A quasisymmetric surface with no rectifiable curves

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June 8, 1998

Abstract

There is a quasiconformal mapping f of R^3 to itself such that the image of $R^2 \times \{0\}$ contains no rectifiable curves.

1991 Mathematics Subject Classification. 30C65.
Keywords: Quasisymmetric maps, quasiconformal mappings, rectifiable curves, Jacobian.

[1]Supported in part by NSF grant # DMS 95-00577.

1 Introduction

A mapping $f : X \to Y$ is called *quasiconformal* if there is an $H < \infty$ so that the linear dilatation

$$H(x) \equiv \lim_{r \to 0} \frac{\max_{|x-y|=r} |f(x) - f(y)|}{\min_{|x-y|=r} |f(x) - f(y)|} \leq H,$$

for every x. The purpose of this note is to prove

Theorem 1.1 *There is a quasiconformal mapping f of R^3 to itself such that*

$$C^{-1} \leq \frac{|f(z) - f(w)|}{\varphi(|z - w|)} \leq C, \quad \forall z, w \in R^2 \times \{0\}, \tag{1}$$

for some $C < \infty$ and some increasing function φ such that

$$\lim_{t \to 0} \varphi(t)/t = \infty. \tag{2}$$

For $E \in R^2 \times \{0\}$, if the image $f(E)$ can be covered by balls $\{B(f(x_j), \varphi(r_j)\}$, then E is covered by balls $B(x_j, Cr_j)$. Then (**??**) easily implies that if $f(E)$ has finite α-dimensional Hausdorff measure then E has zero α-measure. In particular, any curve in $f(R^2)$ must have infinite length, since otherwise its preimage would be a non-degenerate curve of zero length, a contradiction. Recall that a homeomorphism $f : X \to Y$ is called *quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$|x - y| \leq t|x - z| \quad \text{implies} \quad |\mathrm{f(x) - f(y)}| \leq \eta(\mathrm{t})|\mathrm{f(x) - f(z)}|. \tag{3}$$

Quasiconformal maps of R^d, $d \geq 2$, to itself are quasisymmetric (e.g., [**?**]), so Theorem **??** answers a question of S. Rohde (Question 18 of [**?**]) by implying

Corollary 1.2 *There is a quasisymmetric embedding of R^2 into R^3 such that the image $f(R^2)$ contains no rectifiable curves.*

In this case one can avoid the theorem that quasiconformal implies quasisymmetric; our function φ will be constructed so that $\varphi(tr) \leq \varphi(t)\varphi(r)$ and this combined with (**??**) easily implies f is quasisymmetric on R^2.

In [**?**], J. Heinonen and P. Koskela define, for a quasiconformal map f on the upper half space R^n_+,

$$a_f(x) = \exp\{n^{-1}\mathrm{vol}(\mathrm{B}(\mathrm{x},\mathrm{r}))^{-1}\int_{\mathrm{B}(\mathrm{x},\mathrm{r})} \log \mathrm{J_f}(\mathrm{y})\mathrm{dy}\},$$

(J_f denotes the Jacobian of f and r is half the distance of $x = (x_1, \dots, x_n)$ to the boundary of the half space) and

$$E_\infty = \{x \in \partial R^n_+ : 0 < \liminf_{r\to 0} a_f(x + re_n) \leq \limsup_{r\to 0} a_f(x + re_n) = \infty\}.$$

Question 6 of [**?**] asks if there is an example where E_∞ has positive Lebesgue measure and whether it can be the whole boundary. In our example, the Jacobian at (x, y, t) is comparable to $(\varphi(t)/t)^3$, and thus E_∞ is the whole boundary.

It is unknown whether there is an embedding of R^2 into R^3 which satisfies (**??**) with $\varphi(t) = t^\alpha$, for some $\alpha < 1$ (it is easy to see that $\alpha < 2/3$ is impossible). Such maps are easy to find taking R into R^2 (just take the usual parameterization of the von Koch snowflake), and are known to exist for $R^d \to R^n$ if n is large enough with respect to d (see [**?**]).

After seeing my first proof of Corollary **??**, S. Rohde pointed out the map I constructed on R^2 extended to to a quasiconformal map of R^3 and hence gave Theorem **??**. J. Heinonen then observed that this map solved the problem from [**?**]. I thank them for their comments and I thank the referees for numerous suggestions which improved the clarity of the paper.

2 The picture of the non-rectifiable surface

We will start by describing a picture of the surface $f(R^2)$ which gives the main idea of the construction. We will then give a more careful construction of the map on all of R^3.

Let $\psi : R \to R$ be a smooth 1-periodic function with $\psi(0) = 0$ and such that the length of the graph Γ of ψ above $[0, 1]$ is 2. Define a surface $S \subset R^3$ by $S = \Gamma \times [0, 1]$. Then S is a smooth surface of area 2 and is isometric to $[0, 2] \times [0, 1]$ (with respect to the intrinsic path metric).

Start with the unit square Q_0 in R^2. Define a map $f_0 : Q_0 \to R^3$ by first stretching the square in the x direction by a factor of 2 (i.e., apply the map $(x, y) \to (2x, y)$ and then map the image rectangle isometrically onto the surface S. See Figure **??**.

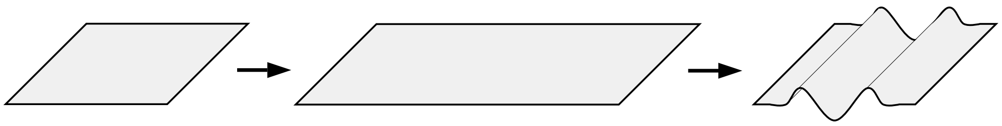

Figure 1: The map F_1: Stretching a square and folding it.

Now choose a large integer k_1. The dyadic subsquares of Q_0 of size 2^{-k_1} are mapped by F_1 to pieces of S which look like small rectangles of size $2^{-k_1+1} \times 2^{-k_1}$, and since F_2 is smooth, the approximation is better and better, the larger k_1 is. Replace each such approximate rectangle by a scaled copy of $[0, 1] \times \Gamma$, now oriented to the "oscillations" of the surface are orthogonal to the first set. See Figure **??**. The scaled copies will not quite meet along their edges, but we can make them do so by bending them with arbitrarily small angle (depending on k_1). Because we have stretched in the y direction this time we see that small enough squares of size r in Q_0 are now mapped onto regions which approximate small squares of size $2r \times 2r$, within a small error.

We now continue by induction. After n steps, we have a smooth surface $f_n(Q_0)$ which maps small squares of size $r = 2^{-k_n}$ to pieces of the surface which approximate a $2^n r \times 2^n r$ square with small error. Next take each dyadic square in Q_0 of some size $\ll r$ and stretch by a factor of 2 in the x direction and map it onto a scaled copy of S, bending it to make the

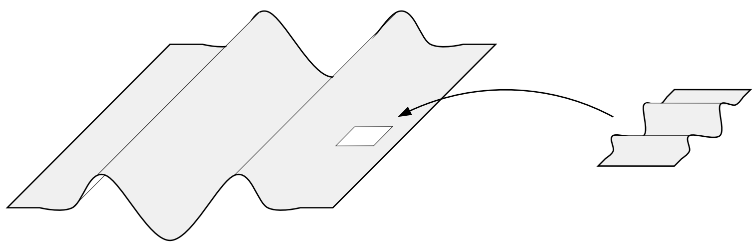

Figure 2: Folding in the orthogonal direction

edges fit together. Then stretch in the y direction on an even smaller scale. If we choose the $\{k_n\}$ to increase quickly enough then (**??**) will hold for $\varphi(t) = t2^{\alpha(t)}$ where $\alpha(t)$ is the minimal n so $2^{-k_n} < t$.

3 The construction

Now we give a more careful construction of the desired map on R^3. It will be the limit of a sequence of compositions $f_n = g_1 \circ \ldots \circ g_n$ where the g_i's are given by the following

Lemma 3.1 *There is a $C < \infty$, so that for any $0 < \delta, \epsilon, \eta \leq 1$, there is a $\nu = 2^{-m} > 0$ and a quasiconformal map $g : R^3 \to R^3$ so that if $z = (x, y, t) \in R^3$, then*

1. *$g(z) = z$ if $|t| > \delta$.*
2. *The linear dilatation of g is $\leq 1 + C(\epsilon)$ in all of R^3, where $C(\epsilon) \to 0$ as $\epsilon \to 0$.*
3. *The linear dilatation of g is $\leq 1 + \eta$ if $|t| < \nu$.*
4. *For $z, w \in R^2 \times \{0\}$, we have*

$$\left.\begin{array}{r} (1+\epsilon)(1-\eta) \\ 1-\eta \\ (1+\frac{C\delta}{|z-w|})^{-1} \end{array}\right\} \leq \frac{|g(z)-g(w)|}{|z-w|} \leq \left\{\begin{array}{ll} 1+\epsilon+\eta, & |z-w| \leq \nu. \\ 1+\epsilon+\eta, & \nu \leq |z-w| \leq \delta. \\ 1+\frac{C\delta}{|z-w|}, & |z-w| \geq \delta. \end{array}\right.$$

First we check that Theorem **??** follows from the lemma. Choose a sequence $\{\epsilon_n\}$ in $(0,1]$, so that $\sum_n \epsilon_n = \infty$. For example, $\epsilon_n = 1$ for all n will correspond to the picture described in the previous section and taking $\epsilon_n \to 0$, will give an asymptotically flat example. Let $\eta_n = 10^{-n}$. Let $\delta_1 = 1$. Now apply the lemma with $\epsilon_1, \delta_1, \eta_1$ to obtain a quasiconformal mapping g_1, and a positive number ν_1. In general, given ν_{n-1}, choose $\delta_n < \min(\nu_{n-1}/8, \delta_{n-1}/100)$, and apply the lemma to the triple $\epsilon_n, \delta_n, \eta_n$, to obtain the map g_n and the number ν_n. Let $f_n = g_1 \circ \ldots \circ g_n$. Then f_n is a finite composition of quasiconformal maps and hence is quasiconformal. It is the identity map on $\{z = (x, y, t) : |t| \geq 1\}$ and to estimate its dilatation at a point z with $|t| < 1$, choose k so that $\delta_{k+1} \leq |t| < \delta_k$. Then $g_{k+1} \circ \ldots \circ g_n$ is the identity at z. The map g_k has linear dilatation at most $1 + C(\epsilon_k)$ everywhere, and hence at z. For each $1 \leq j < k$, the image of z under $g_j \circ \ldots \circ g_k$ is in $\{z = (x, y, t) : |t| < \delta_j\}$ (since the inverse is the identity on on the complement) and hence the dilatation is at most $1 + \eta_j$. Thus the linear dilatation of f_n at z is at most $(1 + \sup_{k \leq n} C(\epsilon_k)) \prod_{k \leq n}(1 + \eta_j) \leq C$. It is easy to check that $\{f_n\}$ is Cauchy with respect to uniform convergence on compacta, so $f = \lim_n f_n$ exists and is quasiconformal.

To prove (**??**) choose $z, w \in R^2$ and choose k so that $2\delta_{k+1} \leq |z - w| \leq 2\delta_k$. For $j = n-1, \ldots, 1$, let $z_j = g_j \circ \ldots \circ g_n(z)$ and similarly for w_j. For $k < j \leq n-1$, induction and the fact that $\prod(1 - (\delta_j/\delta_{k+1})) > 1/2$ imply

$$|z_j - w_j| \geq (1 - \eta_j)|z_{j+1} - w_{j+1}| \geq \frac{1}{2}|z - w| \geq \delta_{k+1} \geq \delta_j.$$

Thus case 3 of part 4 of the lemma holds and

$$(1 + C\frac{\delta_j}{\delta_{k+1}})^{-1} \leq \frac{|z_{j-1} - w_{j-1}|}{|z_j - w_j|} \leq (1 + C\frac{\delta_j}{\delta_{k+1}}).$$

For $j = k$ we have (no matter which case holds),

$$(1 - \eta_k)|z_{k+1} - w_{k+1}| \leq |z_k - w_k| \leq 2(1 + \eta_k)|z_{k+1} - w_{k+1}|$$

For $j = k-1$, $|z_{k-1} - w_{k-1}| \leq 4|z - w| \leq 8\delta_k \leq \nu_{k-1}$ so case 1 holds in part 4 of the lemma. Futhermore, for $j = k-2, \ldots, 1$, $|z_{j-1} - w_{j-1}| \leq 4|z_j - w_j|$

and $\nu_j < \delta_j \leq \nu_j/4$. Thus $|z_j - w_j| \leq \nu_j$ implies $|z_{j-1} - w_{j-1}| \leq \nu_{j-1}$. Thus for $j < k$, case 1 of part 4 of Lemma **??** applies and so

$$1 - \eta_j \leq \frac{|z_{j-1} - w_{j-1}|}{(1+\epsilon_j)|z_j - w_j|} \leq 1 + \eta_j.$$

Putting all these together and telescoping we get,

$$\begin{aligned}\prod_{j=k+1}^{n}(1 + C\frac{\delta_j}{\delta_{k+1}})^{-1}\prod_{j=1}^{k}(1-\eta_j) \quad &\leq \quad \frac{|z_1 - w_1|}{|z-w|\prod_{j=1}^{k}(1+\epsilon_j)} \\ &\leq 2\prod_{j=k+1}^{n}(1 + C\frac{\delta_j}{\delta_{k+1}})\prod_{j=1}^{k}(1+\eta_j).\end{aligned}$$

Since the products converge (independent of k and n) and $f(z) = z_1$, $f(w) = w_1$, we have

$$C^{-1} \leq \frac{|f(z) - f(w)|}{\varphi(|z-w|)} \leq C,$$

where $\varphi(t) = t\prod_{j=1}^{\alpha(t)}(1+\epsilon_j)$, with $\alpha(t)$ defined by $\alpha(t) = \max\{k : t \leq 2\delta_k\}$. This is (**??**). Since $\sum \epsilon_n = \infty$, we also have (**??**).

Thus Lemma **??** proves Theorem **??**. Now we prove the lemma. We will build g as a composition $g = g_1 \circ g_2 \circ g_3$. Let $\psi : R \to R$ be a smooth 1-periodic function with $\psi(0) = 0$ and such that the length of the graph Γ of ψ above $[0,1]$ is $1+\epsilon$. Define a surface $S \subset R^3$ by $S = \Gamma \times R$. Let $L(x) = \int_0^x \sqrt{1 + |\psi'(t)|^2}dt$, be the arclength of the graph over $[0,x]$ and let J be the inverse, L^{-1}. Define a map $F : R^2 \to S \subset R^3$ by

$$F : (x,y) \to (J((1+\epsilon)x), y, \psi(J((1+\epsilon)x))). \tag{4}$$

Now extend F to a function G defined on all of R^3 by setting $G(x,y,t) = (x,y,t)$ if $|t| \geq 1$, G is quasiconformal with dilatation at most $1 + C(\epsilon)$, and the dilatation of G tends to 1 as $t \to 0$. For example, we can do this by making sure the $\partial_t G(x,y,0)$ has norm 1 and is perpendicular to $S = F(R^2)$, e.g., set

$$G(x,y,t) = F(x,y) + t \cdot N(x,y),$$

for small enough $t < t_0$, where

$$N(x,y) = \frac{(-\psi'(J((1+\epsilon)x), 0, 1)}{\sqrt{|\psi'(J((1+\epsilon)x))|^2 + 1}},$$

is the normal vector to $F(R^2)$ at the point $F(x,y)$.

To define g_1, let $g_1(x,y,t) = \frac{\delta}{2} G(\frac{2}{\delta}(x,y,t))$. Thus the tangent map near $R^2 \times \{0\}$ looks like an affine stretch by a factor of $1+\epsilon$ in the x direction, followed by a rotation in the (x,t) plane.

To define the map g_2 we do a similar thing, but at a much smaller scale and reversing the roles of x and y. Choose ν_1 so small that g_1, restricted to the set $\{(x,0,t) : t < \nu_1\}$ has linear dilatation $\leq 1 + \eta/10$. Now let $g_2(x,y,t) = \nu_1 G(\nu_1^{-1}(y,x,t))$. Choose ν_2 so that g_2 has linear dilatation $\leq 1 + \eta/10$ on $\{(0,y,t) : t < \nu_2\}$. Thus near R^2, the tangent map looks like an affine stretch in the y direction, followed by a rotation in the (y,t) plane.

Finally define g_3 by

$$g_3(x,y,t) = \begin{cases} (x,y,t), & \text{if} \quad |\mathrm{t}| \geq \nu_2/2, \\ (x,y,\nu_2 h(t/\nu_2)), & \text{if} \quad \nu_2/8 \leq |\mathrm{t}| \leq \nu_2/2, \\ (x,y,(1+\epsilon)t), & \text{if} \quad \mathrm{t} \geq \nu_2/8, \end{cases}$$

where h smoothly interpolates between $(1+\epsilon)/8$ and $1/2$ on $[\frac{1}{8}, \frac{1}{2}]$ and satisfies $|h' - 1| \leq C(\epsilon)$. Thus near R^2, the map is an affine stretch in the t direction and g_3 is $1 + C(\epsilon)$ quasiconformal.

We claim that $g = g_1 \circ g_2 \circ g_3$ satisfies the lemma with $\nu = \nu_2/16$. Most of the conditions are clear from the construction. For example, it is clear that g is quasiconformal and is the identity on $\{(x,y,t) : |t| > \delta\}$. Moreover, it has dilatation bounded everywhere by $1 + C(\epsilon)$ (since each of the three component maps does). Finally, for points in $\{(x,y,t) : |t| < \nu\}$, by considering the tangent maps of the composition, it is easy to see that the map acts as a conformal dilation of size $1+\epsilon$ with error at most η. This also proves the expansion estimates on the small and intermediate scales (i.e., $|z-w| \leq \delta$). For large scales (i.e., $|z-w| \geq \delta$), we just use the fact that

$|G(z)-G(w)| = |z-w| + O(1)$ which implies $|g(z)-g(w)| = |z-w| + O(\delta)$. This completes the proof of the lemma.

Added in proof: G. David and T. Toro have improved Corollary

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