant $K \approx 82.6$. The purpose of this paper is to prove

Theorem 1.1. *For any simply connected domain Ω, there is a 7.82-quasiconformal map σ of Ω to S which extends to the identity on $\partial\Omega = \partial S$. This map is also biLipschitz from the hyperbolic metric on Ω to the path metric on S. There is a 13.3-biLipschitz map from Ω with its hyperbolic metric to S with its path metric.*

Strictly speaking, this result is not an improvement of the Epstein-Marden estimates since our construction is not conformally natural. It follows from our proof that there is also a conformally natural map (with a possibly larger, but uniformly bounded constant). This is because if $\Phi : \mathbb{D} \to \Omega$ is conformal, and G is a group of Möbius transformations acting on Ω, then $\iota \circ \sigma \circ \Phi$ is quasiconformal from the disk to itself and hence has quasisymmetric boundary values $h : \mathbb{T} \to \mathbb{T}$ (with bounds depending only on the quasiconformal constant of σ). Moreover, this boundary map h conjugates the Fuchsian group $G_1 = \Phi^{-1} \circ G \circ \Phi$ to the Fuchsian group $G_2 = \iota \circ G \circ \iota^{-1}$. Hence by the Douady-Earle extension theorem [3], there is a conformally natural extension $\tilde{h}$ of h to the disk with quasiconformal constant depending only on h (and hence only on the quasiconformal constant of σ). Thus $\Phi^{-1} \circ \tilde{h} \circ \iota^{-1} : \Omega \to S$ is the desired conformally natural, quasiconformal map. It is not clear whether the best constant for Sullivan's theorem is the same if we also require the map to be conformally natural.

This paper is a companion to [2]. That paper gives a short proof of Sullivan's theorem with an unspecified constant $K < \infty$ and discusses the connection of Sullivan's theorem to properties of conformal mappings. In particular, it shows that if Sullivan's theorem holds for a domain Ω with a constant K, then there is a K-quasiconformal map F of Ω to the unit disk which has bounded derivative. Moreover, Corollary 1.7 of [2] and Theorem 1.1 imply

Corollary 1.2. *Any conformal map $f : \mathbb{D} \to \Omega$ can be written as $f = g \circ h$ where h is a 7.82-quasiconformal self-map of the disk and $|g'|$ is bounded away from zero.*

I also prove in [2] that is we could prove this with $K = 2$, the well known Brennan conjecture would follow. This provides some extra motivation for finding the best constant in Sullivan's theorem. See [2] for further results, remarks and questions regarding Sullivan's theorem and its applications.

I would like to thank John Garnett, Al Marden and Michel Zinsmeister for reading an earlier version of this paper and for their numerous helpful comments and suggestions. The reader familiar with Epstein and Marden's paper [4] will see the obvious debt this paper owes to it; although many of the details vary, the construction we give here is motivated by their ideas.

2. The construction

Epstein and Marden's proof of Sullivan's theorem in [4] comes in two steps: first showing the nearest point retraction from Ω to the ϵ-distance surface S_ϵ of S is biLipschitz and then averaging the nearest point retraction map from S_ϵ to S over certain intervals in a foliation of S_ϵ. Our construction is motivated by the second step, although we shall apply it to construct a map from S directly to Ω.

2.1. **Overview of the construction.** Before beginning the details, we will describe the overall plan of the construction.

1. The first step is to reduce the case of a general Ω to the special case of "finitely bent" by showing that every simply connected domain has finitely bent approximations and that if we prove the estimate with uniform constants for these approximations, we can pass to the limit for Ω.
2. Next we assume Ω is finitely bent. This means that the corresponding convex hull surface S is a finite union of geodesic triangular faces $\{F_j\}$. Moreover, Ω itself can be written as a finite union of "triangles" $\{T_j\}$ (ideal hyperbolic triangles for certain disks D_j inside Ω) with vertices on $\partial\Omega$, "interior crescents" $\{C_j\}$ (regions bounded by two intersecting circles, each the boundary of a triangle) and "boundary crescents" $\{B_j\}$ (bounded by two orthogonal circles, one the boundary of a triangle, the other a boundary arc of Ω). We will define a metric $\tilde{\rho}$ on each piece which is comparable to the usual hyperbolic metric ρ on Ω.
3. We divide each triangle T_j into a central triangle and three "cusp regions" (to be described below). We define a subregion $\Omega_2 \subset \Omega$ be removing all the central triangles for the T_j's. This region has a natural foliation E as in the paper of Epstein and Marden [4]. We also define $\Omega_0 = \cup_j T_j \cup B_j$ and a region $\Omega_1 = \Omega_0 \cap \Omega_2 = \Omega_2 \setminus \cup_j C_j$. The desired map $F : S \to \Omega$ will be a composition of maps
$$S \xrightarrow{f_0} \Omega_0 \xrightarrow{f_1} \Omega_1 \xrightarrow{f_2} \Omega_2 \xrightarrow{f_3} \Omega \xrightarrow{f_4} \Omega.$$
4. The map $f_0 : S \to \Omega_0$ is simply the inverse of the nearest point retraction from faces on S to the triangles T_j and boundary crescents B_j. It is isometric on each face of S, so has quasiconformal constant $K_0 = 1$.
5. To describe the map $f_1 : \Omega_0 \to \Omega_1$, we need to decompose each triangle into pieces in two different ways. For simplicity, assume our triangle is in the upper half-plane with vertices at 0, 1 and ∞, i.e., $T = \{z = x+iy : 0 < x < 1, |z-\frac{1}{2}| > \frac{1}{2}$. First we cut T into three isomorphic pieces (see Figure 1),
$$U_1 = \{z : 0 < x < 1, |z| > 1, |z-1| > 1\},$$
$$U_2 = \{z : 0 < x < \frac{1}{2}, |z - \frac{1}{2}| > \frac{1}{2}, |z| < 1\},$$
$$U_3 = \{z : \frac{1}{2} < x < 1, |z - \frac{1}{2}| > \frac{1}{2}, |z-1| < 1\}.$$
The three domains meet at the "center" of the triangle $u = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The second

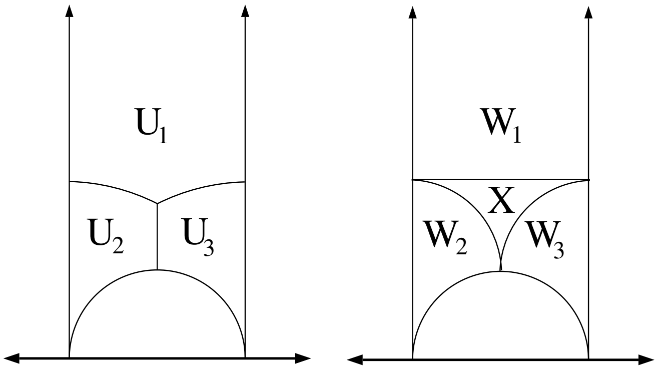

FIGURE 1. Subdividing the triangles

decomposition of T is into four pieces: three isometric "cusps"

$$W_1 = \{z : 0 < x < 1, |y| > 1, \},$$
$$W_2 = \{z : 0 < x < \frac{1}{2}, |z - \frac{1}{2}i| < \frac{1}{2}, |z - \frac{1}{2}| \geq \frac{1}{2}\},$$
$$W_3 = \{z : \frac{1}{2} < x < 1, |z - 1 - \frac{1}{2}i| < \frac{1}{2}, |z - \frac{1}{2}| > \frac{1}{2}\}$$

and a "central triangle"

$$X = T \setminus (W_1 \cup W_2 \cup W_3) = \{z : y < 1, |z - \frac{1}{2}i| > \frac{1}{2}, |z - 1 - \frac{1}{2}i| > \frac{1}{2}\}.$$

See Figure 1. The map $f_1 : T \to T$ is a quasiconformal map from each U_i to W_i, $i = 1, 2, 3$ which is the identity on ∂T. Let $g : U_1 \to W_1$ be the unique conformal map which takes $\infty \to \infty$, $i \to i$ and $1 + i \to 1 + i$ and let h be the map $h(x, y) = (x, g^{-1}(yi))$. Then we define $f_1 = h \circ g$ on U_1 and define it symmetrically on U_2 and U_3. Then f is the identity on ∂T and its quasi-constant on each U_i, $i = 1, 2, 3$, is the same as for h, which we will show is $K_1 = \max_{1 \geq t} |g'(it)|$ (Lemma 2.7). The map g can be written explicitly in terms of the modular function λ and using this we will show that

$$K_1 = \max_{t \geq 1} |g'(it)| = |g'(i)| \approx 1.20656$$

6. The map $f_2 : \Omega_1 \to \Omega_2$ is defined by sliding points along leafs of the foliation E. We will define this more precisely later, but roughly the idea is to fix a parameter $0 < s \leq \frac{1}{2}$ and look at the interval I in a leaf of E so that $I \cap \Omega_2$ has length $2s$ and is "centered" at x. The map f_2 slides x along the leaf by an amount which depends on width and position of the crescents crossed by I. The

quasiconformal constant of this map will be bounded by an explicit function $K_2(s)$. For example, we will compute $K_2(1/2) = 6.93808$.

7. The map $f_3 : \Omega_2 \to \Omega$ is the identity on the crescents and the inverse of f_1 on the triangles. Thus its quasi-constant is also $K_1 \approx 1.20655$.
8. The final map $f_4 : (\Omega, \tilde{\rho}) \to (\Omega, \rho)$ is the identity map. The metric $\tilde{\rho}$ is chosen so this map is conformal, and hence can be ignored for computing the quasiconformal dilatation of F, but it needs to be considered to compute the biLipschitz constant of F.
9. The map F constructed in this way has quasiconformal constant $K = K_1^2 K_2 \approx 10.1002$. We can do a little better using two observations. First, the places where the maps f_1 and f_2 attain their maximum dilatations are different and thus the constant for the composition is a little better than just the product $K_1 K_2$. Second, the map f_3 is conformal on the crescents, but K_1-quasiconformal on the triangles. Thus we could modify it to make it K_1-quasiconformal on the crescents as well, without having to pay any extra "price". Using these ideas, we get $K \approx 7.8127$.

We now start describing each step of the construction in more detail.

2.2. **Passing to a limit.** We will only do the construction explicitly for certain "finitely bent" cases. To pass to the limit we use the following easy lemma (which is left to the reader).

Lemma 2.1. *Suppose $\Omega \subset \mathbb{C}$ is simply connected, $0 \in \Omega$ and $\Omega \neq \mathbb{C}$. Suppose $0 \in \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega = \cup_n \Omega_n$ and for each n there is an onto map $f_n : \Omega_n \to \mathbb{D}$ which is K-quasiconformal and hyperbolically M-biLipschitz and such that $f_n(0) = 0$. Then there is a $f : \Omega \to \mathbb{D}$ which is also K-quasiconformal and hyperbolically M-biLipschitz.*

It is also convenient to record the following known result about quasiconformal mappings (e.g., Theorem 35.1 [8]). Using this, it is enough to estimate the dilatation of our map on the faces of S and check that it is continuous across the boundaries of these pieces.

Lemma 2.2. *Suppose $\mathbb{D} = \cup \Omega_n$ where $\{\Omega_n\}$ is a finite collection of regions with disjoint interiors and rectifiable boundaries. If $f : \mathbb{D} \to \Omega$ is a homeomorphism* *which is K-quasiconformal on the interior of each Ω_n then f is a K-quasiconformal mapping on the whole disk.*

2.3. **Approximation by triangulated polygons.** Given a simply connected Ω we will describe how to approximate it by "finitely bent" regions. A different method is described in [4]. Our first step is to approximate Ω from within by a polygonal region P. Fix some small $\epsilon > 0$. The vertices of P should be a finite set $E = \{z_k\}_0^N$ with $z_0 = z_N$ which satisfy

1. $\text{dist}(z_k, z_{k+1}) < \epsilon \text{dist}(z_k, \partial\Omega)$
2. The angle formed by any three consecutive points is within ϵ of π.

(We can easily create such a polygon by first approximating Ω by a smooth region and then taking vertices on the boundary of this region, very close to each other in comparison to the distance to $\partial\Omega$). Next we wish to triangulate P, i.e., write $P = \cup_j T_j$ where $\{T_j\}$ is a union of Euclidean triangles which satisfy

1. they have pairwise disjoint interiors,
2. every vertex of T_j is an element of E,
3. the open Euclidean disk D_k whose boundary contains the three vertices of T_j does not intersect E.

The last condition is not obvious, so we will show that such a triangulation is always possible.

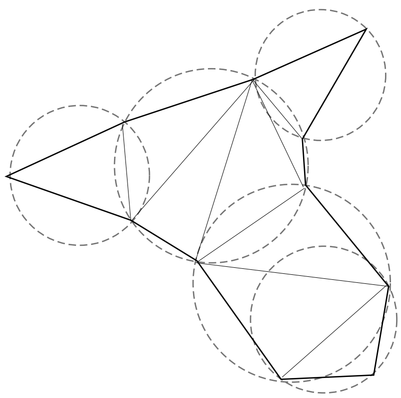

FIGURE 2. Triangulation with disks

First, for each pair of consecutive points z_k, z_{k+1} let $I_k = [z_k, z_{k+1}]$. If I_k is not already the side of a previously chosen triangle then we choose a triangle with I_k as one side as follows. Consider the family of circles containing these two points and centered on the perpendicular bisector L of the segment I_k. When the center of the circle is the midpoint of I_k then clearly z_k and z_{k+1} are the only points of E on the circle. Move the center of the circle continuously along L into P until the first time the circle meets three (or more) points of E. If it meets just three points of E then the corresponding triangle satisfies all our conditions and is put into our triangulation. If it meets $n \geq 4$ or more points of E, the resulting convex polygon is easily divided into $n-2$ triangles with the desired properties and each is put in our triangulation.

To see that two triangles arising in this way must have disjoint interiors, consider two such triangles T_1 and T_2 and let D_1 and D_2 be the corresponding disks determined by the vertices. If the disks are disjoint, then so are the triangles so there is nothing to do. If $D_1 = D_2$ then T_1 and T_2 had to be formed by splitting a convex n-gon, $n > 3$, into triangles, and hence are disjoint by choice. Finally suppose D_1 and D_2 intersect but are not identical. Let L be the Euclidean line determined by the two points of $\partial D_1 \cap \partial D_2$. Since E does not intersect these disks the vertices of T_1 must lie on $\partial D_1 \setminus D_2$ and the vertices of T_2 must lie on $\partial D_2 \setminus D_1$ and hence the triangles lie on opposite sides of L. Thus the two triangles can share at most an edge or vertex in common.

Thus the triangles we have constructed satisfy all the desired conditions, except that their union may not be all of P. To complete the triangulation of P take a triangle T from our collection and an edge I of T which borders an "uncovered" part of P. Let L be the perpendicular bisector of I and let $c \in L$ be the center of the disk D corresponding to T. Deform D by moving its center c along L in the direction towards I and keeping the endpoints of I on its boundary. Continue until the boundary of disk hits three or more points of E. Clearly, the new boundary points lie in the half-plane determined by L which does not contain c, so the convex polygon determined by these points has I as one side and covers some points not previously covered by the triangulation. Triangulate the resulting convex polygon if necessary and add the triangles to our collection. The proof of disjointness works just as before, so all the desired properties hold.

Now repeat this construction until all of P is covered (since there are only a bounded number of triangles in any triangulation of P and we we add at least one new triangle each time, the process must terminate in a finite number of steps). We will let T denote the collection of Euclidean triangles we have constructed.

2.4. **Approximation by finitely bent regions.** We now wish to modify the polygon P by replacing its straight line edges by circular arcs, in such a way that the resulting region has a finitely bent hyperbolic convex hull, i.e., the boundary component S of the convex hull of $\partial\Omega$ facing Ω will consist of a finite number of geodesic faces joined along infinite geodesics.

Suppose $\{I_k\}$ is an enumeration of the boundary segments of P and let T_k be the unique triangle in our triangulation which has I_k as an edge and let D_k be the corresponding disk (determined by having the three vertices of T_k on its boundary). Let J_k be the smaller arc of ∂D_k whose endpoints are the same as the endpoints of I_k. If ϵ in our choice of E is close to zero, then the arcs J_k are very close to I_k (they make angle $\leq \epsilon$ with I_k at the endpoints) and hence they are pairwise non-intersecting and their union is a Jordan curve bounding a simply connected region Ω'.

From this point on, we will only work with Ω' and never the original Ω, so for convenience we will assume that $\Omega = \Omega'$.

We claim that the corresponding convex hull boundary S is finitely bent, i.e., is a finite union of geodesic triangles.

First we show that every D_k is in Ω. Suppose D_1 is a disk which is not. Then ∂D_1 crosses $\partial\Omega$. Since D_1 does not contain any point of E, ∂D_1 must cross some arc $J_k \subset \partial\Omega$ twice. Thus three vertices defining D_1 must lie in D_k, which is impossible since D_k does not contain any points of E. The contradiction implies $D_1 \subset \Omega$, as desired.

Let $W \subset \mathbb{H}^3$ be the union of Euclidean hemispheres with bases $\{D_k\}$. By the previous paragraph each disk is in Ω so that W is contained in a complementary component of the convex hull of $\partial\Omega$. We claim that it is the whole component, i.e., $\partial W \cap \mathbb{H}^3 = S$.

To prove this, it is enough to show that given any disk $D \subset \Omega$, the hemisphere with base D is contained in W. Let $x \in D$ and choose a $T_k \in T$ such that $x \in T_k$. Then the interiors of D and D_k intersect since they both contain x. If $D = D_k$, there is nothing to do. Otherwise, $\partial D \cap \partial D_k$ consists of two points. Let L be the line through these two points and let H be the halfplane bounded by L such that

$D_k \cap H \subset D \cap H$. The triangle T_k must lie in H (otherwise it would have a vertex in $\partial D_k \setminus H \subset D$ contrary to the assumption that $D \subset \Omega$ and hence contains no points of E). Hence $x \in H$ and hence above x, the hemisphere for D lies under the hemisphere for D_k, as desired. This completes the proof that S is finitely bent.

2.5. **The nearest point retraction from Ω to S.** The nearest point retraction from Ω to S is now quite simple to describe. Given a point $x \in \Omega$ one takes a horoball in $\mathbb{H}^3$ based at x and expand it until it first hits S. Since S is the boundary of a convex set, there is a unique point of S hit and this is the "nearest point" to Ω.

The surface S consists of a finite number of geodesic triangles, one corresponding to each element of our triangulation. These are called the faces of S and will be denoted by $\{F_k\}$. The face F_k is the part of S which projects vertically to T_k. On each of these faces we define a map $F_k \to D_k \subset \Omega$ by following the unique geodesic which is orthogonal to F_k at a point x in the direction towards D_k. The image of F_k, is an triangle S_k in the hyperbolic geometry of D_k. More precisely, if T_k is a boundary triangle then S_k is bounded by $J_k \subset \partial D_k$ on one side and the other two sides are circles orthogonal to ∂D_k which connect the endpoints of J_k to the third vertex of T_k. If T_k is an interior triangle, then ∂S_k consists of three circular arcs orthogonal to ∂D_k and connecting the vertices of T_k. It is easy to verify that the nearest point retraction map takes S_k back to F_k and hence different S_k's are disjoint (since they map to different sets). Moreover, the retraction map is isometric from the hyperbolic of D_k restricted to S_k to the hyperbolic metric of $\mathbb{H}^3$ restricted to F_k. Note that in general we do not have $S_k \subset T_k$; this happens iff T_k contains the center of the disk D_k.

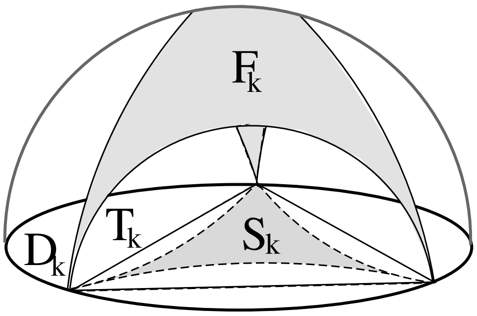

FIGURE 3. Geodesic face projects to triangles.

The union of of the S_k's may not be all of Ω (in fact, it won't be unless Ω is a disk). If T_j and T_k are triangles which share and edge I then the corresponding "hyperbolic" triangles both have vertices at the endpoints of I, but the corresponding edges I_j, I_k need not be the same. The region C_{jk} between these edges we will call an "interior crescent" and to each such crescent we will associate a number $0 \leq \alpha \leq \pi$ which is the angle between I_j and I_k at either common endpoint. Under the nearest point retraction onto S this crescent maps to the geodesic G_{jk} in S which is the common boundary of the adjacent faces F_j and F_k. The number α is also equal to the exterior angle the two faces make along their common boundary and is called the bending angle (or bending measure) of this geodesic.

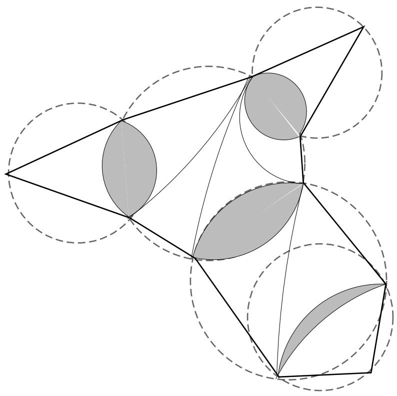

FIGURE 4. Hyperbolic triangles and crescents (shaded).

2.6. **Foliating the crescents.** The retraction map collapses each crescent onto a geodesic and the preimage of a single point is a circular arc in the crescent which is orthogonal to both boundary curves. This gives a foliation of the crescent. There is an equally obvious orthogonal foliation by circular arcs connecting the two endpoints of the crescent. Following Epstein and Marden we call the first foliation E and the second G. See Figure 5.

If we map the crescent by a Möbius transformation sending the endpoints to 0 and ∞, the crescent becomes a wedge of angle α bounded by straight lines through the origin and the foliation is just the union of circular arcs centered at the origin. By taking logarithms we can map the wedge to an infinite horizontal strip of width α and the foliation E is mapped to vertical line segments and G to horizontal lines.

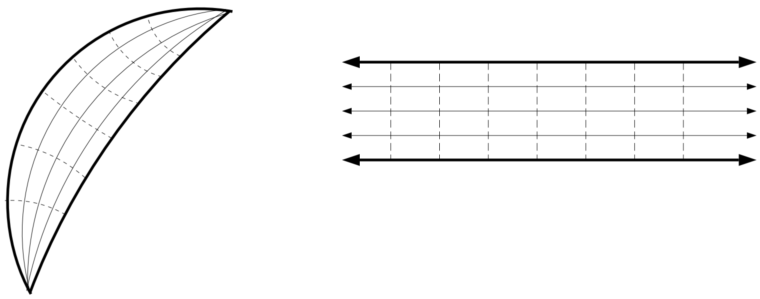

FIGURE 5. The foliations in crescents

We define a metric $\tilde{\rho}$ on the original crescent by mapping the Euclidean metric on the strip back to the crescent. We will prove a little later that this metric is comparable (with absolute constants) to the hyperbolic metric on Ω restricted to the crescent. See Lemma 2.3.

2.7. **Foliating the triangles.** Next we would like to extend these foliations into the triangles (or more precisely, into the triangles minus their central triangles).

If S is a boundary triangle with boundary edge I then divide it into two pieces by the geodesic in D joining the endpoints of I. One piece is a crescent with angle $\alpha = \pi/2$ (which we will call a boundary crescent) and the other is an ideal triangle in D. On the boundary crescent take the foliation E as before by mapping to a infinite horizontal strip and taking inverse images of vertical line segments. The foliation G is again by circular arcs connecting the two endpoints.We define the metric $\tilde{\rho}$ on boundary crescents by restricting the hyperbolic metric on D.

Thus it only remains to describe the foliation in an ideal triangle T. We will do this by cutting the triangle T into three cusp pieces W_1, W_2, W_3 and a central triangle X as described earlier. See Figure 6. Each W_j can be conformally mapped in a unique way to $W_1 = \{z = x + iy : 0 < x < 1, y > 1\}$ and in W_1 we define the foliation E by horizontal lines and the foliation G by vertical lines. We define a metric $\tilde{\rho}$ on W_j by pulling back the hyperbolic metric of the upper half plane restricted to W (this is the same as restricting the hyperbolic metric on the disk D corresponding to T to W_j).

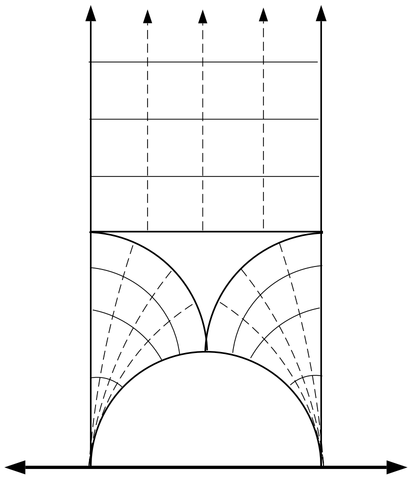

FIGURE 6. The foliations in triangles (E is solid, G is dashed)

2.8. **The metric $\tilde{\rho}$ is comparable to ρ.** In this section we show the metric $\tilde{\rho}$ we have defined on Ω is comparable to the usual hyperbolic metric ρ on Ω. First we review a few basic facts about the hyperbolic metric.

Suppose $\Omega_1 \subset \Omega_2$ and ρ_{Ω_1} and ρ_{Ω_2} are the corresponding hyperbolic metrics. Then $d\rho_{\Omega_2} \leq d\rho_{\Omega_1}$. A well know consequence of the Koebe $\frac{1}{4}$-theorem that if $D = D(z_0, r)$ is a disk and Ω is a simply connected domain containing D and such that $\partial\Omega \cap \partial D \neq \emptyset$, then

$$d\rho_\Omega \leq d\rho_D \leq 4\rho_\Omega. \tag{1}$$

If $\Omega = H$ is a half-plane then we have $d\rho_H(z_0) = \frac{1}{2} d\rho_D(z_0)$ (just map $D \to H$ by a Möbius transformation and note the derivative has absolute value 2 at z_0). Thus if Ω is simply connected and contains a half-plane H and $x \in \partial H \cap \partial\Omega$, then

$$d\rho_\Omega(z) \leq d\rho_H(z) \leq 2d\rho_\Omega(z), \tag{2}$$

for any point $z \in H$ which is on the line perpendicular to ∂H at x.

Lemma 2.3. *Let ρ be the hyperbolic metric on Ω and let $\tilde{\rho}$ be the metric on Ω defined in the previous two sections. Then on interior crescents we have*

$$\frac{1}{2} \leq \frac{d\rho}{d\tilde{\rho}} \leq 1,$$

on triangles we have

$$\frac{\sqrt{3}}{4} \leq \frac{d\rho}{d\tilde{\rho}} \leq 1,$$

and boundary crescents we have

$$\frac{1}{\sqrt{2}} \leq \frac{d\rho}{d\tilde{\rho}} \leq 1.$$

Proof. First consider interior crescents. By construction, each interior crescent C has sides which are orthogonal to the boundaries of the disks $D_1, D_1 \subset \Omega$ and $C \subset D_1 \cup D_1$. Suppose the crescent has angle α. Then if we map the endpoints of the crescent to $0, \infty$ so that the crescent is mapped to $V_1 = \{z : |\arg(z)| < \alpha/2\}$, the domain $D_1 \cup D_2$ is mapped to $V_2 = \{z : |\arg(z)| < (\alpha+\pi)/2\}$ and Ω is mapped to a domain Ω^* containing this wedge.

The metric $\tilde{\rho}$ on V_1 is given by $d\tilde{\rho} = \frac{1}{|z|}|dz|$. If $z_0 \in V_1$, let $H = \{z : |\arg(z) - \arg(z_0)| < \pi/2\}$. This is a half-plane and $V_1 \subset H \subset V_2$ and $d\rho_H(z_0) = d\tilde{\rho}(z_0)$. By (2), $d\rho_{\Omega^*} \leq d\rho_H \leq 2d\rho_{\Omega^*}$, which proves the desired inequality.

Next, consider a triangle T and corresponding disk $D \subset \Omega$. Since $D \subset \Omega$ we have $\tilde{\rho} \geq \rho$. To prove the other inequality, map D to the upper half plane $\mathbb{H}$ by a Möbius transfomation so T is mapped to the triangle with vertices $0, 1, \infty$ and Ω is mapped to some domain Ω^* containing the upper half plane. By symmetry, we need only consider points in $U = \{z = x + iy : 0 < x < 1/2, |z| > 1\}$. Fix a point $z = x + iy \in U$ and let $D_1 = D(z, y)$ and $D_2 = D(z, \mathrm{dist}(z, \partial\Omega^*)) = D(z, r_2)$. Since $0 \in \partial\Omega^*$ we know $y \leq r_2 \leq \sqrt{x^2 + y^2}$. Thus

$$d\tilde{\rho}(z) = d\rho_{\mathbb{H}}(z) = \frac{1}{2} d\rho_{D_1}(z) = \frac{1}{2}\frac{r_2}{y} d\rho_{D_2}(z) \leq 2\frac{r_2}{y} d\rho_{\Omega^*}(z) = 2\frac{r_2}{y} d\rho(z).$$

Since $x \leq 1/2$ and $y \geq \sqrt{3}/2$, this gives $d\tilde{\rho}(z) \leq \frac{4}{\sqrt{3}} d\rho$, as desired.

Finally we consider boundary crescents. As in the case of triangles assume we have a boundary crescent $B \subset D \subset \Omega$. Just as above $\tilde{\rho} \geq \rho$ because of the containment $D \subset \Omega$. Again map D to the upper half plane $\mathbb{H}$ sending B to the half disk with base $[0, 1] \subset \partial\Omega^*$. For $z = x + iy$ in this half disk, we have $\mathrm{dist}(z, \Omega) = y$ and so $d\tilde{\rho} = d\rho_{\mathbb{H}} \leq 2d\rho_{\Omega^*} = 2d\rho$, as desired. $\square$

The following is now immediate

Corollary 2.4. *The identity map $(\Omega, \tilde{\rho}) \to (\Omega, \rho)$ is $4/\sqrt{3}$-biLipschitz.*

2.9. **Mapping U to W.** In this section we define a (1.20656)-quasiconformal map from U_1 to W_1 which is the identity on the boundary. If one is willing to take a larger constant, then this section may be skipped. In Section 2.14 we will describe an much easier map which gives the constant 1.76759.

As noted in the beginning of the section, the map f_1 is defined using the conformal map $g : U_1 \to W_1$. It slightly easier to consider the map $g : U \to W$ where

$$U = \{z : 0 < x < 1/2, |z| > 1\},$$

$$W = \{z : 0 < x < 1/2, |y| > 1, \},$$

and g is uniquely determined by the requirement that $g(i) = i$, $g(\infty) = \infty$ and $g(u) = \frac{1}{2} + i$ (where $u = \frac{1}{2} + i\sqrt{3}/2$). If we can construct this map, then the map from U_1 to W_1 can be obtained by reflecting over the line $x = \frac{1}{2}$. In this section we gather together some facts we will need about the map g.

Fortunately, the map g can be written as a composition of maps with explicit formulas. The first element of this composition is the well known modular function $\lambda(z)$, which gives a conformal map from the region $T = \{z = x + iy : 0 < x < 1, |z - \frac{1}{2}| > 1/2\}$ onto the upper half-plane. This function can be explicitly written as $\lambda(z) = \lambda_1(z)/\lambda_2(z)$, where

$$\lambda_1(z) = \sum_{n=-\infty}^{\infty} (\frac{1}{\cos^2(\pi(n-\frac{1}{2})z)} - \frac{1}{\sin^2(\pi(n-\frac{1}{2})z)}),$$

$$\lambda_2(z) = \sum_{n=-\infty}^{\infty} (\frac{1}{\cos^2(\pi n z)} - \frac{1}{\sin^2(\pi(n-\frac{1}{2})z)}).$$

The fact that λ is given by this formula and maps T to the upper half-plane is proven in Section 7.3.5 of Ahlfors' book [1].

Since our region U is strictly contained in the upper half-plane, these series representations converge exponentially fast on U, so the truncations will give good approximations. In fact, if we let

$$\lambda_{1,N}(z) = \sum_{n=-N}^{N} (\frac{1}{\cos^2(\pi(n-\frac{1}{2})z)} - \frac{1}{\sin^2(\pi(n-\frac{1}{2})z)}),$$

$$\lambda_{2,N}(z) = \sum_{n=-N}^{N} (\frac{1}{\cos^2(\pi n z)} - \frac{1}{\sin^2(\pi(n-\frac{1}{2})z)}),$$

then

Lemma 2.5. *Suppose $z = x + iy$. Then for all x and all $y \geq \frac{1}{2}$, and $j = 1, 2$,*

$$|\lambda_j(z) - \lambda_{j,N}(z)| \leq 200e^{-\pi N}. \tag{3}$$

For $y \geq 1$, $j = 1, 2$,

$$|\frac{d}{dy}\lambda_j(z) - \frac{d}{dy}\lambda_{j,N}(z)| \leq 400e^{-\pi N}. \tag{4}$$

Proof. Since $\cos(iy) = \cosh(y)$ and $\sin(iy) = \sinh(y)$, we have

$$|\cos(z)|, |\sin(z)| \geq \frac{1}{4}e^y, \qquad y \geq \frac{1}{2}$$

Thus each of the terms

$$\cos^2(\pi(n - \frac{1}{2})iy), \sin^2(\pi(n - \frac{1}{2})iy), \cos^2(\pi niy), \sin^2(\pi(n - \frac{1}{2})iy)$$

is bounded below by $(4e)^{-2}e^{2\pi ny}$. Thus for $y \geq 1/2$,

$$|\lambda_j(z) - \lambda_{j,N}(z)| \leq 16e^2 \sum_{n=N+1}^{\infty} e^{-\pi n} = 16e^2 \frac{e^{-\pi N}}{1 - e^{-\pi}} \leq 200e^{-\pi N},$$

as desired. The second estimate follows from the first by the usual Cauchy estimate for the derivative of an analytic function. □

Thus if we use $N = 10$, the lemma says the truncations will be accurate to within $400e^{-10\pi} \leq 10^{-11}$. Using *Mathematica* to compute the truncated series we get

$$\lambda_1(i) \approx 6.87519, \qquad \lambda_1'(i) \approx 22.9211i,$$

$$\lambda_2(i) \approx 13.7504, \qquad \lambda_2'(i) \approx 13.7504i.$$

This gives

$$|\lambda'(i)| = |\frac{\lambda_1'(i)\lambda_2(i) - \lambda_1(i)\lambda_2'(i)}{\lambda_2(i)^2}| \approx 1.09422.$$

As noted above, λ maps the region T to the upper half-plane with $\lambda(0) = 1$, $\lambda(1) = \infty$ and $\lambda(\infty) = 0$ (see Theorem 7, page 281 of [1]). Using symmetry it is easy to see that $\lambda(i) = 1/2$ and $\lambda(u) = u$ and the region $U \subset T$ is mapped univalently onto $V = \{z = x+iy : 0 < x < \frac{1}{2}, y > 0, |z-1| < 1\}$. This region, in turn, is mapped to the lower half-plane by the function

$$J(z) = \frac{4}{27}\frac{(\frac{3}{4} + (\frac{1}{2} - z)^2)^3}{z^2(1 - z)^2}.$$

See page 282 of [1]. (We are abusing notation; usually the composition $J \circ \lambda$ is called the "J-function".) Moreover, $J(0) = \infty$, $J(\frac{1}{2}) = 1$ and $J(u) = 0$.

Next note that $z \to i + \frac{1}{2\pi}\cos^{-1}(2z-1)$ is a univalent map of the lower half-plane to the half-strip W taking $\infty \to \infty$, $0 \to i$ and $1 \to 1+i$. Thus, we see that the desired map is

$$g(z) = i + \frac{1}{2\pi}\cos^{-1}(2J(\lambda(z)) - 1).$$

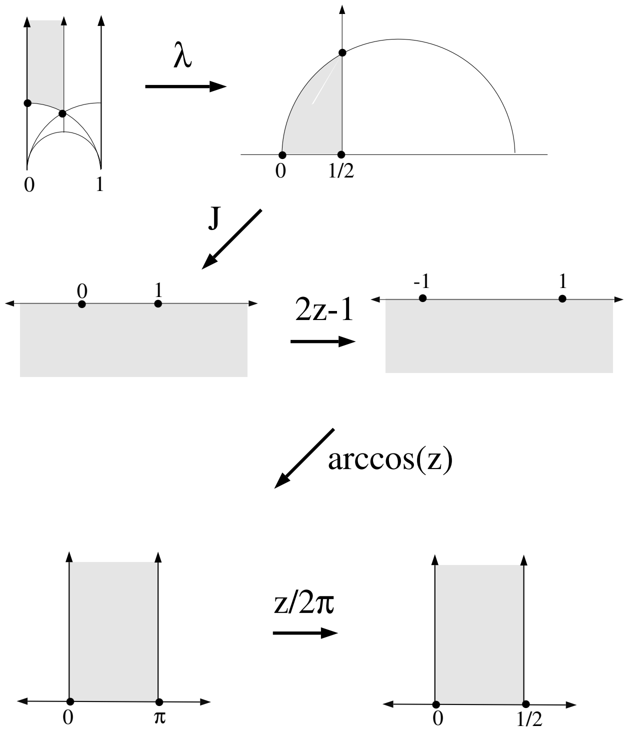

FIGURE 7. The map g as a composition

Using the chain rule and the series expansions

$$J(z) = 1 + 12(z - \frac{1}{2})^2 + O(|z - \frac{1}{2}|^4),$$

$$\cos(z) = 1 - \frac{1}{2}z^2 + O(|z|^4),$$

it is easy to show that

$$|g'(i)| = |\lambda'(i)|\frac{2\sqrt{3}}{\pi}.$$

Combined with our earlier estimate for $|\lambda'(i)|$ this gives $|g'(i)| \approx 1.20655$. The other fact that we will need about g is

Lemma 2.6. *For all $y \geq 1$, $|g'(iy)|$ is decreasing in y (in particular, $|g'(iy)| \leq |g'(i)|$ for $y \geq 1$).*

Proof. Let U' be a polygonal approximation to U in which the circular arc connecting i and u is replaced by a polygon which is convex down and for which the segment containing i is horizontal. Then by the Schwarz-Christoffel formula, the conformal mapping F_1 from the upper-half plane to W which sends $0 \to i$, $1 \to 1+i$ and $\infty \to \infty$ has derivative

$$F_1'(z) = z^{-1/2}(z-1)^{-1/2},$$

and the conformal mapping F_2 from the upper half-plane to U' sending $0 \to i$, $1 \to u$ and $\infty \to \infty$ has derivative

$$F_2'(z) = z^{-1/2}(z-1)^{-1/2-\sum_j \alpha_j} \prod_j (z-x_j)^{\alpha_j}$$

for some choice of $\{x_j\} \in (0,1)$ and $\{\alpha_j\} > 0$.

The conformal map h from U' to W sending $\infty \to \infty$, $i \to i$ and $u \to 1+i$ can be written as $h = F_1 \circ F_2^{-1}$ and so its derivative at iy is F_1'/F_2' at $t = F_2^{-1}(iy) < 0$. Since $0 < x_j < 1$, and

$$\frac{F_1'(t)}{F_2'(t)} = \prod_j (\frac{t-1}{t-x_j})^{\alpha_j},$$

we see that this is an increasing function on $(-\infty, 0]$ since each $(t-1)/(t-x_j)$ is increasing there. Since the map $y \to F_2^{-1}(iy)$ is clearly decreasing, we see that $|h'(iy)|$ is decreasing in y. The conformal map $g : U \to W$ can be written as a uniform limit of maps h as above (by reflection the maps can be extended to an open neighborhood of $\{z = x+iy : y > 1\}$) and hence g' is a limit of the derivatives. A limit of non-decreasing functions is non-decreasing, so the lemma is proven. □

Lemma 2.7. *Let*

$$U_1 = \{z : 0 < x < 1, |z| > 1, |z-1| > 1\},$$

$$W_1 = \{z : 0 < x < 1, |y| > 1, \}.$$

Then there is a K_1 quasiconformal map from U_1 to W_1 which is the identity on $\partial U_1 \cap \partial W_1 = \{z = x+iy : x \in \{0,1\}, y \geq 1\}$. One can take $K_1 = 1.20656$.

Proof. Let $g : U \to W$ be the conformal map described above and by reflection extend it to a conformal map from U_1 to W_1. Define $h : W_1 \to W_1$ by $h(x,y) = h(x, |g^{-1}(iy)|)$. It is very easy to see that a map of the form $(x,y) \to (x, f(y))$ has quasiconformal dilatation $\max(|f'(y)|, |f'(y)|^{-1}|)$ at the point (x,y) (for a more general case, see Lemma 2.8 below).

Thus $h \circ g$ maps U_1 to W_1, is the identity on $\partial U_1 \cap \partial W_1$ and has quasiconformal constant

$$K_1 = \max_{y \geq 1} |g^{-1}(iy)'|^{-1} = |g'(i)|,$$

by Lemma 2.6. The value of K_1 then follows from our earlier computation. □

2.10. **Computing the dilatation.** In this section we record a fairly straightforward computation for the quasiconformal dilatation of certain maps which we will need later. Given a mapping G defined on a domain U and a point $z \in U$ we define

$$L_1(G,z) = \liminf_{w \to z} \frac{|G(z) - G(w)|}{|z-w|},$$

$$L_2(G,z) = \limsup_{w \to z} \frac{|G(z) - G(w)|}{|z-w|},$$

Lemma 2.8. *Suppose $G : (x,y) \to (x, g(x,y))$ is differentiable. Then*

$$\frac{1}{2}(\sqrt{Y} - \sqrt{X}) = L_1(G,z) \leq L_2(G,z) = \frac{1}{2}(\sqrt{Y} + \sqrt{X}), \tag{5}$$

where

$$X = 1 + (g_x)^2 + (g_y)^2 - 2|g_y|, \qquad Y = 1 + (g_x)^2 + (g_y)^2 + 2|g_y|.$$

Thus the quasiconformal dilatation of G is

$$\limsup_{r \to 0} \frac{\max_{|z-w|=r} |G(z) - G(w)|}{\min_{|z-w|=r} |G(z) - G(w)|} = \frac{(\sqrt{Y} + \sqrt{X})}{(\sqrt{Y} - \sqrt{X})}.$$

Proof. The tangent map for G is $(x,y) \to (x, xg_x + yg_y) = (u,v)$. Thus the preimage of the circle $u^2 + v^2 = 1$ is the ellipse given by $x^2 + (xg_x + yg_y)^2 = 1$. If r_1, r_2 are lengths of the semiminor and semimajor axes respectively of this ellipse then (5) is equivalent to the claim that

$$\frac{1}{r_1} = \frac{1}{2}(\sqrt{Y} + \sqrt{X}), \qquad \frac{1}{r_2} = \frac{1}{2}(\sqrt{Y} - \sqrt{X}).$$

To prove these, rewrite the equation for the ellipse as

$$(1 + (g_x)^2)x^2 + 2g_x g_y xy + (g_y)^2 y^2 = Ax^2 + Bxy + Cy^2 = 1.$$

It is well known from the theory of conic sections that

$$r_1 r_2 = \frac{2}{\sqrt{4AC - B^2}}$$

$$\frac{r_1^2 + r_2^2}{r_1^2 r_2^2} = \frac{1}{r_1^2} + \frac{1}{r_2^2} = A + C$$

(since $A + C$ and $4AC - B^2$ are invariant under rotations of the ellipse, and these equations hold when $B = 0$, i.e., the axes are parallel to the coordinate axes). Thus

$$\begin{aligned}
r_1^2 + r_2^2 &= 4\frac{A+C}{4AC - B^2} \\
\frac{1}{r_2^2}\frac{4}{4AC - B^2} + r_2^2 &= 4\frac{A+C}{4AC - B^2} \\
\frac{1}{r_2^2} + \frac{1}{4}(4AC - B^2) r_2^2 &= A + C \\
\frac{1}{u} + \alpha u &= \beta.
\end{aligned}$$

Since r_1^2 and r_2^2 play symmetric roles, they must be the two solutions of this equation. Using the quadratic formula to solve it, we get

$$\begin{aligned}
r_1^2, r_2^2 &= \frac{\beta \pm \sqrt{\beta^2 - 4\alpha}}{2\alpha} \\
&= 2\frac{(A+C) \pm \sqrt{(A+C)^2 - (4AC - B^2)}}{4AC - B^2}.
\end{aligned}$$

Setting

$$\begin{aligned}
X &= A + C - \sqrt{AC - B^2} \\
&= 1 + (g_x)^2 + (g_y)^2 - \sqrt{4(1 + (g_x)^2)(g_y)^2 - 4(g_x g_y)^2} \\
&= 1 + (g_x)^2 + (g_y)^2 - 2|g_y|,
\end{aligned}$$

and similarly

$$Y = A + C + \sqrt{AC - B^2} = 1 + (g_x)^2 + (g_y)^2 + 2|g_y|,$$

we get

$$\begin{aligned} r_2^2 &= \frac{\frac{1}{2}(X+Y)+\sqrt{XY}}{\frac{1}{2}\frac{1}{4}(Y-X)^2} \\ &= 4\frac{Y+2\sqrt{XY}+X}{(Y-X)^2} \\ &= 4\frac{(\sqrt{Y}+\sqrt{X})^2}{(\sqrt{Y}-\sqrt{X})^2(\sqrt{Y}+\sqrt{X})^2} \\ &= \frac{4}{(\sqrt{Y}-\sqrt{X})^2}, \end{aligned}$$

which clearly gives the desired result for r_2. The calculation for r_1 is similar. $\square$

First note that for a map of the form $(x, y) \to (x, y + g(x, y))$ with $g_y \geq 0$, the formulas for X and Y become

$$X = (g_x)^2 + (g_y)^2, \qquad Y = (g_x)^2 + (2 + g_y)^2. \tag{6}$$

Note that $\sqrt{Y} + \sqrt{X}$ clearly increases if either $|g_x|$ or g_y increases. Thus to find an upper bound for $\sqrt{Y} + \sqrt{X}$ it suffices to find upper bounds for $|g_x|$ and g_y.

On the other hand, by differentiating $\sqrt{x^2 + (2 + y)^2} - \sqrt{x^2 + y^2}$, it is easy to check that $\sqrt{Y} - \sqrt{X}$ decreases if $|g_x|$ increases or g_y decreases. Thus to find a lower bound for $\sqrt{Y} - \sqrt{X}$ we need an upper bound for $|g_x|$ and a lower bound for g_y (which will usually be 0).

Finally, we claim that the dilatation

$$K = \frac{\sqrt{Y} + \sqrt{X}}{\sqrt{Y} - \sqrt{X}},$$

increases if either $|g_x|$ or g_y increases. To see this, consider

$$\mu = \frac{K-1}{K+1} = \sqrt{Y}/\sqrt{X} = \sqrt{g_x^2 + (2 + g_y)^2}/\sqrt{g_x^2 + g_y^2}.$$

This is clearly increasing in both $|g_x|$ and g_y (just view it as $|g_y + ig_x|/|2 + g_y + ig_x|$). Since the map $\mu \to K$ is increasing, this proves the claim. Thus to find an upper bound on K it suffices to find upper bounds on $|g_x|$ and g_y.

2.11. **The averaging map.** In this section we define the map $f_2 : \Omega_1 \to \Omega_2$. Recall that every point $z \in \Omega_1 \subset \Omega_2$ is on a leaf of the foliation E and the map f_2 will be defined by "sliding" the point a certain distance along the leaf (i.e., f_2 will preserve leaves of E).

Given an interval I on a leaf of the foliation E we will measure its length in two ways. One is just the length of I with respect to the metric $\tilde{\rho}$. The other is the $\tilde{\rho}$ length of $I \cap \Omega_2$, i.e., we only measure the length of I in the triangles and boundary crescents and ignore the length contributed by interior crescents. This length will also be called the S-length because it is the length of $R(I)$ on S (recall R is the nearest point retraction from Ω to S).

Given a point $z \in \Omega_1$, choose an orientation for the leaf L which contains it. For a real number s with $|s| \leq 1/2$, let $I(z, s)$ be the closed interval in L of S-length $|s|$ with one endpoint z and other endpoint in the positive direction if $s > 0$ and the negative direction if $s < 0$. Let $L(z, s)$ be the signed length of $I(z, s)$ in E and let $b(z, s) = L(z, s) - s$. Another way to define $b(z, s)$ is to travel along the leaf of E containing z in the direction $\text{sign}(s)$ until we have traversed distance s in Ω_1 (so we don't count the distance crossing crescents). Then $b(s, z)$ is $\text{sign}(s)$ times the sum of the widths of the crescents we have crossed. (If the stopping point is on the boundary of a crescent we count that crescent in the sum.) We shall prove later (Lemma 2.9) that for $s < 1$, $b(s, z)$ is bounded independently of z and Ω; thus in some sense a leaf of E must spend a fixed fraction of its length outside interior crescents.

For $z \in \Omega_1$ and $0 < s \leq \frac{1}{2}$, define

$$B(z, s) = \frac{1}{2s} \int_{-s}^{s} b(z, s) ds,$$

and we define $f_2(z)$ to be the point in Ω_2 which is on the same leaf L as z and (signed) distance $B(z, s)$ from it. In other words, we define $f_2(z)$ by sliding z (signed) distance $B(z, s)$ along L. If L is regular this define a single point. If L is singular, there may be two points distance $B(z, s)$ from z, but these two points then must be on the boundary of a central triangle and must correspond to each other via the map f_1.

If $z_1, z_2 \in \partial\Omega_1$ are points on the same leaf of E which are separated by a single interior crescent of width α, we claim that $f_2(z_1) = f_2(z_2)$. To see this, suppose z_2 is in the positive direction from z_1 and note that the crescent contributes $+\alpha/2$ to $B(z_1, s)$ and contributes $-\alpha/2$ to $B(z_2, s)$. Since z_1 and z_2 are α apart, this proves the claim. Thus the composition $f_2 \circ f_1 \circ f_0$ is continuous across the bending geodesics in S. Combined with the previous paragraph, this shows $f_3 \circ f_2 \circ f_1 \circ f_0$ is continuous from S to Ω.

Now we have to show the map f_2 is piecewise quasiconformal. Since our map f_2 preserves leaves of E it is locally of the form $f:(x,y)\to(x,y+g(x,y))$, so by Lemma 2.8, the quasiconformal dilatation is

$$K = \frac{\sqrt{(g_x)^2+(2+g_y)^2}+\sqrt{(g_x)^2+(g_y)^2}}{\sqrt{(g_x)^2+(2+g_y)^2}-\sqrt{(g_x)^2+(g_y)^2}}.$$

We will write this a little more compactly as $k(g_x,g_y)$, where

$$k(u,v) = \frac{\sqrt{u^2+(2+v)^2}+\sqrt{u^2+v^2}}{\sqrt{u^2+(2+v)^2}-\sqrt{u^2+v^2}}. \tag{7}$$

We will give upper bounds for g_y and $|g_x|$ in terms of the "bending function" B_1 which is defined as follows. For any $z\in\Omega$ on a leaf L of E and $0<s\leq 1$ let I be the interval in L centered (with respect to S-length) at z of S-length $2s$ and let $B_1(z,s,\Omega)=L(z,s)$. Then

$$B_1(s) = \sup_{\Omega}\sup_{z\in\Omega} B_1(z,s,\Omega).$$

In other words, for any finitely bent Ω and any interval I of length $2s$ in the foliation $R(E)\subset S$ on the corresponding convex hull surface, the total bending along I is less than $B_1(s)$. Of course, it is not obvious that there is a finite value of $B_1(s)$ with this property, but we will prove that there is in the next section.

Now for the upper bounds on $|g_y|$ and $|g_x|$. Since we only need bounds that hold almost everywhere, we may assume z in the interior of triangle. First, by the fundamental theorem of calculus

$$g_y(z) = \frac{1}{2s}[b(z,s)-b(z,-s)] \leq \frac{1}{2s}B_1(s). \tag{8}$$

Also note that since $b(z,s)\geq 0$ and $b(z,-s)\leq 0$ we have that $g_y(z)\geq 0$ for all z.

Since b is a step function

$$g(z) = \sum_j \alpha_j^+(s-d_j^+(z)) - \sum_k \alpha_k^-(s-d_k^-(z))$$

where $\{\alpha_j^+\}$ are the crescent widths hit by I in the positive direction and $\{\alpha_k^-\}$ are in the negative direction. The function $d_j^\pm(x,y)$ is the S-distance from the crescent C_j to the G-leaf containing z. Then $d_j^\pm(z)=w_0^\pm(z)+\sum_{n=1}^j w_n^\pm(z)$ where $w_n^\pm(z)$ is the length of the interval connecting C_n to C_{n+1} on the E-leaf containing z and $w_0^\pm(z)$ is the length of the interval connecting z to C_1 .

It is easy to compute that $|\frac{d}{dx}w_j^{\pm}(z)| = w_j^{\pm}(z)$. To do this, map the triangle to the ideal triangle $0, 1, \infty$ in the upper half plane so that the leaf of E in question is a horizontal line. Then at (Euclidean) height y, the (hyperbolic) length of the leaf is $w = 1/y$ and its rate of change is $\frac{dw}{dy} = -1/y^2$. Since hyperbolic distance in the vertical direction is $h = \log y$, we see from the chain rule that $\frac{dw}{dh} = \frac{dw}{dy}/\frac{dh}{dw} = (-1/y^2)/(1/y) = -1/y = -w$; in other words, the rate of change of the width has absolute value equal to width, as desired.

Since $d_j^{\pm}(z) \leq s$ this implies

$$|g_x(z)| \leq \frac{1}{2s}s(\sum_j \alpha_j + \sum_k \alpha_k) \leq \frac{1}{2}B_1(s). \tag{9}$$

Thus we get the estimate

$$K_2(s) \quad \leq \quad k(\frac{1}{2}B_1(s), \frac{1}{2s}B_1(s)), \tag{10}$$

where k is as in (7).

2.12. **A bound on bending.** In this section, the E and G foliations will always refer to the foliations on the convex hull surface S corresponding to E and G in Ω via the retraction map R. As before, let $B_1(s)$ be the maximum sum of angles crossed by any S-length $2s$ segment of the E foliation.

Lemma 2.9. *With B_1 defined as above, and $0 \leq s < 1$,*

$$B_1(s) \quad \leq \quad B_2(s) \equiv \frac{\pi + \log \frac{1+s}{1-s}}{\sqrt{1-s^2}}.$$

Proof. The proof follows the extremely clever idea of the proof of Proposition 2.14.2 of Epstein and Marden's paper [4], although some of the details vary. The basic idea it to take a foliation path γ_1 of S-length $2s$ and center $x \in S$ and approximate it by another path γ_2 and then compare the angles crossed by the two paths. It is also interesting to note that the proof of this lemma in the only place in this proof of Sullivan's theorem which is really three dimensional: we will use in an essential way that S is the boundary of a convex region in hyperbolic space and I do not know how to derive the same estimates from merely two dimensional arguments (although this may be possible).

Without loss of generality we assume x is on a face of S and not a bending geodesic so that γ_1 is differentiable at x. Let v_0 be a tangent vector to S at x which is perpendicular to γ_1 at x. The new path is $\gamma_2 = P \cap S$ where P is the unique hyperbolic plane in $\mathbb{H}^3$ which contains x and which is orthogonal to the vector v_0 at the point x. We can estimate the angles crossed by γ_2 using the following lemma from [4].

Lemma 2.10 (Lemma A.6.2, [4]). *Let Q be a noncompact convex polygon with a finite number of geodesic sides in the hyperbolic plane. Let A be an arc of length s in ∂Q. Then the sum of the exterior angles of Q met by A (including the endpoints) is at most $\pi + s$.*

The elegant proof of this given in [4] is attributed to Francis Bonahon. Briefly, it consists of replacing (if necessary) Q by the convex hull of A and a vertex p of Q at ∞. Then connect p to the finite vertices of Q, thus writing Q as a union of triangles. Each such has one finite length side, and this length is an upper bound for the area of the triangle. The Gauss-Bonet theorem implies the sum of the exterior angles is less than $2\pi + s$ and subtracting the exterior angle π at p gives the result.

Since S is the boundary of a convex set (namely the convex hull of $\partial\Omega$) and is finitely bent, $\gamma_2 = P \cap S$ is the boundary of a convex polygon Q in P. Moreover, this polygon is noncompact because otherwise the base of P (i.e. the corresponding disk in the boundary of hyperbolic space) would lie inside Ω and P would have to be tangent to S wherever they intersected, contrary to our choice of P. Thus Q is compact and the lemma applies to γ_2.

So we now have to turn our estimate for the bending along γ_2 into an estimate for bending along γ_1. To do this we will not use the hyperbolic arclength parameterization of γ_2, but instead will use the parameterization given by the leaves of the foliation G, i.e., parameterize γ_1 by arclength and parameterize γ_2 so that $\gamma_1(t)$ and $\gamma_2(t)$ lie on the same leaf of G. We want to estimate two quantities: first the distance $D(t)$ between $\gamma_1(t)$ and $\gamma_2(t)$ (since leaves of G are geodesics in S this is the same as the distance along the leaf containing both points) and second the angle $A(t)$ between $\gamma_2(t)$ and the foliation E. We claim that both are bounded by the quantities we obtain when γ_1 is the horizontal line segment $\{(t, 1) : 0 \le t < 1\}$ in $\mathbb{H}$ and γ_2 is the circular arc $\{(t, \sqrt{1-t^2}) : 0 \le t < 1\}$ (note that the parameterizations are related as above if G consists of vertical lines). More concretely, we claim

$$D(t) \leq D_0(t) \equiv \log \frac{1}{\sqrt{1-t^2}}, \tag{11}$$

$$A(t) \leq A_0(t) \equiv \arcsin(t). \tag{12}$$

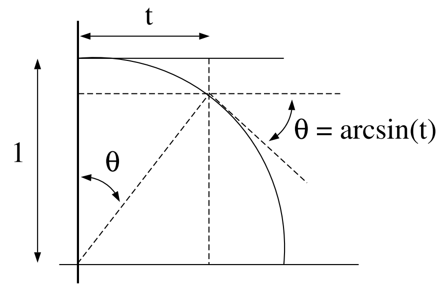

FIGURE 8. The definition of D_0 and A_0

The angle $A(t)$ is a little difficult to deal with directly since it may be discontinuous at bending geodesics (since the tangents to both γ_2 and E have discontinuities there). To get around this problem, we again follow Epstein and Marden by bounding $A(t)$ by another angle which does vary continuously.

For two tangent vectors v, w in $\mathbb{H}^3$ with the same base point, we let $\text{ang}(v, w)$ denote the angle between these vectors. Given a point $x \in S$ and a tangent vector v at x we let $\text{ang}(v, E)$ denote the minimal angle between v and the two possible tangent vectors to E at x. Similarly for $\text{ang}(v, G)$.

Let $v(t)$ be the unit tangent vector of $\mathbb{H}^3$ at $\gamma_2(t)$ which is orthogonal to P. Since leaves of G are parallel to the bending geodesics, the tangent direction to G (ignoring signs) is continuous along γ_2, even when we cross a bending geodesic. Since $v(t)$ is obviously continuous along γ_2, we see that $A_2(t) = \text{ang}(v(t), G)$ is continuous.

By construction $v(t)$ is always perpendicular to $\gamma_2'(t)$ (since this vector is tangent to P) and γ_2' lies in the plane generated by E and G at $\gamma_2(t)$. We claim that

$$\text{ang}(\gamma_2'(t), E) \leq \text{ang}(v(t), G),$$

i.e., $A(t) \leq A_2(t)$. To prove this, fix G, E and γ_2' and rotate $v(t)$ around the axis parallel to γ_2'. It is easy to see that the smallest angle with G occurs when $v(t)$ is the plane containing G, E and γ_2' and that this angle equals the angle between E and

γ_2'. Thus for $v(t)$ in its original position, the angle with G must have been larger, proving the claim. See Figure 9. Thus we only need to prove (12) with A_2 in place of A.

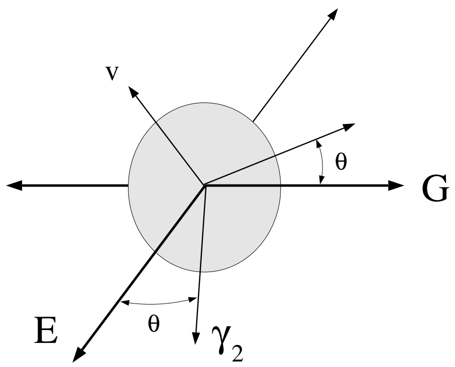

FIGURE 9. Proof that $\text{ang}(\gamma_2'(t), E) \leq \text{ang}(v(t), G)$

To prove (11) and (12) we will enumerate the triangles of S hit by the paths and consider the intersections $\tilde{\gamma}_1 = \gamma_1 \cap T$ and $\tilde{\gamma}_2 = \gamma_2 \cap T$. Fix one such triangle T. Then $\tilde{\gamma}_1$ connects two edges of T and using a Möbius transformation we may place the common vertex at ∞. Thus we may assume T lies in a vertical plane and that $\tilde{\gamma}_1$ corresponds to a horizontal line segment at height 1 (so G consists of vertical lines and E of horizontal lines in T). See Figure 10.

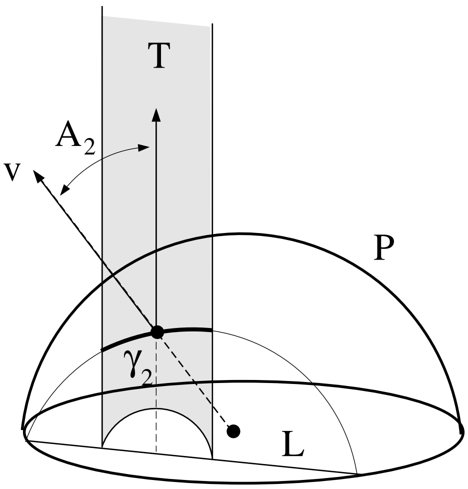

FIGURE 10. The angle between γ_2 and G

The plane P then corresponds to some Euclidean hemisphere and v is the normal vector to this hemisphere. Moreover, $\tilde{\gamma}_2$ is the circular arc $P \cap T$. It is clear that the angle v makes with the vertical and height of $\tilde{\gamma}_2$ both change most rapidity if $\tilde{\gamma}_2$ is radial (i.e., its vertical projection onto the base of P is a radius of that disk). Thus we need only consider that case and the desired estimates become the following claims about circles in the plane.

Lemma 2.11. *Suppose* $0 \leq a < b < 1$ *and let* $L_1 = \{(a, y) : 0 \leq y < \infty\}$ *and* $L_2 = \{(b, y) : 0 \leq y < \infty\}$. *Suppose* x *is a point on the real axis and* y *is a point on* L_1 *such that*

$$\sqrt{1-a^2} \leq y \leq 1/\sqrt{1-a^2} \tag{13}$$

and

$$\theta_a = \arcsin(|x-a|/|x-y|) \leq \arcsin(a). \tag{14}$$

Let $z \in L_2$ *be such that* $|x-z| = |x-y| \equiv r$. *Then*

$$\sqrt{1-b^2} \leq z \leq 1/\sqrt{1-b^2} \tag{15}$$

$$|\theta_b| = |\arcsin(|x-b|/r)| \leq \arcsin(b). \tag{16}$$

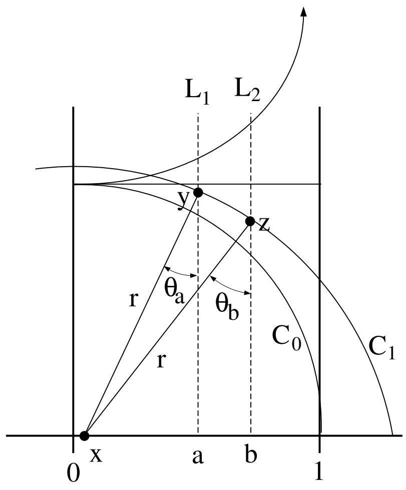

FIGURE 11. The proof of Lemma 2.11

Proof. This is elementary geometry and calculus and breaks into several cases. First we prove the left side of (15). Let C_0 be the circle of radius 1 around 0 and let C_1 be the circle of radius r around x. Since

$$r = |x - y| = \sqrt{y^2 + x^2 + 2ax + a^2} \geq |x - 1|,$$

the circle C_1 contains the point 1 in its interior disk. Thus C_1's intersection with L_1 and $L_3 = \{(1, y) : y \geq 0\}$ are above C_0's intersections with these lines and hence C_1's intersection with L_2 is above C_0's intersection with this line, as desired (otherwise C_1 and C_0 would have to intersect at least 4 times which is impossible).

Next we prove the right side of (15). The arc of C_1 between L_0 and L_1 is convex down whereas the upper bound is convex up. By assumption the circle has smaller derivative at $t = a$, and hence remains below the upper bound on the whole interval $[a, b]$.

Now we prove (16). First suppose $x \leq 0$. Then $r \geq 1$. Consider $\theta_t = \arcsin((|x| + t)/r)$ as a function on $[a, b]$. At $t = a$ it is less than $\arcsin(t)$ by assumption and since $r > 1$ is has smaller derivative than $\arcsin(t)$ and hence it remains smaller on the whole interval, and hence is smaller at $t = b$, as desired. Next suppose $0 \leq x \leq b$. Since $|x - b| \leq b$ and $z \geq \sqrt{1 - b^2}$ we get

$$\theta_b = \arctan(|x - b|/z) \leq \arctan(b/\sqrt{1 - b^2}) = \arcsin(b),$$

as desired. Finally if $x \geq b$ then (16) is obvious since $|x - t|/r$ as t goes from a to b angle decreases from a to b. $\square$

Conditions (13) and (15) are equivalent to (11) and conditions (14) and (16) are equivalent to (12). Thus the lemma says that if (11) and (12) hold when γ_2 enters T they still hold when it leaves (and at all points in $\gamma_2 \cap T$). Thus by induction we have now proven the desired estimates (11) and (12).

Finally we must estimate the exterior angle of γ_1 when it crosses a bending geodesic L in terms of the exterior angle for γ_2 when it crosses L. Since γ_1 is perpendicular to L, its exterior angle at L is the same as the exterior angle made by the two faces F_1 and F_2 of S which meet along the geodesic L. Let α denote this exterior angle. Let β be the exterior angle made by γ_2 when it crosses L. Choose t so that $\gamma_2(t) \in L$.

We claim that

$$\alpha \le \frac{\beta}{\cos(A_0(t))}. \tag{17}$$

To prove this we need the following simple lemma about planes in Euclidean 3-space.

Lemma 2.12. *Suppose two half-planes H_1, H_2 meet along a line L with exterior angle α. Suppose $0 \in L$ and v is a unit vector such that* $\operatorname{ang}(v, L) \le \theta$*. Let P be the plane perpendicular to v. Let v_i be the unit vector in the direction of the ray $P \cap H_i$ for $i = 1, 2$. Let $\beta = \pi -$* $\operatorname{ang}(v_1, v_2)$ *be the exterior angle made by these vectors. Then $\alpha \le \beta/\cos(\theta)$.*

Proof. We may as well assume L is the z-axis, and

$$H_1 = \{(x, y, z) : x > 0, y = \sin(\alpha/2)x\},$$

$$H_2 = \{(x, y, z) : x < 0, y = \sin(\alpha/2)|x|\}.$$

Also, because $\cos(\theta)$ is decreasing for $0 \le \theta \le \pi/2$ it suffices to prove the estimate when $\operatorname{ang}(v, L) = \theta$, and without loss of generality we may assume the z-coordinate of v is positive.

We want to show $\beta \ge \cos(\theta)\alpha$. Think of β as a function of v. We claim that β will be minimized when v lies in the xz-plane. This is the same as saying $\operatorname{ang}(v_1, v_2)$ will be maximized when v is in the xz-plane. Let $P_1 = \{(x, y, z) : y = 1\}$ and let z_i be the z-coordinate of the point $P \cap H_i \cap P_1$ for $i = 1, 2$. It is easy to see that $|z_1 - z_2|$ is maximized when v is in the xz-plane (since the line $P \cap P_1$ has steepest slope in this case) and that $z_1 = -z_2$ in this case. It is also easy to see that for a given displacement $|z_1 - z_2|$ the angle between v_1 and v_2 is maximized when $z_1 = -z_2$. This implies the claim that β is minimized when v is in the xz-plane. See Figure 12.

Thus we may also assume the y-coordinate of v is 0 and its x-coordinate is negative. It is now straight forward to compute β in terms of α. See Figure 13. Take points as in the figure with coordinates

$$A = (\cot(\theta), 0, 0), \qquad B = (\cot(\theta), 0, 1),$$

$$C = (\cot(\theta), \cot(\theta)\tan(\alpha/2), 0), \qquad D = (\cot(\theta), \cot(\theta)\tan(\alpha/2), 1).$$

The the line $[0, D]$ lies in $H_1 \cap P$, hence it is parallel to the vector v_1. The segment $[OB]$ also lies in P. Thus the angle DOB will be half the exterior angle formed by

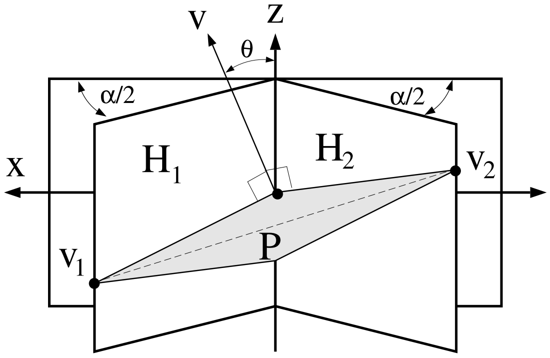

FIGURE 12. β is minimized when v is in the xz-plane

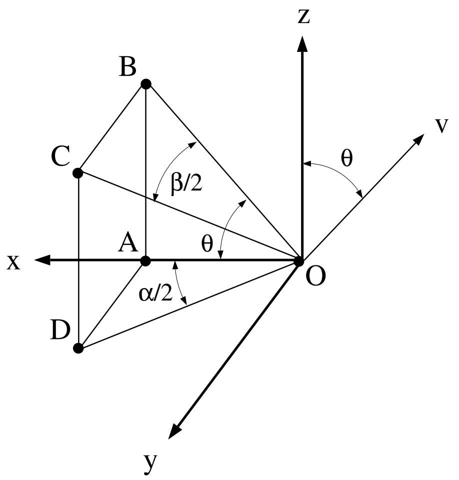

FIGURE 13. Computing relation between α, β, θ

v_1 and v_2, i.e. is $\beta/2$. Thus

$$|BD| = \tan(\beta/2)|OB| = \tan(\beta/2)/\sin(\theta),$$

$$|AC| = \tan(\alpha/2)|OA| = \tan(\alpha/2)/\tan(\theta).$$

Since $|AC| = |BD|$, we get

$$\tan(\alpha/2) = \tan(\beta/2)/\cos(\theta).$$

Since $\arctan(x)$ is increasing and convex down on $x \geq 0$, We have

$$\frac{\arctan(cx)}{cx} \leq \frac{\arctan(x)}{x},$$

if $c \geq 1$, or equivalently $\arctan(cx) \leq c \arctan(x)$. Thus

$$\alpha = 2\arctan(\frac{1}{\cos(\theta)}\tan(\beta/2)) \leq \frac{2}{\cos(\theta)}\arctan(\tan(\beta/2)) = \frac{\beta}{\cos(\theta)},$$

as desired. □

The lemma implies (17) by taking $v = v(t)$ and H_i to be the tangent plane to F_i at $\gamma_2(t)$ for $i = 1, 2$. We have now proved that if γ_2 crosses a bending geodesic L at parameter t with exterior angle β, then the angle between the corresponding faces is at most $\alpha = \beta/\cos(A_2(t))$.

Since $|A_2(t)| \leq A_0(t) = \arcsin t$, we have $\cos(A_2(t)) \geq \sqrt{1-t^2}$, and hence the sum of exterior angles crossed by γ_1 in the interval $[-s, s]$ is at most

$$\sum_j \frac{\beta_j}{\sqrt{1-t_j^2}} \leq \frac{1}{\sqrt{1-s^2}} \sum_j \beta_j$$

where we assume γ_2 has exterior angle β_j at parameter value t_j in the interval $[-s, s]$. The hyperbolic length of the path γ_2 is bounded by the length of the arc of $|z| = 1$ from $x = -s$ to $x = s$ which is easily computed to be

$$\int_{-s}^{s} \frac{1}{1-t^2} dt = \log \frac{1+s}{1-s}.$$

Thus the amount of bending in γ_2 is at most $\pi + \log \frac{1+s}{1-s}$ and this proves Lemma 2.9. □

The estimate is certainly not sharp for s near 1. For such values γ_2 has hyperbolic length greater than π and in this case, not all the bending can be concentrated at the two endpoints. Moreover, for $s > 1$ one can deduce a finite bound for $B_1(s)$ by cutting into subintervals of length < 1 and adding the estimates for these subintervals.

2.13. **A quasiconformal constant for Sullivan's theorem.** The desired map $S \to \Omega$ is a composition of maps $F = f_4 \circ f_3 \circ f_2 \circ f_1 \circ f_0$. The map f_0 is piecewise conformal and the maps f_1 and f_3 are piecewise quasiconformal with constant $K_1 = 1.20656$. The map f_2 is piecewise quasiconformal with constant K_2 given by (10). Combining this with Lemma 2.9 we get

$$K_2(s) = k(\frac{1}{2}B_2(s), \frac{1}{2s}B_2(s)),$$

and setting $s = \frac{1}{2}$, we can use *Mathematica* to compute $K_2(1/2) = 6.93808$. Thus the map F has quasi-constant at most

$$K = K_1 K_2 K_1 = 10.1002$$

There are two observations we can use to improve this a bit. The first is that the maps f_1 and f_2 take on their maximum dilatations in different places and so the dilatation for the composition should be strictly less than $K_1 K_2$. More precisely, the map f_1 has its largest dilatation close to the central triangles and the dilatation tends to 1 as we go out the cusps. On the other hand, the map f_2 has a better estimate near the central triangles because for such z's the corresponding E-leaf must cross the triangle and hence has large length not in a crescent. For example, if I is an E-interval of S-length $2s < 1$ centered at $z \in T$ and z is less than hyperbolic distance $r = -\log 2s$ from a central triangle then $I \setminus T$ has at most one component and this component has S-length at most $s/2$. Thus the bending along I is at most $B_2(s/2)$ instead of $B_2(s)$. This improves our estimate of the dilatation of f_2 at such points to

$$K_3(s) = k(\frac{1}{2}B_2(\frac{s}{2}), \frac{1}{2s}B_2(\frac{s}{2})).$$

where k is given by (7).

If $z = x + iy \in W_1$ is more than hyperbolic distance r from a central triangle then $y \geq \frac{1}{2s}$ and we estimate the dilatation of f_2 using $B_2(s)$ just as before, but for f_1 we have the improved estimate

$$K_1(z) = |g'(ig^{-1}(\frac{1}{2s}))|,$$

at such points (since we know $|g'|$ is decreasing along the positive imaginary axis). Since $|g'(iy)| \leq |g'(i)| \leq K_1 = 1.20656$ we know that

$$|g^{-1}(\frac{1}{2s})| \geq h(s) = 1 + (\frac{1}{2s} - 1)/K_1.$$

Thus

$$K_1(z) \leq |g'(h(s)i)|,$$

which we can estimate from our explicit formula for g. Thus for any given $0 < s \leq \frac{1}{2}$, F will be quasiconformal with constant at most

$$K_4(s) = K_1 \max(K_2(s)|g'(h(s)i)|, K_3(s)K_1).$$

Sampling different values of s leads to taking $s = .39$ which gives $K_4 = 9.18558$.

The second idea for improving our estimate concerns the map $f_3 : \Omega_2 \to \Omega$. On triangles this map is just f_1^{-1} and it is the identity on the crescents. Thus it is $K_1 = 1.20656$ quasiconformal on the triangles, but conformal on the crescents. We can take advantage of this by redefining the metric on the crescents in Ω_2 by shrinking distances along the E-foliation by a factor of K_1 and then defining f_3 on the crescents to be the map which expands distances along the E-foliation by a factor of K_1. Thus f_3 is still a K_1-quasiconformal map. However, since the crescents in Ω_2 are "thinner" now, our estimate of $B_2(s)$ is reduced by a factor of K_1. Therefore in our estimate of the dilatation of f_2 we may replace each occurance of $B_2(s)$ by $B_2(s)/K_1$. Thus for any $0 < s \leq 1/2$, F has dilatation at most

$$K_5(s) = K_1 \max(K_7(s)|g'(h(s)i)|, K_6(s)K_1),$$

where

$$K_6(s) = k(\frac{1}{2K_1}B_2(\frac{s}{2}), \frac{1}{2sK_1}B_2(\frac{s}{2})).$$

and

$$K_7(s) = k(\frac{1}{2K_1}B_2(s), \frac{1}{2sK_1}B_2(s)).$$

Searching values of s leads to $s = .39$ which gives $K_5(.39) = 7.8127$.

2.14. **A biLipschitz constant for Sullivan's theorem.** In this section we will note the map constructed above is biLipschitz from the path metric on S to the hyperbolic metric on Ω, although we will not explicitly compute the constant. We will compute the constant for another map in which f_1 is replaced by a simpler map.

As we shall see below, the maps f_0, f_2 and f_4 are all bilipschitz. However, the $f_1 : U_1 \to W_1$ we constructed is not; it has zero derivative at the point $u = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The mapping f_1 is biLipschitz outside a neighborhood of this point (with constant depending on the neighborhood). If the map f_2 is chosen with a parameter $s < \frac{1}{2}$, then $f_3 \circ f_2 \circ f_1$ is the identity on a neighborhood of the center of each triangle. Since f_1 (and hence f_3) are biLipschitz outside this neighborhood, we see that our map F is biLipschitz with a constant depending on s (with a little more work one can show it is also biLipschitz for $s = 1/2$).

In order to give a map with an explicit constant we will replace f_1 and f_3 by other maps obtained by repeating the construction of f_1, but instead of the map $g : U \to W$

defined earlier, we use the map $g_0 : U \to W$ given by

$$(x, y) \to (x, y + 1 - \sqrt{1 - x^2}).$$

Then by using (6) and the bounds $g_y = 0$, $|g_x| \leq 1/\sqrt{3}$, we get g_0 is quasiconformal with constant

$$K_{g_0} = k(\frac{1}{\sqrt{3}}, 0) \approx 1.76759.$$

If we let

$$k_1(x, y) = \frac{1}{2}(\sqrt{x^2 + (y+2)^2} - \sqrt{x^2 + y^2}),$$
$$k_2(x, y) = \frac{1}{2}(\sqrt{x^2 + (y+2)^2} + \sqrt{x^2 + y^2}),$$

then

$$L_1(f_1) = k_1(\frac{1}{\sqrt{3}}, 0)(1 - \sqrt{3/4}) \approx .651388 \approx \frac{1}{1.53518}$$
$$L_2(f_1) = k_2(\frac{1}{\sqrt{3}}, 0) \approx 1.32951 \approx \frac{1}{.752158}.$$

The extra factor of $(1 - \sqrt{3/4})$ occurs in L_1 because k_1 only computes L_1 with respect to the Euclidean metric. When a point at height y is shifted upwards by t the hyperbolic metric decreases by a factor of $\frac{y}{y+t}$. For g_0 the worse case occurs at the point u where $y = 1 - \sqrt{3/4}$ and $y + t = 1$.

Using Lemma 2.8 and our bounds for f_2 gives for $s = 1/2$,

$$L_1(f_2) = k_1(B_2(s)/2, B_2(s)/2s) \approx .356553,$$

and

$$L_2(f_2) = k_2(B_2(s)/2, B_2(s)/2s) \approx 6.39594.$$

Since f_3 is the identity on interior and boundary crescents and the inverse of f_1 on triangles, we get

$$L_1(f_3) = \frac{1}{L_2(f_1)} \approx .752158,$$
$$L_2(f_3) = \frac{1}{L_1(f_1)} \approx 1.53518.$$

Finally, by Lemma 2.3

$$L_1(f_4) = \frac{\sqrt{3}}{4} \approx .433013, \quad L_2(f_4) = 1.$$

and hence the whole composition satisfies

$$L_1(f) = L_1(f_1)L_1(f_2)L_1(f_3)L_1(f_4) \approx .0756439 \approx \frac{1}{13.2198},$$

$$L_1(f) = L_2(f_1)L_2(f_2)L_2(f_3)L_2(f_4) \approx 13.0544.$$

As in the previous section, we could probably reduce the constants by a more careful analysis of where the worse case occurs in each step of the composition, but we will not do this here.

2.15. **Computer code.** For the reader's convenience we give *Mathematica* code for evaluating the functions described in the last two sections.

```
lambda1[z_]:= Pi^2 Sum[ Cos[Pi(n-.5)z]^{-2}
                          - Sin[Pi(n-.5)z]^{-2},{n,-10,10}];
lambda2[z_]:= Pi^2 Sum[ Cos[Pi(n)z]^{-2}
                           -  Sin[Pi(n-.5)z]^{-2},{n,-10,10}];
lambda[z_]:= lambda1[z]/lambda2[z];
dlambda[z_]=D[lambda[z],z];
J[z_]:= (4/27)(1-z+z^2)^3/(z^2(1-z)^2);
K1= N[Abs[ dlambda[I] 2 Sqrt[3] /Pi]];   (* = 1.2065... *)
g[z_]:= ArcCos[ 2 J[lambda[z]]-1]/(2Pi);
dg[z_]=Abs[D[g[z],z]];
L1[x_,y_]:= (1/2)(Sqrt[x^2 + (2+y)^2] -Sqrt[x^2+y^2]);
L2[x_,y_]:= (1/2)(Sqrt[x^2 + (2+y)^2] +Sqrt[x^2+y^2]);
k[x_,y_]:= L2[x,y]/L1[x,y];
B2[t_]:= (Pi+Log[(1+t)/(1-t)])/Sqrt[1-t^2];
K2[s_]:= k[B2[s]/2,B2[s]/(2s)];
K3[s_]:= k[B2[s/2]/2,B2[s/2]/(2s)];
h[s_]:= 1+ ((2s)^{-1} -1)/1.20656;
K4[s_]:= K1 Max[ K2[s] dg[ h[s] I ], K3[s]  K1];
K6[s_]:= k[B2[s/2]/(2 K1),B2[s/2]/(2s K1)];
K7[s_]:= k[B2[s]/(2 K1),B2[s]/(2s K1)];
K5[s_]:= K1 Max[ K7[s] dg[ h[s] I],  K6[s]  K1];
Kg0=k[1/Sqrt[3],0];
L1f1=L1[1/Sqrt[3],0]Sqrt[3]/2;
L2f1=L2[1/Sqrt[3],0];
L1f2[s_]:=L1[B2[s]/2,0];
L2f2[s_]:=L2[B2[s]/2,B2[s]/(2s)];
L1F= L1f1 L1f2[.5] (L2f1)^{-1}  Sqrt[3]/4;
L2F= L2f1 L2f2[.5] (L1f1)^{-1};
```

References

[1] L.V. Ahlfors. *Complex Analysis*. McGraw Hill, 1979.

[2] C.J. Bishop. Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture. 1999. Preprint.

[3] A. Douady and C. J. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.*, 157(1-2):23–48, 1986.

[4] A. Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, pages 113–253. Cambridge Univ. Press, Cambridge, 1987.

[5] C. Rourke. Convex ruled surfaces. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, pages 255–272. Cambridge Univ. Press, Cambridge, 1987.

[6] D. Sullivan. Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur S^1. In *Bourbaki Seminar, Vol. 1979/80*, pages 196–214. Springer, Berlin, 1981.

[7] W.P. Thurston. *The geometry and topology of 3-manifolds.* Geometry Center, University of Minnesota, 1979.

[8] Jussi Väisälä. *Lectures on n-dimensional quasiconformal mappings.* Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 229.

C.J. Bishop, Mathematics Department, SUNY at Stony Brook, Stony Brook, NY 11794-3651

E-mail address: `bishop@math.sunysb.edu`