CONFORMAL REMOVABILITY IS HARD

CHRISTOPHER J. BISHOP

Abstract. A planar compact set E is called conformally removable if every homeomorphism of the plane to itself that is conformal off E is conformal everywhere, and hence linear. Characterizing such sets is notoriously difficult and in this paper, we partially explain this by showing that the collection of conformally removable subsets of $S = [0,1]^2$ is not a Borel subset of the space of compact subsets of S with its Hausdorff metric. We give some similar results for other classes of removable sets and pose a number of open problems related to removability and conformal welding, using the language of descriptive set theory.

1. Introduction

Several well known problems in classical complex analysis have remained open for nearly a century and seem intractable. Two of these are to characterize the compact planar sets that are removable for conformal homeomorphisms, and to characterize conformal welding homeomorphisms among all circle homeomorphisms. The purpose of this paper is to partially explain the difficulty of these problems by proving that the collection of conformally removable sets is not a Borel subset of the space of all planar compact sets with the Hausdorff metric. Much of the paper is a survey of the relevant ideas from complex analysis and descriptive set theory, and a recasting of known results into new forms. However, we also present a new result regarding two special classes of removable Jordan curves, and we discuss several new open problems at the interface of classical complex analysis and descriptive set theory. We start by recalling some relevant definitions.

A planar compact set E is called removable for a property P if every function with property P on $\Omega = E^c = \mathbb{C} \setminus E$ is the restriction of a function on $\mathbb{C}$ with this property. For example, if P is the property of being a bounded holomorphic function, then E is

Date: June 16, 2020; revised Dec 26, 2025.

1991 *Mathematics Subject Classification.* Primary: 30C35, 28A05 Secondary: 30H05 .

Key words and phrases. conformally removable, conformal welding, Borel sets, analytic sets, well-founded trees,

The author is partially supported by NSF Grant DMS 2303987.

removable iff every bounded holomorphic function on its complement extends to be bounded and holomorphic on the whole plane (and hence is constant by Liouville's theorem). A standard result in many introductory complex variable classes is the Riemann removable singularity theorem, that says single points are removable in this sense. While there are a wide variety of properties that could be considered, most attention has been devoted to the following cases:

- H^∞-removable: $P =$ bounded and holomorphic,
- A-removable: $P = H^\infty$ and extends continuously to E,
- S-removable: $P =$ holomorphic and 1-to-1 (also known as conformal or schlicht),
- CH-removable: $P =$ conformal and extends to a homeomorphism of $\mathbb{C}$.

For an excellent survey of what is known about each of these classes, see Malik Younsi's 2015 paper [64].

The basic problem is to find "geometric" characterizations of removable sets. For example, Xavier Tolsa has given a characterization of H^∞-removable sets in terms of the types of positive measures supported on the set (see Section 2). Ahlfors and Beurling [1] gave a characterization of S-removable sets as "NED sets" (negligible sets for extremal distance; the precise definition will be given at the end of Section 7). On the other hand, although there are various known sufficient conditions and necessary conditions, e.g., [30], [32], [33], there is no simple characterization of A-removable or CH-removable sets. Thus it appears that characterizing these sets is "harder" than characterizing H^∞-removable or S-removable sets. The following is a precise formulation of this idea (G_δ and Borel sets will be defined later in this section; for the moment think of G_δ as "relatively simple" and not Borel as "very complicated").

Theorem 1.1. *Let $S = [0, 1]^2$ be the unit square in $\mathbb{C}$ and let 2^S denote the hyperspace of S, i.e., the compact metric space consisting of all compact subsets of S with the Hausdorff metric. Within this metric space, the collection of*

1. *H^∞-removable subsets is a G_δ,*
2. *S-removable subsets is a G_δ,*
3. *A-removable subsets is not Borel,*
4. *CH-removable subsets is not Borel.*

Thus, in some sense, removability for conformal homeomorphisms is distinctly more complicated than for bounded holomorphic functions. It turns out that the proof of parts (1) and (2) are fairly elementary, and that parts (3) and (4) follow from well known results in descriptive set theory and complex analysis.

Given a closed Jordan curve Γ with bounded complementary component Ω and unbounded component Ω^*, there are conformal maps $f : \mathbb{D} = \{|z| < 1\} \to \Omega$ and $g : \mathbb{D}^* = \{|z| > 1\} \to \Omega^*$. Both these maps extend homeomorphically to the circle $\mathbb{T} = \partial\mathbb{D} = \{|z| = 1\}$, so $h = g^{-1} \circ f$ is a homeomorphism of the circle to itself. Such a map is called a conformal welding. A single curve Γ can give rise to several weldings due to different choices of the conformal maps f and g but all such maps are related by compositions with Möbius transformations of the circle. Similarly, two curves that are Möbius images of each other will have the same set of associated weldings. In fact, this is true for any image of a curve Γ under a homeomorphism of the sphere that is conformal off Γ. (For brevity, we call this a CH-image of Γ.) For a CH-removable curve, such a map must be a Möbius transformation, so conformally removable curves (modulo Möbius transformations of the 2-sphere) are uniquely determined by their welding (modulo Möbius transformations of the circle).

It is very tempting to claim that a non-removable curve is not uniquely determined by its welding, but this is still open; it is possible that there is some non-removable curve Γ so that any CH-image of Γ is also a Möbius image. Very likely there is no such curve. Indeed, an even stronger conjecture is that any conformally non-removable curve has a CH-image of positive area. Combined with the measurable Riemann mapping theorem (e.g., Theorem V.B.1 of [2], or Theorem 5.3.2 of [4]), this conjecture would imply that every non-removable curve has a CH-image that is not a Möbius image. We will say more about these problems in Section 10.

It is known that not all circle homeomorphisms are weldings, e.g., examples are given in [9] and [49], and these examples are described in Section 10. Thus the map from curves to circle homeomorphisms is not onto. However, weldings form a "large" subset in several senses. For example, conformal weldings are dense in all circle homeomorphisms. This is easy for the uniform metric, since every circle diffeomorphism is a welding, but they are also dense in a much stricter sense: for any $\epsilon > 0$, any circle homeomorphism can be altered on set of length ϵ to become a conformal welding. See Theorem 1 of [9]. Moreover, weldings generate all circle homeomorphisms, i.e., any circle homeomorphism is the composition of two conformal weldings, [55]. It follows from a result of Pugh and Wu that conformal weldings contain a residual set in the space of all circle homeomorphisms (see Section 10 for details). However, it is not known if weldings are a Borel subset of circle homeomorphisms. It follows from general results about Borel sets (to be stated more precisely in Section 3), that if the map from curves to weldings were injective, then conformal weldings would be a Borel subset of circle homeomorphisms. Thus the question of whether conformal weldings are a Borel subset is closely linked to understanding the failure of injectivity for this map, and it seems likely that injectivity fails exactly for CH-non-removable curves, creating a strong link between these problems. Moreover, the collection of non-removable curves is quite complex, as indicated by the following result.

Theorem 1.2. *As above, let $S = [0,1]^2$ be the unit square in $\mathbb{C}$ and let 2^S denote the hyperspace of S, i.e., the compact metric space consisting of all compact subsets of S with the Hausdorff metric. Within this metric space, the collection of A-removable closed Jordan curves is not Borel. Similarly, the collection of CH-removable Jordan curves is not Borel.*

Next, we define a few terms that we have been using. Given a compact set K, we define the Hausdorff distance between compact subsets K_1, K_2 as

$$d_H(K_1, K_2) = \inf\{\epsilon : K_2 \subset K_1(\epsilon), K_1 \subset K_2(\epsilon)\},$$

where $K_j(\epsilon) = \{z : \text{dist}(z, K_j) < \epsilon\}$ is the open ϵ-neighborhood of K_j, $j = 1, 2$. This defines a compact metric space consisting of all compact subsets of K, called the Hausdorff hyperspace of K and denoted 2^K (e.g., see Theorem A.2.2 of [10]). In this paper, we mainly deal with three examples of K: the unit interval $I = [0,1] \subset \mathbb{R}$, the unit square $S = [0,1]^2 \subset \mathbb{R}^2 = \mathbb{C}$, or the Riemann sphere $\mathbb{S}$. The collection of Borel sets is the smallest σ-algebra containing the open sets (a σ-algebra is closed under countable unions, countable intersections and complements). An F_σ set is a countable union of closed sets; a G_δ is a countable intersection of open sets (this terminology originates with Hausdorff in 1914). These are the lowest level of a hierarchy of Borel sets, indexed by the countable ordinals. A Borel map is one for which the preimage of any open set is a Borel set.

Analytic sets (also known as Suslin sets) are continuous images of Borel sets, but they need not be Borel themselves (more about this later). The complement of an analytic set is called co-analytic. The sets in parts (3) and (4) of Theorem 1.1, and in Theorem 1.2, turn out to be co-analytic complete, a condition we will define in Section 5, and that implies that they are non-Borel in a strong sense.

The removable sets in the first three cases of Theorem 1.1 all form σ-ideals of compact sets, i.e., they are closed under taking compact subsets and under compact countable unions. The subset property is obvious, and the fact that a compact set that is a countable union of compact removable sets is also removable is proven in [64] for each of these three classes. The dichotomy theorem for co-analytic σ-ideals (e.g., Theorem IV.33.3 in [35]) then says these collections must be either G_δ or co-analytic complete in 2^S. Theorem 1.1 indicates which possibility occurs in each case. It is not known whether the CH-removable sets form a σ-ideal; indeed, it is not even known if the union of two overlapping CH-removable sets is CH-removable. If the sets are disjoint, then this is true, but is remains open even if both sets are Jordan arcs sharing a single endpoint. The proof of Theorem 1.1 shows that the collection of CH-removable sets is co-analytic complete, and this fact adds some additional evidence that these sets may form a σ-ideal.

Although it is a basic theorem of descriptive set theory that every uncountable Polish space X contains analytic and co-analytic sets that are not Borel (see Section 4), it is very interesting to obtain "natural" examples. For example, if $X = C([0,1])$ (continuous functions on $[0,1]$ with the supremum norm) the following subsets of functions are all known to be co-analytic complete, and hence non-Borel:

- everywhere differentiable [47],
- differentiable except on a finite set [57] or countable set [27],
- nowhere differentiable [46],
- everywhere convergent Fourier series [3].

For the space $C([0,1])^{\mathbb{N}}$ of sequences of continuous functions on $[0,1]$ the space CN of everywhere convergent sequences is co-analytic complete, as is the space CN_0 of sequences converging to zero everywhere. See Theorem IV.33.11 of [35] by Kechris. A famous result of Hurewicz [31] says that the collection of countable, compact subsets of $I = [0,1]$ is co-analytic but not Borel in 2^I with its Hausdorff metric. See Theorem 5.6. Other known examples of non-Borel subsets of 2^I are:

- sets of uniqueness [36],

• sets of strict multiplicity [34].

A closed set $E \subset \mathbb{T}$ is a set of uniqueness if any trigonometric series that converges to zero everywhere off E must be the all zeros series. E is a set of strict multiplicity if it supports a measure whose Fourier coefficients tend to zero; the Fourier series of such a measure shows that its support is not a set of uniqueness in a strong way. These particular examples have an intimate connection to the foundations of modern mathematics: Cantor showed that finite sets are sets of uniqueness, and the problem of extending this to infinite sets led him to the creation of set theory. For more about this fascinating episode in the history of mathematics, see e.g., [17], [18], [44], [58]. For further "natural" examples of non-Borel sets from analysis and topology, see [6] by Howard Becker.

This note was prompted by email discussions with Guillaume Baverez, in which he proposed a possible characterization of CH-removable Jordan curves in terms of their conformal weldings. I doubted such a concise criterion could be given, and eventually I found a counterexample to his conjecture, but the interchange raised the question of quantifying the difficulty of the problem. This paper was written in the hope that gathering the basic facts needed from descriptive set theory might be of interest to fellow complex analysts, and perhaps motivate some of them to attack other variants of these problems, e.g., those discussed in Sections 7, 10 and 11.

I thank Alex Rodriguez for carefully reading the manuscript and locating many typos and small errors that I had missed (any remaining mistakes are my own responsibility). Also many heartfelt thanks to Dimitrios Ntalampekos for many detailed and very helpful comments that improved this paper in various ways. In addition to spotting a number of typos and minor errors, he suggested shorter proofs of some statements, and strengthened the statement of Theorem 1.2. The original version of this result only claimed that A-removable curves formed a non-Borel set, but Dimitrios observed that the same proof also works for CH-removable curves, if we make use of a result of Jang-Mei Wu [63] that I was unaware of. This is a substantial improvement of the paper, answering a question posed in the original version.

2. H^∞-REMOVABILITY IS "EASY"

As we shall explain below, identifying removable sets isn't exactly easy in the usual sense, but in terms of descriptive set theory the collection of such sets is pretty simple.

Lemma 2.1. *The collection of H^∞-non-removable subsets of $S = [0,1]^2$ is an F_σ subset of 2^S. The H^∞-removable sets are therefore a G_δ subset.*

Proof. Suppose $E \subset [0,1]^2$ is non-removable for H^∞. Then there is a non-constant, bounded holomorphic function f defined on the complement of E. Near infinity, f has a Laurent expansion

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots$$

and this has at least one non-zero coefficient c_k for some $k \geq 1$. If $c_1 = 0$, the function

$$f_1(z) = z(f(z) - c_0) = \frac{c_2}{z} + \frac{c_3}{z_2} + \dots$$

is also bounded, non constant and holomorphic off E. Continuing in this way, we see that we eventually obtain a bounded holomorphic function on $\Omega = \mathbb{C} \setminus E$ that has non-zero coefficient c_1 in its Laurent expansion.

Let X_n be the collection of non-removable sets in $[0,1]^2$ whose complements support a holomorphic function whose absolute value is bounded by 1 and whose Laurent coefficient satisfies $|c_1| \geq 1/n$. We claim X_n is a closed set in 2^S. Fix n and suppose $\{K_j\} \subset X_n$ are compact sets converging to K in the Hausdorff metric. Assume f_j is the holomorphic function on K_j^c attesting to its membership in X_n. Each compact disk D in the complement of K is eventually contained in the complements of the K_j for j large enough. Since $|f_j| \leq 1$ for all j, Montel's theorem (e.g. Theorem 10.13 in [45]) implies that we may extract a subsequence that converges to a holomorphic function f_D on D. Covering K^c by a countable union of such disks and applying a diagonalization argument, we may extract a subsequence converging to a holomorphic function f bounded by 1. Applying the Cauchy integral formula to a fixed circle surrounding $[0,1]^2$ we see that the Laurent coefficients of f_j converge to the Laurent coefficients of f and hence $|c_1(f)| \geq 1/n$. Thus $K \in X_n$. Since every non-removable set is in some X_n, the collection of all non-removable sets is an F_σ in 2^S. □

The proof that S-removable sets form a G_δ is very similar, but now the trick of replacing $f(z)$ by $z(f(z) - c_0)$ to get $|c_1| > 0$ might not give a 1-to-1 map. Instead, we may assume the map is conformal off E and has an expansion $f(z) = z + c_1/z + c_2/z^2 + \dots$ so that $c_k \neq 0$ for some k. Thus it suffices to prove each member of the countable family $K_{n,k}$ where $|c_k| \geq 1/n$ is closed. We proceed as before, but now we justify the use of Montel's theorem slightly differently. Since f_j is univalent on $\{|z| > 2\}$ and is normalized so that $f'(\infty) = 1$, Koebe's distortion theorem (Theorem I.4.1 of [24]) implies $f_j(\{|z| > r\})$ contains $\{|z| > 4r\}$ for sufficiently large r. Thus f_j is uniformly bounded on $\{|z| \leq r\}$ for any $r > 0$, and hence is uniformly bounded on any compact disk $D \subset \mathbb{C}$. Thus we can apply Montel's theorem on D, and complete the proof as before.

Of course, just because H^∞-non-removable sets are Borel in 2^S does not mean that it is an easy task to find an elegant characterization of them. Indeed, it is a deep result of Xavier Tolsa [61] that E is non-removable for bounded holomorphic functions if and only if it supports a positive measure μ of linear growth, i.e.,

$$\mu(D(x,r)) \leq Mr, \tag{2.1}$$

(for some $M < \infty$ and all $x \in \mathbb{R}^2$ and $r > 0$) and it has finite Menger curvature in the sense that

$$c^2(\mu) = \int\int\int c^2(x,y,z)d\mu(x)d\mu(y)d\mu(z) < \infty, \tag{2.2}$$

where $c(x,y,z)$ is the reciprocal of the radius of the unique circle passing through x, y and z (linear growth implies $(d\mu)^3$ gives zero measure to the set were two or more of x, y, z agree).

3. Analytic sets

A topological space X is called Polish if it is separable (has a countable dense set) and has a compatible metric that makes it complete (Cauchy sequences converge). Standard examples include Euclidean space $\mathbb{R}^n$, the continuous functions on $[0,1]$ with the supremum norm, $C([0,1])$, and the collection of compact subsets of a compact set $K \subset \mathbb{R}^n$ with the Hausdorff metric. Another important example is the Baire space $\mathbb{N}^\mathbb{N}$ of infinite sequences of positive integers equipped with the metric given by $d((a_n),(b_n)) = e^{-m}$, where $m = \max\{n \geq 0\colon a_k = b_k \text{ for all } 1 \leq k \leq n\}$. One can show $\mathbb{N}^\mathbb{N}$ is homeomorphic to the irrational numbers (with the usual topology) although they are different as metric spaces (one is complete and the other is not). Every Polish space is the continuous image of the Baire space (Lemma B.1.2, [10]).

As stated in the introduction, the Borel sets in a topological space form the smallest σ-algebra (i.e., closed under complements and countable unions and intersections) that contains the open sets. A map is called Borel if the inverse image of any open set is a Borel set. If follows that the preimage of any Borel set under a Borel map is also Borel, and hence that the composition of Borel maps is a Borel map.

If X is a Polish space, then $A \subset X$ is called analytic if there is another Polish space Y and a Borel set $E \subset X \times Y$ so that A is the projection of E onto A, i.e.,

$$A = \{x \in X : \exists\, y \in Y \text{ such that } (x, y) \in E\}.$$

Clearly, any Borel set $B \subset X$ is a projection of the Borel set $B \times X \subset X \times X$, so Borel sets are clearly analytic. However, it is known that any uncountable Polish space contains an analytic set that is not Borel (see Lemma 4.1), and several explicit examples were already mentioned in Section 1.

Analytic sets are closed under countable unions and intersections (see [35] or Appendix B of [10]) but are generally not closed under taking complements, thus they do not usually form a σ-algebra. If $A \subset X$ is analytic, then $A^c = X \setminus A$ is called co-analytic. Borel images and preimages of analytic sets are also analytic. In descriptive set theory, analytic sets are denoted $\mathbf{\Sigma}_1^1$ and co-analytic sets $\mathbf{\Pi}_1^1$ (using light-faced characters refers to something else). These form the simplest elements of the projective hierarchy of sets, much as closed and open sets are the simplest sets of the Borel hierarchy. Analytic and co-analytic sets can be quite complicated, e.g., although every uncountable analytic set contains a perfect subset, Gödel [28] showed that this question for co-analytic sets is undecidable (similar to his results for the Axiom of Choice and the Continuum Hypothesis). Similarly, all analytic sets are Lebesgue measurable, but proving general projective sets are measurable requires additional axioms, e.g., the assumption that certain "large cardinals" exist, e.g., see Steel's article [59].

There are several equivalent characterizations of analytic sets, including (see Section 11.3 of [11])

(1) A is the projection of a closed set in $X \times \mathbb{N}^{\mathbb{N}}$,
(2) A is the continuous image of $\mathbb{N}^{\mathbb{N}}$,

(3) A is a continuous image of a Polish space,
(4) A is the continuous image of a Borel subset of a Polish space,
(5) A is the Borel image of a Borel subset of a Polish space.

In comparison, Borel subsets of a Polish space are characterized as follows (see Theorem 11.12 of [11])

(1) a continuous 1-to-1 image of $\mathbb{N}^{\mathbb{N}}$,
(2) a continuous 1-to-1 image of a Borel subset of a Polish space,
(3) a 1-to-1 projection of a closed set in $X \times \mathbb{N}^{\mathbb{N}}$,
(4) both a co-analytic and analytic set (see below).

Analytic sets are also known as Suslin sets in honor of Mikhail Yakovlevich Suslin, who proved that a set is Borel if and only it is both analytic and co-analytic. While a research student of Lusin in 1917, Suslin constructed a Borel set in the plane whose projection on the real axis is not Borel, contradicting a claim in a 1905 paper of Lebesgue (Cooke [17] refers to Lebesgue's error as "one of the most fruitful mistakes in all the history of analysis"). Suslin died of typhus in 1919 at the age of 24, having published just one 4-page paper while alive, and one posthumously with Sierpinski. His work was further developed by Lusin[1], Sierpinski[2] and others, and Suslin's legacy remains very active a century later.

To prove that the conformally non-removable subsets of $S = [0,1]^2$ form an analytic subset of the hyperspace of S, we first record a few simple facts.

Lemma 3.1. *For any Borel map $f : X \to Y$ between Polish spaces, the graph of f is a Borel set in $X \times Y$.*

Proof. It suffices to prove the complement of the graph is Borel. Since Y is separable, there is a countable basis $\{B_k\}$ for the topology. Thus given any $x \in X$ and $y \in Y$ so that $y \neq f(x)$ there is a basis element B_k so that $f(x) \in B_k$ and $y \notin B_k$. In other words, (x, y) is contained in the Borel product set $f^{-1}(B_k) \times (Y \setminus B_k) \subset X \times Y$ and

[1]In 1936 Lusin was the victim of a political attack that included charges of taking credit for Suslin's work and publishing too much in Western journals. Lusin survived the incident and was officially rehabilitated in 2012. See [21], [42]. However, Lusin's thesis advisor, Egorov, died in 1931 following a hunger strike in prison after similar attacks.

[2]According to [17], although Sierpinski was technically under arrest in Moscow during World War I as an Austrian citizen, he was allowed to participate in the academic life of Moscow University.

this set is disjoint from the whole graph of f. Thus the complement of the graph of f is a countable union of Borel sets, and hence it is Borel itself. □

Lemma 3.2. *Suppose $K \subset \mathbb{C}$ is a compact set and suppose A is an analytic subset of 2^K (i.e, A is a collection of compact subsets of K). Then the collection of compact subsets of K that each contain some element of A (i.e., the collection of supersets of A) is also an analytic subset of 2^K.*

Proof. Since A is analytic, it is the continuous image of some Polish space X, say $A = f(X)$. Define a map $\phi : X \times 2^K \to 2^K \times 2^K \to 2^K$ by $(x, E) \mapsto (f(x), E) \mapsto f(x) \cup E$. The first map in the composition is continuous since f is assumed to be continuous. The second map is continuous since it is easy to check that taking unions is a continuous map from $2^K \times 2^K \to 2^K$. Thus $\phi(X \times 2^K)$ is the continuous image of a Polish space (because products of Polish spaces are also Polish), and hence it is an analytic subset of 2^K. However, the image is exactly the collection of all possible unions of sets in A with compact subsets of K, and hence it is precisely the collection of all supersets elements of A (compact subsets of K containing an element of A). □

For a compact set $K \subset \mathbb{C}$, we say $U \subset K$ is relatively open in K if $U = K \cap V$ for some open set $V \subset \mathbb{C}$.

Lemma 3.3. *Suppose X is a Polish space. Suppose $K \subset \mathbb{C}$ is compact and that each relatively open $U \subset K$ is associated to a closed set $X(U) \subset X$. Moreover, assume that $\cap_\alpha X(U_\alpha) = X(\cup_\alpha U_\alpha)$ for any collection of relatively open subsets $\{U_\alpha\}$ of K. Then the map Λ from points of X to compact subsets of K defined by*

$$\Lambda : x \to K_x = K \setminus \cup\{U : x \in X(U)\},$$

is a Borel map from X to 2^K.

Proof. Note that if $V \subset W$ are relatively open sets, then $V \cup W = W$, and hence

$$X(V) \supset X(V) \cap X(W) = X(V \cup W) = X(W),$$

so our map has a "reverse monotone" property. For each closed set $E \subset K$ and $\epsilon > 0$ consider the open ball in 2^K

$$B(E, \epsilon) = \{F \subset K : d_H(F, E) < \epsilon\}$$

These form a basis of the topology of the hyperspace 2^K, so it suffices to show preimages of such sets are Borel. Each such set is a countable union of closed balls

$$\overline{B}(E,\delta) = \{F \subset K : d_H(F,E) \leq \delta\},$$

for some sequence of δ's tending up to ϵ. Thus it suffices to show that sets of the form $\Lambda^{-1}(\overline{B}(E,\delta))$ are Borel, i.e., $\{x \in X : d_H(K_x, E) \leq \delta)\}$ is a Borel subset of X.

Let $\overline{N}(E,\delta) = \{y \in \mathbb{C} : \text{dist}(y,E) \leq \delta\}$ and similarly for $\overline{N}(K_x,\delta)$. It is easy to check that the condition $d_H(K_x,E) \leq \delta$ holds for some $x \in X$ if and only if x is in the intersection of the sets $Y_1 = \{x : K_x \subset \overline{N}(E,\delta)\}$ and $Y_2 = \{x : E \subset \overline{N}(K_x,\delta)\}$. Hence it suffices to show both Y_1 and Y_2 are Borel.

First consider Y_1. We claim that $x \in Y_1$ if and only if $x \in X(U)$ where $U = \{z : \text{dist}(z,E) > \delta\}$. Suppose $x \in X(U)$. Then K_x is in the complement of U, and hence every point of K_x is within distance δ of E, i.e., $K_x \subset \overline{N}(E,\delta)$. Hence $x \in Y_1$. Conversely, suppose $x \in Y_1$. Then any point $y \in U$ is strictly more than distance δ from E and so y cannot be in K_x. Therefore y is in one of the relatively open sets (call it U_y) that was subtracted from K in the definition of K_x, and hence $x \in X(U_y)$. Thus $x \subset \cap_{y\in U} X(U_y) = X(\cup_{y\in U} U_y)$. Since every point of U is in this union, we have $U \subset \cup_{y\in U} U_y$, so $\cap_{y\in U} X(U_y) \subset X(U)$ by the reverse monotone property. By assumption, $X(U)$ is a closed subset of X, so Y_1 is closed, and hence it is Borel.

Next we consider Y_2. The complement $X \setminus Y_2$ consists of points x so that E contains some point y that is strictly more than distance δ from K_x, i.e., K_x misses some closed disk $D' = \{z : |z \in \mathbb{C} : |z - y| \leq \delta\}$. Thus the compact set K_x is a positive distance from D' and hence it also misses some closed disk $D \supset D'$ that is centered at a rational point of the plane and that has rational radius $> \delta$. For each point $z \in D \cap K$, $z \notin K_x$ implies $x \in X(U_z)$ for some relatively open set $U_z \subset K$ containing z, hence $x \in \cap_{z\in D\cap K} X(U_z) = X(\cup_{z\in D\cap K} U_z) = X(V_D)$ where V_D is some relatively open set containing $D \cap K$ but disjoint from K_x. For each rational closed disk chosen in this way, the corresponding set $X(V_D)$ is closed. If $x \in X \setminus Y_2$, then it is in one of these closed sets and hence $X \setminus Y_2$ is contained in the union of these countably many closed sets. Conversely, if x is in some $X(V_D)$, then K_x omits D and hence every point of K_x is strictly more than distance δ from some point of E. Thus $X \setminus Y_2 = \cup_D X(V_D)$ is F_σ, and hence Y_2 is also Borel, as desired. □

Next we want to specialize to the case when X is the space of homeomorphisms of the 2-sphere to itself that are holomorphic off $S = [0,1]^2$ and normalized to be $h(z) = z + O(1/|z|)$ at infinity. The space of homeomorphisms of a compact Polish space (like the 2-sphere) is always a Polish space itself, but in this case we can be more explicit and take the metric $d(f,g) = \sup|f-g| + \sup|f^{-1} - g^{-1}|$, where distances are measured in the spherical metric. It is not completely trivial to find a countable dense subset, but we sketch a proof, leaving a few details for the reader to verify.

Lemma 3.4. *Let X denote the collection of homeomorphisms of the 2-sphere $\mathbb{S}^2$ to itself that are holomorphic off $S = [0,1]^2$ and normalized to equal $z + O(1/|z|)$ at infinity. Then X contains a countable dense subset, i.e., any element of X can be uniformly approximated by elements of this subset.*

Proof. First, by replacing $f(z)$ by $f(rz)$ where $r > 1$ is very close to 1, we may assume f is also holomorphic on a neighborhood of ∂S, and thus each edge of ∂S maps to an analytic arc under f. For n a positive integer, consider the vertices V_n of a $(1/n) \times (1/n)$ square grid G_n inside S. If n is sufficiently large, then the points $f(V_n \cap \partial S)$, taken in order around ∂S, define the vertices of a simple closed polygon P, and the Riemann map g from the exterior of S to the exterior of P (fixing ∞) uniformly approximates f. By perturbing the vertices of this polygon slightly, we may assume these vertices have rational coordinates, and that the map g still uniformly approximates f. The fact that g uniformly approximates f uniformly on compact sets outside S follows from the Carathéodory kernel convergence theorem (e.g., Theorem 8.11 of [45]), and uniform convergence up to the boundary follows from Rado's theorem (e.g., Theorem II.5.2 of [29]).

Next consider the vertices of the grid G_n that are in the interior of S. Choose $\eta \ll 1/n$, and within an η-neighborhood of each interior vertex v, perturb f so that $f(v)$ has rational coordinates, and the new map (still called f) approximates the old one. For each edge e of G_n connecting two vertices v and w of G_n we want to approximate $f(e)$ by a finite polygonal path from v to w whose vertices all have rational coordinates, and then map the edges of G_n to their corresponding polygonal arcs. If n is large enough, then for any fixed $\epsilon > 0$ the image under f of each square in G_n has diameter less than $\epsilon/2$. Thus we if can approximate each arc $f(e)$ to within

$\epsilon/4$ by a polygonal arc, then any homeomorphic extension of f that maps the edges of G_n to their corresponding polygonal arcs will approximate f to within ϵ.

If n is large enough and η is small enough, we may assume all the image vertices are distinct, and are pairwise separated by a distance of at least some $\delta > 0$. For each vertex of G_n let C_v denote the circle of radius $\delta/100$ around $f(v)$. Each interior vertex is the endpoints of four edges e_1, e_2, e_3, e_4 of G_n, and for each $j = 1, 2, 3, 4$, we choose the last point p_j of e_j on C_v (here, the "last point" means the last time we hit the circle as we travel along e_j from v to the other endpoint of e_j). We can then connect v to each of the four points p_1, p_2, p_3, p_4 by line segments that meet only at v. We do the same for vertices on ∂S, but now there may only be two or three adjacent edges to consider.

We then approximate the subarc of e_k from p_k to the corresponding point q_j on the circle around the other endpoint w of e_k. These subarcs are all compact and pairwise disjoint, so they are all a positive distance from each other. Thus we can approximate each in the Hausdorff metric by pairwise disjoint polygonal arcs, all lying outside all of the circles C_v. Having done this, we can then slightly perturb the arcs to assure that the vertices all have rational coordinates (we do not change the coordinates corresponding to images of vertices of G_n, as these are already rational).

Now map each edge of the grid G_n homeomorphically to the corresponding polygonal using the map that agrees with our previous choices on the vertices of G_n, and that multiplies arclength by a constant factor. Finally, extend this map on the edges of G_n to a homeomorphism of each square of G_n to the corresponding polygonal region defined by the images of the edges, e.g., using conformal maps, we can reduce to extending a circle homeomorphism h to a homeomorphism of the interior disk, which is trivial by the "radial extension" $z \to h(z/|z|) \cdot |z|$. Our mappings on adjacent squares agree on the common boundary segments, so they define a homeomorphism of S that agrees on ∂S with our holomorphic approximation. The final homeomorphism might not have the precise normalization $z + O(1/z)$ near infinity, but it is very close to this, and we can impose this form with a small dilation and translation. The resulting collection of maps is countable, since each map is determined by a finite collection of rational numbers and a forced renormalization. We leave to the reader the verification that the inverse of the homeomorphisms we have constructed approximate the inverse of f. □

Lemma 3.5. *The CH-non-removable subsets of $S = [0,1]^2$ form an analytic subset of the hyperspace of $[0,1]^2$. Thus the removable sets are co-analytic.*

Proof. Let X be the space of homeomorphisms of the 2-sphere to itself that are holomorphic off $S = [0,1]^2$ and normalized to be $h(z) = z + O(1/|z|)$ at infinity. For each open set $U \subset \mathbb{C}$ let $X(U)$ be the elements of X that are holomorphic on U. Since uniform limits of holomorphic functions are holomorphic, this is a closed subset of X. Moreover, if h is holomorphic on each set in a collection $\{U_\alpha\}$, then it is holomorphic on the union so $X(\cup_\alpha U_\alpha) = \cap_\alpha X(U_\alpha)$. All the functions in this set may be holomorphic on a strictly larger set, e.g., if the union has removable complement, but this equality still holds, and simply gives an example where $X(V) = X(W)$ even if V is strictly contained in W.

For each $h \in X$, and let $U_h = \mathbb{C}\backslash K_h$ be the largest open set so that h is holomorphic on some neighborhood of every $z \in U_h$ (alternatively, U_h is the interior of the set of points where $h'(z)$ exists). Lemma 3.3 says that $h \mapsto K_h$ from X to $Y = 2^K$ is a Borel map, and Lemma 3.1 says its graph $\{(h, K_h)\}$ is a Borel set in $X \times Y$. Hence the projection onto the second coordinate gives an analytic set $A = \{K_h : h \in X\}$ (projections of Borel sets are analytic). By definition, a compact subset of K is conformally non-removable if and only if it contains a non-empty set in A. Removing a point from an analytic set gives another analytic set, so by Lemma 3.2 the supersets of non-empty elements of A form another analytic set. Thus conformally non-removable sets are analytic in 2^K. □

Lemma 3.6. *The A-removable subsets of $S = [0,1]^2$ are co-analytic in 2^S.*

Proof. This is exactly the same as the proof of Lemma 3.5, except that now we work in the Polish space of all continuous functions on the Riemann sphere that are holomorphic off $[0,1]^2$, normalized to have supremum norm 1. This space is complete with the usual supremum metric, and a countable dense set is not hard to construct, e.g., one can copy the proof of Lemma 3.4 up to the point where we approximate by a function taking rational values on the vertices of the grid G_n, then triangulate these vertices and and use affine maps on the triangles. (This is much easier than before, because we do not need to produce 1-to-1 maps.)

As before, the map sending each such function to the complement of the set where it is holomorphic is a Borel mapping of this Polish space into 2^S, and the projection of its graph onto the second coordinate gives an analytic subset of 2^S. Taking all supersets of all non-empty projections gives all A-non-removable sets, and shows this collection is analytic. □

4. Analytic non-Borel sets exist

The following is standard result, but we include the simple proof for completeness. We follow the argument in Section 11.5 of [11].

Lemma 4.1. $\mathbb{N}^\mathbb{N}$ *contains an analytic set that is not Borel. Thus the complement of this set is co-analytic and not Borel.*

Proof. This is a diagonalization argument. We claim it that suffices to show there is an analytic subset $X \subset \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ so that every analytic subset $A \subset \mathbb{N}^\mathbb{N}$ occurs as a slice $A = X_y = \{x \in \mathbb{N}^\mathbb{N} : (x, y) \in X\}$, for some y. Given such a set X, then

$$B = \{x \in \mathbb{N}^\mathbb{N} : (x, x) \in X\}$$

is the projection of the intersection of X with the (closed) diagonal of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ and hence is the continuous image of an analytic set, and therefore is itself analytic. The complementary set $B^c = \{x \in \mathbb{N}^\mathbb{N} : (x, x) \not\in X\}$ is automatically co-analytic, and if B^c were also analytic, then it would be equal to a slice X_y of X for some y. Thus,

$$X_y = \{x : (x, y) \in X\} = B^c = \{x : (x, x) \not\in X\}.$$

However, in this case

$$y \in B \quad \Rightarrow \quad (y, y) \in X \quad \Rightarrow \quad y \in X_y = B^c$$

and

$$y \in B^c \quad \Rightarrow \quad (y, y) \not\in X \quad \Rightarrow \quad y \not\in X_y = B^c \quad \Rightarrow \quad y \in B,$$

so assuming either $y \in B$ or $y \in B^c$ both lead to contradictions. Thus B^c can't be analytic, and hence neither B nor B^c is Borel (since Borel sets are closed under complements, and all Borel sets are analytic). Thus we have reduced proving the existence of a non-Borel analytic set to finding an analytic set $X \subset \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ which has every analytic subset of $\mathbb{N}^\mathbb{N}$ as a slice.

First we show this is possible for closed slices. The idea is that we can encode any closed set by the list of open basis elements it misses. More precisely, if Y is a Polish with a countable basis $\{B_k\}$ for the topology, and if $y \in Y$, then let $S(y) \subset \mathbb{N}$ be the set of all natural numbers k with $y \notin B_k$. Then set $T(y) \subset \mathbb{N}^\mathbb{N}$ to be the collection of all the sequences with elements in $S(y)$, i.e., $T(y) = S(y)^\mathbb{N}$.

Consider the set $Z = \{(y, s) \subset Y \times \mathbb{N}^\mathbb{N} : y \in Y, s \in T(y)\}$. First, we claim that every closed set $F \subset Y$ occurs as a slice of Z. To prove this, let $S(F) \subset \mathbb{N}$ be the set indices k of basis elements B_k missing F. Fix the second coordinate of Z to be some sequence $s \in \mathbb{N}^\mathbb{N}$ whose union of elements is exactly the countable set $S(F)$. If (y, s) is any point in this slice, then we must have $s \in S(y)$, and so y misses every open basis set B_k that misses F (and possible others), so $y \in F$. Conversely, if $y \in F$, then $S(F) \subset S(y)$, so $s \in S(F)^\mathbb{N} \subset S(y)^\mathbb{N}$, and hence (y, s) is in Z. This proves every closed set $F \subset Y$ occurs as a slice of Z.

Next, we claim Z is a closed subset of $Y \times \mathbb{N}^\mathbb{N}$. Consider $y_n \to y$ in Y and $z_n \in T(y_n)$ with $z_n \to z$ in $\mathbb{N}^\mathbb{N}$. We need to show $z \in T(y)$. If $y_n \to y$ and $y_n \notin B_k$ for large n, then $y \notin B_k$, since B_k^c is closed. Hence an integer is in $S(y)$ if it is in $S(y_n)$ for all sufficiently large n (the converse need not be true). Since $z_n \to z$ in $\mathbb{N}^\mathbb{N}$, it converges coordinate-wise, and so if the kth coordinate of z_n is in $S(y)$ for all large enough n, the same is true for z, i.e., $z \in T(y)$, as desired, proving Z is closed.

Finally, to obtain every analytic subset of $\mathbb{N}^\mathbb{N}$ as a slice, we apply the previous argument to $Y = \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ to get a closed set $X \subset Y \times \mathbb{N}^\mathbb{N} = (\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}) \times \mathbb{N}^\mathbb{N}$ so that every closed subset of $(\mathbb{N}^\mathbb{N})^2$ occurs as a slice of X. Hence every analytic subset of $\mathbb{N}^\mathbb{N}$ occurs when we project X onto the first coordinate. Projections of analytic sets are analytic, so projecting X onto the first and third coordinates gives an analytic subset of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$, whose first coordinate ranges over all analytic subsets of $\mathbb{N}^\mathbb{N}$. □

Note that this implies the cardinality of the analytic subsets of a Polish space is at most the cardinality of $\mathbb{N}^\mathbb{N}$, i.e., the same as $\mathbb{R}$, the continuum c. Since single points are analytic sets, the analytic subsets of $\mathbb{R}$ have cardinality exactly c. In particular, the collection of all Borel subsets of $\mathbb{R}$ also has cardinality c.

5. Co-analytic complete sets

A co-analytic subset $A \subset X$ of a Polish space is called co-analytic complete if for any co-analytic set B of $\mathbb{N}^{\mathbb{N}}$ there is a Borel map $f : \mathbb{N}^{\mathbb{N}} \to X$ so that $f(y) \in A$ iff $y \in B$, i.e., $B = f^{-1}(A)$. Thus membership in any such B can be reduced to checking membership in A.

Lemma 5.1. *If $f : X \to Y$ is a Borel map between Polish spaces, if A is co-analytic, and if $f^{-1}(A)$ is co-analytic complete in X, then A is co-analytic complete in Y.*

Proof. If $B \subset \mathbb{N}^{\mathbb{N}}$ is co-analytic, then there is a Borel map $g : \mathbb{N}^{\mathbb{N}} \to X$ so that $B = g^{-1}(f^{-1}(A))$, since $f^{-1}(A)$ is co-analytic complete. Thus $h = f \circ g$ is a Borel map from $\mathbb{N}^{\mathbb{N}}$ to Y and $B = h^{-1}(A)$. Hence any co-analytic set B in $\mathbb{N}^{\mathbb{N}}$ is a Borel preimage of A, and hence A is co-analytic complete. □

Lemma 5.2. *A co-analytic complete set cannot be Borel.*

Proof. Let $B \subset \mathbb{N}^{\mathbb{N}}$ be a non-Borel, co-analytic set (such exist by Lemma 4.1). If $A \subset Y$ is co-analytic complete, then, by definition, there is a Borel $f : \mathbb{N}^{\mathbb{N}} \to Y$ so that $B = f^{-1}(A)$. But Borel inverse images of Borel sets are Borel, so A cannot be Borel since B is not Borel. □

Therefore a simple strategy for proving $A \subset X$ is not Borel is to find a Borel map $f : Y \to X$ so that $B = f^{-1}(A) \subset Y$ is a known co-analytic complete set in Y. If A is co-analytic, then Lemmas 5.1 and 5.2 imply A is not Borel. If A is not co-analytic, then it is automatically not Borel (all Borel sets are both analytic and co-analytic). To make this work, we need one co-analytic complete set to start from. A standard choice is the collection of well-founded trees, which we define next.

Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers (including the empty sequence). A tree T is a subset of $\mathbb{N}^*$ that is closed under removing the final element, i.e., if a finite sequence is in T, so is every initial segment, including the empty one (this labels the root vertex of T). An infinite branch of T is an element of $\mathbb{N}^{\mathbb{N}}$, all of whose finite initial segments belong to T. The set of all infinite branches of T is denoted $[T]$ (this is also sometimes called the boundary of T and denoted ∂T, but we will not use this alternate notation). A tree is well-founded if it has no infinite branches. Finite trees are obviously well-founded, and the infinite set of finite sequences $(n, n-1, n-2, \dots, 1)$ with $n \in \mathbb{N}$, together with all initial segments of these sequences, form an infinite well-founded tree. See Figure 1.

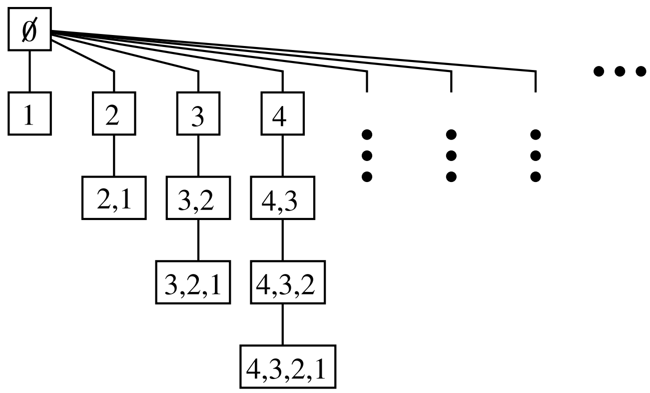

FIGURE 1. An example of a well-founded tree. It is an infinite tree, but has no infinite branches.

The sequence spaces $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ each have a product topology which is metrizable with the metric

$$d(\{a_k\}, \{b_k\}) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{2^k(1 + |a_k - b_k|)}.$$

Since $\mathbb{N}^*$ is countable and a subset can be identified with its indicator function, any tree can be identified with a point of $2^{\mathbb{N}}$, i.e., the Cantor set of infinite binary sequences. In fact, the set of all trees corresponds to a closed subset of $2^{\mathbb{N}}$, that we will denote X_T. Thus X_T is a Polish space itself (it is also a Cantor set, since no tree is isolated in the induced topology). However, we will show that the collection of well-founded trees is co-analytic complete, and hence non-Borel, in this space. To prove this, we will use the following result (Lemma 11.22 of [11]).

Lemma 5.3. *Every closed set in $\mathbb{N}^{\mathbb{N}}$ is of the form $[T]$ for some tree T. For every analytic set $A \subset \mathbb{N}^{\mathbb{N}}$ there is a tree T so that $a = (a_1, a_2, \dots) \in A$ if and only if there is some $b = (b_1, b_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ so that the "weaving map" satisfies*

$$W(a, b) = (a_1, b_1, a_2, b_2, \dots) \in [T].$$

Proof. The first part is straightforward (this argument was suggested by Dimitrios Ntalampekos, shortening the original proof). Suppose $K \subset \mathbb{N}^{\mathbb{N}}$ is closed, and let T be the tree of all finite initial segments of all elements in K. By definition, each element of K corresponds to an infinite branch of the tree T. Conversely, if $x = (x_1, \dots, x_n, \dots) \in \mathbb{N}^{\mathbb{N}}$ corresponds to an infinite branch of T, then we wish to show that $x \in K$. By the definition of T, for each fixed $n \in \mathbb{N}$ we can find a sequence $y(n) \in K$ whose first n entries agree with $(x_1, \dots, x_n)$ (because each initial segment has "children"). Thus, $y(n)$ converges to x in the product topology of $\mathbb{N}^{\mathbb{N}}$. Since K is closed, we have $x \in K$.

To prove the second part of the lemma, note that $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ by the 1-1, continuous map that interweaves sequences:

$$W : (a_1, a_2, \dots) \times (b_1, b_2, \dots) \mapsto (a_1, b_1, a_2, b_2, \dots).$$

Thus, if A has the form given in the lemma, then it is the projection onto the first coordinate of the closed set $W^{-1}([T]) \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, and hence A is analytic (note that $W^{-1}([T])$ is closed since $[T]$ is closed and W is a homeomorphism).

Conversely, if A is analytic, then it is a continuous image $A = f(\mathbb{N}^{\mathbb{N}})$ and hence A is the projection of the closed set $(f(x), x) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ (recall that graphs of continuous functions are closed sets). Since W is a homeomorphism, the W-image of this closed graph gives a closed set in $\mathbb{N}^{\mathbb{N}}$. Applying the first part of this lemma gives a tree T corresponding to A that satisfies the interweaving condition in the lemma. □

Note that we have actually proved something stronger than was claimed: A is analytic if and only if there exists a tree T so that A is the projection of $W^{-1}([T])$ to the first coordinate. Also note that the weaving map $W : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ used above can also be defined as a map $W : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^* \to \mathbb{N}^*$ by truncating $\{a_n\}$ as follows:

$$W(\{a_k\}_1^\infty, \{b_k\}_1^n) = (a_1, b_1, a_2, b_2, \dots, a_n, b_n) \in \mathbb{N}^*.$$

We will use this definition in the proof Lemma 5.5 below.

Lemma 5.4. *The well-founded trees are a co-analytic subset of X_T.*

Proof. It suffices to prove the ill-founded trees (those containing an infinite branch) form an analytic set. Consider the set $Z = \{(T, x)\} \subset X_T \times \mathbb{N}^{\mathbb{N}}$ such that x is an infinite branch of T. The projection of Z onto the first coordinate gives all ill-founded trees, so these trees will form an analytic set if Z is closed in $X_T \times \mathbb{N}^{\mathbb{N}}$. Suppose $T_n \to T \in X_T$ and $x_n \to x \in \mathbb{N}^{\mathbb{N}}$ in the product topologies. Then any initial segment of x is an initial segment of x_n for all sufficiently large n. Thus this segment is a vertex of T_n for all large enough T_n and hence also of T. Since every initial segment of x is a vertex of T, x is an infinite branch of T. Thus Z is closed, ill-founded trees are analytic and well-founded trees are co-analytic. □

Lemma 5.5. *The well-founded trees are a co-analytic complete subset of X_T.*

Proof. Recall that $X_T \subset 2^{\mathbb{N}}$ denotes the set of trees. By Lemma 5.4, the well founded trees are co-analytic, so it suffices to verify the other part of the definition: given any co-analytic set $B \subset \mathbb{N}^{\mathbb{N}}$, there is a Borel map of $\mathbb{N}^{\mathbb{N}}$ to X_T so that B is the inverse image of the well founded trees.

Let $A = B^c$. By definition, A is analytic, so by Lemma 5.3 there is a tree T so that $a = (a_1, a_2, \dots) \in A$ iff $W(a, b) \in [T]$ for some $b = (b_1, b_2, \dots) \in \mathbb{N}^{\mathbb{N}}$. Using T, we define a map $\mathbb{N}^{\mathbb{N}} \to X_T$ as follows. For $a = (a_1, a_2, \dots)$, we let $T(a)$ denote the collection of all finite sequences $\{b_k\}_1^n$ (including the empty sequence) so that $W(a, b) \in T$ (W as defined just before Lemma 5.4). Clearly $T(a)$ is a tree. Moreover, a sequence $a \in \mathbb{N}^{\mathbb{N}}$ belongs to $B = A^c$ if and only if $W(a, b) \notin [T]$ for all $b \in \mathbb{N}^{\mathbb{N}}$. Thus $a \in A^c$ if and only if $T(a)$ has no infinite branches, i.e., if and only if $T(a)$ is a well-founded tree.

To finish the proof, we verify that the map $a \mapsto T(a)$ is Borel. Recall that $X_T \subset 2^{\mathbb{N}}$, and that a basis for the topology consists of specifying a finite initial segment of a sequence, and allowing the remaining elements to be free. The inverse image of such a basis element is the collection of all infinite sequences $a \in \mathbb{N}^{\mathbb{N}}$, so that

(1) interweaving the initial elements of a with the specified elements of the basis gives a finite string in T, and
(2) there is some continuation of the specified elements to an infinite sequence so that interweaving is a branch of T.

Thus a is simply the sequence of odd coordinates of branches of T that passes through the specified vertex. The collection of all such sequences is a closed set in $\mathbb{N}^{\mathbb{N}}$. Thus the inverse image of a general open set in $2^{\mathbb{N}}$ is a countable union of closed sets, and hence the mapping $a \to T(a)$ is Borel. □

Theorem 5.6 (Hurewicz, [31])**.** *The compact countable subsets of $I = [0, 1]$ are co-analytic complete in 2^I.*

Proof. First we must show this collection is co-analytic or, equivalently, that the uncountable compact subsets of I form an analytic subset of 2^I. We use the fact that every compact, uncountable set K supports a non-atomic probability measure μ, and hence the function $f(x) = \mu([0,x])$ is continuous, increasing, $f(0) = 0$, $f(1) = 1$ and is constant on each connect component of $[0,1] \setminus K$. Consider the set $Z = \{(K,f)\} \subset 2^I \times C([0,1])$, where f and K are related as above: f is continuous, $f(0) = 0$, $f(1) = 1$ and f is constant on the complementary components of K. Projection onto the first coordinate gives all uncountable compact sets, so it suffices to show Z is closed. Thus we need to show that if $K_n \to K$ in the Hausdorff metric, and if $f_n \to f$ uniformly, then f is constant on the complementary components of K. However, any two points $x < y$ in such a component define a compact interval $[x,y]$ that is a positive distance from K, and hence is outside of K_n for large enough n, and thus $f_n(x) = f_n(y)$ for all large enough n. Taking limits gives $f(x) = f(y)$, as desired.

Next we show the collection of compact, countable sets is co-analytic complete. By Lemma 5.1, it suffices to show that there is a continuous map from the space of trees, X_T, into 2^I, so that the image of a tree T is a countable subset of I if and only if T is well-founded. For each $n = 1, 2, \dots$, let $A_n = \{x \in [0,1] : \frac{1}{2n+1} \leq |x - \frac{1}{2}| \leq \frac{1}{2n}\}$. Then the A_n are all disjoint and each set consists of two compact intervals. For any $S \subset \mathbb{N}$, define

$$A_S = \{\frac{1}{2}\} \cup \bigcup_{n \in S} A_n.$$

This is a compact subset of $[0,1]$, and equals $\{1/2\}$ if and only if S is empty.

Suppose we are given a tree T. The root vertex (labeled by the empty string) is associated to $E_0 = I_\emptyset = [0,1]$. In general, suppose E_n is a compact subset of $[0,1]$ whose connected components are a countable number of points labeled by strings of length $< n$, and a countable number of non-trivial closed intervals I_s labeled by strings of length n. All strings that occur as labels of intervals in E_n correspond to labels of vertices in level n of T, and for each such label, 2^n intervals in E_n will have that label. To construct E_{n+1} from E_n, we keep every point component from E_n (and leave the label the same) and replace each interval component J_s labeled by a string s of length n by $L_S(A_S)$, where S is the set of integers that can be appended to S to give a length $n+1$ string in T (i.e., these correspond to the edges leading out of vertex s), where A_S is as above, and where L_S is a linear map from J to J_s. Since each A_n

consists of two intervals, each nth generation interval with a given label gives rise to two intervals in the next generation with identical labels. Let $E_T = \cap E_n$. Since the E_n are nested compact sets, this is a non-empty compact subset of $[0, 1]$.

If T has an infinite branch, then following this branch through the construction gives a Cantor subset of E, hence E is uncountable. Conversely, if E is uncountable, then $E \cap J_1$ must be uncountable for one of the countably many connected components of E_1. Then $E \cap J_2$ must be uncountable for one of the countably many components of E_2 contained in J_1. Continuing in this way, we obtain nested, non-degenerate components $J_1 \supset J_2 \supset J_3 \supset \dots$ whose labels form an infinite branch of T, so T is not well-founded. It is easy to check that the map from trees to sets, described above, is continuous: if two trees are very close, then the construction of the corresponding sets is the same, except inside a union of intervals, each of which have small length, so the sets are close in the Hausdorff metric. □

The endpoints of all the components of E_n in the previous proof are rational numbers. Thus we could reformulate the result to say that compact subsets of $\mathbb{Q} \cap I$ are co-analytic complete in 2^I (one first uses Lemma 3.2 to show that the collection of compact sets containing at least one irrational number is analytic in 2^I, so the compact subsets of $\mathbb{Q}$ is co-analytic).. Theorem 5.6 also gives a rather concrete example of a non-Borel set in $[0, 1]$. Let $\{r_n\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$ and for $K \in 2^I$ define

$$f(K) = \sum_{r_n \notin K} 3^{-n}.$$

Clearly f is 1-to-1 (since distinct sums of powers of 3 are distinct). The sets $\{K : f(K) > \alpha\}$ are easily checked be open in 2^I, so f is Borel. Thus

$$X = \{f(K) : K \subset \mathbb{Q} \cap [0, 1] \text{ and is compact } \} \subset [0, 1]$$

cannot be Borel. An earlier "explicit" non-Borel set, given in terms of continued fractions expansions, is due to Lusin [43]

Dimitrios Ntalampekos pointed out that the existence of a non-Borel set depends on the Axiom of Choice, and he asked where we have used this. In fact, we have utilized it from the beginning, as many of the basic facts about analytic sets depend on choice, e.g., Suslin's proof that a set that is both analytic and co-analytic must be Borel. Without choice, it is consistent with Zermelo-Fraenkel set theory that the real numbers are a countable union of countable sets, in which case every subset of the reals is Borel. For a discussion (and citations) of "how much" choice is needed to construct non-Borel sets, see the Math Overflow discussion [13]. Also see [22] for a related development of measure theory without the Axiom of Choice.

6. A-removable sets are co-analytic complete

We start with a well known fact from complex analysis.

Lemma 6.1. *If $E \subset [0,1]$ has positive length, then it is H^∞-non-removable.*

Proof. If E is an interval, then we simply apply the Riemann mapping theorem to conformally map the complement of E (on the sphere) to the unit disk. This gives a non-constant bounded holomorphic function on the complement.

The general case was proven by Ahlfors and Beurling in [1] (or see Section I.6 of Garnett's book [23]). Note that if $w = u + iv$

$$\begin{aligned} F(w) &= \int_E \frac{dz}{z-w} = \int_E \frac{dt}{t-(u+iv)} \\ &= \int_E \frac{(t-u+iv)dt}{(t-u-iv)(t-u+iv)} \\ &= \int_E \frac{t-u}{(t-u)^2+v^2}dt + i\int_E \frac{v}{(t-u)^2+v^2}dt \end{aligned}$$

is holomorphic on $\Omega = E^c$, has imaginary part in $[-\pi, \pi]$, and Laurent expansion $\ell(E)/z + c_2/z^2 + \dots$ near infinity. Thus $G = \exp(F/2)$ takes values in the right half-plane, $(G-1)/(G+1)$ maps Ω holomorphically into the disk, and one can compute its leading Laurent coefficient to be $c_1 = \ell(E)/4 > 0$. □

Extending this result from subsets of $\mathbb{R}$ to subsets of graphs $\Gamma = \{(x, f(x)\} \subset \mathbb{R}^2$ of real Lipschitz functions f was a major breakthrough by Alberto Calderón [14], when he proved the L^p boundedness of the Cauchy integral operator on Lipschitz graphs. This led to many important developments in in harmonic analysis and geometric measure theory over the last fifty years, including Tolsa's result discussed in Section 2. For some of the related history, see [20], [50], [60], [62].

The following is stated and proved on page 117 of Carleson's 1951 paper [15]:

Theorem 6.2. *If $E_1, E_2 \subset [0,1]$ are compact and if E_2 has positive Lebesgue measure, then $E = E_1 \times E_2$ is A-removable iff E_1 is countable.*

Proof. For completeness, we recall the proof of both directions. If E_1 is countable, the removability of $E_1 \times E_2$ is due to Besicovitch [7], but we give a short proof suggested by Dimitrios Ntalampekos that foreshadows remarks in the last section of this paper.

If E_1 is countable, then $E_1 \times [0,1]$ is a compact set that is the countable union of vertical slits. Each isolated slit is removable; this is a simple consequence of Morera's theorem (e.g. Theorem 4.19 of [45]). Removing those isolated slits, one ends up with a new compact set $E_1' \times [0,1]$. The set E_1', if non-empty, is also countable, so must have isolated points. Then one proceeds with transfinite induction on the rank of the countable set E_1 to get the removability.

Conversely, if E_2 has positive length, then by Lemma 6.1 there is a non-constant bounded analytic function f on the complement of iE_2 with a positive Laurent coefficient c_1. If E_1 is uncountable, then it supports a non-atomic, positive, finite measure μ. Therefore $F(z) = \int f(z+x)d\mu(x)$ is continuous on the sphere and holomorphic off $E = E_1 \times E_2$. The fact that

$$\frac{1}{z-x} = \frac{1}{z} + (\frac{1}{z-x} - \frac{1}{z}) = \frac{1}{z} + \frac{x}{z(z-x)},$$

implies F also has non-zero Laurent coefficient c_1 and hence is non-constant. Therefore E is A-non-removable. □

Corollary 6.3. *The A-removable compact subsets of $S = [0,1]^2$ are co-analytic complete in 2^S, hence not Borel.*

Proof. We already know this set is co-analytic by Corollary 3.6. To prove co-analytic completeness, by Lemma 5.1 it suffices to show that the mapping $E \mapsto E \times [0,1]$ is continuous between the respective Hausdorff metrics and hence reduces the set of countable compact subsets of $[0,1]$ to the set of A-removable sets. Since the former is co-analytic complete by Theorem 5.6, so is the latter. □

7. CH-removable sets are co-analytic complete

The following is due to Fred Gehring [25] in 1960. We include a proof for the reader's convenience.

Lemma 7.1. *For compact sets $E \subset [0,1]$, $E \times [0,1]$ is CH-non-removable if and only if E is uncountable.*

Proof. First suppose E is compact and uncountable. Then E supports a a positive, finite, non-atomic measure μ. By restricting μ to an appropriate subset E_0 of zero Lebesgue measure and multiplying by an appropriate constant we may assume μ is singular to Lebesgue measure, is supported in an interval $J = [a, b] \subset [0, 1]$, and has total mass equal to half the length of J. For a fixed constant $c \in [0, 1]$ define $h_c(x) = x$ outside J and

$$h_c(x) = x + c\left(\int_0^x d\mu(t) - \frac{x-a}{2}\right),$$

inside J. It is easy to check this is a homeomorphism that is linear with slope $1 - \frac{c}{2}$ on each component of $J \setminus E_0$. On the other hand, h_c maps E_0 to a set of length $c\ell(J)/2 > 0$. Let $g(y) = \max(0, \frac{1}{2} - |y - \frac{1}{2}|)$ and define

$$F(x, y) = (h_{g(y)}(x), y).$$

See Figure 2. This is a homeomorphism of the plane that is the identity off $J \times [0, 1]$, and for any component K of $J \setminus E_0$, F is a skew linear map on $K \times [0, \frac{1}{2}]$ and $K \times [\frac{1}{2}, 1]$ with uniformly bounded dilatation. Thus F is quasiconformal off $E_0 \times [0, 1]$. It is not quasiconformal on the whole plane because the zero length set $E_0 \times \{y\}$ is mapped to a set of positive length for each $0 < y < 1$, and thus $E_0 \times [0, 1]$ is a set of zero area that is mapped to positive area; this is impossible for quasiconformal maps, see e.g., [2]. Using the measurable Riemann mapping theorem, we can find a quasiconformal mapping φ of the whole plane so that $\varphi \circ F$ is conformal off $E \times [0, 1]$ but not quasiconformal everywhere, hence not conformal everywhere. Thus $E \times [0, 1]$ is CH-non-removable.

Conversely, note that if E is CH-non-removable with witness f and if $z_0 \notin E$, then

$$g(z) = (f(z) - f(z_0))/(z - z_0)$$

is continuous, non-constant, and bounded on the plane and holomorphic off E, so E is also A-non-removable. Thus A-removable sets are also CH-removable. Hence by Theorem 6.2 if E is countable, then $E \times [0, 1]$ is CH-removable. (One could also directly apply the same transfinite induction argument as given in the proof of Theorem 6.2). □

Corollary 7.2. *CH-removable sets in $S = [0, 1]^2$ are co-analytic complete in 2^S.*

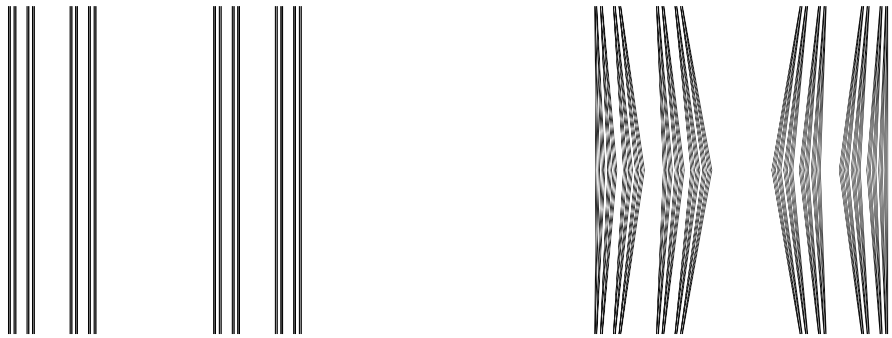

FIGURE 2. If E is a Cantor set, then there is homeomorphism h of $\mathbb{C}$ that is quasiconformal off $E \times [0, 1]$ and maps $E \times [0, 1]$ to a set of positive area. This can't happen if E has zero length and h is quasiconformal on the whole plane.

The proof is the same as for A-removable sets, except using Gehring's result in place of Carleson's.

Recently, Dimitrios Ntalampekos [48] has suggested a characterization of CH-removable sets that is closely related to the characterization of S-removable sets due to Ahlfors and Beurling. Given two continua $F_1 and F_2$ inside an open planar domain Ω, we consider the family Γ of rectifiable paths connecting F_1 to F_2. Given set $E \subset \mathbb{C}$, we can consider the sub-family Γ_E of Γ consisting of paths that miss E. If for every Ω, F_1, and F_2 as above, the extremal length of Γ_E is the same as the extremal length of Γ, then we say E is negligible for extremal distances, or "NED" for brevity. Ahlfors and Beurling proved that a compact set E is S-removable if and only if it is NED (see Theorems 6 and 9 of [1]).

Ntalampekos calls a set CNED (countably negligible for extremal distances) if Γ always has the same extremal length as the sub-family consisting of paths that hit E in at most countably many distinct points (we do not care how often each point of E is hit by a path). In [48] he shows that several known families of CH-removable sets are special cases of CNED sets, and conjectures that closed CNED sets are the same as CH-removable sets. Corollary 4.4 of [48] says that if a closed set $X \subset \mathbb{C}$ is CNED, then for any $\epsilon > 0$, and for any two points $x, y \in \mathbb{R}^2$, there is a path γ connecting x and y of length at most $|x - y| + \epsilon$ so that $\gamma \cap X$ is countable (ignoring multiplicities).

This clearly fails if $X = E \times [0,1]$, and $E \subset \mathbb{R}$ is uncountable. Thus the proof of Lemma 7.2 and the remarks following Lemma 5.2 show that the collection of CNED sets is not Borel in 2^S, where $S = [0,1]^2$. Moreover, it would be co-analytic complete if it is co-analytic. Is this the case?

8. A-removable Jordan curves are co-analytic complete

A case of particular interest among compact planar sets are the closed Jordan curves. Let $\text{Homeo}(X,Y) \subset C(X,Y)$ denote the 1-to-1 continuous maps of X into Y. It is easy to see that this subset is neither open nor closed in $C(X,Y)$. However, a map $f : \mathbb{T} \to \mathbb{C}$ is 1-to-1 if and only if any two disjoint closed dyadic intervals have disjoint images (an open condition) and hence $\text{Homeo}(\mathbb{T},\mathbb{C})$ is a G_δ set in $C(\mathbb{T},\mathbb{C})$.

We can think of closed Jordan curves as elements of $\text{Homeo}(\mathbb{T},\mathbb{C})/\text{Homeo}(\mathbb{T},\mathbb{T})$, i.e., modulo re-parameterizations. Thus $f, g \in \text{Homeo}(\mathbb{T},\mathbb{C})$ are equivalent if $f = g \circ \rho$ for some $\rho \in \text{Homeo}(\mathbb{T},\mathbb{T})$. We can define a metric between equivalence classes as

$$d([f],[g]) = \inf\{\|f - g \circ \rho\|_\infty : \rho \in \text{Homeo}(\mathbb{T},\mathbb{T})\},$$

although Jordan curves are not complete in this metric. A complete metric on Jordan curves separating 0 and ∞ is described by Pugh and Wu in [52], by choosing a particular parameterization of each curve. They attribute the idea to Thurston: one takes conformal maps of $\mathbb{S}\setminus\mathbb{T}$ to $\mathbb{S}\setminus\Gamma$ normalized to fix 0 and ∞ respectively and that have positive derivative at these points, and then use the supremum metrics between conformal maps.

Theorem 8.1. *The collection of A-removable Jordan curves contained in $S = [0,1]^2$ is co-analytic complete in 2^S.*

Proof. As in previous proofs, we first verify that the collection is co-analytic by showing its complement is analytic. Consider the set Z of pairs (γ, f) where γ is a non-removable closed Jordan curve and f is a continuous function on the 2-sphere that is holomorphic off γ and has Laurent coefficient $c_1 = 1$ (to confirm that it is non-constant). Again, as before, it suffices to show Z is closed, so it suffices to show that if $\gamma_n \to \gamma$ and $f_n \to f$ then f is holomorphic off γ. This follows since uniform limits of holomorphic functions are holomorphic, and any closed disk that misses γ will miss γ_n for all sufficiently large n.

As noted in Lemma 5.1, it suffices to construct a Borel map from some Polish space into the space of Jordan curves, so that the preimage of the A-removable curves is a known co-analytic complete set B. We will take the Polish space to be the set of trees X_T (defined in Section 5), and the preimage set to be the collection of well-founded trees. The latter is co-analytic complete by Lemma 5.5.

To simplify some formulas, we work in $[-1,1]^2$ instead of $[0,1]^2$. We start with a map from trees to compact subsets of $[-1,1]$ that maps well-founded trees into countable sets, using a slightly different map than we did in the proof of Theorem 5.6. For $n \in \mathbb{N}$, we define

$$A_n = \{x : \frac{1}{4} + \frac{1}{2n+1} \leq |x| \leq \frac{1}{4} + \frac{1}{2n}\},$$

and for $S \subset \mathbb{N}$

$$A_S = \{\pm\frac{1}{4}\} \cup \bigcup_{n \in S} A_n \subset [-1,1].$$

This is similar to what we did in the proof of Theorem 5.6, except that now the pairs of intervals A_n converge to two different points $\pm 1/4$, instead of a single point. However, the rest of the construction is the same, and associates to each tree T a compact set E_T that is countable if and only if T is well-founded. Recall that each string s of length n is associated to 2^n intervals which we label I_s^j, $j = 1, \dots 2^n$. We assume these are numbered left to right.

Next we construct a Cantor set $K = \cap_n K_n \subset K_0 = [-1,1]$ of positive Lebesgue measure where $K_0 \supset K_1 \supset \cdots \supset K$ and each K_n is a union of 2^n disjoint closed intervals which we denote $\{K_n^k\}$, $k = 1, \dots, 2^n$. We assume that for a fixed n, the components $\{K_n^k\}_1^{2^n}$ are numbered left to right and that their maximum length $\ell_n = \max_k |K_n^k|$ tends to zero with n. For the current proof, we may assume that for each n, every K_n^k has length $\ell_n = 2^{-n-1}(1 + 1/n)$, so K has length $1/2$.

Our Jordan curves will be constructed using templates that are closed sets G_J, where the index $J \in \{K_n^k\}$ is one of the component intervals in the construction of the Cantor set K. The largest J is $J = [-1,1] = K_0$ and we denote $G_{[-1,1]}$ by G_0 for brevity. It is illustrated in Figure 3. In general, G_J consisting of countable union of polygonal arcs, rectangles and copies of K. The rectangles are all of the form $I \times J'$ where each I is some A_n, i.e., one of the component intervals of $A_{\mathbb{N}}$, and J' is a component of one of the sets $J \cap K_m$ where $m > j$ if $J = K_j^k$.

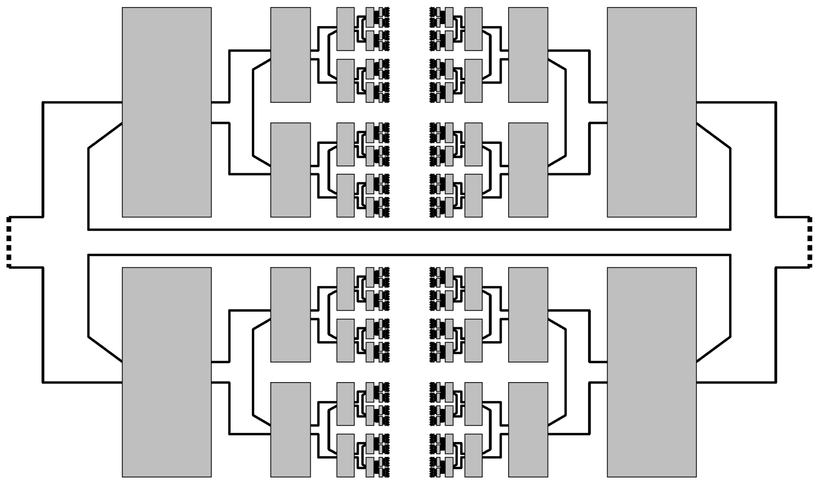

FIGURE 3. The basic template G_0 for the construction. Each symmetric pair of columns of rectangles corresponds to a positive integer. The two dashed vertical segments (far left and far right) are only used in the first step of the construction, to give a closed curve.

We attempt to describe Figure 3 in words. G_0 has two copies of the Cantor set K, positioned in the vertical lines $\{x = \pm 1/4\}$, near the center of the picture. There are countably many rectangles, arranged in vertical columns which accumulate on the two Cantor sets from the left and right respectively. Each positive integer k corresponds to 2^{k+1} rectangles arranged in two columns. The integer 1 corresponds to the two leftmost and two rightmost rectangles in Figure 3. The integer 2 corresponds to the eight rectangles in the two columns adjacent to the first two, and so on. More precisely, the 2^{k+1} rectangles associated to the integer k are the components of $A_k \times K_n$. The set A_k has two components and K_n has 2^n components, giving the correct number of rectangles in the product. Each rectangle is then connected to three other rectangles in the two adjacent columns, and to one other rectangle in the same column, all as shown in Figure 3. (Slightly different arcs are used to connect the outermost rectangles to each other, as shown by dashed vertical segments at the far left and far right of Figure 3.) For templates G_J other than G_0 the construction is exactly the same, except that K_n is replaced by $K_{n+j} \cap J$ if $J = K_j^k$. This is done so that the limiting Cantor sets are all translates of the same fixed set K.

Given these templates, we construct a Jordan curve Γ as an intersection $\Gamma = \cap_n \Gamma_n$ of compact connected sets each consisting of a countable union of rectangles, polygonal arcs and copies of the Cantor set K. The steps of the construction are controlled by the choice of a rooted tree T in $X_T \subset 2^{\mathbb{N}}$, and is designed so that Γ will be A-removable if and only if T is well-founded.

So suppose T is fixed. The construction always starts with a copy of G_0 that has two short polygonal arcs added at the far left and far right, to join the upper and lower halves of the template set, making it connected. These are shown as dashed segments in Figure 3, but occur as solid lines in several of the following figures.

We will induct over levels of the tree, starting at the root vertex (labeled by the empty string) and at each stage of the construction, we will have a set Γ_n consisting of a countable collection of rectangles joined by polygonal arcs and accumulating on translates of the set K. At the nth stage, each rectangle $R = I \times J$ is labeled by a n-long string of positive integers that is a label of some vertex v of the tree T. To go from Γ_n to Γ_{n+1}, we replace each rectangle R in Γ_n by a rescaled copy of the template G_J (rescaled affinely to exactly fit into R). If vertex v is a leaf of T (i.e., it has no children), then every rectangle R' in the rescaled copy of the template is replaced by a pair of horizontal line segments that connect the vertical sides of R' exactly at the points where arcs of the template connect R' to other rectangles in the template. If v is not a vertex then there is a set of positive integers that when appended to the label of v give labels of its children. For the template rectangles corresponding to these integers we leave the rectangle alone. For the other integers (those that do not correspond to children of v), we replace the corresponding rectangles with horizontal line segments, as above. Doing this for every rectangle in Γ_n gives a closed connected subset $\Gamma_{n+1} \subset \Gamma_n$.

The simplest case is when the tree T has only one vertex (labeled by the empty string). Then every rectangle of the template G_0 is replaced by pair of horizontal segments. The result is illustrated in Figure 4. Here, Γ is a closed Jordan curve that is a countable union of polygonal arcs and two copies of the Cantor set K, and is clearly an A-removable set.

The next easiest case is when we have a rooted tree with two vertices, say with root labeled by the empty string and the single leaf labeled by "1". If we replace the

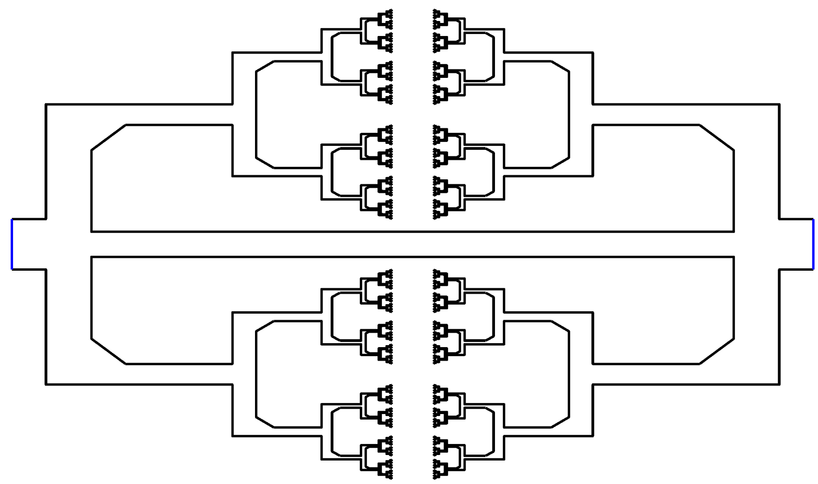

FIGURE 4. The curve corresponding to the single vertex rooted tree, labeled by the empty string. This is a countable union of line segments and two linear Cantor sets and hence is A-removable. It is the "simplest" curve in our collection.

four rectangles in G_0 that correspond to the integer "1" with rescaled copies of G_0, the result is shown in Figure 5. Any curve corresponding to a tree that contains the edge connecting the root to vertex "1", will be a subset of the illustrated set. When T consists only of this one edge, then every rectangle in Figure 5 is replaced by a pair of horizontal edges, giving the closed Jordan curve shown in Figure 6. If the second vertex was labeled "k" instead, the replacements would occur in corresponding columns of the template.

Finally, we have to observe that the resulting curve is A-removable if and only if the associated tree T is well-founded. If T is well-founded, then the final curve is a countable union of line segments and linear Cantor sets and hence is A-removable by one direction of Carleson's theorem. If T has an infinite branch then the curve contains a copy of $E \times K$, where E is a Cantor set depending on the branch, and thus it is non-A-removable by other direction of Carleson's theorem.

Next, we will verify that the map from trees to curves is continuous from the product topology on $X_T \subset 2^{\mathbb{N}}$ to the Hausdorff metric on 2^S. Recall that each tree is encoded by a binary sequence in $2^{\mathbb{N}}$ whose nth coordinate indicates whether nth

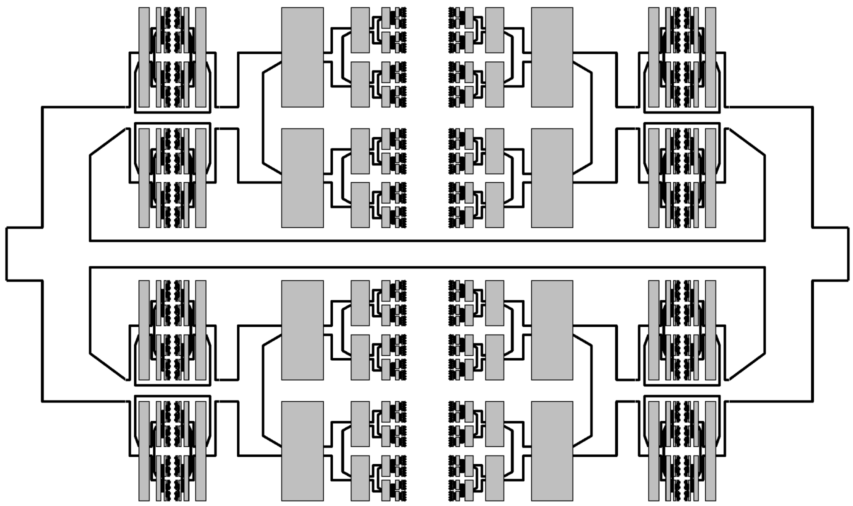

FIGURE 5. The four rectangles corresponding to "1" in the template have been replaced by rescaled copies of the template. Any curve containing the vertices $\{\emptyset, 1\}$ will contain these arcs.

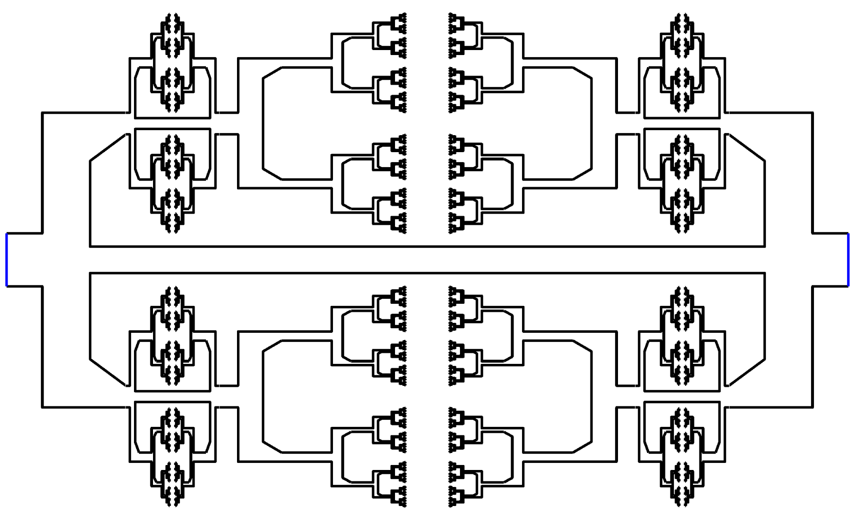

FIGURE 6. The curve corresponding to the tree with vertices $\{\emptyset, 1\}$.

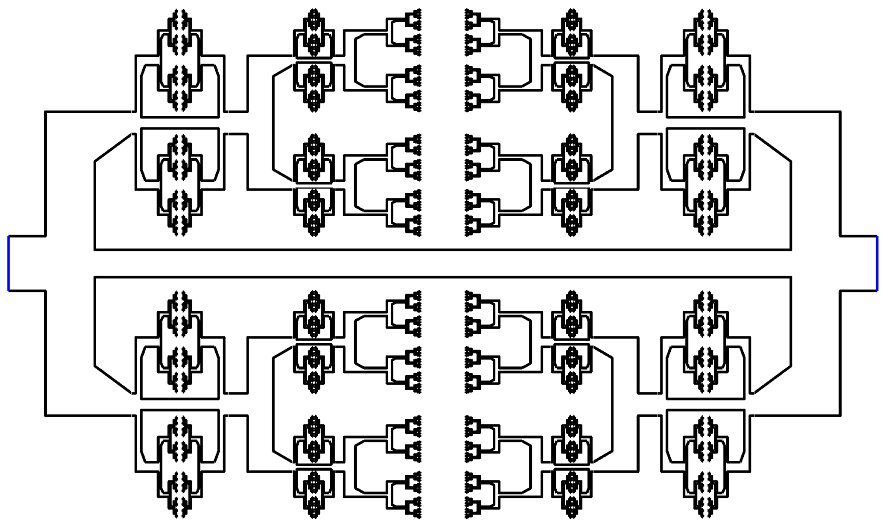

FIGURE 7. The curve corresponding to the tree with vertices $\{\emptyset, 1, 2\}$. There are countably many segments and 10 copies of the linear Cantor set K.

string (according to some fixed enumeration of $\mathbb{N}^*$) is the label of some vertex of the tree. If the encodings of two trees T_1, T_2 agree for the first N places, then the two corresponding curves share the same templates for these vertices, and can only disagree within the rectangles that are filled in later in the construction. However, each time we apply a template, the rectangles the occur inside the replaced rectangle have smaller diameter (tending to zero with both the length of the corresponding string label, and the size of the last entry of the string). Thus the curves corresponding to T_1 and T_2 agree except within a union of disjoint rectangles that each have small diameter, and so that each curve contains some point in each rectangle. Thus the Hausdorff distance between the curves is at most ℓ_N. Therefore, the set of well-founded trees is the preimage of the set of A-removable curves under a continuous map from X_T into the hyperspace of $[-1, 1]^2$. Hence this collection of A-removable curves is co-analytic complete and, in particular, it is not Borel. $\square$

9. CH-removable Jordan curves are co-analytic complete

The logarithmic capacity, $C_{\log}(E)$, of a set $E \subset [0,1]$ is defined as the supremum of the masses of positive measures μ supported on E so that the convolution $\mu * \log \frac{1}{x}$ is bounded above by 1 on E (hence everywhere). This agrees with the definition in Carleson's book [16], but it disagrees with some other sources, such as [24], that define logarithmic capacity as $\exp(-1/C_{\log}(E))$, and call $1/C_{\log}(E)$ the Robin's constant of E. Both definitions give the same sets of zero capacity, but we prefer Carleson's approach here, as his version is sub-additive and the other is not. In [1], Ahlfors and Beurling show that $K \subset [0,1]$ is S-non-removable iff $C_{\log}([0,1] \setminus K) < C_{\log}([0,1])$, and that this implies that K has positive length (but not conversely). This result is the basis for the following theorem.

Theorem 9.1. *The collection of CH-removable Jordan curves contained in $S = [0,1]^2$ is co-analytic complete in 2^S.*

Proof. In [63] Jang-Mei Wu proves that if $K \subset [0,1]$ is a Cantor set with the property that the logarithmic capacity of $[0,1] \setminus K$ is strictly less than the logarithmic capacity of $[0,1]$, and if E is any Cantor set, then $E \times K$ is CH-non-removable. By the result of Ahlfors and Beurling noted above, this is same as saying K is S-non-removable.

Since any uncountable closed set contains a Cantor set, we can use Wu's theorem and half of Lemma 6.2 to deduce that for a closed set $E \subset [0,1]$, the product $E \times K$ is CH-removable if and only if E is countable. Thus if we use such a Cantor set K in the proof of Theorem 8.1, we obtain Theorem 9.1. (We also need to show these curves form a co-analytic set, but this is essentially the same as for A-removable curves.)

To finish the proof, we construct a Cantor set K with the desired properties. Start with $K_0 = [0,1]$ and remove an open interval of length a_0 centered at $1/2$, leaving two closed intervals as K_1. In general, remove a centered, open interval of length a_n from each component of K_n to obtain K_{n+1}, and let $K = \cap_n K_n$. One can easily show the logarithmic capacity of an interval of length r is comparable to $1/|\log r|$, so the sub-additivity of logarithmic capacity (see Lemma 4 of [16]), implies the logarithmic capacity of $[0,1] \setminus K$, is at most $O\left(\sum_{n=0}^{\infty} 2^n/|\log a_n|\right)$, which is as small as we wish if a_0 is small and we take $a_n \searrow 0$ fast enough. $\square$

As noted at the end of the introduction, this result is due to Dimitrios Ntalampekos, who pointed out that the proof of Theorem 8.1 applies to CH-removable curves if we simply take the Cantor set K to be one of the CH-non-removable sets constructed by Jang-Mei Wu in [63].

The proof of Wu's result in the general case is perhaps too long to replicate fully here, but for the sake of completeness, we will sketch the construction of a single Cantor set K with the property that $E \times K$ is CH-non-removable for any Cantor set E. This is sufficient for a self-contained proof of Theorem 9.1. We roughly follow Wu's proof in [63] for the general case, but several steps simplify for our set (and we do not need to recall as much potential theory).

Lemma 9.2. *There is a Cantor set $K \subset [0,1]$ so that for a compact set E, the product $E \times K$ is CH-non-removable if and only if E is uncountable.*

Proof. If E is countable, then the product is removable by Lemma 7.1, so we only need to prove the other direction.

We start by building a sequence of nested compact sets $H_0 \subset H_1 \subset \cdots \subset \mathbb{C}$, so that each set consists of finite number of horizontal line segments, each centered on the y-axis. For each segment I in H_n, there will be a segment J on the y-axis centered where I crosses the y-axis, and so that J hits no other points of H_n. To begin the construction, we let H_0 be just the single segment $I = [-1, 1] \subset \mathbb{R}$ and let the associated vertical segment be $J = [-i/2, i/2]$. In general, given a non-trivial segment I in H_n for $n \geq 2$, we can define its associated vertical segment J as follows. If I hits $i\mathbb{R}$ at iy, and if $\delta_I = \text{dist}(I, H_n \setminus I)$, then we can take $J = i \cdot [y - \delta_I, y + \delta_I]$.

Next, let J' be the vertical segment concentric with J and one third the length. Note that the collection of these smaller intervals J' from a single generation is pairwise disjoint, and any two of them are separated by a open interval at least as long as the longer of the two. Let R denote the rhombus that is the convex hull of $I \cup J'$, and add $2n$ horizontal segments with endpoints on the boundary of R, with heights evenly spaced over the top and bottom halves of J'. The process is illustrated in Figure 8. We define H to be the closure of $\cup_{n=0}^{\infty} H_n$. Note that the vertical intervals corresponding to two adjacent horizontal segments in H_n are not just disjoint, but are separated by a non-trivial open interval which does not hit H_m for any m, and thus misses H. Thus $i\mathbb{R} \setminus H$ is open and dense, so the horizontal projection of H onto $i\mathbb{R}$ is a Cantor set.

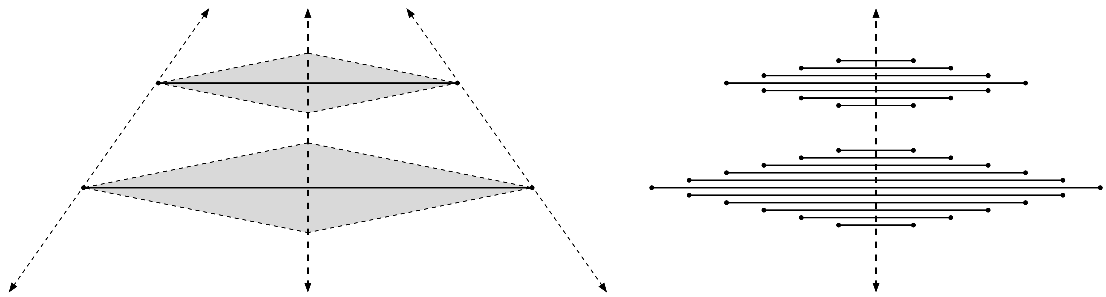

FIGURE 8. The left side shows two adjacent horizontal segments of H_n. The vertical dashed line is the y-axis, the two diagonal dashed lines are edges of a rhombus from the previous generation. The shaded rhombuses are those corresponding to the two segments. On the right, we show the horizontal segments of H_{n+1} defined using these rhombuses. In the limit, we obtain a set H which contains uncountably many horizontal line segments, all of which have zero harmonic measure from ∞.

Let $\mathbb{H}_r = \{x+iy : x > 0\}$ denote the right half-plane, and let $\mathbb{H}_l = \{x+iy : x < 0\}$ be the left half-plane. Define $\Omega = \mathbb{H}_r \setminus H$. This domain is simply connected, and so, by the Riemann mapping theorem, it can be mapped to $\mathbb{H}_r$ by a conformal map f that fixes ∞ and $\pm i$. Let $f_n : \mathbb{H}_r \to \mathbb{H}_r \setminus H_n$. Then $f_n \to f$ uniformly on compact sets of $\mathbb{H}_r$, and so the same is true for their derivatives. It is easy to check that if $f_n = u_n + iv_n$, then $v_n(iy)$ is increasing on the preimages of $i\mathbb{R} \setminus H_n$ and constant on the preimage of each segment in H_n. Thus u_n has normal derivative > 0 on the former set and $= 0$ on the latter. This implies $\mathrm{Re}(f_n') > 0$ on $\mathbb{H}_r$, and hence the same is true for f (the limit can't be zero, for then f would be constant by Harnack's inequality). By Schwarz reflection, f can be extended to a conformal map of $\Omega = \mathbb{C} \setminus H$ to a domain W whose boundary is a Cantor set K on the y-axis. This is the set K we are seeking.

We claim that K is a Cantor set. It is clear that K is closed and has no isolated points, so we need only show its connected components are all points. To prove this, suppose I is a component of H_n. It is easy to check from the definitions that the distance from I to the closest distinct component of H_n is less than $1/n$ (actually it is much smaller, less than $1/n!$), and that adjacent components of H_n have length differing by at most $1/n$. Thus each component I of H_n can be separated from ∞ in $\mathbb{H}_r$ by a crosscut γ_I of $\mathbb{H}_r$ that lies inside $\Omega_n = \mathbb{C} \setminus H_n$, and such that γ_I can be separated from ∞ by a crosscut σ_I of Ω_n with diameter at most $4/n$. For point components x of H_n we can take $\sigma_x = \gamma_x$, and for segment components I, we choose σ_I to connect endpoints of components of H_n that are adjacent to I, as illustrated in Figure 9.

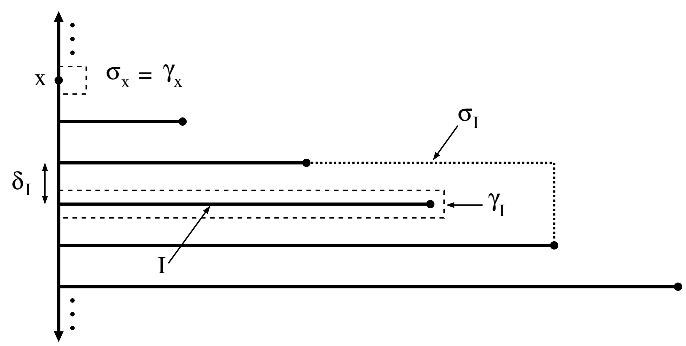

FIGURE 9. Each component I of H_n is separated from infinity by a crosscut γ_I of small harmonic measure, which implies its f-image has small harmonic measure, hence small diameter in $\mathbb{H}_r$. Reflecting γ_I across the y-axis gives a loop around I in $\mathbb{C} \setminus H_n$, and an infinite nested family of such loops defines a unique connected component of H. The f-images of these loops have diameters tending to zero, implying every connected component of K is a point, and giving a 1-to-1 correspondence between points of K and connected components of H.

By the maximum principle, the harmonic measure of γ_I (from the point 2) is at most the harmonic measure of σ_I, and the latter is bounded $O(\sqrt{\mathrm{diam}(\sigma_I)}) = O(n^{-1/2})$ by corollary of Beurling's projection theorem (e.g., Corollary III.9.3 of [24]). Thus by conformal invariance, K can be separated from $f(2)$ in $\mathbb{H}_r$ by a finite set of crosscuts in $\mathbb{H}_r$ with endpoints outside K (namely, the f-images of γ_I), each of which has small harmonic measure, hence small diameter. This implies K only has point components, proving the claim.

We let $F = f^{-1}$ denote the inverse conformal map from $W = \mathbb{C} \setminus K$ to Ω. The argument in the previous paragraph also shows that F associates each point of K to a component of H. More precisely, if $z_n \to iy \in K$ then $F(z_n)$ can only accumulate on the associated component of H. In particular, $\{\mathrm{Im}(F(z_n))\}$ has a well defined limit, even if the real parts do not. Note that by symmetry

$$\mathrm{Im}(F(x+iy)) = \mathrm{Im}(F(-x+iy)). \tag{9.1}$$

In particular, F restricted to $\mathbb{R}\setminus\{0\}$ is a continuous, strictly increasing function, and

$$-\lim_{x\nearrow 0} F(x) = \lim_{x\searrow 0} F(x) = 1, \tag{9.2}$$

i.e., F has a jump of size 2 at the origin. The fact that F associates points of K to components of H means that at other points of the y-axis, F can only have a non-negative jump in the following sense: for any real number M,

$$\lim_{x\searrow 0} \mathrm{Im}\left[F(x+iMx+iy)) - F(-x-iMx+iy)\right] = 0. \tag{9.3}$$

The existence of F implies K is S-non-removable, and hence K has positive length by the result of Ahlfors and Beurling mentioned just before Theorem 9.1. However, we can give a direct proof of this as follows. Inside $\Omega^+ = \mathbb{H}_r \setminus H$, the set H has positive harmonic measure (the choice of base point in Ω^+ is unimportant, but to be concrete, we take $z_0 = 2$). To see this, observe that $I = [1/2, 1]$ has positive harmonic measure in $\mathbb{H}_r \setminus I$, and since $I \subset H$, the maximum principle implies H has positive harmonic measure in Ω. Thus by conformal invariance, K has positive harmonic measure in $\mathbb{H}_r$, and thus it has positive length.

Suppose E is any Cantor set. We claim that $E \times K$ is CH-non-removable. By restricting to a subset and translating we may assume $E \subset [0, 1/10]$, and that E has zero length (if E has positive length, there is nothing to do since $E \times K$ has positive area, and so it is CH-non-removable by the measurable Riemann mapping theorem).

Since E is uncountable, it supports a non-atomic probability measure μ. Let

$$G(z) = \int_E F(z+t)d\mu(t).$$

It is easy to verify that G is continuous everywhere (it is the convolution of a locally bounded function and a non-atomic measure) and is holomorphic off $E \times K$.

If G were a homeomorphism, and if $E\times K$ were CH-removable, then G extends to a conformal homeomorphism of the plane to itself, and hence it would be a linear map. However, it follows from (9.2) that F restricted to $\mathbb{R}$ can be written as the sum of a continuous, strictly increasing function F_1 and the jump function $F_2(x) = 1+\mathrm{sign}(x)$. The convolution of μ with the step function F_2 is continuous and singular: it maps the zero length E to a set of positive length. Adding the strictly increasing convolution of μ with F_1 preserves this property, hence $G = F * \mu$ is not a linear map. This contradiction completes the proof of non-removability, once we know that G is a homeomorphism.

It suffices to show that if $z \neq w$, then $G(z) \neq G(w)$. This follows if

$$\mathrm{Re}\left(\frac{G(z)-G(w)}{z-w}\right) > 0.$$

In fact, we will prove a uniform estimate

$$\mathrm{Re}\left(\frac{G(z)-G(w)}{z-w}\right) \geq \eta(|z-w|) > 0.$$

where the lower bound only depends on the distance between z and w. Since G is continuous, we only need to prove such a bound for a dense set of pairs so we may assume $\mathrm{Re}(z) \neq \mathrm{Re}(w)$ and neither z nor w is in $E \times \mathbb{R}$. Since $G(z)$ is a convex combination of the values $\{F(z+t)\}$, $t \in E$, it suffices to show that

$$\mathrm{Re}\left(\frac{F(z)-F(w))}{z-w}\right) \geq \eta(|z-w|) > 0$$

when $\mathrm{Re}(z) < \mathrm{Re}(w)$ and neither real part is zero. If the segment $I = [z, w]$ connecting z and w does not hit K, then F is analytic on a neighborhood of I and

$$\mathrm{Re}\left(\frac{F(z)-F(w)}{z-w}\right) = \int_0^1 \mathrm{Re}(F'(z+t(w-z)))dt. \tag{9.4}$$

The integral on the right is positive since $\mathrm{Re}(F') > 0$ off K. Moreover, the integral is bounded uniformly away from zero depending only on the length of the segment I. (This uses that $\mathrm{Re}(F')$ is positive off the Cantor set K, and that it has a positive limit at ∞.) The other possibility is that S crosses the imaginary axis at $iy \in K$. Set $I_x = I \setminus \{|\mathrm{Re}(z)| < x\}$. Then we have

$$\begin{aligned}\mathrm{Re}\left(\frac{F(z)-F(w)}{z-w}\right) \quad &\geq \quad \lim_{x \searrow 0} \int_{I_x} \mathrm{Re}(F'(z+t(w-z)))dt \\ &\quad + \liminf_{x \searrow 0} [F(x+iMx+iy)) - F(-x-iMx+iy)]\end{aligned}$$

where M is the slope of S (note that $M \neq \infty$ since we assumed $\mathrm{Re}(z) \neq \mathrm{Re}(w)$). The integral over I_x is bounded away from zero for x small, since one of its two components has length greater than $|I|/4$ for small x, and by (9.4) this gives a positive lower bound. By (9.3) the limit infimum in the second term is a non-negative real number. This gives the desired lower bound, and therefore G is a homeomorphism, completing the proof that $E \times K$ is CH-non-removable. $\square$

Dimitrios Ntalampekos observed that the arguments of Ahlfors and Beurling in [1] can be used to prove that the complement S-non-removable compact set $K \subset i\mathbb{R}$ can be conformally mapped to the complement of a compact set H whose connected components are either points or non-trivial horizontal slits, and that some non-trivial slits must occur. Therefore, the proof given above for a single, explicit K could be used to prove Wu's theorem in general, although this would require invoking the aforementioned results of Ahlfors and Beurling.

10. How hard is conformal welding?

We recall some definitions from the introduction. If Γ is a closed Jordan curve in the plane, the Riemann mapping theorem gives conformal maps f and g from the inside and outside of the unit circle to the inside and outside of Γ. By Carathéodory's theorem[3] these maps extend to be homeomorphisms of $\mathbb{T}$ to Γ. Thus $h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$ is a homeomorphism, and circle homeomorphisms that arise in this way are called conformal weldings.

Not every homeomorphism is a welding. In [49], Oikawa proved that if $h : \mathbb{T} \to \mathbb{T}$ is given by $h(\exp(i\theta)) = \exp(i(2\pi)^{1-\alpha} \cdot \theta^\alpha)$ for some $0 < \alpha < 1$ and $0 \leq \theta < 2\pi$, then h is not a conformal welding of the circle.

A more geometric example can be described as follows. Consider the graph of $\sin(1/x)$ for $x \neq 0$, together with the limiting segment $[-i, i]$. See Figure 10. This is closed set X dividing the plane into two simply connected domains and one can show that the conformal maps form either side of $\mathbb{T}$ to either side of X still define a circle homeomorphism h. Moreover, we can choose f and g so that $1 \in \mathbb{T}$ corresponds to the prime end $[-i, i]$ under both maps, and hence h fixes this point.

However, h cannot correspond to any Jordan curve Γ; if it did, one could conformally map the two sides of X to the two sides of Γ so that the maps agree along the graph of $\sin(1/x)$. Since this curve is removable for conformal homeomorphisms the map extends to be conformal from the complement $[-i, i]$ to the complement of a point. Since the complement of the segment is conformally equivalent to the

[3]This result was actually first proven by Carathéodory's student Marie Torhorst in her 1918 doctoral dissertation using Carathéodory's theory of prime ends, so perhaps it is more appropriate to call it the Carathéodory-Torhorst theorem; see [54] for some of the history.

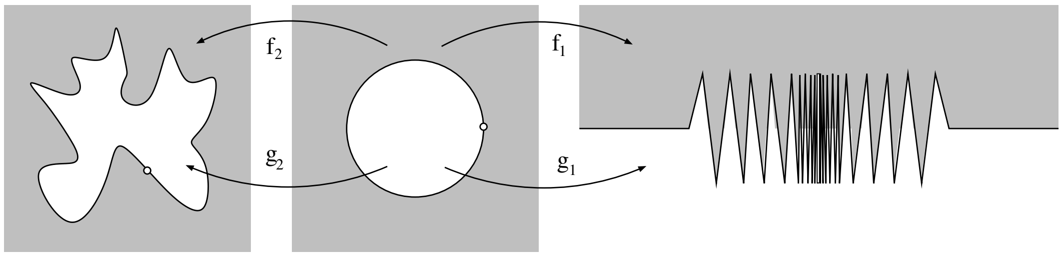

FIGURE 10. An example of a non-welding homeomorphism. If f_1, g_1 map the two sides of $\mathbb{T}$ to the two sides of a $\sin(1/x)$ curve γ, then $h = g_1^{-1} \circ f_1$ is a homeomorphism, but is not a conformal welding, as explained in the text.

unit disk, we would get conformal map between the disk and the plane, which would violate Liouville's theorem. Thus this homeomorphism is not a conformal welding.

It is a long standing, and apparently very difficult, problem to characterize conformal weldings among circle homeomorphisms. We explained in Section 8 that circle homeomorphisms are a G_δ set in $C(\mathbb{T}, \mathbb{T})$, and hence a Polish space.

Question 1. *Are conformal weldings Borel in the space of circle homeomorphisms? Are non-weldings co-analytic complete?*

It is not hard to prove that weldings form analytic subset of circle homeomorphisms, so non-weldings are co-analytic. The difficult part seems to be to construct a Borel map from a Polish space into circle homeomorphisms, so that the preimage of the non-weldings consists of known co-analytic complete set. For example, is there a Borel map from compact sets of $\mathbb{T}$ to circle homeomorphisms so that the preimage of the non-weldings are the countable compact sets? We saw above that it is possible to construct a circle homeomorphisms that has an "obstruction" at just one point. Also, it is known (see Theorem 2 of [9]) that any circle homeomorphism h can be written as $h = f^{-1} \circ g$ where f and g are conformal maps of $\mathbb{D}$ and $\overline{\mathbb{D}}^c$ onto disjoint simply connected domains, with equality holding everywhere except on a set $E \subset \mathbb{T}$ so that E and $h(E)$ both have logarithmic capacity zero. Can this be improved to a countable exceptional set? If so, can this be used to give a map from non-weldings to countable subsets of $\mathbb{T}$? Such a map is going in the wrong direction to prove that non-weldings are co-analytic complete, but it would still be very interesting to understand if such a map exists and what its properties are.

The best known sufficient condition for being a conformal welding (due to Pfluger [51]) is quasisymmetry (QS for brevity): $h : \mathbb{T} \to \mathbb{T}$ is M-quasisymmetric if

$$\frac{1}{M} \leq \frac{|h(I)|}{|h(J)|} \leq M,$$

whenever I, J are adjacent arcs on $\mathbb{T}$ of the same length, and $|I|$ denotes the length of an arc. A map is quasisymmetric if it is M-quasisymmetric for some finite M. For a fixed M, M-quasisymmetry is clearly a closed condition (with respect to uniform convergence), so taking $M \to \infty$ along the integers shows quasisymmetric homeomorphisms are a F_σ set inside $\mathrm{Homeo}(\mathbb{T}, \mathbb{T})$. Quasisymmetric weldings correspond precisely to closed curves that are quasicircles, i.e., images of the unit circle under quasiconformal maps of the plane. There are numerous characterizations of this class of curves, including the following: any two points $z, w \in \gamma$ are connected by a subarc with diameter bounded by $O(|z - w|)$.[4] It is easy to see M-quasisymmetric maps are nowhere dense, so the set of quasisymmetric homeomorphisms is meager the space of all circle homeomorphisms. A set is meager if it is a countable union of nowhere dense sets. Such sets are also called "first category", although this usage is becoming less common. A set is called residual if it is the complement of a meager set. Trivially, subsets of meager sets are meager, and supersets of residual sets are residual.

A more recent (and somewhat less well known) sufficient condition to be a conformal welding is for h to be log-singular, i.e., that there exist a set $E \subset \mathbb{T}$ of logarithmic capacity zero so that $\mathbb{T} \setminus f(E)$ also has logarithmic capacity zero. See [9]. Quasisymmetric and log-singular circle homeomorphisms are easily seen to be disjoint sets (e.g., QS homeomorphisms preserve sets of zero logarithmic capacity). Recently, Alex Rodriguez proved that any circle homeomorphism is the composition of two log-singular homeomorphisms, and hence any circle homeomorphism is the composition of two conformal weldings [55]. However, his proof decomposes even "nice" homeomorphisms as the composition of two highly singular maps. Is this necessary?

[4]According to page 84 of Lehto's biography [41] of Alhfors, this was first proved in Martti Tienari's 1962 dissertation, and independently by Ahlfors. Lehto quotes Ahlfors as saying "I have to confess that when I first proved the result, I thought it was too good to be true"

Can a homeomorphism with some given modulus of continuity be decomposed into welding with similar estimates?

Question 2. *Is any bi-Hölder circle homeomorphism the composition of bi-Hölder welding maps?*

Some other classes of circle homeomorphisms that are known to be conformal weldings were described by David [19] and Lehto [40] (actually they describe conditions on a measurable function μ on the unit disk so that the Beltrami equation $f_{\overline{z}} = \mu f_z$ has a homeomorphic solution, and the boundary values of these solutions are the circle homeomorphisms I am referring to). See also Chapter 20 of [4].

Question 3. *Is the collection of David homeomorphisms a Borel set within the space of circle homeomorphisms? The collection of Lehto homeomorphisms?*

If γ is a closed Jordan curve with complementary components Ω_1, Ω_2, we say $x \in \gamma$ is rectifiably accessible from Ω_k, for $k = 1, 2$, if it is the endpoint of a rectifiable curve in Ω_k. By a result of Gehring and Hayman (see [26] or Exercise III.16 of [24]) this occurs iff a hyperbolic geodesic ray ending at x has finite Euclidean length. A result of Charles Pugh and Conan Wu [52] says there is a residual set of closed curves γ so that no point on γ is rectifiably accessible from both sides at once. In their terminology, γ is not pierced by any rectifiable arc. See [12] for an explicit construction of an extreme example of such a curve γ (any rectifiable curve crossing γ intersects γ in positive length). By a result of Beurling, the set of points that are not rectifiably accessible from Ω_k, $k = 1, 2$ is the image of a zero logarithmic capacity set on $\mathbb{T}$ under any conformal map $\mathbb{D} \to \Omega_k$ (see [8], Exercise III.23 of [24], or [5]). If γ is not pierced by any rectifiable curve, let E be the set of boundary points that are rectifiably accessible from Ω_1. Then no point of E can be rectifiably accessible from Ω_2, and by Beurling's theorem, the image of E has zero logarithmic capacity under the Riemann map from Ω_2 to the disk, and $\gamma \setminus E$ has zero capacity under the Riemann map for Ω_1. Therefore every rectifiably non-pierceable curve of Pugh and Wu has a conformal welding that is log-singular. Theorem 3 of [9] states that h is log-singular if and only if the corresponding curve is flexible; this means that the set of curves corresponding to h is dense in the space of all closed curves with the Hausdorff metric. See [9] for the precise definition. Thus the set of curves with a given log-singular welding is dense in the space of all closed Jordan curves, and hence is CH-non-removable in a strong way. Therefore we have the following result.

Theorem 10.1. *The collection of CH-non-removable closed curves is residual in the space of all closed Jordan curves.*

Very recently, Rodriguez [56] has proven that every log-singular circle homeomorphism is the welding of a collection of curves that includes curves of every Hausdorff dimension on $[1, 2]$, and even a curve of positive area. If a curve has positive area, then by scaling a non-zero dilatation supported on the curve, we can use the measurable Riemann mapping theorem to produce a 1-parameter family of non-removable curves, none of which is a Möbius image of the others. In particular, this gives uncountably many curves with the same welding, so that no two of them are Möbius images of each other. Given the result above for curves, it is natural to ask the analogous question for circle homeomorphisms.

Question 4. *Is the set of log-singular homeomorphisms residual in the space of all circle homeomorphisms?*

Question 5. *What is the Borel complexity of the log-singular homeomorphisms?*

It is not hard to show that both sets are analytic: h is log-singular if for every $n \in \mathbb{N}$ there is a compact set such that both E and $h(E^c)$ have logarithmic capacity less than $1/n$ (Lemma 11 of [9]). Thus the log-singular maps are a countable intersection of projections of the Borel sets $\{(h, E) : \text{cap}(E), \text{cap}(h(E^c)) < 1/n\}$ in $\text{Homeo}(\mathbb{T}, \mathbb{T}) \times 2^{\mathbb{T}}$. Can analytic be improved to Borel?

Recall that we say Γ' is a CH-image of Γ if $\Gamma' = f(\Gamma)$ where f is a homeomorphism of the sphere that is conformal off Γ. We will say this is a strict CH-image if f is not a Möbius transformation, and say it is a very strict CH-image if $f(\Gamma)$ is not a Möbius image of Γ. It is tempting to say that a strict image is also very strict, but this might not be true. Maxime Fortier Bourque pointed out that the image of Γ under a non-Möbius homeomorphism of the sphere might coincidentally agree with its image under some Möbius map. Moreover, using log-singular weldings, Malik Younsi [65] constructed a curve with a strict CH-image that agrees with itself. In Younsi's example, there are also very strict CH-images that are not Möbius images, so it is still possible that a very strict image exists whenever a strict image does.

Question 6. *Is the map from (equivalence classes of) curves to (equivalence classes of) conformal weldings* 1*-to-*1 *exactly on the CH-removable curves?*

I expect this is true. The following is a stronger version.

Question 7. *Does every CH-non-removable curve have a CH-image of positive area?*

More generally, does this hold for all CH-non-removable sets? It does for all examples known to the author. Various other questions about weldings and CH-removable curves remain open.

Question 8. *Is the map from equivalence classes of curves to equivalence classes of weldings always either* 1*-to-*1 *or uncountable-to-*1*?*

Question 9. *Are CH-images of a curve a connected set in the Hausdorff metric?*

Question 10. *Is there a 1-parameter family of zero-area, non-CH-removable curves that is continuous in the Hausdorff metric, so that no element is a Möbius image of any other member of the family?*

Question 11. *The CH-images of a flexible curve are dense in the space of closed Jordan curves, and hence are not a closed set. Is this set of curves Borel? (It must be analytic.) Is it connected? Can it be totally disconnected? (Not if the answer to Question 7 is yes.)*

11. What are natural ranks for removable sets?

This section requires greater familiarity with the transfinite ordinals than did earlier sections. Very briefly, each ordinal is a well ordered set (each element has a successor, although some elements have no predecessor). The ordinals themselves are well ordered and there is a first well ordering of an uncountable set, which is denoted ω_1. Every ordinal that becomes before ω_1 is, by definition, the well ordering of some countable set. The continuum hypothesis is the claim that $\omega_1 = c$, where c is the cardinality of $\mathbb{R}$, and is well known to be independent of ZFC.

If X is Polish and $A \subset X$ is co-analytic, then there is always a co-analytic rank on A. This is a function ρ on X that assigns each point of X to some ordinal $\leq \omega_1$ and such that

(1) $A = \{x \in X : \rho(x) < \omega_1\}$,
(2) $\{(x, y) \in A \times A : \rho(x) < \rho(y)\}$ is co-analytic in $X \times X$,
(3) $\{(x, y) \in A \times A : \rho(x) \leq \rho(y)\}$ is co-analytic in $X \times X$.

Given such a function ρ, one can show that for every countable ordinal α, every set $A_\alpha = \{x \in A : \rho(x) \leq \alpha\}$ is a Borel set, and every analytic subset of A is contained in some A_α. Moreover, A is Borel if and only if every co-analytic rank of A is bounded above by some countable ordinal.

The standard example (dating back to Cantor and motivating his invention of transfinite ordinals) involves the derived sets of a compact set in $\mathbb{R}$. Given a compact K, the derived set K' is K with its isolated points removed; this is a compact subset of K, with at most countably many points removed. If K was finite then $K' = \emptyset$, and otherwise we can repeat the process to get the second derived set K''. Continuing, we get a nested sequence of sets that either becomes empty after $n < \infty$ steps (in which case we set $\rho(K) = n$) or we get an infinite, strictly decreasing sequence of nested compact sets whose intersection is a non-empty compact set K^ω. If the derived set of K^ω is empty, then set $\rho(K) = \omega$, and otherwise continue as before. We proceed with this using transfinite induction. If K is countable, then since we remove at least one point at each stage, we must reach the empty set at some countable ordinal, and take this ordinal to be the rank of K. Since we remove only countably many points at each stage, starting with an uncountable set never gives the empty set at any countable ordinal. For such sets the rank is defined to be ω_1. This defines a rank for the co-analytic set of countable, compact subsets of $[0, 1]$.

In [37] Kechris and Woodin describe a natural rank on the set of everywhere differentiable functions in $C([0, 1])$. See also [38], [39], [53], for comparisons between their rank and other ranks on the same set. A thesis of [37] is that "natural" co-analytic sets should have natural ranks.

Question 12. *What is a natural rank on the space of conformally removable sets?*

For the special case of product sets $E \times [0, 1]$ with E countable, we can just take the usual rank on countable compact sets described above using derived sets.

Question 13. *Can the derived set rank on $E \times [0, 1]$ be extended to a co-analytic rank on all removable sets in $S = [0, 1]^2$?*

References

[1] Lars Ahlfors and Arne Beurling. Conformal invariants and function-theoretic null-sets. *Acta Math.*, 83:101–129, 1950.

[2] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard.

[3] Miklos Ajtai and Alexander S. Kechris. The set of continuous functions with everywhere convergent Fourier series. *Trans. Amer. Math. Soc.*, 302(1):207–221, 1987.

[4] Kari Astala, Tadeusz Iwaniec, and Gaven Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*, volume 48 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2009.

[5] Zoltan Balogh and Mario Bonk. Lengths of radii under conformal maps of the unit disc. *Proc. Amer. Math. Soc.*, 127(3):801–804, 1999.

[6] Howard Becker. Descriptive set-theoretic phenomena in analysis and topology. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 1–25. Springer, New York, 1992.

[7] Abram S. Besicovitch. On Sufficient Conditions for a Function to be Analytic, and on Behaviour of Analytic Functions in the Neighbourhood of Non-Isolated Singular Points. *Proc. London Math. Soc. (2)*, 32(1):1–9, 1931.

[8] Arne Beurling. Ensembles exceptionnels. *Acta Math.*, 72:1–13, 1940.

[9] Christopher J. Bishop. Conformal welding and Koebe's theorem. *Ann. of Math. (2)*, 166(3):613–656, 2007.

[10] Christopher J. Bishop and Yuval Peres. *Fractals in probability and analysis*, volume 162 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.

[11] Andrew M. Bruckner, Judith B. Bruckner, and Brian S. Thomson. *Real Analysis, Second Edition*. ClassicalRealAnalysis.com, 2008.

[12] Jack Burkart. A Jordan curve that cannot be crossed by rectifiable arcs on a set of zero length. 2020. Preprint, https://arxiv.org/abs/2011.04831.

[13] A.E. Caicedo, J.D. Hamkins, and G. Lowther. Non-Borel sets without axiom of choice. Math Overflow. URL: https://mathoverflow.net/questions/32720/non-borel-sets-without-axiom-of-choice.

[14] Alberto P. Calderón. Cauchy integrals on Lipschitz curves and related operators. *Proc. Nat. Acad. Sci. U.S.A.*, 74(4):1324–1327, 1977.

[15] Lennart Carleson. On null-sets for continuous analytic functions. *Ark. Mat.*, 1:311–318, 1951.

[16] Lennart Carleson. *Selected problems on exceptional sets*. Van Nostrand Mathematical Studies, No. 13. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[17] Roger Cooke. Uniqueness of trigonometric series and descriptive set theory, 1870–1985. *Arch. Hist. Exact Sci.*, 45(4):281–334, 1993.

[18] Joseph W. Dauben. The trigonometric background to Georg Cantor's theory of sets. *Arch. History Exact Sci.*, 7(3):181–216, 1971.

[19] Guy David. Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 13(1):25–70, 1988.

[20] Guy David. Analytic capacity, Calderón-Zygmund operators, and rectifiability. *Publ. Mat.*, 43(1):3–25, 1999.

[21] Sergei S. Demidov and Boris V. Lëvshin, editors. *The case of academician Nikolai Nikolaevich Luzin*, volume 43 of *History of Mathematics*. American Mathematical Society, Providence, RI, 2016. Translated from the 1999 Russian original by Roger Cooke, Research and commentary by N. S. Ermolaeva (Minutes), A. I. Volodarskii and T. A. Tokareva (Appendices).

[22] David H. Fremlin. *Measure theory. Vol. 5. Set-theoretic measure theory. Part II.* Torres Fremlin, Colchester, 2015. Corrected reprint of the 2008 original.

[23] John Garnett. *Analytic capacity and measure.* Lecture Notes in Mathematics, Vol. 297. Springer-Verlag, Berlin-New York, 1972.

[24] John B. Garnett and Donald E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2005.

[25] Frederick W. Gehring. The definitions and exceptional sets for quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 281:28, 1960.

[26] Frederick W. Gehring and Walter K. Hayman. An inequality in the theory of conformal mapping. *J. Math. Pures Appl. (9)*, 41:353–361, 1962.

[27] Szymon Głąb. On the complexity of continuous functions differentiable on cocountable sets. *Real Anal. Exchange*, 34(2):521–529, 2009.

[28] Kurt Gödel. The consistency of the axiom of choice and of the generalized continuum-hypothesis. *Proceedings of the National Academy of Sciences*, 24(12):556–557, 1938.

[29] G. M. Goluzin. *Geometric theory of functions of a complex variable.* Translations of Mathematical Monographs, Vol. 26. American Mathematical Society, Providence, RI, 1969.

[30] Juha Heinonen and Pekka Koskela. Definitions of quasiconformality. *Invent. Math.*, 120(1):61–79, 1995.

[31] Witold Hurewicz. Zur theorie der analytischen mengen. *Fund. Math.*, 15:4–16, 1930.

[32] Peter W. Jones. On removable sets for Sobolev spaces in the plane. In *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)*, volume 42 of *Princeton Math. Ser.*, pages 250–267. Princeton Univ. Press, Princeton, NJ, 1995.

[33] Peter W. Jones and Stanislav K. Smirnov. Removability theorems for Sobolev functions and quasiconformal maps. *Ark. Mat.*, 38(2):263–279, 2000.

[34] Robert Kaufman. Fourier transforms and descriptive set theory. *Mathematika*, 31(2):336–339 (1985), 1984.

[35] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[36] Alexander S. Kechris and Alain Louveau. *Descriptive set theory and the structure of sets of uniqueness*, volume 128 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[37] Alexander S. Kechris and W. Hugh Woodin. Ranks of differentiable functions. *Mathematika*, 33(2):252–278 (1987), 1986.

[38] Haseo Ki. The Kechris-Woodin rank is finer than the Zalcwasser rank. *Trans. Amer. Math. Soc.*, 347(11):4471–4484, 1995.

[39] Haseo Ki. On the Denjoy rank, the Kechris-Woodin rank and the Zalcwasser rank. *Trans. Amer. Math. Soc.*, 349(7):2845–2870, 1997.

[40] Olli Lehto. Quasiconformal mappings and singular integrals. In *Symposia Mathematica, Vol. XVIII*, pages 429–453. Academic Press, London-New York, 1976. (Convegno sulle Transformazioni Quasiconformi e Questioni Connesse, INDAM, Rome, 1974),.

[41] Olli Lehto. *Lars Ahlfors.* American Mathematical Society, Providence, RI, 2015. At the summit of mathematics, Translated from the 2013 Finnish original [MR3222281] by William Hellberg.

[42] George G. Lorentz. Who discovered analytic sets? *Math. Intelligencer*, 23(4):28–32, 2001.

[43] Nikolai Lusin. Sur les ensembles analytiques. *Fundamenta Mathematicae*, 10(1):1–95, 1927.

[44] Jerome H. Manheim. *The genesis of point set topology.* Pergamon Press, Oxford-Paris-Frankfurt; The Macmillan Co., New York, 1964.

[45] Donald E. Marshall. *Complex analysis.* Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 2019.

[46] R. Daniel Mauldin. The set of continuous nowhere differentiable functions. *Pacific J. Math.*, 83(1):199–205, 1979.
[47] Stefan Mazurkiewicz. Über die menge der differenzierbaven functionen. *Fund. Math.*, 27:244–249, 1936.
[48] Dimitrios Ntalampekos. CNED sets: countably negligible for extremal distances. *Selecta Math. (N.S.)*, 30(4):Paper No. 61, 57, 2024.
[49] Kôtaro Oikawa. Welding of polygons and the type of Riemann surfaces. *Kodai Math. Sem. Rep.*, 13:37–52, 1961.
[50] Hervé Pajot. *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral*, volume 1799 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.
[51] Albert Pfluger. Ueber die Konstruktion Riemannscher Flächen durch Verheftung. *J. Indian Math. Soc. (N.S.)*, 24:401–412 (1961), 1960.
[52] Charles Pugh and Conan Wu. Jordan curves and funnel sections. *J. Differential Equations*, 253(1):225–243, 2012.
[53] Taje I. Ramsamujh. Three ordinal ranks for the set of differentiable functions. *J. Math. Anal. Appl.*, 158(2):539–555, 1991.
[54] Lasse Rempe. On prime ends and local connectivity. *Bull. Lond. Math. Soc.*, 40(5):817–826, 2008.
[55] Alex Rodriguez. Every circle homeomorphism is the composition of two weldings. 2025. Preprint, https://arxiv.org/pdf/2501.06347.
[56] Alex Rodriguez. Flexible curves and Hausdorff dimension. 2025. Preprint.
[57] Nikolaos Efstathiou Sofronidis. The set of continuous piecewise differentiable functions. *Real Anal. Exchange*, 31(1):13–21, 2005/06.
[58] Shashi Mohan Srivastava. How did Cantor discover set theory and topology? *Resonance*, pages 977–999, 2014.
[59] John R. Steel. What is ... a Woodin cardinal? *Notices Amer. Math. Soc.*, 54(9):1146–1147, 2007.
[60] Xavier Tolsa. Analytic capacity, rectifiability, and the Cauchy integral. In *International Congress of Mathematicians. Vol. II*, pages 1505–1527. Eur. Math. Soc., Zürich, 2006.
[61] Xavier Tolsa. *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory*, volume 307 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2014.
[62] Joan Verdera. L^2 boundedness of the Cauchy integral and Menger curvature. In *Harmonic analysis and boundary value problems (Fayetteville, AR, 2000)*, volume 277 of *Contemp. Math.*, pages 139–158. Amer. Math. Soc., Providence, RI, 2001.
[63] Jang-Mei Wu. Removability of sets for quasiconformal mappings and Sobolev spaces. *Complex Variables Theory Appl.*, 37(1-4):491–506, 1998.
[64] Malik Younsi. On removable sets for holomorphic functions. *EMS Surv. Math. Sci.*, 2(2):219–254, 2015.
[65] Malik Younsi. Removability and non-injectivity of conformal welding. *Ann. Acad. Sci. Fenn. Math.*, 43(1):463–473, 2018.

C.J. Bishop, Mathematics Department, Stony Brook University, Stony Brook, NY 11794-3651

Email address: `bishop@math.stonybrook.edu`