Packing Dimension and Cartesian Products

Christopher J. Bishop[1] Yuval Peres[2]

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Abstract

We show that for any analytic set A in $\mathbf{R}^d$, its packing dimension $\dim_{\mathrm{P}}(A)$ can be represented as $\sup_B\{\dim_{\mathrm{H}}(A\times B)-\dim_{\mathrm{H}}(B)\}$, where the supremum is over all compact sets B in $\mathbf{R}^d$, and $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. (The lower bound on packing dimension was proved by Tricot in 1982). In contrast, we show that the dual quantity $\inf_B\{\dim_{\mathrm{P}}(A\times B)-\dim_{\mathrm{P}}(B)\}$, is at least the "lower packing dimension" of A, but can be strictly greater. (The lower packing dimension is greater or equal than the Hausdorff dimension.)

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[1]Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794-3651. Supported in part by NSF grant # DMS 9204092 and by an Alfred P. Sloan Foundation Fellowship.

[2]Department of Statistics, University of California, Berkeley, California 94720. Research partially supported by NSF grant # DMS-9404391.

1 Introduction

Marstrand's product theorem (Marstrand (1954)) asserts that if $A, B \subset \mathbf{R}^d$ then

$$\dim_{\mathrm{H}}(A \times B) \geq \dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B)\,,$$

where "$\dim_{\mathrm{H}}$" denotes Hausdorff dimension. Refining earlier results of Besicovitch and Moran (1945), Tricot (1982) showed that

$$(\dim_{\mathrm{H}}(A \times B) - \dim_{\mathrm{H}}(B)) \leq \dim_{\mathrm{P}}(A)\,, \tag{1}$$

where "$\dim_{\mathrm{P}}$" denotes packing dimension (see the next section for background). Tricot (1991) and also Hu and Taylor (1994) asked if this is sharp, i.e., given A, can the left-hand side of (1) be approximated arbitrarily well by appropriate choices of B on the right-hand side? Our first result states that this is possible.

Theorem 1.1 *For any analytic set A in $\mathbf{R}^d$*

$$\dim_{\mathrm{P}}(A) = \sup_B \{\dim_{\mathrm{H}}(A \times B) - \dim_{\mathrm{H}}(B)\}\,, \tag{2}$$

where the supremum is over all compact sets $B \subset \mathbf{R}^d$.

The proof actually shows that, when $\dim_{\mathrm{P}}(A) < d$, the supremum is attained at some compact set B of Hausdorff dimension $d - \dim_{\mathrm{P}}(A)$. We remark that there are compact sets $A \in \mathbf{R}^d$ for which the supremum is attained *only* with sets B of dimension $d - \dim_{\mathrm{P}}(A)$.

The sets which have equal Hausdorff and packing dimensions have been singled out by Tricot (1982) (who called them "dimension-regular") and by Taylor (1986) because of their good behavior in Cartesian products and probabilistic applications. Theorem 1.1 shows that these are the only compact sets which have these good properties universally.

A related question concerns the defect of packing dimension under products. Tricot (1981) showed that

$$\dim_{\mathrm{P}}(E \times F) \geq \dim_{\mathrm{H}}(E) + \dim_{\mathrm{P}}(F)\,. \tag{3}$$

Hu and Taylor (1994) asked if all sets $E \subset \mathbf{R}$ satisfy

$$\inf_F \{\dim_{\mathrm{P}}(E \times F) - \dim_{\mathrm{P}}(F)\} = \dim_{\mathrm{H}}(E)\,? \tag{4}$$

The answer turns out to be more delicate. For a bounded set E in a metric space, Denote by $N(E,\epsilon)$ the maximal cardinality of a collection of disjoint closed balls of radius ϵ with centers in E. Let

$$\underline{\dim}_{\mathrm{M}}(E) = \liminf_{\epsilon \to 0} \frac{\log N(E,\epsilon)}{|\log \epsilon|} \tag{5}$$

be the *lower Minkowski dimension* of E. Following Mattila (1994), we call the "regularization" of this index the **lower packing dimension**, denoted $\underline{\dim}_{\mathrm{P}}$, although it is not constructed using packing measures. It is defined, for any set E in a metric space, by

$$\underline{\dim}_{\mathrm{P}}(E) = \inf_{E \subset \cup_j E_j} \sup_j \underline{\dim}_{\mathrm{M}}(E_j)\,, \tag{6}$$

where the infimum is over all countable collections of bounded sets $\{E_j\}$ whose union contains E.

It is easy to see that $\dim_{\mathrm{H}}(E) \leq \underline{\dim}_{\mathrm{P}}(E)$ for any set E; examples where the inequality is strict were given by Tricot (1982).

Proposition 1.2 *For any compact set E in $\mathbf{R}^d$,*

$$\inf_F \{\dim_{\mathrm{P}}(E \times F) - \dim_{\mathrm{P}}(F)\} \geq \underline{\dim}_{\mathrm{P}}(E) \tag{7}$$

where the infimum is over all compact sets $F \subset \mathbf{R}^d$, and there exist compact subsets E of $\mathbf{R}$ for which the inequality in (7) is strict.

2 Background on packing dimension and trees

For background on Hausdorff measures and dimension we refer to Falconer (1990). A more naive notion is the *upper Minkowski dimension* (sometimes called the upper box dimension) defined for any set E in a metric space by

$$\overline{\dim}_{\mathrm{M}}(E) = \limsup_{\epsilon \to 0} \frac{\log N(E,\epsilon)}{|\log \epsilon|}\,. \tag{8}$$

(The lower and upper Minkowski dimensions are infinite if E is not totally bounded). Tricot (1980, 1982) introduced **packing dimension**, which plays a dual role to Hausdorff dimension in many settings. For our present purpose, the representation of packing dimension which is convenient to use as a definition, is as a regulariziation of upper Minkowski dimension:

$$\dim_{\mathrm{P}}(A) = \inf_{A \subset \cup_j A_j} \sup_j \overline{\dim}_{\mathrm{M}}(A_j)\,. \tag{9}$$

(See Tricot (1982), Proposition 2, or Falconer (1990), Proposition 3.8.)

Part (i) of the next lemma is due to Tricot (1982) (see also Falconer 1990); Part (ii) is in Falconer and Howroyd (1994) and in Mauldin and Mattila (1994).

Lemma 2.1 *(i) Let E be a closed set in a complete metric space. If any open set V which intersects E satisfies $\overline{\dim}_{\mathrm{M}}(E \cap V) \geq \alpha$, then $\dim_{\mathrm{P}}(E) \geq \alpha$.*

(ii) Let E be a subset of a seperable metric space, with $\dim_{\mathrm{P}}(E) > \alpha$. Then there is a (relatively closed) nonempty subset $\widetilde{E}$ of E, such that $\dim_{\mathrm{P}}(\widetilde{E} \cap V) > \alpha$ for any open set V which intersects $\widetilde{E}$.

(iii) The analogues of (i) and (ii) with $\underline{\dim}_{\mathrm{M}}$ and $\underline{\dim}_{\mathrm{P}}$ in place of $\overline{\dim}_{\mathrm{M}}$ and $\dim_{\mathrm{P}}$ respectively, are valid.

Proof : *(i)* See Tricot (1981) or Falconer (1990), Proposition 3.6.
(ii) Define $\widetilde{E}$ to be the set of points $x \in E$ such that every neighborhood W of x satisfies $\dim_{\mathrm{P}}(E \cap W) > \alpha$. Clearly $\widetilde{E}$ is relatively closed in E. If $\widetilde{E}$ was empty, then E would be covered by relatively open sets $E \cap W$ of packing dimension at most α. Passing to a countable subcover (using seperability) would yield a contradiction. To verify that $\widetilde{E}$ has the required property, Let V be any open set which intersects $\widetilde{E}$. By the definition of $\widetilde{E}$, the intersection $V \cap (E \setminus \widetilde{E})$ is covered by the union of the open sets W such that $\dim_{\mathrm{P}}(E \cap W) \leq \alpha$, and using seperability again,

$$\alpha < \dim_{\mathrm{P}}(E \cap V) \leq \max\{\dim_{\mathrm{P}}(\widetilde{E} \cap V), \alpha\}.$$

This completes the proof.
(iii) The proofs of the previous two parts transfer. □

Tree Notation: We will represent closed subsets of the cube by infinite trees, and use finite trees as tools in our constructions. A (rooted) **tree** T is a connected acyclic graph with a distinguished vertex ρ designated as the root. It may be finite or infinite, but every vertex must have finite degree. For every vertex v we write $|v|$ for its **level**, i.e., the number of edges between it and the root. Neighbours of v at level $|v|+1$ are called **children** of

v. More generally, if v is on the path from the root to a vertex w, then w is called a **descendant** of v. If a vertex of T has no children, it is called a **leaf** of T. If all the leaves of a finite tree T are at level n, we say that T has **uniform depth** n.

Now we focus on *infinite* trees. An infinite self-avoiding path in a tree T, starting at the root of T, is called a **ray**. The set of all rays is called the **boundary** of T and denoted ∂T. For two distinct rays $x, y \in \partial T$ we define their *distance* to be

$$\operatorname{dist}(x,y) = 2^{-n} \text{ if } x \text{ and } y \text{ have exactly } n \text{ edges in common.} \tag{10}$$

This makes ∂T a compact metric space. For any vertex v, denote by $[v]$ the set of rays going through v. There is a one-to-one correspondence between measures on ∂T and flows on T, since the additivity axiom for measures, $\mu([v]) = \sum\{\mu([w]) \,:\, w \text{ a child of } v\}$ corresponds to Kirckhoff's law for flows. Probability measures correspond to **unit flows**. In the sequel, we write $\mu(v)$ instead of $\mu([v])$. A set of vertices Π that intersects every ray of T is called a **cut-set**. Equivalently, Π is a cut-set iff $\{[v] : v \in \Pi\}$ is a cover of ∂T.

The closed dyadic subcubes of the unit cube in $\mathbf{R}^d$ have a natural tree structure when ordered by inclusion, and the resulting tree is regular with 2^d children to every vertex. Given any compact set $K \in [0,1]^d$, the dyadic cubes which intersect K form a subtree $T(K)$ of this regular tree. Conversely, for any subtree T of the regular 2^d-ary tree, the boundary ∂T can be regarded as a closed subset of the sequence space on the 2^d symbols $\{0,1\}^d$, and then binary representation yields a compact set $K \subset [0,1]^d$ such that $T = T(K)$.

Given a closed set $K \subset [0,1]^d$, let $|T_n(K)|$ denote the number of closed dyadic cubes of side length 2^{-n} which intersect K. Equivalently, $|T_n(K)|$ is the number of vertices of the tree $T(K)$ at level n. It is a standard fact that the upper and lower Minkowski dimensions can be calculated by counting dyadic cubes; comparing this with the the definition (10) of the metric on ∂T gives

$$\overline{\dim}_{\mathrm{M}}(K) = \limsup_{n\to\infty} \frac{1}{n} \log_2 |T_n(K)| = \overline{\dim}_{\mathrm{M}}(\partial T(K))$$

and

$$\underline{\dim}_{\mathrm{M}}(K) = \liminf_{n\to\infty} \frac{1}{n} \log_2 |T_n(K)| = \underline{\dim}_{\mathrm{M}}(\partial T(K)).$$

The β-dimensional Hausdorff content of a set E is

$$\mathcal{H}^\beta_\infty(E) = \inf\{\sum_j (\text{diam } E_j)^\beta : E \subset \cup_j E_j\},$$

the infimum being over all countable covers of E. The Hausdorff dimension of E is

$$\dim_{\text{H}}(E) = \inf\{\beta : \mathcal{H}^\beta_\infty(E) = 0\}.$$

For purposes of computing the Hausdorff dimension, it suffices to estimate the Hausdorff content using coverings by dyadic squares. Such a covering corresponds to a cut-set of the tree $T(K)$ and the Hausdorff dimension becomes

$$\inf_\beta \{\inf_\Pi \sum_{v\in\Pi} 2^{-\beta|v|} < \infty\}.$$

Call the inner sum a "*β-dimensional cut-set sum.*" The mass distribution principle (see Proposition 4.2 in Falconer (1990)), says that if K supports a positive measure μ satisfying a Hölder estimate $\mu(F) \le C(\text{diam}F)^\beta$ for any set F and some constant C, then $\dim_{\text{H}}(K) \ge \beta$. It suffices to assume the Hölder estimate for dyaic cubes; In terms of the tree, this estimate means that $T(K)$ supports a flow μ so that the flow going through any vertex v satisfies $\mu(v) \le C' 2^{-\beta|v|}$.

The product tree: Given two compact sets A, B in $[0,1]^d$, we can construct the tree corresponding to the Cartesian product $A \times B \subset [0,1]^{2d}$ from the trees $T(A)$ and $T(B)$. In general, given two trees T^1, T^2, the product tree $T^1 * T^2$ has the vertex set

$$\{(v_1, v_2) \ : \ v_i \in T^i \ \text{ for } \ i = 1,2 \ \text{ and } \ |v_1| = |v_2|\},$$

the root $(\text{root}(T^1),\ \text{root}(T^2))$, and the adjacency relation:

the vertex (v_1, v_2) is adjacent to (u_1, u_2) iff v_i is adjacent to u_i for $i = 1,2$.

It is easy to see that $T(A \times B) = T(A) * T(B)$.

3 Proof of Theorem 1.1

Given an analytic set A in $\mathbf{R}^d$, denote $\alpha = \dim_{\text{P}}(A)$. We will first prove the theorem under the assumption $0 < \alpha < d$. We will construct a compact set B in $\mathbf{R}^d$ of Hausdorff dimension $d - \alpha$ so that $\dim_{\text{H}}(A \times B) = d$.

Let $\widetilde{\alpha} < \alpha$. Recent results of Joyce and Preiss (1993) imply that A contains a subset A_0 which is closed in $\mathbf{R}^d$ with $\dim_{\mathrm{P}}(A_0) > \widetilde{\alpha}$. By partitioning and translating if necessary, we may assume that A_0 is contained in the unit cube.

Applying Lemma 2.1(ii) to the boundary of the tree $\partial T(A_0)$, we obtain a subtree $T(\widetilde{A})$ with $A_0 \subset A$ such that any open set V on the boundary of this subtree has packing dimension greater than $\widetilde{\alpha}$. This implies that for any vertex v of the tree $T(\widetilde{A})$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 |T_n(\widetilde{A}, v)| \geq \alpha \tag{11}$$

where $|T_n(\widetilde{A}, v)|$ is the number of descendants w of v such that $|w| = n$. (I.e., the dyadic cube $Q(w)$ corresponding to each such vertex w is contained in $Q(v)$ and has side length 2^{-n}.)

We will consider truncations $T^k(\widetilde{A})$ of the tree $T(\widetilde{A})$ to certain levels N_k, and use them to inductively construct a nested increasing sequence of trees Γ^k, whose limit Γ corresponds to an approximation $\widetilde{B}$ of the promised set B. Simultaneously, we will describe directed unit flows from the root to the boundaries of the product trees $T^k(\widetilde{A}) * \Gamma^k$ which satisfy uniform Hölder estimates of orders d_k close to d. This will suffice to establish the variational principle (2). A further limiting argument will be needed to show that the supremum there is attained.

First step: By (11), there is a level N_1 such that $T(\widetilde{A})$ has more than $2^{\widetilde{\alpha} N_1}$ vertices at level N_1. Construct Γ^1 of uniform depth N_1, where every vertex in the first $\lceil N_1(1-\alpha/d)\rceil$ levels has 2^d children, and every vertex in the remaining $\lfloor N_1\alpha/d \rfloor$ levels has a single child (see Figure 1). The leaves of Γ^1 form a cut-set Π_1 of cardinality at most $2^{N_1(d-\alpha)+d}$, so the corresponding $d-\alpha$ dimensional cut-set sum satisfies

$$\sum_{v\in\Pi_1} 2^{(\alpha-d)|v|} \leq 2^d\,.$$

Now consider the **uniform unit flow** μ on the product tree $T^1(\widetilde{A}) * \Gamma^1$ which assigns each vertex at level N_1 the same mass.

Let (v, w) be a vertex at level $j \leq N_1$ of this product tree. Observe that v has at most $2^{d(N_1-j)}$ descendants at level N_1 of $T(\widetilde{A})$. If $j > \lceil N_1(1-\alpha/d)\rceil$ then w has only one

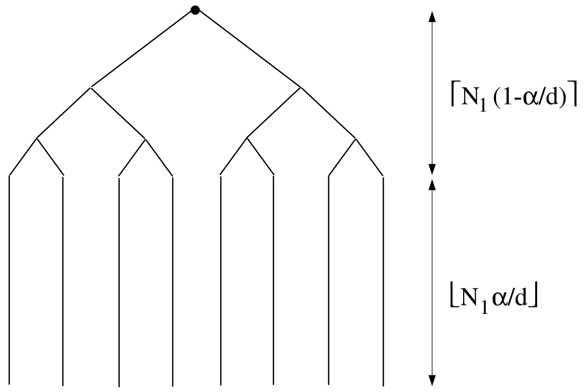

Figure 1: Full expansion followed by linear growth

descendant at level N_1 of Γ^1, so that

$$\mu(v,w) \leq 2^{d(N_1-j)} \cdot 1 \cdot 2^{-\widetilde{\alpha}N_1} \cdot 2^{-dN_1(1-\alpha/d)} = 2^{-dj+(\alpha-\widetilde{\alpha})N_1} . \tag{12}$$

Denoting

$$\widetilde{d} := d - \frac{\alpha - \widetilde{\alpha}}{1-\alpha/d} , \tag{13}$$

we deduce from (12) that the Hölder estimate

$$\mu(v,w) \leq 2^{-\widetilde{d}\,|(v,w)|} \tag{14}$$

holds if $j = |(v,w)| > \lceil N_1(1-\alpha/d) \rceil$. On the other hand, an improved Hölder estimate (with d in place of $\widetilde{d}$) clearly holds for smaller values of j, since Γ_1 has full branching at these levels. We conclude that (14) is valid for all vertices (v,w) of $T^1(\widetilde{A}) * \Gamma^1$. For use in the induction below, we define $N_0 = 0$ and $d_0 = d_1 = \widetilde{d}$.

The inductive step: We assume that we have constructed a tree Γ^k of uniform depth N_k, in which every vertex has at most 2^d children. Also, we assume there is a unit flow μ on the product tree $T^k(\widetilde{A}) * \Gamma^k$ which satisfies $\mu(v,w) \leq 2^{-d_{k-1}j}$ for any $N_{k-1} \leq j \leq N_k$ and any vertex (v,w) at level j of the product tree; for leaves (v,w) of Γ^k, we assume the estimate $\mu(v,w) \leq 2^{-d_k N_k}$ holds.

The construction of Γ^{k+1} from Γ^k is done in four stages:

1. To each leaf of Γ^k attach a full 2^d-ary tree of depth kN_k to obtain a tree $\Gamma^{k,1}$ of uniform depth $(k+1)N_k$.

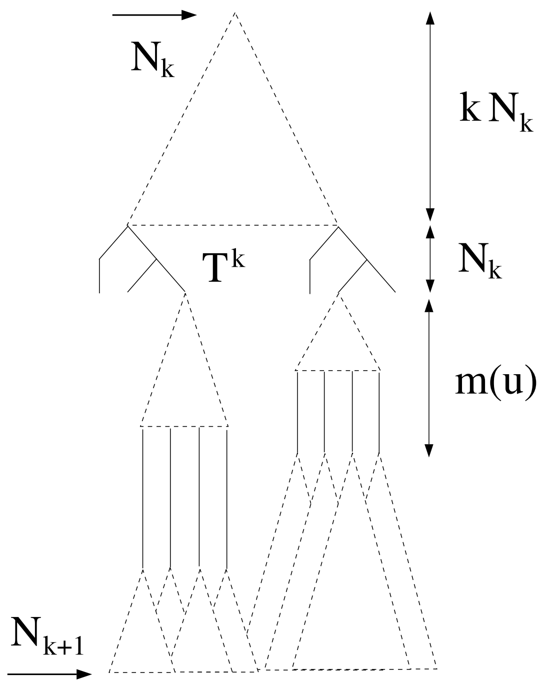

Figure 2: The four stages of the construction

2. To each leaf of $\Gamma^{k,1}$ attach a copy of $T^k(\widetilde{A})$. This yields a tree $\Gamma^{k,2}$ of uniform depth $(k+2)N_k$, where each leaf of $\Gamma^{k,2}$ corresponds to some leaf of $T^k(\widetilde{A})$ (this is a many-to-one correspondence).

3. Now we mimic the first step of the induction. For each leaf u of $T^k(\widetilde{A})$, choose an integer $n(u)$ such that u has at least $2^{n(u)\widetilde{\alpha}}$ descendants at level $|u| + n(u)$, and $n(u) \geq (k^2 + k + 1)N_k$. This is possible by (11). For future reference, we denote by $T^{k,3}$ the subtree of $T(\widetilde{A})$ obtained by appending to every vertex u at level N_k its descendants in the next $n(u)$ levels. To each vertex of $\Gamma^{k,2}$ corresponding to u, append a tree of uniform depth $m(u) = n(u) - (k+1)N_k$, where every vertex in the first $\lceil m(u)(1 - \alpha/d) \rceil$ levels has 2^d children, and every vertex in the remaining $\lfloor m(u)\alpha/d \rfloor$ levels has a single child. This defines $\Gamma^{k,3}$. The leaves of $\Gamma^{k,3}$ form a cut-set Π_{k+1} such that the corresponding β-dimensional cut-set sum satisfies

$$\sum_{w \in \Pi_{k+1}} 2^{-\beta|w|} \leq 2^{d(k+1)N_k} \cdot \sum_{u \in \partial T^k(\widetilde{A})} 2^{m(u)(d-\alpha)+d} \cdot 2^{-\beta(N_k + n(u))}, \tag{15}$$

where the first factor $2^{d(k+1)N_k}$ is a bound on the number of leaves of $\Gamma^{k,1}$. The number of leaves of $T^k(\widetilde{A})$ is at most 2^{dN_k}. Using this and the inequality

$m(u) \le n(u)$, we see that the cut-set sum (15) is bounded by

$$2^{d(k+2)N_k+d} \cdot 2^{n(u)(d-\alpha-\beta)} . \tag{16}$$

For any fixed $\beta > d - \alpha$, these bounds tend to zero as $k \to \infty$, because $n(u) > k^2 N_k$.

4. Finally, denote by N_{k+1} the maximum depth of $\Gamma^{k,3}$, and to each leaf w of $\Gamma^{k,3}$ append a tree of uniform depth $N_{k+1} - |w|$ with 2^d children at every level. This yields Γ^{k+1}.

The contruction is illustrated in Figure 2. The dashed boxes represent subtrees with full growth (2^d children at each vertex).

Now that we have constructed the tree Γ^{k+1} we want to construct a flow μ on the product tree $T^{k+1}(\widetilde{A}) * \Gamma^{k+1}$ which extends the flow μ constructed previously on $T^k(\widetilde{A}) * \Gamma^k$. (Rather than call the flow constructed at each step μ_k, we simply refer to them all as μ since the definitions are consistent.) It is most convenient to do this in four stages following the construction of the tree.

1. Suppose (v, w) is a leaf of $T^k(\widetilde{A}) * \Gamma^k$. By our induction hypothesis the flow satisfies the Hölder estimate
$$\mu(v, w) \le 2^{-d_k N_k}.$$
Let $T^{k,1}$ be the truncation of $T(\widetilde{A})$ to level $(k+1)N_k$ (so it has the same depth as $\Gamma^{k,1}$). We define the flow on $T^{k,1} * \Gamma^{k,1}$ by dividing the mass assigned by μ to each vertex of the smaller tree $T^k(\widetilde{A}) * \Gamma^k$ equally among its descendants at level $(k+1)N_k$ of the extended tree $T^{k,1} * \Gamma^{k,1}$. Since $\Gamma^{k,1}$ has full branching at these levels (i.e., each vertex has 2^d children) it is clear that the resulting flow satisfies
$$\mu(v, w) \le 2^{-d_k j},$$
for vertices in level j such that $N_k \le j \le (k+1)N_k$.

2. This is the only stage where we do not divide the mass equally among descendants. Let $T^{k,2}$ be the truncation of $T(\widetilde{A})$ to level $(k+2)N_k$ (so it has the same depth as $\Gamma^{k,2}$). For each leaf (v, w) of $T^{k,1} * \Gamma^{k,1}$, let u be the leaf of $T^k(\widetilde{A})$ which is an ancestor of v. Divide the mass of (v, w) evenly among the leaves (v', w') of $T^{k,2} * \Gamma^{k,2}$ which satisfy:
v' is a descendant of v and w' is the unique descendant of w corresponding to u.

At worst, we have concentrated all the mass of (v,w) onto a single descendant, so for leaves (v',w') of $T^{k,2} * \Gamma^{k,2}$ we get

$$\mu(v',w') \le \mu(v,w) \le 2^{-dkN_k} \le 2^{-d_{k+1}(k+2)N_k},$$

where we define

$$d_{k+1} := \min\{\frac{dk}{k+2}, \tilde{d}\}\,. \tag{17}$$

Thus at level $(k+2)N_k$, the flow μ satisfies a Hölder estimate of order d_{k+1}.

3. Next, we extend the definition of μ to the product tree $T^{k,3} * \Gamma^{k,3}$. The two trees in the product do not have uniform depth. Let v be a leaf of $T^{k,2}$ which is a descendant of a leaf u of T^k, and let w be a leaf of $\Gamma^{k,2}$ which corresponds to u. Divide the mass $\mu(v,w)$ equally among the descendants of (v,w) at level $N_k + n(u)$. Since the flow μ is Hölder with exponent $d_{k+1} \le \tilde{d}$ at (v,w), the argument in step 1 of the induction implies that the same estimate holds for all descendants of (v,w) down to level $N_k + n(u)$.

4. Finally, to define the flow on $T^{k+1} * \Gamma^{k+1}$, use equal division again; since Γ^{k+1} has full branching here the Hölder estimate established at the previous stage persists.

This completes the inductive step.

The limit of the trees Γ^k is a tree Γ which corresponds to a compact set $\tilde{B}$ in the cube $[0,1]^d$. Each cut-set of Γ corresponds to a cover of B by dyadic cubes, so the cut-set sum estimates (16) in stage 3 of the inductive step imply that

$$\dim_{\mathrm{H}}(\tilde{B}) \le d - \alpha. \tag{18}$$

The flow μ extends to the product tree $T(\tilde{A}) * \Gamma$ and corresponds to a measure μ' on $\tilde{A} \times B$ which satisfies the estimate $\mu'(Q) \le \operatorname{diam}(Q)^{d_k}$ for any dyadic cube Q of side-length at most 2^{-N_k}. Since $d_k = \tilde{d}$ for all large k by (17), the mass distribution principle (see, e. g., Proposition 4.2 in Falconer (1990)) implies that

$$\dim_{\mathrm{H}}(\tilde{A} \times \tilde{B}) \ge \tilde{d}. \tag{19}$$

Under our standing assumption $\alpha = \dim_{\mathrm{P}}(A) < d$, we can get $\tilde{d}$ arbitrarily close to d by choosing $\tilde{\alpha}$ close to α and using (13). Thus (18) and (19), combined with Tricot's upper bound (1), prove the variational formula (2) in this case.

Now we show that the supremum in (2) is **attained** when $\alpha < d$. For each integer m, taking $\widetilde{\alpha} \geq \alpha - 1/m$, the construction above yields a compact set $\widetilde{B}_m$ in $\mathbf{R}^d$ such that

$$\dim_{\mathrm{H}}(\widetilde{B}_m) \leq d - \alpha \quad \text{and} \quad \dim_{\mathrm{H}}(A \times \widetilde{B}_m) \geq \widetilde{d}_m = d - \frac{d}{m(d-\alpha)}\,. \tag{20}$$

By translating and scaling, we may assume that each $\widetilde{B}_m$ contains the origin and has diameter at most $1/m$. Defining $B := \cup_{m=1}^{\infty} \widetilde{B}_m$, we see by (20) that

$$\dim_{\mathrm{H}}(B) \leq d - \alpha \quad \text{and} \quad \dim_{\mathrm{H}}(A \times B) \geq d\,. \tag{21}$$

By Tricot's upper bound (1), both inequalities in (21) must, in fact, be equalities. Thus $\dim_{\mathrm{P}}(\widetilde{A}) = \dim_{\mathrm{H}}(\widetilde{A} \times B) - \dim_{\mathrm{H}}(B)$, as desired.

For analytic sets $A \subset \mathbf{R}^d$ with $\dim_{\mathrm{P}}(A) = d$, the variational formula (2) follows easily from the validity of this formula for any closed subset A_0 of A with $d - \epsilon < \dim_{\mathrm{P}}(A_0) < d$, by taking ϵ small. (The existence of such subsets A_0 is a consequence of the work of Joyce and Preiss (1993).) By a more involved recursive argument, we can show the supremum is attained in this case also, but since including this argument would lengthen the paper considerably, we omit it. □

4 Packing dimension of product sets: Proof of Proposition 1.2

We must show that any two compact sets E and F in $\mathbf{R}^d$ satisfy $\dim_{\mathrm{P}}(E \times F) \geq \underline{\dim}_{\mathrm{P}}(E) + \dim_{\mathrm{P}}(F)$. Assume that

$$\alpha < \underline{\dim}_{\mathrm{P}}(E) \quad \text{and} \quad \beta < \dim_{\mathrm{P}}(F)\,. \tag{22}$$

Applying Lemma 2.1(ii) to the boundary of the tree $\partial T(F)$, we obtain a subtree $T(\widetilde{F})$ with $\widetilde{F} \subset F$ such that for any vertex w of the tree $T(\widetilde{F})$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 |T_n(\widetilde{F}, w)| > \beta\,, \tag{23}$$

where $|T_n(\widetilde{F}, w)|$ is the number of descendants u of w such that $|u| = n$. Similarly, by part (iii) of Lemma 2.1, there is a subset $\breve{E}$ of E such that any vertex v of $T(\breve{E})$ satisfies

$$\liminf_{n\to\infty} \frac{1}{n} \log_2 |T_n(\breve{E}, v)| \geq \alpha\,. \tag{24}$$

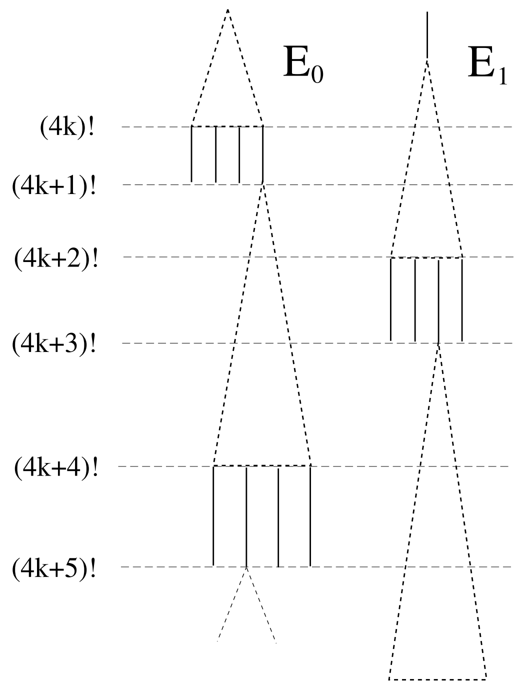

Figure 3: Linear growth on disjoint scales

Consider any vertex (v, w) of the product tree $T(\breve{E}) \times T(\widetilde{F})$. For $n > |v| = |w|$ we have

$$|T_n(\breve{E} \times \widetilde{F}, (v, w))| = |T_n(\breve{E}, v)| \cdot |T_n(\widetilde{F}, w)|,$$

so the asymptotics (24) and (23) imply that

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 |T_n(\breve{E} \times \widetilde{F}, (v, w))| \geq \alpha + \beta\,.$$

Invoking part (i) of Lemma 2.1, we obtain $\dim_{\mathrm{P}}(E \times F) \geq \alpha + \beta$. Since α and β are only restricted by the assumption (22), this completes the proof of (7).

Now we construct a compact set in $[0, 1]$ for which the inequality in (7) is strict. For $i = 0, 1$, let E_i be a compact set in $[0, 1]$ such that every vertex at level n of the tree $T(E_i)$ has

$$\begin{cases} \text{one child} & \text{for} \quad n \in \cup_k [(4k + 2i)!, (4k + 2i + 1)!) \\ \text{and two children} & \text{otherwise.} \end{cases}$$

Loosely speaking, The sets E_0 and E_1 are large at most scales, and the scales where they are small are disjoint. More precisely, if v is a vertex of $T(E_0)$ with $|v| \leq (4k + 1)!$, and if $n \in [(4k + 2)!, (4k + 4)!)$, then $|T_n(E_0, v)| \geq 2^{n-(4k+1)!}$. Defining $S_0 = \cup_{k=1}^{\infty}[(4k + 2)!, (4k + 4)!)$, we obtain

$$\liminf_{n\in S_0} \frac{1}{n} \log_2 |T_n(E_0, v)| = 1\,. \tag{25}$$

for any vertex v of $T(E_0)$. Similarly, defining $S_1 = \cup_{k=1}^{\infty}[(4k)!, (4k+2)!)$, we get

$$\liminf_{n \in S_1} \frac{1}{n} \log_2 |T_n(E_1, v)| = 1\,, \tag{26}$$

Let $E = E_0 \cup E_1$. We claim that

$$\dim_{\mathrm{P}}(E \times F) \geq 1 + \dim_{\mathrm{P}}(F) \quad \text{for any compact set} \quad F \subset [0,1]\,. \tag{27}$$

(In fact this is an equality, but we only need one direction.)

To verify this claim, let $\beta < \dim_{\mathrm{P}}(F)$. As before, there is a tree $T(\widetilde{F})$ with $\widetilde{F} \subset F$ such that for any vertex w of the tree $T(\widetilde{F})$, we have the asymptotic relation (23) holds. Now we distinguish two cases.

Case I: All vertices w of $T(\widetilde{F})$ satisfy

$$\limsup_{n \in S_0} \frac{1}{n} \log_2 |T_n(\widetilde{F}, w)| > \beta\,. \tag{28}$$

In this case, the relation (25) implies that any vertex (v, w) of the product tree $T(E_0 \times \widetilde{F})$ satisfies $\limsup_{n \in S_0} \frac{1}{n} \log_2 |T_n(E_0 \times \widetilde{F}), (v,w)| > \beta$. By Lemma 2.1$(i)$, this yields $\dim_{\mathrm{P}}(E_0 \times \widetilde{F}) \geq 1 + \beta$, establishing the claim (27).

Case II: There is a vertex w of $T(\widetilde{F})$ that does not satisfy (28).

In this case, by (23) every descendant u of w must satisfy

$$\limsup_{n \in S_1} \frac{1}{n} \log_2 |T_n(\widetilde{F}, u)| > \beta\,. \tag{29}$$

since the union $S_0 \cup S_1$ contains all integers greater than 4!. Let $T(\widetilde{F}, w)$ be the subtree of $T(\widetilde{F})$ which consists of all ancestors and descendants of w. Applying Lemma 2.1(i) to the boundary of $T(E_1) * T(\widetilde{F}, w)$ yields that $\dim_{\mathrm{P}}(E_1 \times \widetilde{F}) \geq 1 + \beta$, completing the proof of the claim (27).

Since clearly $\underline{\dim}_{\mathrm{M}}(E_i) = 0$, for $i = 0, 1$, the definition of lower packing dimension gives $\underline{\dim}_{\mathrm{P}}(E) = 0$. Comparing this with the claim (27) shows that the inequality in Proposiution 1.2 is strict for the set E constructed here. □

Remark: Hu and Taylor (1994) used the notation "aDim(E)" for $\inf_F \{\dim_{\mathrm{P}}(E \times F) - \dim_{\mathrm{P}}(F)\}$. It is easy to see that the sets E_0, E_1 in the preceding proof have aDim$(E_i) = 0$ for $i = 0, 1$, so the inequality aDim$(E) \geq 1$ established above (which is really an equality) shows that the index aDim is not stable under finite union.

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