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ORTHOGONAL FUNCTIONS IN H^∞

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ABSTRACT. We construct examples of H^∞ functions f on the unit disk, so that the push forward of Lebesgue measure on the circle is a radially symmetric measure μ_f in the plane and characterize which symmetric measures can occur in this way. Such functions have the property that $\{f^n\}$ is orthogonal in H^2, and provide counterexamples to a conjecture of W. Rudin, originally disproved by Carl Sundberg. Among the consequences is that there is f in the unit ball of H^∞ so that the corresponding composition operator maps the Bergman space isometrically into a closed subspace of the Hardy space.

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Date: August 20, 1998.
1991 *Mathematics Subject Classification.* Primary: 30H05 Secondary: 30D35, 30D55, 47B38 .
The author is partially supported by NSF Grant DMS 98-00924.

1. Introduction

Let H^∞ denote the algebra of bounded holomorphic functions on the unit disk, $\mathbb{D}$, let $\mathcal{U}$ be the closed unit ball of H^∞ and let $\mathcal{U}_0 = \{f \in \mathcal{U} : f(0) = 0\}$. If $f \in H^\infty$ then it has radial boundary values (which we also call f) almost everywhere on the unit circle, $\mathbb{T}$. We say that f is *orthogonal* if the sequence of powers, $\{f^n\}$, $n = 0, 1, \dots$ is orthogonal, i.e., if

$$\int_{\mathbb{T}} f^n \bar{f}^m d\theta = 0,$$

whenever $n \neq m$. In this paper we will characterize orthogonal functions in H^∞ in terms of the Borel probability measure $\mu_f(E) = |f^{-1}(E)|$, where $|\cdot|$ denotes Lebesgue measure on $\mathbb{T}$, normalized to have mass 1. Moreover, we will determine exactly which measures arise in this way. More precisely,

Theorem 1.1. *If $f \in \mathcal{U}_0$ then $\{f^n\}$, $n = 0, 1, 2, \dots$ is an orthogonal sequence iff μ_f is a radially symmetric probability measure supported in the closed unit disk and satisfies*

$$\int_{|z|\leq 1} \log \frac{1}{|z|} d\mu_f(z) < \infty. \tag{1.1}$$

Moreover, given any measure μ satisfying these conditions there is a $f \in \mathcal{U}_0$ such that $\mu = \mu_f$.

The result is motivated by the observation that if f is an inner function (i.e., $f \in H^\infty$ and $|f| = 1$ a.e. on $\mathbb{T}$) with $f(0) = 0$ then μ_f is normalized Lebesgue measure on $\mathbb{T}$ (Lemma 2.3) and f is orthogonal since (if $m > n$),

$$\int_{\mathbb{T}} f^n \bar{f}^m d\theta = \int_{\mathbb{T}} f^{n-m} d\theta = 2\pi f^{n-m}(0) = 0.$$

At a 1988 MSRI conference Walter Rudin asked if the converse is true, i.e., are multiplies of inner functions the only orthogonal bounded holomorphic functions on the disk? In other words, is normalized Lebesgue measure on the circle the only radially symmetric measure which can occur as a μ_f? Our characterization shows that many other symmetric measures can occur and hence provide counterexamples to Rudin's "orthogonality conjecture". The conjecture was originally and independently disproved by Carl Sundberg.

The simplest example of a measure satisfying Theorem 1.1 (other than Lebesgue measure on a circle) is to take μ to be Lebesgue measure on the union of two circles $\{z : |z| = \frac{1}{2}\} \cup \{z : |z| = 1\}$, normalized to give each mass $\frac{1}{2}$. The corresponding function f is orthogonal by the theorem, but is clearly not inner since $|f| = \frac{1}{2}$ on a subset of $\mathbb{T}$ of positive measure.

A more interesting example of a radially symmetric measure satisfying (1.1) is normalized area measure on the disk. Thus there is an $f \in \mathcal{U}_0$ so that μ_f is normalized area measure. We will show (Lemma 6.1) that for any holomorphic g on the disk, and $f \in \mathcal{U}_0$ orthogonal,

$$\|g \circ f\|_{H^p}^p = \int_{\mathbb{D}} |g|^p d\mu_f + \mu_f(\mathbb{T}) \|g\|_{H^p}^p, \tag{1.2}$$

and hence

Corollary 1.2. *There is an $f \in \mathcal{U}_0$ so that for any analytic g on $\mathbb{D}$, g is in the Bergman space, A^p, iff $g \circ f$ is in the Hardy space, H^p (and the norms are equal).*

Thus the subspace M_f spanned by the powers of f in H^2 is isomorphic to the Bergman space and multiplication by f on M_f is isomorphic to multiplication by z on the Bergman space. Since both spaces are Hilbert spaces, of course one is isomorphic to a subspace of the other, but it is perhaps a little surprising that this isomorphism can be accomplished with a composition operator. Similar statements can be made for Bergman spaces with respect to radial weights $wdxdy = d\mu$ of finite mass which satisfy (1.1).

More generally, it would be interesting to know for which pair of spaces X, Y of analytic functions on $\mathbb{D}$ there is an $f \in \mathcal{U}_0$ so that $g \in X$ iff $g \circ f \in Y$, and to characterize such f's when they exist. The latter problem is interesting even when $X = Y$, e.g., see [7]. In Corollary 6.3 we characterize orthogonal functions with this property when $X = Y = H^p$ (it is true iff $\mu_f(\mathbb{T}) > 0$). In particular, all inner functions have this property (as claimed in [7]).

Paul Bourdon has pointed out that (1.2) implies that orthogonal functions f such that $\mu_f(\mathbb{T}) > 0$ give examples of composition operators with closed range. See [9] and [25] for characterizations of such functions.

The radial symmetry of a "Rudin counterexample" has also been noted by Paul Bourdon [4]. He showed that f is orthogonal iff the Nevanlinna counting function,

$$N_f(w) = \sum_{f(z)=w} \log(\frac{1}{|z|})$$

is almost everywhere constant on each circle centered on the origin. He also showed the answer to Rudin's question is "yes" if f is univalent and that if f is orthogonal, the closure of the range of f is a disk (since the range of f equals the set where N_f is positive). The Nevanlinna function N_f is related to μ_f by the formula

$$N_f(w) = \log\frac{1}{|w|} - \int \log\frac{1}{|z-w|} d\mu_f(z),$$

(except possibly on a set of logarithmic capacity zero). This is due to W. Rudin in [14] but we shall give a proof for completeness (Lemma 3.1).

Corollary 1.3. *If $f \in \mathcal{U}_0$ is non-constant and orthogonal then $N_f(w) = N(|w|)$ for all w outside an exceptional set of zero logarithmic capacity, where*

$$N(r) = \int_r^1 \frac{1-\mu(t)}{t} dt,$$

for some increasing function μ on $[0,1]$ with $\mu(0) = 0$, $\mu(1) = 1$ and $\int_0^1 \mu(t)\frac{dt}{t} < \infty$ (in fact, $\mu(r) = \mu_f(D(0,r))$). Moreover, for every such N there is an $f \in \mathcal{U}_0$ so that $N_f(w) = N(|w|)$ except possibly on a set of logarithmic capacity zero.

The "$\Rightarrow$" direction of this is due to Paul Bourdon [5]. The condition on N in the previous result has many equivalent formulations; for example, it holds iff $M(r) = N(e^r)$ on $(-\infty, 0]$ is concave up, has $M(0) = 0$ and $\sup_{r<0} M(r) + r < \infty$, or if $N(|z|)$ is subharmonic on $\mathbb{D} \setminus \{0\}$ and $N(|z|) + \log|z|$ is bounded above. The behavior of the composition operator $C_f : g \to g \circ f$ can often be expressed in terms of N_f, e.g., [18], [20], [21]. The result above provides radial examples with any desired rate of decay faster than $1 - r$ as $r \to 1$.

If f is orthogonal, then $f(0) = 2\pi \int f d\theta = 0$, so f cannot be an outer function. However, our construction can be modified to give

Corollary 1.4. *There is an orthogonal f so that $f(z)/z$ is a non-constant outer function.*

Thus, not only are there orthogonal functions which are not inner, there are examples with only the most trivial possible inner factor. I do not know whether there is an example so that $f(z)/z$ is bounded away from zero on $\mathbb{D}$ or which symmetric measures μ are of the form μ_f with $f(z)/z$ outer.

One can also construct examples with other properties. For example, $f \in \mathcal{U}_0$ is said to be in the hyperbolic little Bloch class $\mathcal{B}_0^h$ if

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2} = 0.$$

(This is contained in the usual little Bloch space, where only the numerator is required to go to zero.) We will show (Lemma 5.2) that if g is inner and $f \in H^\infty$ then $\mu_{f\circ g} = \mu_f$. Thus taking g to be an inner function in the hyperbolic little Bloch class (which exists by a result of Wayne Smith [19] and independently of Aleksandrov, Anderson and Nicolau, [1]; also see Cantón's paper [6]), we can deduce that

Corollary 1.5. *Any of the measures in Theorem 1.1 is μ_f for some $f \in \mathcal{B}_0^h$.*

Cima, Korenblum and Stessin [8] also identified symmetric properties of orthogonal functions and showed the answer to Rudin's question is "yes" if f is Hölder of order $\alpha > \frac{1}{2}$ on $\mathbb{T}$. I do not know if there is an (non-inner) orthogonal function which is continuous up to the boundary, but expect that it might be possible to build one by modifying the construction in this paper. If there is a continuous orthogonal function, it would be very interesting to know if the result of Cima, Korenblum and Stessin is sharp, and if not, what the best modulus of continuity for such a function could be. What other natural conditions on an orthogonal function imply that it is actually inner?

I would like to thank Joseph Cima for helpful conversations about Rudin's problem and composition operators. I am extremely grateful to Paul Bourdon for his carefully reading of an earlier version and the many helpful comments, suggestions and references he provided. Also thanks to Carl Sundberg for discussing his results with me.

The remaining sections are organized as follows:

Section 2: We describe some elementary properties of μ_f and prove it is radial iff f is orthogonal.

Section 3: We prove Corollary 1.3, (given Theorem 1.1).

Section 4: We prove some results concerning the convergence of μ_f.

Section 5: We prove Corollary 1.5 (given Theorem 1.1).

Section 6: We prove Corollary 1.2 (given Theorem 1.1).

Section 7: We construct a symmetric μ_f which is supported on two circles.

Section 8: We construct all examples supported in $\{\frac{1}{2} \leq |z| \leq 1\}$.

Section 9: We complete the proof of Theorem 1.1.

Section 10: We prove Corollary 1.4.

2. Elementary properties of μ_f

We begin by recalling a few simple facts about analytic functions f and the corresponding measures μ_f. Many of these are well known but we include them for the convenience of the reader.

Lemma 2.1. *If $f \in H^\infty$ then μ_f satisfies*

$$\int \log \frac{1}{|z|} d\mu_f(z) < \infty.$$

Proof. If f has a zero of order n at the origin, then $g(z) = f(z)/z^n$ is holomorphic on the disk and $|g| = |f|$ on $\mathbb{T}$, so $\mu_g(A) = \mu_f(A)$ for any annulus $A = \{z : r_1 \leq |z| \leq r_2\}$. Thus

$$\int \varphi(z) d\mu_f(z) = \int \varphi(z) d\mu_g(z),$$

for any radial function φ. Using Fatou's lemma and the fact that $\log|g(z)|^{-1}$ is superharmonic on the disk ([11], page 35), we deduce

$$\begin{aligned}\int \log\frac{1}{|z|} d\mu_f(z) &= \int \log\frac{1}{|z|} d\mu_g(z)\\ &= \frac{1}{2\pi}\int \log|g(e^{i\theta})|^{-1} d\theta\\ &= \frac{1}{2\pi}\int \lim_{r\to 1}\log|g(re^{i\theta})|^{-1} d\theta\\ &\le \frac{1}{2\pi}\lim_{r\to 1}\int \log|g(re^{i\theta})|^{-1} d\theta\\ &\le \log|g(0)|^{-1}\\ &< \infty,\end{aligned}$$

as desired. □

A similar estimate is true for other points, e.g.,

$$\int \log\frac{1}{|z-a|} d\mu_f(z) < \infty.$$

In particular, this implies that the set where f has radial limit a must have measure zero (a well known fact).

Given an arc $I \subset \mathbb{T}$ we define the Carleson box with base I as

$$Q = Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1-|z| \le |I|\}.$$

A positive measure μ is a Carleson measure if there is a $C < \infty$ so that $\mu(Q_I) \le C|I|$, for every arc $I \in \mathbb{D}$.

Lemma 2.2. *If $f \in \mathcal{U}_0$ then μ_f is a Carleson measure with constant independent of f.*

Proof. Define $\varphi(z) = \omega(z, Q, \mathbb{D}\setminus Q)$ for $z \in \mathbb{D}\setminus Q$ and $\varphi(z) = 1$ for $z \in Q$. It is easy to see that $\omega(z, I, \mathbb{D}) \ge M^{-1} > 0$ for every $z \in \partial Q \cap \mathbb{D}$ and some $M < \infty$

(independent of I and $z \in \partial Q$), so the maximal principle implies

$$\varphi(0) \leq M\omega(0, I, \mathbb{D}) \leq M|I|.$$

Let $f_r(z) = f(rz)$. Note that $\lim_{r\to 1} \varphi(f(rx)) = \varphi(f(x))$ for almost every $x \in \mathbb{T}$, because φ is continuous on the closed disk, except at two points and the set where f has a radial limit equal to one of these is measure zero (by the remark following Lemma 2.1). So by the Lebesgue dominated convergence theorem,

$$\mu_f(Q) \leq \int \varphi d\mu_f = \frac{1}{2\pi}\int \varphi \circ f d\theta = \lim_{r\to 1} \frac{1}{2\pi}\int \varphi \circ f_r d\theta. \tag{2.1}$$

Since φ is superharmonic on $\mathbb{D}$, $\varphi \circ f$ is superharmonic on $\mathbb{D}$, so the right hand side of (2.1) is $\leq \varphi(f(0)) = \varphi(0) \leq M|I|$. $\square$

If $f(0) \neq 0$ then μ_f is still a Carleson measure, but with norm depending on $|f(0)|$.

One can think of the previous lemma as a weak version of the Littlewood subordination principle: that if f is an analytic self map of the disk then $g \in H^p$ implies $g \circ f \in H^p$ (with smaller or equal norm). Formally, this implies,

$$\int |g| d\mu_f \leq \frac{1}{2\pi}\int_{\mathbb{T}} |g \circ f| d\theta = ||g \circ f||_{H^p} \leq ||g||_{H^p}$$

for every $g \in H^p$. This implies $d\mu_f$ is a Carleson measure with norm independent of f (e.g., Theorem I.5.6 of [11]).

The following result appears in many places (e.g. [12], Lemma 1 of [13], page 405 [16], Theorem VIII.30 [23]) and is sometimes called "Löwner's lemma". See [10] and its references for various generalizations.

Lemma 2.3. *If f is an inner function such that $f(0) = 0$, then μ_f is normalized Lebesgue measure on the unit circle.*

Proof. It is enough to check that $\mu_f(I) = |I|$ for arcs. Let I be an arc on the unit circle and let $\varphi(z) = \omega(z, I, \mathbb{D})$. Then $\varphi \circ f$ is bounded, harmonic, takes only radial boundary values 1 and 0 almost everywhere f has radial boundary values in I or its complement respectively (since f is inner). Thus

$$|I| = \varphi(0) = \varphi(f(0)) = \frac{1}{2\pi}\int_{f^{-1}(I)} d\theta = \mu_f(I),$$

as desired. $\square$

As noted before, the following is similar to results in [4] and [8].

Lemma 2.4. *Suppose $f \in H^\infty$. Then the measure μ_f is radially symmetric iff $\{f^n\}$ is orthogonal.*

Proof. If μ_f is radially symmetric, it can be written so that

$$\int g(z)d\mu_f(z) = \int_0^{2\pi}\int_0^\infty g(re^{i\theta})d\theta d\nu(r),$$

for every $g \in C_c(\mathbb{R}^2)$ (continuous functions of compact support) and for some measure ν on $(0,\infty)$. Thus

$$\int_{\mathbb{T}} f^n \bar{f}^m d\theta = \int_{\mathbb{C}} z^n \bar{z}^m d\mu_f(z) = \int_0^\infty \int_0^{2\pi} r^{n+m}e^{i(n-m)\theta}d\theta d\nu(r) = 0,$$

if $n \neq m$, so f is orthogonal. Conversely, if f is orthogonal, then μ_f satisfies

$$\int_{\mathbb{C}} z^n \bar{z}^m d\mu_f(z) = \int_0^{2\pi}\int_0^\infty r^{n+m}e^{i(n-m)\theta}d\mu_f(re^{i\theta}) = 0,$$

for $n \neq m$. Thus for any polynomial $P(z,\bar{z}) = \sum_{n,m} a_{n,m}z^n\bar{z}^m$ in z and $\bar{z}$,

$$\int_{\mathbb{C}} P(z,\bar{z})d\mu_f(z) = \int_{\mathbb{D}} \sum_n a_{n,n}r^{2n}d\mu_f(z),$$

and hence for any $|\lambda| = 1$,

$$\int_{\mathbb{C}} P(\lambda z, \bar{\lambda}\bar{z})d\mu_f(z) = \int_{\mathbb{C}} P(z,\bar{z})d\mu_f(z).$$

Since polynomials in z and $\bar{z}$ are dense in the continuous functions on the closed unit disk, we deduce that for any $g \in C_c(\mathbb{R}^2)$,

$$\int_{\mathbb{D}} g(z) d\mu_f(z) = \int_{\mathbb{D}} g(\lambda z) d\mu_f(z),$$

for any $|\lambda| = 1$. This implies μ_f is radially symmetric. $\square$

The following lemma greatly simplifies the construction of the basic example (where μ_f is supported on two circles). It says that if we can construct an example so that μ_f is radially symmetric on the smaller circle, then it automatically looks like Lebesgue measure on the larger one.

Lemma 2.5. *Suppose $f \in \mathcal{U}_0$ and μ_f is supported on the circles $C_{1/2} \cup C_1 = \{|z| = \frac{1}{2}\} \cup \{|z| = 1\}$. If μ_f restricted to $C_{1/2}$ is a multiple of Lebesgue 1-dimensional measure, then so is μ_f restricted to C_1.*

Proof. Suppose u is any bounded harmonic function on $\mathbb{D}$. Then $v(z) = u(f(z))$ is also bounded and harmonic on $\mathbb{D}$ and $u(0) = v(0)$. Thus

$$\begin{aligned} u(0) = v(0) &= \frac{1}{2\pi} \int_0^{2\pi} u(f(e^{i\theta})) d\theta \\ &= \int u(z) d\mu_f(z) \\ &= \int_{C_{1/2}} u(z) d\mu_f(z) + \int_{C_1} u(z) d\mu_f(z) \\ &= \mu_f(C_{1/2}) u(0) + \int_{C_1} u(z) d\mu_f(z). \end{aligned}$$

Hence $\int_{C_1} u d\mu_f = \mu_f(C_1) u(0)$, for any bounded harmonic function u on $\mathbb{D}$. This easily implies μ_f restricted to C_1 is a multiple of Lebesgue measure on C_1. $\square$

The same proof gives the following generalization of Lemma 2.5.

Lemma 2.6. *Suppose $f \in \mathcal{U}_0$. Then μ_f restricted to the unit circle is of the form $\frac{1}{2\pi}(1-g(\theta))d\theta$, where g is the balayage of μ_f onto the circle, i.e.,*

$$g(\theta) = \int_{\mathbb{D}} P_z(\theta) d\mu_f(z),$$

where $P_z(\theta)$ is the Poison kernel for $\mathbb{D}$ with respect to the point z.

3. The Nevanlinna counting function

If $f \in H^\infty$, the Nevanlinna counting function is defined as

$$N_f(w) = \sum_{f(z)=w} \log \frac{1}{|z|}.$$

If $f \in \mathcal{U}_0$ then $N_f(w) \leq \log |w|^{-1}$. Clearly this is just the Greens function for the Riemann surface associated to f (projected to the plane by summing over sheets). Since μ_f is the projection of harmonic measure for the Riemann surface, the following is analogous to the standard result for Greens functions of planar domains. As noted in the introduction the following is due to W. Rudin in [14].

Lemma 3.1. *If $f \in \mathcal{U}_0$ then $\Delta N_f = -\delta_0 + \mu_f$ in the sense of distributions ($\Delta = \partial_x^2 + \partial_y^2$ denotes the Laplacian and δ_0 is the Dirac mass at the origin). Moreover,*

$$N_f(w) = \log \frac{1}{|w|} - \int \log \frac{1}{|z-w|} d\mu_f(z), \tag{3.1}$$

for all w, except possibly for an exceptional set E of logarithmic capacity zero where "$<$" holds.

The exceptional set is required. For example, if f is the universal covering map of $\mathbb{D}$ minus a compact set E of zero logarithmic capacity, then f is an inner function, μ_f is normalized Lebesgue measure on the circle and $N_f(z) = \chi_{\mathbb{D}\setminus E}(z) \log \frac{1}{|z|}$.

Proof. For $0 < r < 1$, let $f_r(z) = f(rz)$ and let $\gamma_r = f_r(\mathbb{T})$. If we choose r so that f' never vanishes on the circle of radius r, then γ_r is a smooth curve and it is easy to check using Green's theorem that $\Delta N_{f_r} = -\delta_0 + \mu_{f_r}$. To see that (3.1) holds for μ_{f_r}, note that both sides of the equation have the same distributional Laplacian, so they differ by a harmonic function. N_{f_r} vanishes outside the unit disk by definition, and the right side of (3.1) vanishes there because μ_{f_r} evaluates harmonic functions at 0. Hence the difference between the left and right sides is the constant zero function.

For any smooth φ with compact support,

$$\int N_{f_r} \Delta\varphi dxdy = -\varphi(0) + \int \varphi d\mu_{f_r}.$$

We shall see later that μ_{f_r} converges to μ_f weakly (Corollary 4.4) and clearly $N_{f_r} \nearrow N_f$ as $r \nearrow 1$. Thus taking $r \to 1$ and applying the monotone convergence theorem we get

$$\int N_f \Delta\varphi dxdy = -\varphi(0) + \int \varphi d\mu_f.$$

This proves the first claim of the lemma. Next we verify (3.1).

We already know that if we replace f by f_r then we have equality in (3.1) for all z and as $r \to 1$, and we know $N_{f_r}(z) \nearrow N_f(z)$ for all z. Thus the question reduces to whether

$$U_r(w) \to U_1(w), \quad r \to 1 \tag{3.2}$$

for all w except a set E of logarithmic capacity zero, where

$$U_r(w) = \int \log \frac{1}{|z-w|} d\mu_{f_r}(z).$$

Note that by superharmonicity of $\log|f|^{-1}$, U_r is decreasing in r and that U_1 is bounded below by $-\log 2$ (since $|z-w| < 2$ for points in the unit disk).

To prove (3.2) we follow the proof of Frostman's theorem (e.g., Theorem II.6.4 in [11]). Suppose σ is a measure so that $V(z) = \int \log|z-w|^{-1} d\sigma(z)$ is bounded.

It suffices to show $\sigma(E)=0$. By Fatou's lemma

$$\lim_{r\to 1}\int \log\frac{1}{|z-w|}d\mu_{f_r}(z)\geq \int \log\frac{1}{|z-w|}d\mu_f(z),$$

so

$$\lim_{r\to 1} U_r(w)\geq U_1(w),$$

for all w. On the other hand, by Fatou's lemma, Fubini's theorem and the Lebesgue dominated convergence theorem

$$\begin{aligned}
\int_E \lim_{r\to 1} U_r(w)d\sigma(w) &\leq \lim_{r\to 1}\int_E U_r(w)d\sigma(w)\\
&= \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} V(f(re^{i\theta}))d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} V(f(e^{i\theta}))d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_E \log\frac{1}{|f(e^{i\theta})-w|}d\sigma(w)d\theta\\
&= \int_E U_1(z)d\sigma(w)
\end{aligned}$$

Thus we must have $\lim_{r\to 1} U_r(w)=U_1(w)$ except on a set of zero σ measure. $\square$

Thus we we see that $\{f^n\}$ is an orthogonal sequence iff μ_f is radial iff N_f is radial except on a set of logarithmic capacity zero. This gives an alternate approach to the results of Bourdon in [4].

We can also compute exactly which radial functions can occur as N_f for some $f\in\mathcal{U}_0$. Note that

$$\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{|re^{i\theta}-w|}d\theta=\begin{cases}\log\frac{1}{|w|}, & r\leq |w|,\\ \log\frac{1}{r}, & r\geq |w|.\end{cases}$$

Thus if μ_f is radially symmetric and we set $\mu(r)=\mu_f(D(0,r))$, then

$$N_f(w)=\log\frac{1}{|w|}-\int\log\frac{1}{|z-w|}d\mu_f(z)=\int_{|w|}^1\frac{1-\mu(r)}{r}dr.$$

Moreover, the integral condition

$$\int_{\mathbb{D}}\log\frac{1}{|z|}d\mu<\infty,$$

becomes

$$\int_0^1 \mu(r)\frac{dr}{r} < \infty.$$

Thus Theorem 1.1 implies,

Corollary 3.2. *Suppose* $N(r) = \int_r^1 \frac{(1-\mu(t))}{t} dt$ *for some increasing function* μ *so that* $\mu(0) = 0$, $\mu(1) = 1$ *and* $\int_0^1 \mu(r)dr/r < \infty$. *Then there is a* $f \in \mathcal{U}_0$ *so that* $N_f(z) = N(|z|)$ *except on a set of zero logarithmic capacity.*

For example, if μ_f is normalized area measure on the unit disk then $\mu(r) = r^2$ and $N_f(z) = \log \frac{1}{r} - (1-r) \approx (1-r)^2$ as $r \to 1$.

4. Weak convergence of μ_f

We will obtain the functions f in Theorem 1.1 by a "cut and paste" construction of the corresponding Riemann surface. What this means is that we shall build a sequence of nested Riemann surfaces $R_0 \subset R_1 \subset R_2 \subset \cdots \subset \cup R_n = R$ by identifying subdomains of the unit disk along common boundary arcs. The projection of R into the unit disk is a bounded holomorphic function on R, and hence R must be hyperbolic, i.e., its universal covering space is the unit disk, $\mathbb{D}$. The desired map will be the covering map $f : \mathbb{D} \to R$ followed by the projection into the disk and the corresponding measure μ_f is simply the harmonic measure for the surface R, projected into the plane. In fact, we shall abuse notation and consider the covering map $f : \mathbb{D} \to R$ as actually mapping into the complex numbers (i.e., we identify the covering map and this map followed by the projection into the plane). By a similar abuse we shall think of harmonic measure on R and the corresponding projected measure μ_f as the same. Similarly, we will fix a point in R_0 which projects to 0 and call it 0 as well. All our covering maps will be chosen to map 0 in the disk to

0 on the surface. See [2], [22], where a similar procedure has been used in different problems.

The main point we must be careful about is to show that the harmonic measure for R is the limit of the measures for R_n. To see that there might be a problem in general, consider what can happen when the surfaces are not nested, e.g., R_n is the unit disk minus the points $\{z_k\} = \frac{1}{2}\exp(i2\pi k 2^{-n})$, $k = 1, \dots, 2^n$. Then the universal covering map $f_n : \mathbb{D} \to R_n$ is an inner function (the isolated boundary points do not have any harmonic measure, so all the measure lives on the part of the boundary above the unit circle) and hence μ_{f_n} is Lebesgue measure on the unit circle. However, one can show (with some work) that $f_n(z) \to \frac{1}{2}z$ uniformly on compact sets of $\mathbb{D}$ so that μ_f is Lebesgue measure on the circle of radius $1/2$. However, if the Riemann surfaces are nested by (increasing) inclusion, then we will show the corresponding measures converge weakly, i.e.,

$$\lim_{n\to\infty} \int g d\mu_n = \int \varphi d\mu,$$

for any $g \in C_c(\mathbb{R}^2)$.

Lemma 4.1. *Suppose $\epsilon > 0$ and $D(0,\epsilon) = R_0 \subset R_1 \subset \dots$ are obtained by identifying subdomains of the unit disk along boundary arcs. Let $R = \cup_{n=1}^\infty R_n$. Choose covering maps $f_n : \mathbb{D} \to R_n$ and $f : \mathbb{D} \to R$ so that $f_n(0) = f(0) = 0$. Then μ_{f_n} converges weakly to μ_f on the closed unit disk.*

The easiest way to see this is using Brownian motion; we shall first sketch such a proof and then give a more classical proof without using Brownian motion.

Let $\mathcal{W}$ be the Weiner space of continuous paths in $\mathbb{C}$ starting at the origin. If R is a Riemann surface constructed as above then we can think of the paths as taking values in R and for each path $w \in \mathcal{W}$, we define the stopping time t_w as the first time t such that $w(t) \notin R$. Then $w \to t_w$ is measurable and the harmonic measure for R is simply the push forward of Wiener measure on $\mathcal{W}$ under the map given by $w \to w(t_w)$. Given a sequence of nested surfaces $R_0 \subset R_1 \subset \dots$ as in the lemma, we get a corresponding sequence of maps $g_n : \mathcal{W} \to \mathbb{C}$. Moreover, if $R = \cup_n R_n$ and $g : \mathcal{W} \to \mathbb{C}$ is the corresponding map, then $g(w) = \lim_n g_n(w)$; this is because the inclusions imply that for any continuous path in the plane, the first time it leaves R is the limit of the first times it left R_n. Thus for any bounded, continuous function φ on the plane, $\varphi(g_n(w)) \to \varphi(g(w))$ for all w, so the Lebesgue dominated convergence theorem implies

$$\int_{\mathcal{W}} \varphi(g(w))dw = \lim_n \int_{\mathcal{W}} \varphi(g_n(w))dw,$$

which is the desired weak convergence.

The sketch above is simple and explains why the result is true, but uses the existence of Wiener measure and deep connections between it and harmonic measure. It therefore seems desirable to provide a second proof which uses only function theory. Moreover, we will need some corollaries of the following classical proof for our applications to composition operators.

Let $\{R_n\}$, R, $\{f_n\}$ and f be as in the lemma and let $\Omega_n = f^{-1}(R_n) \subset \mathbb{D}$. Then $\Omega_0 \subset \Omega_1 \subset \dots$ and $\cup_n \Omega_n = \mathbb{D}$. Let ω_n be the harmonic measure for Ω_n with respect to the origin and let φ be any continuous function on the plane. We want to show that

$$\lim_n \int \varphi(f(z))d\omega_n(z) = \int_{\mathbb{T}} \varphi(f(e^{i\theta}))d\theta/2\pi.$$

We start by proving the much easier fact that ω_n converges weakly to normalized Lebesgue measure on the circle. (Since f need not be continuous up to the boundary, $\varphi \circ f$ need not be continuous either, so weak convergence of ω_n is not, by itself, enough to prove weak convergence of μ_{f_n}.)

Lemma 4.2. *If $\{0\} \in \Omega_0 \subset \Omega_1 \subset$ is a sequence of subdomains so that $\cup_n \Omega_n = \mathbb{D}$, and $\omega_n = \omega(0, \cdot, \Omega_n)$ is the corresponding harmonic measure with respect to the origin, then $\{\omega_n\}$ converges weakly to (normalized) Lebesgue measure on $\mathbb{T}$. Moreover, the measures ω_n are all Carleson with a uniform constant.*

Proof. The Carleson condition follows from Lemma 2.2 applied to the covering map onto Ω_n, so we need only prove weak convergence. Since $\cap_n(\overline{\mathbb{D}} \setminus \Omega_n) = \mathbb{T}$, there is a sequence $\{r_n\} \nearrow 1$ such that $D_n = \{z : |z| < r_n\} \subset \Omega_n \subset \mathbb{D}$. Suppose $I \subset \mathbb{T}$ is an open arc and let Q be the corresponding Carleson square, $Q = \{z \in \mathbb{D} : z/|z| \in I, 1 - |z| \leq |I|\}$. To show ω_n converges weakly to normalized Lebesgue measure, it is clearly enough to show that $\omega_n(Q) \to |I|$.

Let $U_n = D_n \cup Q$ and $V_n = \mathbb{D} \setminus (Q \setminus D_n)$. First observe that $\omega(0, I, U_n) \to |I|$. To see this simply note that

$$\omega(0, I, U_n) \quad = \quad |I| - \int_{\partial U_n \setminus I} [\int_I P_z(\theta) d\theta] d\omega(0, \cdot, U_n),$$

and that $\int_I P_z(\theta) d\theta \to 1/2$, as $n \to \infty$ when z is in one of the radial part of ∂U_n and $\to 0$ when z is in the interior of $\partial U_n \cap D_n$. Since the radial parts of ∂U_n have length tending to zero as $n \to \infty$ their harmonic measure also tends to 0 as $n \to \infty$ (e.g., by Lavrentiev's estimate of harmonic measure on rectifiable domains (e.g. page 80 of [3]; one could also use Beurling's estimate that a set of diameter δ in a simply connected domain has harmonic measure $\lesssim \sqrt{\delta}$). Thus the Lebesgue dominated convergence theorem implies $\omega(0, I, U_n) \to |I|$. A similar argument shows $\omega(0, Q \cap V_n, V_n) \to |I|$ as $n \to \infty$.

Thus by the monotonicity of harmonic measure,

$$\omega_n(Q) \geq \omega(0, \partial\Omega_n \cap Q, U_n) \geq \omega(0, I, U_n) \to |I|,$$

so $\liminf_n \omega_n(Q) \geq |I|$. On the other hand,

$$\omega_n(Q) \leq \omega(0, Q \cap \partial V_n, V_n) \to |I|,$$

which proves $\omega_n(Q) \to |I|$. This proves the lemma. □

Lemma 4.3. *Suppose g is a bounded, continuous function on $\mathbb{D}$ which has non-tangential limit $g(x)$ almost everywhere on $\mathbb{T}$ and that ν_n is a sequence of probability measures on $\overline{\mathbb{D}}$ which converge weakly to (normalized) Lebesgue measure on the circle and which are all Carleson measures with a uniform constant. Then*

$$\lim_n \int_{\overline{\mathbb{D}}} g(z) d\nu_n(z) = \frac{1}{2\pi} \int_{\mathbb{T}} g(z) d\theta.$$

Proof. We may assume that $\|g\|_\infty = 1$. Fix some $\epsilon > 0$. Since g has non-tangential limits almost everywhere, given almost any $x \in \mathbb{T}$ there is a $\delta(x) > 0$ so that if I is any interval containing x with length less than $\delta(x)$, then g is within $\epsilon/2$ of $g(x)$ on the top half of the corresponding Carleson box Q. Fix a complex number a and a $\delta > 0$ and assume $E_a = \{x \in \mathbb{T} : |g(x) - a| \leq \epsilon/2, \delta(x) > \delta\}$ has positive Lebesgue measure. Using the Lebesgue density theorem choose a dyadic interval I of length less than δ so that $|I \cap E_a| \geq (1-\epsilon)|I|$ and let Q_k be the collection of maximal dyadic subsquares with bases $\{I_k\} \subset I$, so that $|g(z) - g(x)|| > \epsilon$ for some z in the top half of Q_k. Let Q be the Carleson square with base I and let $W = Q \setminus \cup Q_k$. Then g is within ϵ of a constant on W, and $|\partial W \cap I| \geq (1-\epsilon)|I|$.

We claim that for these domains $\lim_n \nu_n(W) = |\partial W \cap \mathbb{T}|$. To prove this, we use the weak convergence of $\{\nu_n\}$ to deduce

$$\begin{aligned}
\lim_n \int_W d\nu_n &= \lim_n [\nu_n(Q) - \sum_k \nu_n(Q_k)] \\
&= |I| - \lim_n \sum_k \nu_n(Q_k) \\
&= |I| - \sum_k \lim_n \nu_n(Q_k) \\
&= |I| - \sum |I_k| \\
&= |\partial W \cap \mathbb{T}|,
\end{aligned}$$

where we used the Lebesgue dominated convergence theorem on the sequence space ℓ^1 to interchange the limit and the infinite sum (our assumption that the measures are uniformly Carleson gives that $\nu_n(Q_k) \leq C|I_k|$, independent of n; this gives the ℓ^1 upper bound).

Moreover, the intervals I with these properties form a Vitali cover of $\mathbb{T}$ (see e.g., Section 7.3 [24]), so we can form disjoint cover of almost every point of $\mathbb{T}$ using such intervals. Thus we can construct a finite number of disjoint domains $W_j = Q_j \setminus \cup_k Q_k^j$, where

1. Q_j is a Carleson square with base I_j and $|\partial W_j \cap I_j| \geq (1-\epsilon)|I_j|$,
2. g is within ϵ of a constant c_j on each W_j,
3. $\sum_j |\partial W_j \cap \mathbb{T}| \geq 1 - \epsilon$.

Set $W = \cup_j W_j$ be this finite union. Note that the weak convergence of $\{\nu_n\}$ implies that

$$\limsup_{n\to\infty} \nu_n(\mathbb{D} \setminus W) \leq \epsilon,$$

and so

$$\begin{aligned}
|\lim_n \int g d\nu_n - \int_{\mathbb{T}} g d\theta/2\pi| &\leq \lim_{n\to\infty} |\int_W g d\nu_n - \frac{1}{2\pi}\int_{\partial W\cap\mathbb{T}} g d\theta| + \int_{\mathbb{D}\setminus W} |g| d\nu_n \\
&\qquad + \frac{1}{2\pi}\int_{\mathbb{T}\setminus\partial W} |g| d\theta \\
&\leq \epsilon \sum_j |\partial W_j \cap \mathbb{T}| + 2|\mathbb{T}\setminus \cup \partial W_j| \\
&\leq 4\epsilon.
\end{aligned}$$

Taking $\epsilon \to 0$ proves Lemma 4.3 and thus our function theoretic proof of Lemma 4.1. $\square$

A very special (and easier) case of Lemma 4.3 is

Corollary 4.4. *Suppose $f \in \mathcal{U}_0$ and let $f_r(z) = f(rz)$, for $r < 1$. Then μ_{f_r} converges weakly to μ_f as $r \to 1$.*

Corollary 4.5. *If f is inner and $f(0) = 0$ then μ_{f_r} converges weakly to normalized Lebesgue measure on $\mathbb{T}$.*

5. A change of variables

The following result was suggested by Paul Bourdon and simplifies certain arguments from an earlier version of the paper.

Lemma 5.1. *Suppose g is a positive, continuous function on $\mathbb{D}$ and has non-tangential boundary values almost everywhere on $\mathbb{T}$. Then for any $f \in \mathcal{U}$,*

$$\int g(z) d\mu_f(z) = \frac{1}{2\pi}\int_0^{2\pi} g(f(e^{i\theta}))d\theta.$$

The integral on the left requires some interpretation since g is not necessarily continuous on the support of μ_f. On the interior of the disk, g is continuous and positive so the integral is well defined (possible infinite). On the circle, μ_f is absolutely continuous with respect to Lebesgue measure and the boundary values of g are Borel, so the integral on the circle is also well defined.

Proof. Using the monotone convergence theorem we can reduce to the case when g is bounded (just truncate and let the truncation tend to ∞). So assume g is bounded by M. For any $\epsilon > 0$ we can easily construct a sawtooth region W so that $|\mathbb{T} \cap \partial W| > 1 - \epsilon$ and g extends continuously to the closure of W. Thus we can write $g = (g - h) + h$ where h is continuous, bounded by M and $g - h$ is zero on W. The lemma is true for continuous functions by the definition of μ_f, and

$$\int (g-h) d\mu_f \leq 2M\mu_f(\overline{\mathbb{D}} \setminus W) \leq 2MC\epsilon,$$

since μ_f is Carleson with a uniform constant. Similarly

$$\int (g-h) \circ f(e^{i\theta}) d\theta \leq 2MC\epsilon,$$

so taking $\epsilon \to 0$ proves the lemma. □

The following is now immediate.

Lemma 5.2. *If $g \in H^\infty$ and f is inner with $f(0) = 0$ then $\mu_g = \mu_{g\circ f}$.*

The hyperbolic little Bloch space, $\mathcal{B}_0^h$, is defined as those holomorphic maps $f \in \mathcal{U}$ so that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2} = 0,$$

and is contained in the usual little Bloch space, $\mathcal{B}_0$. Schwarz's inequality implies the left side is bounded by 1 for any analytic self map of the disk, and from this it is easy to verify that g, f are both holomorphic selfmaps of the disk and f is hyperbolic little Bloch then so is $g \circ f$. It is far from obvious that there is an inner function in the hyperbolic little Bloch space, but they do exist (see [1], [6], [19]). This and Lemma 5.2 thus imply,

Corollary 5.3. *If $g \in \mathcal{U}$, then there is a $f \in \mathcal{B}_0^h$ so that $\mu_f = \mu_g$.*

Recall that the Hardy space, H^p , is the set of holomorphic functions g such that

$$\|g\|_{H^p} = \lim_{r \to 1} (\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta)^{1/p} < \infty.$$

Such a function has radial boundary values almost everywhere on $\mathbb{T}$ which we also denote by g. If we know $g \in H^p$ for $p > 1$, then the radial maximal function of g is in L^p and so on can use the dominated convergence theorem to deduce that

$$\|g\|_{H^p} = \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^p d\theta.$$

In general, however, the right hand side might be finite but $g \notin H^p$ (there exist non-zero holomorphic functions on the disk that have radial value zero almost everywhere, hence are not in H^p). If $f \in \mathcal{U}$ then μ_f restricted to $\mathbb{T}$ is absolutely continuous with respect to Lebesgue measure, so $\int_{\overline{\mathbb{D}}} |g|^p d\mu_f$ makes sense.

As another application of Lemma 5.1 we can show

Lemma 5.4. *Suppose $g \in H^p$ on the unit disk and $f \in \mathcal{U}_0$. Then for any $0 < p < \infty$,*

$$\|g \circ f\|_{H^p}^p = \lim_{r \to 1} \int_{\mathbb{D}} |g|^p d\mu_{f_r} = \int_{\overline{\mathbb{D}}} |g|^p d\mu_f. \tag{5.1}$$

Proof. The first equality is the definition of the H^p norm, so we only have to prove the second. If $g \in H^p$ and $f \in \mathcal{U}_0$ then by a result of Ryff [17], $g \circ f \in H^p$ with smaller or equal norm. Thus $|g|^p$ is positive, continuous function on the disk which has non-tangential boundary values almost everywhere and so Lemma 5.1 applies to show

$$\int |g(z)|^p d\mu_f = \frac{1}{2\pi} \int_0^{2\pi} |g(f(e^{i\theta}))|^p d\theta,$$

and since we already know $g \circ f \in H^p$, we can deduce that the right hand side equals $\|g \circ f\|_{H^p}$. ∎

6. Mapping the Bergman space into the Hardy space

For our applications to composition operators, we need a version of Lemma 5.4 that works without the assumption that $g \in H^p$. The proof given above doesn't work in general because if $g \notin H^p$ we can't say that $\|g\|_{H^p} = \int_0^{2\pi} |g|^p d\theta/2\pi$. In fact, we will not even assume g has boundary values on the circle so that this integral is not necessarily defined.

Lemma 6.1. *Suppose g is holomorphic on the open unit disk, $f \in \mathcal{U}_0$ and μ_f is radially symmetric. Then for any $0 < p < \infty$,*

$$\|g \circ f\|_{H^p}^p = \lim_{r \to 1} \int_{\mathbb{D}} |g|^p d\mu_{f_r} = \int_{\mathbb{D}} |g|^p d\mu_f + \mu_f(\mathbb{T}) \|g\|_{H^p}^p. \tag{6.1}$$

Proof. Let $g_s(z) = g(sz)$, for $0 < s < 1$. First we want to show that for any $0 < p < \infty$,

$$\lim_{s \to 1} \int |g(sz)|^p d\mu_f = \int_{\mathbb{D}} |g(z)|^p d\mu_f + \mu_f(\mathbb{T}) \|g\|_{H^p}^p, \tag{6.2}$$

Since g is holomorphic, $|g|^p$, $0 < p < \infty$ is subharmonic (e.g., [11], page 35) and hence $m(r) = \frac{1}{2\pi} \int |g(re^{i\theta})|^p d\theta$, is defined on $[0, 1)$ and is an increasing function of r (Corollary I.6.6, [11]). Therefore we can extend it to be defined at $r = 1$ by $\|g\|_{H^p}^p = m(1) = \lim_{r \to 1} m(r)$. Thus $m_s(r) \equiv m(sr)$ increases to $m(r)$ as $s \to 1$ for all $r \in [0, 1]$. Let ν be the measure on $[0, 1]$ defined by $\nu(E) = \mu_f(\{z : |z| \in E\})$. Since μ_f is radial we have

$$\int \varphi d\mu_f = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \varphi(re^{i\theta}) d\theta d\nu(r).$$

Thus by the monotone convergence theorem,

$$\lim_{s \to 1} \int |g_s|^p d\mu_f = \lim_{s \to 1} \int m_s(r) d\nu = \int_{[0,1]} m(r) d\nu = \int_{\mathbb{D}} |g|^p d\mu_f + \mu_f(\mathbb{T}) m(1).$$

This is (6.2).

We will break the proof of (6.1) into three cases: (1) $\int_{\mathbb{D}} |g|^p d\mu_f = \infty$, (2) $\int_{\mathbb{D}} |g|^p d\mu_f < \infty$ and $\mu_f(\mathbb{T}) = 0$ and finally (3) $\int_{\mathbb{D}} |g|^p d\mu_f < \infty$ and $\mu_f(\mathbb{T}) > 0$.

Case 1: First suppose $\int |g|^p d\mu_f = \infty$. For any $M > 0$ choose $0 < t < 1$ so that $\int_{|z|<t} |g|^p d\mu_f > 2M$ and write $|g|^p = g_1 + g_2$ where g_1, g_2 are non-negative, $g_1 = |g|^p$ on $|z| < t$ and g_1 is continuous and compactly supported in $\mathbb{D}$. Then

$$\int |g|^p d\mu_{f_r} \geq \int g_1 d\mu_{f_r} > \frac{1}{2} \int g_1 d\mu_f \geq M,$$

if r is close enough to 1. Thus $\int |g|^p d\mu_{f_r} \to \infty = \int |g|^p d\mu_f$.

Case 2: Now suppose $\mu_f(\mathbb{T}) = 0$ and $\int |g|^p d\mu_f < \infty$. Since μ_{f_r} converges weakly to μ_f,

$$\lim_{r \to 1} \int |g_s|^p d\mu_{f_r} = \int |g_s|^p d\mu_f,$$

for any fixed $s < 1$. Since $g_s(f(z))$ is holomorphic on the open disk, $|g_s(f(z))|^p$ is subharmonic. Thus $\int |g_s|^p d\mu_{f_r}$ is increasing in r. Thus

$$\int |g_s|^p d\mu_{f_r} \leq \int |g_s|^p d\mu_f.$$

Now take $s \to 1$. For a fixed r, μ_{f_r} is compactly supported in the disk , so $|g_s|^p$ is uniformly bounded on its support and so the left hand side converges to $\int |g|^p d\mu_{f_r}$. Condition (6.2) implies that the right hand side converges to $\int |g|^p d\mu_f$. Thus

$$\int |g|^p d\mu_{f_r} \leq \int |g|^p d\mu_f,$$

for all $r < 1$.

Fix $\epsilon > 0$ and choose $0 < t < 1$ so that $\int_{t<|z|<1} |g|^p d\mu_f < \epsilon$. Write $|g|^p = g_1 + g_2$ as in Case 1. Thus $\int g_2 \mu_f < \epsilon$. Also, if r is close enough to 1 then by weak convergence,

$$|\int g_1 d\mu_f - \int g_1 d\mu_{f_r}| < \epsilon.$$

Thus

$$\int g_2 d\mu_{f_r} \le |\int g_1 d\mu_f - \int g_1 d\mu_{f_r}| + \int g_2 d\mu_f \le 2\epsilon.$$

Hence

$$\begin{aligned} |\int |g|^p d\mu_f - \int |g|^p d\mu_{f_r}| &\le \int g_2 d\mu_{f_r} + |\int g_1 d\mu_f - \int g_1 d\mu_{f_r}| + \int g_2 d\mu_f \\ &\le 4\epsilon, \end{aligned}$$

if r is close enough to 1.

Case 3: Now assume that $\int_{\mathbb{D}} |g|^p d\mu_f < \infty$, $\mu_f(\mathbb{T}) > 0$. If $\lim_{r\to 1} \int |g|^p d\mu_{f_r} = \infty$ then by subharmonicity of $|g \circ f|^p$ we get $\int |g|^p \mu_f = \infty$ so (6.1) holds. Thus we may assume $\lim_{r\to 1} \int |g|^p d\mu_{f_r} < \infty$, i.e., we may assume $g \circ f \in H^p$, and hence that $|g(f(z))|^p$ has a harmonic majorant u on $\mathbb{D}$ (Lemma II.1.1, [11]).

First we show that $g \in H^p$. For $0 < r < 1$ let $D_r = D(0,r)$. Let Ω_r be the component of $f^{-1}(D_r)$ which contains the origin and let ω_r be the harmonic measure on Ω_r with respect to the origin. Let ν_r be the push forward of ω_r under the map f. Then clearly ν_r is supported on $\overline{D_r}$ and for any $E \subset D_r$, $\nu_r(E) \le \mu_f(E)$. By Lemma 2.6 ν_r on $C_r = \partial D_r$, must be $\frac{1}{2\pi} d\theta$ minus the balayage of ν_r restricted to D_r. Since $\nu_r \le \mu_f$, this means that ν_r on C_r is greater or equal $\frac{1}{2\pi} d\theta$ minus the balayage of μ_f restricted to D_r. Since μ_f is radial, its balayage onto C_r is also radial, i.e., is $\frac{1}{2\pi}\mu_f(D_r)d\theta \le \frac{1}{2\pi}(1-\mu_f(\mathbb{T}))d\theta$. Thus on C_r, $\nu_r \ge \frac{1}{2\pi}\mu_f(\mathbb{T})d\theta$. Hence for any g holomorphic on $\mathbb{D}$

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \le \frac{1}{\mu_f(\mathbb{T})}\int |g|^p d\nu_r = \frac{1}{\mu_f(\mathbb{T})}\int |g \circ f|^p d\omega_r.$$

Thus if u is a harmonic majorant of $|g \circ f|^p$ on $\mathbb{D}$,

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \le \frac{1}{\mu_f(\mathbb{T})}\int u\, d\omega_r = \frac{u(0)}{\mu_f(\mathbb{T})} < \infty.$$

In other words, $g \in H^p$ and thus (6.1) follows from Lemma 5.4 $\square$

Recall that the Bergman space, A^p, is defined as the set of holomorphic functions g on the disk $\mathbb{D}$ such that

$$\|g\|_{A^p} = (\frac{1}{\pi}\int_{\mathbb{D}} |g|^p dxdy)^{1/p} < \infty.$$

Corollary 6.2. *If $f \in H^\infty$ is such that $d\mu_f = \frac{1}{\pi}\chi_{\mathbb{D}} dxdy$, then for any analytic function g on the disk, g is in the Bergman space iff $g \circ f$ is in the Hardy space, and $\|g\|_{A^p} = \|g \circ f\|_{H^p}$, i.e., the corresponding composition operator, $C_f : A^p \to H^p$, is an isometry onto its range.*

Proof. Using the Lemma 6.1 we see

$$\begin{aligned}
\|g \circ f\|_{H^p} &= \lim_{r\to 1}(\frac{1}{2\pi}\int_0^{2\pi} |g(f(re^{i\theta}))|^p d\theta)^{1/p} \\
&= \lim_{r\to 1}(\int |g|^p d\mu_{f_r})^{1/p} \\
&= (\int |g|^p d\mu_f)^{1/p} \\
&= \|g\|_{A^p}.
\end{aligned}$$

□

The corollary may seem a little surprising since functions in H^p have non-tangential limits almost everywhere, whereas those in A^p need not, but since f has almost all of its boundary values in the interior of the disk, this is not a contradiction. Of course, it still remains to show (Section 9) that there is an $f \in H^\infty$ so that μ_f is area measure.

The previous corollary obviously holds for any weighted Bergman space where the weight is a radial measure of finite mass satisfying the integral condition (1.1) in Theorem 1.1. If instead of an isometry, we merely want $\|g\|_{A_p} \simeq \|g \circ f\|_{H^2}$ we could take a much bigger class of f's, e.g. $\mu_f = wdxdy$ for some weight w which is bounded above and below on an annulus $\{r < |z| < 1\}$. Constructing such examples only requires the techniques of Section 8, not the full proof of Theorem 1.1.

Similarly, by appropriate choices of μ_f one can construct composition operators on H^p which satisfy conditions like

$$\|C_f(g)\|_{H^p}^p = \frac{1}{2}\|g\|_{H^p}^p + \frac{1}{2}\|g\|_{A^p}^p$$

or

$$\|C_f(g)\|_{H^p}^p = \frac{1}{2}\|g\|_{H^p}^p + \frac{1}{2}\|g_{1/2}\|_{H^p}^p.$$

In [7], a function f is said to have property $(*)$ relative to H^p if for any holomorphic g on $\mathbb{D}$, $g \circ f \in H^p$ implies $g \in H^p$. Paul Bourdon has pointed out that for general $f \in \mathcal{U}$, the condition $\mu_f(\mathbb{T}) = 0$ implies this condition implies $N_f(z) = o(1-|z|)$, which by J. Shapiro's theorem from [18], implies C_f is compact and hence does not have a bounded right inverse. Since f is non-constant, C_f is 1-to-1 and so C_f does not have closed range (this is a consequence of the open mapping theorem, e.g. Corollary 2.12c of [15]). Thus C_f does not have property $(*)$, since any function in $\overline{C_f(H^p)} \setminus C_f(H^p)$ is an H^p function without an H^p preimage. Lemma 6.1 clearly implies

Corollary 6.3. *If $f \in \mathcal{U}_0$ is orthogonal, then f has property $(*)$ relative to H^p iff $\mu_f(\mathbb{T}) > 0$.*

Proof. If $\mu_f(\mathbb{T}) > 0$ then the proof in Case 3 of the proof of Lemma 6.1 shows that $g \circ f \in H^p$ implies $g \in H^p$. Thus f has property $(*)$ with respect to H^p. □

A special case of Corollary 6.3 is when $\mu_f(\mathbb{T}) = 1$, i.e., all inner functions have property $(*)$. It would be very interesting to have a similar characterization of property $(*)$ for general functions in $\mathcal{U}_0$.

7. An example of μ_f supported on two circles

In this section we will construct an $f \in H^\infty$ so that μ_f is supported on the union of two circles $C_{1/2}$ and C_1 (where $C_r = \{z : |z| = r\}$) and is a multiple of Lebesgue measure on each. This example suffices to disprove Rudin's orthogonality conjecture, and introduces the estimates and techniques needed for the general case of Theorem 1.1. In the next section we will show that any radially symmetric probability measure supported in $\{\frac{1}{2} \leq |z| \leq 1\}$ can occur as a μ_f, and in Section 9 we will do the general case of measures supported on $\mathbb{D}$.

Based on Lemmas 4.1 and 2.5 it suffices to build an increasing sequence of Riemann surfaces $\{R_n\}$ so that the corresponding maps $\{f_n\}$ satisfy $f_n(0) = 0$, μ_{f_n} is supported on the two circles $C_{1/2} \cup C_1$ and μ_{f_n} restricted to $C_{1/2}$ is of the form $\frac{1}{2\pi} g_n(\theta) d\theta$ where g_n converges to uniformly to a positive constant.

We start by taking $f_0(z) = \frac{1}{2}z$, i.e, f_0 is the (trivial) Riemann mapping from $\mathbb{D}$ to the disk $R_0 = \{|z| < 1/2\}$. The corresponding measure $\mu_0 = \mu_{f_0}$ is normalized Lebesgue measure on the circle $C_{1/2}$, i.e., it is $\frac{1}{2\pi} g_0(\theta) d\theta$ where $g_0(\theta) = 1$.

Now we describe the idea of the construction of R_1 (we will give the details later). First we replace R_0 by a slightly smaller disk, S_1. We divide the boundary of S_1 into a large number of alternating intervals which we call type I and J. Along each type I interval we attach a copy certain Riemann surface with boundary over $C_{1/2}$ (different copies to different intervals) and along each type J interval we attach copies of certain surfaces with boundary over C_1. This gives the surface R_1. With appropriate choices of the parameters involved we can show that with high probability, the Brownian paths which first hit ∂S_1 at a type I interval go on to hit the part of ∂R_1 over $C_{1/2}$ and the paths which hit the J intervals go on to hit

∂R_1 over C_1. Thus we have "rerouted" a certain fraction of the harmonic measure on $C_{1/2}$ out to C_1. By choosing various parameters correctly, we can make the harmonic measure over $C_{1/2}$ in R_1 be close to any multiple of Lebesgue measure we want (as long as the total mass is < 1). The resulting measure may not be radially symmetric, but by iterating the construction with variable size barriers, we can make harmonic measure as close to a multiple of Lebesgue measure as we wish, obtaining a radial measure in the limit.

Now we give the construction of R_1 in more detail. Choose δ_1 very small and let $S_1 = D(0, r_1)$, where $r_1 = \frac{1}{2} - \delta_1$. Obviously harmonic measure on S_1 is just normalized Lebesgue measure on its boundary. Choose a large integer m_1 and points $\{z_j\}$ $j = 1, \dots, m_1$ equally spaced on the circle C_{r_1}. Choose a continuous function $0 < \eta(x) < 1$ on C_{r_1} and let I_j be an arc of ∂S_1 of angle measure $\eta(z_j)2\pi/m_1$ and let $\{J_j\}$ be the complementary arcs. For the first step of the construction we can take $\eta(x) = \eta_1$ to be a constant for simiplcity, but in later steps we will have to use non-constant η's.

Fix some $0 < \tau_1 < 1$ and for each arc of the form I_j with endpoints $\{p, q\}$, choose a countable collection of points $E = \{w_k^j\} \subset I_j$, accumulating only at the endpoints of I_j, so that for any $z \in I_j$,

$$\mathrm{dist(z, E)} \leq \tau_1 \mathrm{dist(z, \{p, q\})}. \tag{7.1}$$

Let the components of $I_j \setminus E$ be denoted $\{I_k^j\}$. For each I_k^j, consider the (infinitely connected) planar domain $\mathbb{D} \setminus E$ and the universal cover of the domain. Take a copy of the arc I_k^j in the universal cover; it is on the boundary of a disk D in the universal cover which covers $D(0, \frac{1}{2})$. The arc cuts the universal cover into two components and we let R_k^j denote the component which does not contain D. For each interval I_k^j, we attach a copy of R_k^j to S_1 along the arc I_k^j.

For the intervals $\{J_j\}$ we follow the same procedure, defining a set $E \subset J_j$ and sub intervals $\{J_k^j\}$. The only difference is that except that $D(0, \frac{1}{2})$ is replaced by $D(0,1)$, i.e., we attach a component of the universal cover of $D(0,1) \setminus E$, cut along J_k^j. Doing this for all j and k gives the surface R_1. The harmonic measure for R_1 is now supported on $C_{1/2} \cup C_1$, (the rest of the ideal boundary covers a countable set, so has zero measure) so we only need to check that it is still close to radial on $C_{1/2}$.

Now we want to discuss the two main estimates for describing the harmonic measure of R_1. The first says that a continuous convolution of the Poisson kernel is well approximated by a discrete version, if the sample points are sufficiently close together. The second says that the harmonic measure of I intervals is small when viewed from a J interval and vice versa.

Suppose $D(0,r)$ is a disk and g is a continuous function on a smaller circle C_s, $s < r$. The balayage of g onto the circle C_r is

$$Bg(\theta) = \int_0^{2\pi} g(se^{it}) P_{se^{it}}(\theta) dt,$$

where $P_z(\theta)$ is the Poisson kernel for $D(0,r)$ with respect to the point z.

Lemma 7.1. *With the intervals $\{I_j\}$ defined as above, and $F = \cup_j I_j$, for any continuous $0 < g < 1$ on the circle C_s*

$$B(g\chi_F)(\theta) = \int_F g(se^{it}) P_{se^{it}}(\theta) dt \to B(g\eta)(\theta),$$

uniformly as $m_1 \to \infty$.

Proof. Let K_j be the interval on C_s centered at z_j of angle measure $2\pi/m_1$ (choose them to be half-open, so that they form a disjoint cover of the circle). Define the piecewise constant functions $a(x), b(x)$ on C_{r_1} by

$$a(x) = \sum_j \chi_{K_j}(x)\eta(z_j),$$

$$b(x,\theta) = \sum_j \chi_{K_j}(x)g(z_j)P_{z_j}(\theta),$$

and define

$$A(m_1) = \|\eta(z) - a(z)\|_\infty,$$

$$B(m_1) = \|g(x)P_x(\theta) - b(x,\theta)\|_\infty.$$

It is clear that by uniform continuity, both quantities tend to zero as $m_1 \to \infty$. Thus by using the fact that $\chi_F(x) - a(x)$ has mean value zero on each interval K_j where $b(x,\theta)$ is constant in x we get

$$\begin{aligned}
|B(g\chi_F)(\theta) \quad &- \quad B(g\eta)(\theta)| \\
&= \quad |\int_0^{2\pi} (g(se^{it})P_{se^{it}}(\theta) - b(se^{it},\theta) + b(se^{it},\theta))(\chi_F - \eta(se^{it}))dt| \\
&\le \quad B(m_1)\int_0^{2\pi} |(\chi_F - \eta(se^{it})|dt + \int_0^{2\pi} b(se^{it},\theta)|a(se^{it}) - \eta(se^{it})|dt \\
&\le \quad 2\pi B(m_1) + A(m_1)\max|b|.
\end{aligned}$$

This clearly tends to zero as $m_1 \to \infty$, as desired. $\square$

Now for the second estimate. Next we want to show that the harmonic measure of $C_{1/2}$ is much larger than that of C_1 with respect to a point $z \in I_k^j$. More precisely,

Lemma 7.2. *Suppose $z \in I_k^j$ and suppose γ is a circular arc in S_1 with endpoints in the corresponding set E such that* $\mathrm{dist}(\gamma, \mathrm{z}) \simeq \mathrm{dist}(\mathrm{z}, \{\mathrm{p}, \mathrm{q}\})$ *and which separates z from all the J-intervals. Let Ω be the component of $R_1 \setminus \gamma$ which contains z. Then $\omega(z,\gamma,\Omega)$ tends to 0 as τ_1 does.*

Proof. Standard estimates of hyperbolic metric imply that γ is within a bounded hyperbolic distance of a geodesic in R_1 and that the hyperbolic distance from γ to z is at least $C \log \tau_1^{-1}$. Lifted to the disk, this implies the harmonic measure of γ with respect to z is $\leq \exp(C \log \tau_1) \leq \tau_1^{\alpha}$, for some $\alpha > 0$, as desired. Obviously, the same estimate holds if we reverse the rolls of the I and J intervals. $\square$

The previous result has a simple explaination in terms of Brownian motion. Consider a Brownian motion on the Riemann surface started at z and run until it either hits γ or leaves R_1. The path will only hit γ if it stays on the correct sheet of R_1, but this is extremely unlikely because it will cross the arc I_j many times and each time it has a certain chance (which is large if τ is small) of becoming "tangled" and ending up on the wrong sheet.

We can now show that the harmonic measure of R_1 on the circle $C_{1/2}$ can be taken as close to a multiple of Lebesgue measure as we wish (depending on our choices of m_1, τ_1 and η). The harmonic measure of R_1 on the circle $C_{1/2}$ will be the balayage of the harmonic measure of S_1 restricted to the I intervals, with an error bounded by $C\tau_1^{\alpha}$. The harmonic measure is (normalized) angle measure restricted to the I-intervals. Thus if m_1 is large enough the harmonic measure on $C_{1/2}$ will be of the form $\frac{1}{2\pi} g_1(x) d\theta$, with g_1 as close to a constant as we wish. Take $\frac{1}{2} + \frac{1}{10} \leq g_1(x) \leq \frac{1}{2} + \frac{3}{10}$, to be concrete.

Now suppose we have constructed R_{n-1}. To construct R_n, we follow the method above. We start passing to a subsurface $S_n \subset R_{n-1}$ where the boundary circles over $C_{1/2}$ are replaced by boundaries over $C_{1/2-\delta_n}$. The parameter δ_n is chosen so small that every component of $R_{n-1} \backslash S_n$ is a regular cover of the annulus $\frac{1}{2} - \delta_n < |z| < \frac{1}{2}\}$ (which will be possible by the construction of R_{n-1}) and so that harmonic measure

μ_{S_n} on S_n is very close to harmonic measure on R_{n-1}, say

$$|\int \varphi d(\mu_{S_n} - \mu_{R_{n-1}})| \leq 2^{-n}, \tag{7.2}$$

for every smooth φ with gradient bounded by n.

As before we choose m_n equally spaced points $\{z_j^n\}$ points on $C_n = C_{\frac{1}{2}-\delta_n}$ and define intervals $\{I_j^n\}$ of C_n centered at these points of angle measure $2\pi\eta_n(z_j^n)/m_n$, where

$$\eta_n(x) = (\frac{1}{2} + \frac{2}{10^n})/g_{n-1}(x).$$

The complementary intervals are denoted $\{J_j^n\}$. We choose a very small τ_n and sets E in each interval which satisfies (7.1) with τ_n. We then attach copies $D(0, \frac{1}{2}) \setminus E$ to the copies of the I intervals in ∂S_n and copies of $D(0,1) \setminus E$ to the J intervals. Then if we choose δ_n and τ_n small enough and m_n large enough, we can get the harmonic measure of R_n over $C_{1/2}$ to be $g_n(x)\frac{d\theta}{2\pi}$ with g_n as close to $g_{n-1}\eta_n$ as we wish, say

$$\frac{1}{2} + \frac{1}{10^n} \leq g_n \leq \frac{1}{2} + \frac{3}{10^n}.$$

Continuing in this way we can clearly construct a sequence $\{R_n\}$ of Riemann surfaces so that the harmonic measures over $C_{1/2}$ converge to a multiple of Lebesgue measure. This almost finishes the proof, except that the surfaces $\{R_n\}$ are not nested by inclusion. However, the subsurfaces $\{S_n\}$ constructed as part of the induction are nested and their union is also R. Hence their harmonic measures converge to that of R. By (7.2), the weak limit for the measures on $\{S_n\}$ and $\{R_n\}$ must be the same, so we are done.

The same proof shows that we can build an $f \in H^\infty$ so that $\mu_f|_{C_{1/2}} = \frac{1}{2\pi} g d\theta$ for any continuous g with $0 \leq g < 1$ (or any g which is the decreasing limit of such functions). Similarly, the circle can be replaced by any smooth curve γ, and g by a continuous function such that $gds \leq d\omega(0, \cdot, \mathbb{D} \setminus \gamma)$.

The construction in this section clearly generalizes as follows.

Lemma 7.3. *Suppose R is a Riemann surface build by attaching subdomains of $\mathbb{D}$ along boundary arcs. Let Π denote the corresponding projection of R into the plane. Suppose $\Pi(\partial R)$ hits C_r and there is a $\delta > 0$ so that every component of $\Pi^{-1}(C_r)$ in ∂R is the boundary of a domain in R which is a regular cover of the annulus $\{r - \delta < |z| < r\}$ (or $\{r < |z| < r + \delta\}$). Suppose the harmonic measure of R over C_r, projects to a measure of the form $\frac{1}{2\pi} g d\theta$ on C_r where $0 < g < 1$. Choose $s < r$ very close to r (or $s > r$). Suppose we are given N functions $\{\eta_k\}$ so that $0 < \eta_k < 1$. Choose a large integer m and choose mN equally spaced points $\{z_i\}$ on C_s. Let I_j^k be the interval of length $2\pi\eta_k(z_{k+jN})/mN$ centered at z_{k+jN}. Let J_i denote the components of $C_s \setminus \cup_{j,k} I_j^k$. Choose a small τ and choose sets E satisfying (7.1) in every interval. For $k = 0, \ldots, N$, choose $s_k < s < r_k$. For each each arc in ∂R projecting to I_j^k attach a copy of $A_k \setminus E = \{s_k < |z| < r_k\} \setminus E$. To each arc projecting to a J_i attach a copy of $A_0 \setminus E = \{s_0 < |z| < r_0\} \setminus E$. Then if s is close enough to r, m is large enough and τ is small enough, then the projected harmonic measure of the new surface S on $\partial S \setminus R$ is as close to $\sum_k B_k(\eta_k g)$ as we wish, where B_k denotes balayage from C_s onto ∂A_k.*

For the proof of Theorem 1.1, we can always take $s_k = 0$, i.e., we can attach disks instead of annuli. Only for the proof of Corollary 1.4, will we have to attach proper annuli.

8. Theorem 1.1 on an annulus

In this section we will show that any radially symmetric probability measure μ supported in the annulus $\{z : \frac{1}{2} \leq |z| \leq 1\}$ is of the form μ_f for some $f \in \mathcal{U}_0$, and in the next section we will extend this to the general case.

First some notation. For $0 < r < s < 1$ let $A(r,s) = \{z : r \leq |z| < s\}$. When $s = 1$, we let $A(r,1) = \{z : r \leq |z| \leq 1\}$. For $0 < r < 1$, let $\mu(r) = \mu(A(0,r))$. Let $r^0 = \frac{1}{2}$, $r_0^1 = \frac{1}{2}, r_1^1 = \frac{3}{4}$, and more generally, $r_k^n = \frac{1}{2} + k2^{-n-1}$. Let $\mu_k^n = \mu(A(r_k^n, r_{k+1}^n))$. Also let $C_k^n = C_{r_k^n}$.

By rescaling we may assume $\mathbb{T} \subset \text{supp}(\mu) \subset \overline{\mathbb{D}}$ and hence that $\mu^n 2^n - 1 > 0$ for all n.

We will construct a sequence of Riemann surfaces $R_0 \subset R_1 \subset \dots$ such that the corresponding measure μ_n is supported on the union of 2^n circles, $\cup_{k=0}^{2^n-1} C_{r_k^n}$. On $C_{r_k^n}$ the measure μ_n will have the form $\frac{1}{2\pi} g_k^n d\theta$ where

$$\mu_k^n < g_k^n \leq \mu_k^n + \epsilon_n \tag{8.1}$$

for $k = 0, \dots, 2^n - 2$ and any $\epsilon_n > 0$ we choose, and for $k = 2^n - 1$ we have

$$\mu_{2^{n+1}-2}^{n+1} < g_k^n \leq \mu_{2^n-1}^n. \tag{8.2}$$

Recall that since μ_n is a probability measure, if it gives too much mass to the first $2^n - 1$ annuli, then it must give too little to the last one. It is obvious that such measures $\{\mu_n\}$ converge weakly to μ, so by the argument at the end of the previous section, the μ_f corresponding to the limiting surface $R = \cup_n R_n$ must equal μ.

Thus it only remains to construct the surfaces. As in the previous section we start with $R_0 = D(0, \frac{1}{2})$. To construct R_1, we will proceed exactly as in the previous section, except that instead of redirecting harmonic measure to the unit circle, we send it to the circle $C_{3/4}$. The estimates are all the same so we can obtain a surface

R_1 so that the corresponding μ_1 is supported on $C_{1/2} \cup C_{3/4}$ and is of the form $\frac{1}{2\pi} g_0^1 d\theta$ on C_0^1 and $\frac{1}{2\pi} g_1^1 d\theta$ on C_1^1 where

$$\mu_0^1 < g_0^1 < \mu_0^1 + \epsilon_1,$$

$$\mu_2^2 < g_1^1 < \mu_1^1,$$

for any $\epsilon_1 > 0$ we choose.

To construct R_{n+1}, $n \geq 1$, we just make one small change. The mass on the outermost circle $C^n_{2^n-1}$ is redistributed to itself $C^n_{2^n-1} = C^{n+1}_{2^{n+1}-2}$ and the outermost circle of the next stage, $C^{n+1}_{2^{n+1}-1}$. The mass of any other circle C^n_j is redistributed to three circles; itself, $C^n_j = C^{n+1}_{2j}$, the next circle out in the next generation, C^{n+1}_{2j+1} and the outermost circle of the next generation, $C^{n+1}_{2^{n+1}-1}$.

To do this we let $\tilde{C}^n_j$ be the circle of radius $r^n_j - \delta_n$, where $\delta_n < 2^{-n-10}$ is chosen so small that the harmonic measure on S_n (the subsurface of R_n bounded by the lifts of $\tilde{C}^n_j$ which contains 0 and hence contains S_{n-1}) is as close as we wish to harmonic measure on R_{n-1}, i.e., it satisfies (7.2). We now just apply the construction of Lemma 7.3, with $N = 2$, $s_0 = s_1 = s_2 = 0$, $r_0 = r^{n+1}_{2j}$, $r_1 = r^{n+1}_{2j+1}$ and $r_2 = r^{n+1}_{2^{n+1}-1}$. More precisely, suppose we have two continuous functions η_1, η_2 on $\tilde{C}^n_j$ such that $\eta_1 + \eta_2 < 2$ and m_n equidistributed points $\{z_j\}$ on ∂S_n, choose intervals centered at these points. However, instead of having two types of intervals, we will have three; $\{I_j\}$ of angle measure $2\pi\eta_1(\theta)/m_n$ centered at z_j for j even, $\{K_j\}$ of angle measure $2\pi\eta_2(\theta)/m_n$ centered at $z - j$ for j odd, and the remaining intervals $\{J_j\}$. We choose a very small τ_n and a countable set E in each interval which satisfies (7.1). Then along type I intervals we attach a copy of the universal cover of $D(0, r^n_j) \setminus E$; along the type K intervals we attach the universal cover of $D(0, r^{n+1}_{2j+1}) \setminus E$; and along the type J intervals we attach the universal cover of

$D(0, r^{n+1}_{2^{n+1}-1}) \setminus E$. Then if we take m_n large enough and δ_n, τ_n small enough, the harmonic measure of the surface R_{n+1} over C^n_j will be as close to the balayage of $\eta_1 g^n_j$ onto C^n_j as we wish and the harmonic measure over C^{n+1}_{2j+1} will be as close to the balayage of $\eta_2 g^n_j$ onto that circle as we wish, independent of what changes we make at circles other than C^n_j.

Now do a similar construction around each of the circles C^n_j, $j = 0, \ldots 2^n - 2$. At the outer most circle $C^n_{2^n-1}$ we redirect the measure to only two circles; itself and the outermost circle of the next generation, $C^{n+1}_{2^{n+1}-1}$. By construction, condition (8.1) holds with any constant ϵ_n we want. Then by Lemma 2.6, μ_n on the outermost circle must be normalized Lebesgue measure minus the balayage of the measures on the inner circles. Since these measures have total mass is as close to (but larger than) $\mu(A(\frac{1}{2}, r^n_{2^n-1})) = \sum_{j=0}^{2^n-2} \mu^n_j$ as we wish, the mass of the outermost circle is as close to (but less than) $\mu(A(r^n_{2^n-1}, 1) = \mu^n_{2^n-1}$. Moreover, since the measures on the inner circles are as close to radial as we wish, so is their balayage onto the outermost circle and hence so is μ_f restricted to the outermost circle (this condition defines our choice of ϵ_n). This gives condition (8.2). The proof is completed by taking limits just as in the two previous sections.

9. Theorem 1.1 on the whole disk

To complete the proof of Theorem 1.1 we need to show how to obtain any measure satisfying (1.1). As in the last section we can assume $\mathbb{T} \subset \operatorname{supp}(\mu) \subset \overline{\mathbb{D}}$. We can also simplify the situation slightly by observing that it is enough to assume that most of the mass of μ lives away from the origin, i.e.,

$$\int \log \frac{1}{|z|} d\mu \leq \delta. \tag{9.1}$$

This is because for $f \in H^\infty$ the measure μ_{f^d} is the push forward under $z \to z^d$ of the measure μ_f and so

$$\int \log \frac{1}{|z|} d\mu_f = \frac{1}{d} \int \log \frac{1}{|z|} d\mu_{f^d}.$$

By taking d large we can make the right hand side as small as we wish. Thus for any μ on the disk satisfying (1.1), it suffices to construct an f corresponding to the pull back of μ under z^d, i.e., it suffices to consider only measures satisfying (9.1) for any $\delta > 0$ we choose.

Start by taking $R_0 = D(0, \frac{1}{4})$. Let $r_n = 2^{-n}$ for $n = 0, 1, 2, \dots$ and $\mu_n = \mu(A(r_n, r_{n-1}))$. Then

$$\sum_{n>2} (n-1)(\log 2)\mu_n \leq \int \log \frac{1}{|z|} d\mu \leq \delta, \tag{9.2}$$

so

$$\mu_n \leq \frac{\delta}{(\log 2)(n-1)} \leq \frac{\delta'}{n}, \tag{9.3}$$

where δ' is as small as we wish.

We need two simple facts about harmonic measure on an annulus.

Lemma 9.1. *Suppose $A = \{z : s < |z| < r\}$ and $s < t < r$. Then $\omega(z, C_s, A) = u_{s,r}(z) = (\log |z| - \log r)/(\log s - \log r)$ for any z with $|z| = t$.*

Proof. This is immediate since the given function is harmonic in A, equals 1 on C_s and equals 0 on C_r. □

Lemma 9.2. *Suppose s, t, r and A are as in previous lemma. Then if $t \geq 2s$, there is an $M < \infty$ (independent of s, t, r) such that for $|z| = t$, $\omega(z, \cdot, A)$ restricted to C_s has the form $\frac{1}{2\pi} g d\theta$ and g satisfies $\max_{C_s} g \leq M \min_{C_s} g$.*

Proof. Recall that harmonic measure on ∂A is the normal derivative of Green's function G with pole at z. Let $t' = \frac{2}{3}t > s$. By Harnack's inequality there is an M so that $\max_{C_{t'}} G \leq M \min_{C_{t'}} G$, and hence there is a constant CE so that

$$C(1 - u_{s,t'}) \leq G \leq MC(1 - u_{s,t'}),$$

on $\{s < |z| < t'\}$. Since the normal derivative of $u_{s,r'}$ is constant on C_s (since u is radial), this implies the normal derivative of G on C_s is trapped between two constants A and MA, as desired. $\square$

Consider the annulus $A_n = \{2^{-n} < |z| < 2^{-1}\}$, $n = 3, 4, \dots$ and a point z with $|z| = \frac{1}{3}$. The two previous results imply there is an constant B so that harmonic measure for A on the circle $C_{2^{-n}}$ is of the form $\frac{1}{2\pi} g d\theta$ where $g \geq \frac{B}{n}$, for $n \geq 3$. By (9.2) we can assume μ is chosen so that $\sum_n n\mu_n \leq (2B)^{-1}$. Thus $\sum_n Bn\mu_n \leq \frac{1}{2}$. Thus it is possible to choose disjoint, adjacent intervals $\{I_n\}$, $n = 2, 3, 4 \dots$, on $C_{1/4}$ of angle measure $4\pi n\mu_n/B$. In each interval I_n choose a countable set E_n satisfying the "thickness" condition (7.1) with some τ_n, and attach to I_n a copy of the universal cover of $A_{n+1} \setminus E_n$. The resulting Riemann surface has harmonic measures supported over the union of circles $\cup_n C_{2^{-n}}$, $n = 1, 3, 4, 5, \dots$. Moreover, if we choose $\tau_n \to 0$ quickly enough, the harmonic measure of the circles corresponding to $n = 3, 4, 5 \dots$, is of the form $\frac{1}{2\pi} g_n d\theta$ with $g_n > \mu_{n-1}$ (but might not be close to radial).

For each such circle $C_{2^{-n}}$, choose I and J intervals in the usual way and attach copies of $D(0, \frac{1}{2}) \setminus E$ and $D(0, 2^{-n}) \setminus E$ respectively. As we have seen before, we can choose η, m and τ so that the harmonic measure $\frac{1}{2\pi} g_n d\theta$ on $C_{2^{-n}}$ is as close to (but larger than) μ_n as we wish. Using Lemma 2.6 the harmonic measure of $C_{1/2}$ will be as close to (but less than) μ_1 as we wish, in particular, it is larger than

$\mu(\{\frac{1}{2} \leq |z| < \frac{3}{4}\})$ (this is where we use the assumption that $\mathbb{T}$ is in the support of μ).

The rest of the proof is now the same as the previous section. On each annulus we redistribute the harmonic measure from the circle into the annulus, sending any "extra" measure to the outermost circle, $C_{1-2^{-n}}$. In the limit, we obtain the desired measure μ.

10. An example which is almost an outer function

In this section we will construct an orthogonal function f whose only inner factor is the required zero at 0, i.e., $f(z)/z$ is outer. We will construct f so that 0 is the only zero of f; thus $f(z)/z$ has no Blaschke factor. In order to prove it has no singular inner factor, recall that if $f(z)/z = gh$ with g outer and h a non trivial singular inner function then

$$\log|f|^{-1} = \log|g|^{-1} + \log|h|^{-1},$$

and that the first term on the right is the Possion integral of its boundary values on $\mathbb{T}$, but the second term is the Poisson integral of a singular measure on $\mathbb{T}$ and has boundary value zero almost everywhere on $\mathbb{T}$. Let $H_\epsilon = \{z \in \mathbb{D} : |h(z)| < \epsilon\}$ and $F_\epsilon = \{z \in \mathbb{D} : |f(z)| < \epsilon\}$. Since $\log|h(0)|^{-1} = (\log\frac{1}{\epsilon})\omega(0, H_\epsilon, \mathbb{D} \setminus H_\epsilon)$, we deduce

$$\omega(0, H_\epsilon, \mathbb{D} \setminus H_\epsilon) \geq C/\log\frac{1}{\epsilon},$$

where $C = \log|h(0)|^{-1}$ and consequently, since $H_\epsilon \subset F_\epsilon$,

$$\omega(0, F_\epsilon, \mathbb{D} \setminus F_\epsilon) \geq C/\log\frac{1}{\epsilon}. \tag{10.1}$$

We will construct R so that the harmonic measure of $\{z \in R \setminus D(0, \frac{1}{2}) : |z| \leq 2^{-n}\}$ has harmonic measure (in R, with respect to 0) less than λ^n for some $\lambda < 1$. This contradicts (10.1), so the covering map has no singular inner factor.

Since we have already seen several constructions of this type in great detail, I will only sketch the construction. Start with $R_0 = D(0, \frac{1}{2})$. Divide $C_{1/2}$ into a finite collection of intervals $\{I_n\}$ and in each choose a set E satisfying (7.1). Along each interval attach a copy of $\{\frac{1}{4} < |z| < 1\} \setminus E$. This gives R_2.

Lemmas 9.1 and 9.2 of the previous section imply that harmonic measure of R_2 over $C_{1/4}$ is of the form $\frac{1}{2\pi} g d\theta$ where the max of g is bounded by a universal constant times the minimum. Thus there is a constant $c < \min(g)$ and a $\lambda < 1$ so that

$$\int (g - c) d\theta \leq \lambda \int g d\theta.$$

In other words, we can truncate g to be a constant and still retain a fixed fraction of the harmonic measure.

Now do the (by now) standard construction of I and J intervals on $C_{\frac{1}{4}+\delta}$ attaching copies of $\{\frac{1}{8} < |z| < 1\}$ and $\{\frac{1}{4} < |z| < 1\}$ respectively, so that the new harmonic measure on $C_{1/4}$ is very close to radially symmetric (say within ϵ_1 of constant) and has mass at least $(1 - \lambda)$ times the previous mass.

At the next stage we do the construction near both circles $C_{1/4}$ and $C_{1/8}$. At $C_{1/8}$ we repeat the process of the previous paragraph, making the harmonic measure above $C_{1/8}$ as close to radial as we wish, while retaining at least $(1 - \lambda)$ of the total mass, transferring the excess to C_1 and $C_{1/16}$. On $C_{1/4}$ we only make the measure within ϵ_2 of constant (while losing at most ϵ_1 of the mass), the excess being transferred to $C_{1/8}$ and C_1.

We now iterate the process in the obvious way. At the nth stage we have a surface R_n which only covers the origin once, and so that the harmonic measure is supported on the circles $\{C_{2^{-k}}\}$ with the kth circle getting mass at most λ^k. Thus the same is true for the limiting measure μ. Hence the harmonic measure of the set $\{z \in R \setminus R_0 : |z| < 2^{-n}\}$ has harmonic measure less than $C\lambda^n$ in R. This proves that $f(z)/z$ is outer.

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