LOCALLY MINIMAL SETS FOR CONFORMAL DIMENSION

CHRISTOPHER J. BISHOP AND JEREMY T. TYSON

Abstract. We show that for each $1 \leq \alpha < d$ and $K < \infty$ there is a subset X of $\mathbb{R}^d$ such that $\dim(f(X)) \geq \alpha = \dim(X)$ for every K-quasiconformal map, but such that $\dim(g(X))$ can be made as small as we wish for some quasiconformal g, i.e., the conformal dimension of X is zero. These sets are then used to construct new examples of minimal sets for conformal dimension and sets where the conformal dimension is not attained.

1. Introduction

Given a compact metric space X we define its conformal dimension as

$$\operatorname{Cdim}(X) = \inf_f \dim(f(X)),$$

where the infimum is over all quasisymmetric maps of X into some metric space and "dim" denotes Hausdorff dimension. This was introduced by Pansu in [10]. See also [4] and [15]. A set is called *minimal for conformal dimension* if $\operatorname{Cdim}(X) = \dim(X)$, i.e., no quasisymmetric image can have smaller dimension. Such examples obviously exist in integer dimensions for topological reasons, e.g., a line segment is one-dimensional and any image is connected, hence has dimension ≥ 1. Examples with non-integer dimension are much less obvious, but were shown to exist in [10] and [15] (see also Section 5, Remark 1). In this paper, we strengthen this result by showing that "locally minimal" sets for conformal dimension exist in the following sense.

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Theorem 1. *Suppose that $1 \le \alpha < d$ and $K < \infty$. Then there is a compact, totally disconnected set $X \subset \mathbb{R}^d$ of Hausdorff dimension α such that*

1. $\dim(f(X)) \ge \alpha$ *for every K-quasisymmetric map of X to a metric space,*
2. *for any $\epsilon > 0$ there is a quasiconformal map g of $\mathbb{R}^d$ to itself such that* $\dim(g(X)) < \epsilon$. *In particular,* $\mathrm{Cdim}(X) = 0$.

Recall that a map f between metric spaces is quasisymmetric if there is a homeomorphism η of $[0,\infty)$ to itself such that

$$|x-y| \le t|x-z| \quad \Rightarrow \quad |f(x)-f(y)| \le \eta(t)|f(x)-f(z)|. \tag{1}$$

In the case of maps of $\mathbb{R}^d$ onto itself, $d \ge 2$, this class of mappings is the same as the class of quasiconformal maps.

In this paper, we will say that f is K-quasisymmetric if (1) holds for some η with $\eta(1)\,\eta(1/K) \le 1$. (Note that for any quasisymmetric map this is true for some K.) If f is K-quasisymmetric, then for any four points a,b,c,d such that $|c-d| \ge K|a-c| \ge K|a-b|$ we have

$$\begin{aligned} |f(a)-f(b)| &\le \eta(1)\,|f(a)-f(c)| \\ &\le \eta(1)\,\eta(\frac{1}{K})\,|f(c)-f(d)| \le |f(c)-f(d)|. \end{aligned} \tag{2}$$

The existence of locally minimal sets implies the existence of minimal sets since we may take $X = \cup_n X_n$ where each X_n is chosen using Theorem 1 with $\alpha_1 = \alpha_2 = \cdots = \alpha$ and $K_1 < K_2 < \cdots \nearrow \infty$ (to make X compact we may assume that $\mathrm{diam}(X_n) \to 0$ and that $\{X_n\}$ accumulates at some point of X_1). Thus we have

Corollary 2. *For every $\alpha \in [1,d)$, there is a totally disconnected set $X \subset \mathbb{R}^d$ such that* $\mathrm{Cdim}(X) = \dim(X)$, *i.e., X is minimal for conformal dimension.*

All of the previously known examples of sets of minimal conformal dimension either contain nontrivial paths [10], [15] or are nonremovable sets for quasiconformal maps (and hence have dimension $\geq d-1$) [1]. The proof given in [15] makes use of the notion of generalized modulus introduced by Pansu in [10], [11] and developed in [14]. However, a simpler argument can be given which avoids the use of Pansu's generalized modulus, see Theorem 15.9 of [6][1]. Our proof of Theorem 1 is closely modelled on this second argument.

Another application of Theorem 1 is to take $X = \cup_n X_n$ where $\alpha_n \searrow \alpha$ and $K_n \nearrow \infty$. In this case, it is easy to see that $\mathrm{Cdim}(X) = \alpha$, but $\dim(f(X)) > \alpha$ for any quasisymmetric f. Thus we have

Corollary 3. *For every $\alpha \in [1, d)$ there is a set $X \subset \mathbb{R}^d$ so that $\mathrm{Cdim}(X) = \alpha$, but this dimension is not attained for any quasisymmetric image of X.*

This answers a question from [3] where this result was proved for $\alpha = 1$. That paper was motivated by a question of J. Heinonen (Section 15, [6]) as to whether or not the conformal dimension is always attained by some quasisymmetric image.

2. Some background

Before proving Theorem 1, we recall the definition and basic properties of Hausdorff dimension. For $\alpha \geq 0$, the α-dimensional Hausdorff content of a metric space X is defined as

$$\mathcal{H}^\infty_\alpha(X) = \inf\{\sum_j r_j^\alpha : X \subset \cup_j B(x_j, r_j)\}$$

and the Hausdorff dimension is

$$\dim(X) = \inf\{\alpha : \mathcal{H}^\infty_\alpha(X) = 0\}. \tag{3}$$

[1]The second author wishes to acknowledge Mario Bonk for a valuable discussion during which this latter proof was developed.

In general, one gives an upper bound for $\dim(X)$ by finding an explicit cover of X with small Hausdorff sum. One can give a lower bound using the mass distribution principle: if X supports a positive measure μ such that $\mu(B(x,r)) \leq Cr^\alpha$ for every ball $B(x,r)$, then $\dim(X) \geq \alpha$ (since $0 < \mu(X) \leq \sum_j \mu(B(x_j,r_j)) \leq C\sum_j r_j^\alpha$ for any cover of X).

For our purposes, it is useful to note that one need not cover X by balls in order to compute its dimension. We will say a collection of sets $\mathcal{C}$ is a *nice covering collection* if there are constants $C_1, C_2 < \infty$ such that any ball $B(x,r)$ can be covered by at most C_1 elements of $\mathcal{C}$ each of which has diameter at most $C_2 r$. If we define

$$\tilde{\mathcal{H}}_\alpha^\infty(X) = \inf\{\sum_j \operatorname{diam}(E_j)^\alpha : X \subset \cup_j E_j, E_j \in \mathcal{C}\},$$

then it is easy to check that $\mathcal{H}_\alpha^\infty(X) \leq \tilde{\mathcal{H}}_\alpha^\infty(X) \leq C_1 C_2^\alpha \mathcal{H}_\alpha^\infty(X)$. Thus the definition of the Hausdorff dimension (3) is unchanged if we replace $\mathcal{H}_\alpha^\infty$ with $\tilde{\mathcal{H}}_\alpha^\infty$ for some nice covering collection $\mathcal{C}$. The most commonly used collection of this type is the collection of dyadic cubes in $\mathbb{R}^d$.

We will also use the following easy observation. If $\mathcal{C}$ is a nice covering collection for X and $f : X \to Y$ is quasisymmetric then $f(\mathcal{C}) = \{f(E) : E \in \mathcal{C}\}$ is a nice covering collection for Y. Indeed, if B is a ball in Y then by definition $F = f^{-1}(B)$ is contained in a ball of radius $\operatorname{diam}(F)$ and hence can be covered by C_1 sets in $\mathcal{C}$ of diameter at most $C_2\operatorname{diam}(F)$. The images of these C_1 sets under the map f cover B and have diameter at most $\eta(2C_2)\operatorname{diam}(B)$ by quasisymmetry (see Theorem 2.5 of [13]), as desired.

In this paper we will consider sets constructed as follows. We will fix a finite number of collections $\mathcal{F}_j$, $j = 1, \ldots, m$, where $\mathcal{F}_j$ consists of N_j (closed) subcubes of $[0,1]^d$, each with side length ϵ_j and with disjoint interiors and sides parallel to the coordinate axes. We will also fix a partition of the natural numbers $\mathbb{N} = \{0,1,2,\dots\}$ into m sets $E_1,\dots,E_m$. Given this data we define a nested sequence of compact sets $X_0 \supset X_1 \supset \cdots$ inductively, setting $X_0 = [0,1]^d$ and letting X_{n+1} be the set obtained by replacing each component cube in X_n by a scaled copy of the family $\mathcal{F}_j$ for which $E_j \ni n$. Note that X_n is a finite union of cubes all of which have side length $r_n = \prod_{k=0}^{n-1} \epsilon(k)$, where $\epsilon(k) = \epsilon_j$ iff $k \in E_j$. Finally, we let $X = \cap_n X_n$.

If $\mathcal{C}_n$ denotes the collection of component cubes of X_n, then it is easy to see that $\mathcal{C} = \cup_n \mathcal{C}_n$ is a nice covering collection of X as follows. Given a ball $B(x,r)$ in X choose the smallest n such that $r_n \le r$. Then $B(x,r)$ is covered by the union of those cubes in $\mathcal{C}_n$ which meet it. The number of these cubes is uniformly bounded since they have disjoint interiors, are all contained in $B(x,(1+\sqrt{d})r)$, and all have diameter at most ϵr, where $\epsilon = \min_j \epsilon_j$.

It is now easy to compute the Hausdorff dimension of X as

$$\dim(X) = \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} \log N(k)}{\log(1/r_n)} = \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} \log N(k)}{-\sum_{k=0}^{n-1} \log \epsilon(k)}, \tag{4}$$

where $N(k) = N_j$ if and only if $k \in E_j$. To show that $\dim(X)$ is at most the right hand side of (4), one simply takes all nth generation cubes as a cover of X. To prove the opposite direction one applies the mass distribution principle to the measure μ which gives all nth generation cubes equal mass. The details are left to the reader.

If we let $p(n,j) = \frac{1}{n}\#(E_j \cap [0,n])$, where $\#(S)$ denotes the cardinality of the set S, then (4) can be rewritten as

$$\dim(X) = \liminf_{n\to\infty} \frac{\sum_{j=1}^{m} p(n,j)\log N_j}{-\sum_{j=1}^{m} p(n,j)\log \epsilon_j}. \tag{5}$$

3. Construction of the space X of Theorem 1

Let $\alpha \in [1,d)$ and let $K \in [1,\infty)$ be fixed. Choose an even integer N so that $N \geq 8K$ and let $\epsilon = 1/N$. Let e_j denote the unit vector in $\mathbb{R}^d$ in the x_j direction. The construction is based on three families of subcubes of $[0,1]^d$. Let $\mathcal{F}_1$ be the collection of N^d subcubes of side length ϵ with disjoint interiors. Thus $\epsilon_1 = \epsilon$ and $N_1 = N^d$. Let $\mathcal{F}_2 \subset \mathcal{F}_1$ be the collection of N cubes which hit the x_1 axis, so $\epsilon_2 = \epsilon$ and $N_2 = N$.

The third collection of cubes, $\mathcal{F}_3$, will consist of $N_3 = N$ cubes with side length $\epsilon_3 = 1/(\frac{1}{2}N+2) = 2\epsilon/(1+4\epsilon)$ arranged in two parallel rows. The first row consists of cubes of the form $\epsilon_3 Q_0 + j\epsilon_3 e_1$ for $j = 0,3,4,\ldots \frac{1}{2}N+1$. Note that the indices $j=1,2$ are skipped, leaving a gap of size $2\epsilon_3$ in the row. The second row consists of cubes of the form $\epsilon_3 Q_0 + j\epsilon_3 e_1 + 3\epsilon_3 e_2$ for $j = 1,2,\ldots,\frac{1}{2}N$. In this row, the first and last cubes are omitted. See Figure 1 for the picture of $\mathcal{F}_3$ in two dimensions.

We define the set X following the procedure outlined in the previous section. Assume that we have divided the natural numbers $\mathbb{N} = \{0,1,2,3\ldots\}$ into three disjoint sets $\mathbb{N} = E_1 \cup E_2 \cup E_3$. Let $X_0 = [0,1]^d$. In general, if X_n is a finite union of cubes, construct $X_{n+1} \subset X_n$ by replacing each cube in X_n by a scaled copy of $\mathcal{F}_i$ if $n \in E_i$. Then define $X = \cap_n X_n$. For future reference, we will denote the collection of cubes making up X_n as $\mathcal{C}_n$ and let $\mathcal{C} = \cup_n \mathcal{C}_n$.

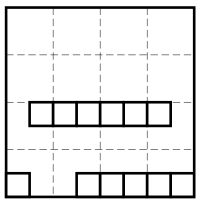

FIGURE 1. The collection $\mathcal{F}_3$ with $N = 12$, $\epsilon_3 = \frac{1}{8}$

Associated to X, we define an auxiliary set $Y = \cap_n Y_n$ which is defined similarly except that we use $\mathcal{F}_1$ if $n \in E_1$ and we use $\mathcal{F}_2$ if $n \in E_2 \cup E_3$ (so $\mathcal{F}_3$ is never used). Clearly $Y = [0,1] \times Z$ is a product set containing horizontal line segments and we will eventually choose the sets E_1, E_2, E_3 so that $\dim(X) = \dim(Y)$. The collection of cubes making up Y_n will be denoted $\mathcal{D}_n$ and we let $\mathcal{D} = \cup_n \mathcal{D}_n$.

Next we want to define a mapping $T : \mathcal{D} \to \mathcal{C}$ with the properties that

(1) $T([0,1]^d) = [0,1]^d$;

(2) $T : \mathcal{D}_n \to \mathcal{C}_n$ is onto;

(3) if $Q' \in \mathcal{D}_{n+1}$, $Q \in \mathcal{D}_n$ and $Q' \subset Q$ then $T(Q') \subset T(Q)$.

This is easy to do by induction. Condition (1) starts the induction and if T has been defined down to level n then we define it at level $n+1$ using the identity map if $n \in E_1 \cup E_2$ (since in this case the same picture occurs in the construction of both X and Y) and if $n \in E_3$ then we choose any surjective map from $\mathcal{F}_2$ to $\mathcal{F}_3$ (there is one since $\mathcal{F}_3$ has the same number of elements as $\mathcal{F}_2$). See Figure 2.

The map T also induces a map from closed sets of Y to closed sets of X as follows. For $K \subset Y$ closed, let

$$T(K) = \bigcap_n \bigcup_{\substack{Q \in \mathcal{C}_n \\ Q \cap K \neq \emptyset}} T(Q).$$

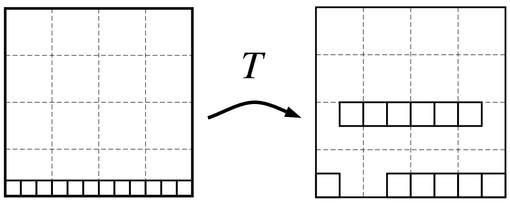

FIGURE 2. Defining the bijection T from $\mathcal{D}$ to $\mathcal{C}$

Later we will be particularly interested in the sets $T(L)$ where $L = [0,1] \times \{z\}$ is a line segment in Y. We will use the fact that for each generational cube $Q \in \mathcal{C}$, the set $T(L) \cap \partial Q$ consists of exactly two points.

Next we want to define the sets E_1, E_2, E_3 so that $\dim(X) = \dim(Y) = \alpha$. First we deal with $\dim(Y) = \alpha$. Let $F = \mathbb{N} \setminus E_1 = E_2 \cup E_3$. If we choose E_1 so that

$$\lim_{n\to\infty} \frac{\#(E_1 \cap [0,n])}{n} = \lim_{n\to\infty} p(n,1) = \frac{\alpha - 1}{d-1},$$

then by (5),

$$\begin{aligned} \dim(Y) &= \lim_{n\to\infty} \frac{p(n,1)\log N_1 + (1-p(n,1))\log N_2}{-p(n,1)\log\epsilon_1 - (1-p(n,1))\log\epsilon_2} \\ &= \lim_{n\to\infty} \frac{-dp(n,1)\log\epsilon - (1-p(n,1))\log\epsilon}{-\log\epsilon} \\ &= \lim_{n\to\infty} 1 + (d-1)p(n,1) = \alpha. \end{aligned}$$

One way to do this explicitly is to put $0 \in E_1$ and in general put $n+1 \in E_1$ iff

$$\alpha_n \equiv \frac{p(n,1)\log N_1 + (1-p(n,1))\log N_2}{-p(n,1)\log\epsilon_1 - (1-p(n,1))\log\epsilon_2} < \alpha,$$

and otherwise put $n+1$ in F, the complement of E_1. In this case it is easy to see that $|\alpha_n - \alpha_{n+1}| = O(1/n)$ and α_{n+1} is either closer to α than α_n is, or it is on the other side of α. Thus

$$|\alpha_n - \alpha| = O(\frac{1}{n}).$$

If we define a measure μ on Y by giving all of the cubes in $\mathcal{D}_n$ the same mass, then the measure of any $Q \in \mathcal{D}_n$ is at most

$$\mu(Q) = \ell(Q)^{\alpha_n} \le \ell(Q)^{\alpha - C/n} \le C_\mu \ell(Q)^\alpha, \tag{6}$$

for some absolute constant C_μ (since $\ell(Q) = \epsilon^n$).

Next we want to split the set $F \subset \mathbb{N}$ into disjoint sets $E_2 \cup E_3$. Do this in any way so that

$$\liminf_{n\to\infty} \frac{1}{n}\#(E_3 \cap [0,n]) = 0, \tag{7}$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\#(E_3 \cap [0,n]) > 0. \tag{8}$$

For example, we can take $E_3 = \{n \in F : (2m)! \leq n < (2m+1)! \text{ for some } m\}$. We will show (7) implies $\dim(X) = \alpha$, while (8) implies $\mathrm{Cdim}(X) = 0$.

Since $N_3 = N_2$ and $\epsilon_3 \geq \epsilon_2$, it is clear from (5) that $\dim(X) \geq \dim(Y) = \alpha$. On the other hand, by taking a sequence $\{n_k\}$ such that $\lim_{k\to\infty} p(n_k, 3) = 0$, we see that (5) implies

$$\dim(X) \leq \lim_{k\to\infty} \frac{p(n_k,1)\log N_1 + p(n_k,2)\log N_2}{-p(n_k,1)\log\epsilon_1 - p(n_k,2)\log\epsilon_2} = \dim(Y).$$

Thus $\dim(X) = \dim(Y) = \alpha$.

4. Proof of Theorem 1

To show that the conformal dimension of X is zero, we simply note that the squares in $\mathcal{F}_3$ come in three connected components U_1, U_2, U_3 which have disjoint $\epsilon_3/2$ neighborhoods V_1, V_2, V_3 respectively (see Figure 3). We may shrink the squares in each component by any factor $\delta > 0$ by a map which is the identity outside $V_1 \cup V_2 \cup V_3$, is linear and conformal in each of U_1, U_2, U_3 and which is

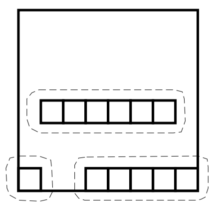

FIGURE 3. Disjoint $\epsilon_3/2$ neighborhoods for the components U_1, U_2, U_3

quasiconformal with constant depending only on δ. The set which we obtain by replacing $\mathcal{F}_3$ by the new pattern and repeating the construction of section 2 is the image of X under a global quasiconformal map g of $\mathbb{R}^d$. This new pattern has N_3 squares of size $\delta\epsilon_3$ and so its dimension by (5) is

$$\dim(g(X)) \leq \liminf_{n\to\infty} \frac{p(n,1)\log N_1 + p(n,2)\log N_2 + p(n,3)\log N_3}{-p(n,1)\log\epsilon_1 - p(n,2)\log\epsilon_2 - p(n,3)(\log\delta + \log\epsilon_3)}.$$

Since $\limsup_{n\to\infty} p(n,3) > 0$ and we can make δ as small as we wish while keeping everything else fixed, we see that $\dim(g(X))$ may be as small as we wish.

Next, we must show that $\dim(f(X)) \geq \alpha$ if f is K-quasisymmetric (in the sense defined before). By rescaling we may assume that

$$\operatorname{dist}(f(X\cap\{x_1=0\}), f(X\cap\{x_1=1\})) = 1. \tag{9}$$

Now suppose ρ is a non-negative Borel function on Y such that

$$\int_0^1 \rho(x,z)\,dx \geq 1$$

for any $z\in Z$; we call such a ρ *admissible*. Then

$$1 \leq \int_0^1 \rho(x,z)^\alpha\,dx$$

for all $z\in Z$ by Hölder's inequality (since $\alpha\geq 1$), which implies

$$1 \leq \int_Y \rho(x,z)^\alpha\,d\mu(z,x) \tag{10}$$

However, if $\dim(f(X)) < \alpha$ then we can find a cover $\{W_k\}$ of $f(X)$ using the nice covering collection $f(\mathcal{C})$ such that $\sum_k \operatorname{diam}(W_k)^\alpha$ is as small as we wish, say $< C_\mu^{-1}$, where C_μ is as in inequality (6). The cover $\{W_k\}$ corresponds to a covering of X by cubes in $\mathcal{C}$, which in turn corresponds to a cover of Y by cubes in $\mathcal{D}$ via the correspondence T. Let $\{Q_k\}\subset\mathcal{D}$ denote this cover of Y and assume (as we may) that the cubes $\{Q_k\}$ have disjoint interiors.

Define a function ρ on Y as $\rho(y) = \mathrm{diam}(W_k)/\ell(Q_k)$ if $y \in Q_k$ (where $\ell(Q)$ denotes the side length of Q). Then ρ is well defined except on a set of μ measure zero and by (6),

$$\int_Y \rho^\alpha \, d\mu = \sum_k \frac{\mathrm{diam}(W_k)^\alpha}{\ell(Q_k)^\alpha} \mu(Q_k) \le C_\mu \sum_k \mathrm{diam}(W_k)^\alpha < 1.$$

If we can show ρ is admissible, then this contradicts (10) and we are done.

Fix a line $L = [0,1] \times \{z\} \subset Y$ and for each $Q \in \mathcal{D}$ let x_Q and y_Q be the two points of $\partial T(Q) \cap T(L)$ (and assume the first coordinate of x_Q is smaller than the first coordinate of y_Q). Let $\mathcal{D}(L)$ be the collection of cubes in our cover of Y which hit L and, for $Q \in \mathcal{D}_n$, let $\mathcal{D}(L,Q)$ be the collection of cubes $Q' \in \mathcal{D}_{n+1}$ which hit L and which are contained in Q. Let $d(Q) = |f(x_Q) - f(y_Q)|$. Then $d(Q) \le \mathrm{diam}(f((T)))$ and we claim that for $Q \in \mathcal{D}_n$

$$\sum_{Q' \in \mathcal{D}(L,Q)} d(Q') \ge d(Q). \tag{11}$$

If we can prove this, then a simple induction shows that

$$\int_0^1 \rho(x,z)\, dx = \sum_{\substack{k \\ Q_k \in \mathcal{D}(L)}} \mathrm{diam}(W_k) \ge d(X_0) = |f(x_{X_0}) - f(y_{X_0})| \ge 1$$

by (9) (recall that $X_0 = [0,1]^d$) and we will be done.

To prove (11) it suffices to consider what happens for each of the three replacement patterns. For $\mathcal{F}_1$ and $\mathcal{F}_2$, there is a chain of adjacent subcubes $Q_1, \ldots, Q_N$ such that $x_Q = x_{Q_1}$, $y_{Q_N} = y_Q$ and $x_{Q_{j+1}} = y_{Q_j}$ for $j = 1, \ldots, N-1$. and so (11) holds by the triangle inequality:

$$d(Q) = |f(x_Q) - f(y_Q)| \le \sum_{j=1}^N |f(x_{Q_j}) - f(y_{Q_j})| = \sum_{j=1}^N d(Q_j).$$

For $\mathcal{F}_3$ assume that the $N/2$ cubes in the first row are numbered from 1 to $\frac{1}{2}N$ ordered by the x_1 coordinate and the second row is similarly labeled $\frac{1}{2}N+1, \ldots, N$.

The same argument as above shows that

$$d(Q) \leq d(Q_1) + |f(y_{Q_1}) - f(x_{Q_2})| + \sum_{j=2}^{N/2} d(Q_j),$$

and so we are done provided that

$$|f(y_{Q_1}) - f(x_{Q_2})| \leq \sum_{j=1+N/2}^{N} d(Q_j).$$

Again by the triangle inequality, this will hold if

$$|f(y_{Q_1}) - f(x_{Q_2})| \leq |f(x_{Q_{1+N/2}}) - f(y_{Q_N})|.$$

However, this is just (2) with

$$a = y_{Q_1}, \quad b = x_{Q_2}, \quad c = x_{Q_{1+N/2}}, \quad d = y_{Q_N},$$

since $|a-b| = 2\epsilon_3$, $2\epsilon_3 \leq |a-c| \leq 4\epsilon_3$ and $|c-d| = \frac{1}{2}N\epsilon_3$.

Thus if f is K-quasisymmetric and $N \geq 8K$ the desired inequality holds which shows that ρ is admissible. This completes the proof of Theorem 1.

5. Additional remarks

Remark 1: The proof also shows that Y is minimal, i.e., $\dim(f(Y)) \geq \dim(Y)$ for any quasisymmetric map f. In fact, one can show $E = [0,1] \times Z$ is minimal for any Borel set $Z \subset \mathbb{R}^{d-1}$. If $\dim(E) = 1$ there is nothing to do since E contains a line segment. If $\dim(E) = \alpha > 1$ and $1 < \beta < \alpha$ then by Frostman's lemma (e.g., Theorem 8.8 of [9]), E supports a product measure μ such that $\mu(B(x,r)) \leq C_\beta r^\beta$. Applying the above proof using coverings by dyadic cubes to define ρ shows that no quasisymmetric image of E can have zero β-content, thus proving that E is minimal. This improves a result from [15], where $[0,1] \times Z$ is shown to be minimal assuming Z is Ahlfors regular.

More generally, what sets $F \subset \mathbb{R}$ have the property that $F \times Z \subset \mathbb{R}^d$ is minimal for any non-empty compact $Z \subset \mathbb{R}^{d-1}$? There are Cantor sets with this property because there are Cantor sets F in $[0,1]$ such that any quasisymmetric image in a metric space has positive 1-dimensional content, and the proof given for $F = [0,1]$ above generalizes to such sets. For example, if $F \subset [0,1]$ is a closed set which is uniformly perfect and has positive Lebesgue measure, and if $[0,1] \setminus F = \cup_j I_j$ is a disjoint union of open intervals satisfying $\sum_j |I_j|^\alpha < \infty$ for all $\alpha > 0$, then any quasisymmetric image of F has positive 1-dimensional content. In the case of quasisymmetric maps of $\mathbb{R}$ to itself, this is Theorem 1.2 of [12]. It is easy to adapt the proof to any quasisymmetric map into a metric space; one only needs to know that such a map is Hölder continuous. For the convenience of the reader, we briefly sketch this argument at the end of the paper.

A set F in $\mathbb{R}$ is called quasisymmetrically thick if any quasisymmetric map from $\mathbb{R}$ to itself (including the identity) sends F to a set of positive Lebesgue measure. As noted in the previous paragraph, Staples and Ward have given some sufficient conditions for a set to be thick in [12]. Do all quasisymmetrically thick sets have the property that all quasisymmetric images (into any metric space) also have positive Hausdorff 1-content?[2] If F is quasisymmetrically thick then is $F \times Z$ always minimal for conformal dimension? If F is not quasisymmetrically thick, then is there a quasisymmetric image of F with dimension < 1?

Remark 2: For sets in $\mathbb{R}^d$, we might consider $\mathrm{QCdim}(X) = \inf \dim(g(X))$ where the infimum is over all quasiconformal maps g of $\mathbb{R}^d$ to itself. This quantity can be strictly larger than the conformal dimension. For example, Antoine's necklace in $\mathbb{R}^3$ (e.g., see [7], [8]) has $\mathrm{QCdim}(X) = 1$ but $\mathrm{Cdim}(X) = 0$. Thus a set in

[2]In Proposition 14.37 of [6], the following result is proved: if $F \subset \mathbb{R}$ is a quasisymmetrically thick set, then every quasisymmetric mapping of $\mathbb{R}$ onto an Ahlfors regular metric space sends F onto a set of positive 1-content. While this result is clearly related to our question, it does not answer it for two reasons: first, the map is assumed to be defined on all of $\mathbb{R}$ rather than just on F and second, the image of $\mathbb{R}$ is assumed to be an Ahlfors regular space.

Euclidean space might be minimal for quasiconformal selfmaps of the space, but not be minimal for quasisymmetric maps which do not extend to the whole space.

Remark 3: Do local maximums occur? Probably not since global maximums are known not to occur [2]; for any compact $X \subset \mathbb{R}^d$ of positive dimension and any $\epsilon > 0$, there is a quasiconformal map f of $\mathbb{R}^d$ to itself such that $\dim(f(X)) > d - \epsilon$. Is it true that for any X of positive dimension and any $K > 1$, there is a K-quasiconformal map such that $\dim(f(X)) > \dim(X)$? This is only interesting for quasiconformal maps in $\mathbb{R}^d$, since if one considers all quasisymmetric maps into metric spaces, then one can obviously increase the dimension by the "snowflake functor", i.e., replacing the metric $|x - y|$ by $|x - y|^\epsilon$ for any $\epsilon < 1$.

Remark 4: Given a set $E \subset \mathbb{R}^2$ we can define D_E to be the set of values of $\dim(f(E))$ as f ranges over all quasiconformal maps of the plane. This must be a single point or an interval since any quasiconformal map can be connected to the identity by a path of maps along which the dimension changes continuously. From [2] and [5], we know that the only possibilities are of the form $\{0\}$, $\{2\}$, $(0, 2)$, $[\alpha, 2)$ and $(\alpha, 2)$ for $0 < \alpha < 2$. The first three are well known to occur, as is $[1, 2)$. This paper shows that $(\alpha, 2)$ and $[\alpha, 2)$ can also occur for every $1 \le \alpha < 2$. The second author conjectured in [15] that the intervals $[\alpha, 2)$, $(\alpha, 2)$ never occur for $0 < \alpha < 1$ and this remains open.

We conclude with a discussion of the fact about quasisymmetrically thick sets mentioned in Remark 1.

Proposition 4. *Let $F = [0, 1] \setminus \cup_j I_j$ be a closed, uniformly perfect set of positive Lebesgue measure, where $\{I_j\}$ is a collection of disjoint open intervals. Assume* *that* $\gamma_p := \sum_j |I_j|^p$ *is finite for all* $p > 0$*. Then the Hausdorff* 1*-content of* $f(F)$ *is positive for all quasisymmetric maps* f *of* F *into any metric space.*

Recall that a set $F \subset [0,1]$ is said to be *uniformly perfect* if there exists a constant $c > 0$ so that for all $x \in F$ and all $0 < r < 1$, there exists a point $y \in F$ with $r/c \leq |x - y| \leq r$. The relevance of this condition in the proposition comes from the fact that quasisymmetric maps on uniformly perfect sets are Hölder continuous (see, for example, Theorem 11.3 of [6]). It is possible that uniform perfectness of a set F as in Proposition 4 may be a consequence of the other hypotheses.

An example of a set $F \subset [0,1]$ satisfying the conditions of Proposition 4 can be constructed as follows: let $I_1 = (\frac{1}{3}, \frac{2}{3})$ and, in general, let I_m, $m \geq 2$, be an interval of length 3^{-m} centered within the largest interval contained in $[0,1] \setminus \cup_{j=1}^{m} I_j$ (if the maximum length is achieved by several intervals, choose any one at random).

Theorem 1.2 of [12] says that if F is chosen as in Proposition 4 and f is any quasisymmetric map of $\mathbb{R}$ to itself, then the Lebesgue measure of $f(F)$ is positive. The proof of Proposition 4 is essentially already contained in the proof of Theorem 1.2 of [12]. The differences between the two results are:

(i) f now takes values in an arbitrary metric space;

(ii) f is defined only on the set F (and not on all of $\mathbb{R}$);

(iii) our conclusion is $\mathcal{H}_1^\infty(f(F)) > 0$ rather than $|f(F)| > 0$.

In what follows, we repeat Staples and Ward's argument from [12], indicating the changes which must be made to deal with these differences.

As mentioned above, if $f : F \to Y$ is a quasisymmetric embedding, then f is $1/s$-Hölder continuous for some $s > 1$. We may assume that the number of intervals I_j

is infinite. If these intervals are ordered so that $|I_1| \geq |I_2| \geq \cdots$, then the sequence $\{|I_j| : j = 1, 2, \dots\}$ satisfies $|I_j| \leq (\gamma_p/j)^{1/p}$ for all $j \in \mathbb{N}$ and all $p > 0$.

Choose a large integer N and let $I = [a, b]$ be the largest subinterval contained in $[0, 1] \setminus \cup_{j=1}^{N-1} I_j$. Then $|I| \geq |F|/N$. We may conjugate $f|_{F \cap I}$ by conformal scalings $g : [0, 1] \to I$ and $h : Y \to \lambda^{-1} Y$ to a mapping $\tilde{f}$ from $\tilde{F} = g^{-1}(F \cap I)$ to the dilated metric space $\lambda^{-1} Y$, where $\lambda = |f(a) - f(b)|$. Then $\tilde{f}$ is again Hölder continuous with exponent $1/s$. We claim that if N is chosen sufficiently large, then

$$\mathcal{H}_1^\infty(\tilde{f}(\tilde{F})) \geq c > 0; \tag{12}$$

which clearly implies that $\mathcal{H}_1^\infty(f(F)) \geq c(\lambda) > 0$.

To prove (12), we note that the collection $\tilde{f}(\mathcal{C})$, where $\mathcal{C}$ is the collection of sets of the form $U \cap F$, U an open subinterval of $[0, 1]$, is a nice covering collection of $\tilde{f}(\tilde{F})$. Let $U_1, U_2, \dots, U_M$ be a finite collection of (disjoint) intervals in $[0, 1]$ covering $\tilde{F}$. For each k, let $a_k = \inf \tilde{F} \cap U_k$ and $b_k = \sup \tilde{F} \cap U_k$. Then, for each k, the (possibly degenerate) interval (b_k, a_{k-1}) is contained in precisely one of the intervals $g^{-1}(I_j \cap I)$, $j = N, N+1, \dots$.

If we denote the metric in $\lambda^{-1} Y$ by $|\cdot|_{\lambda^{-1} Y}$, then

$$\begin{aligned} 1 &= \lambda^{-1} |f(a) - f(b)| = |\tilde{f}(0) - \tilde{f}(1)|_{\lambda^{-1} Y} \\ &\leq \sum_{k=1}^{M} \operatorname{diam}_{\lambda^{-1} Y}(\tilde{f}(U_k)) + \sum_{k=1}^{M-1} |\tilde{f}(a_{k+1}) - \tilde{f}(b_k)|_{\lambda^{-1} Y}. \end{aligned} \tag{13}$$

Set $p = 1/(2s)$. Then

$$\begin{aligned} \sum_{k=1}^{M-1} |\tilde{f}(a_{k+1}) - \tilde{f}(b_k)|_{\lambda^{-1} Y} &\leq C \sum_{k=1}^{M-1} |a_{k+1} - b_k|^{1/s} \leq C \sum_{\substack{j=N \\ I_j \cap I \neq \emptyset}}^{\infty} |g^{-1}(I_j)|^{1/s} \\ &= C \sum_{j=N}^{\infty} \left(\frac{|I_j|}{|I|} \right)^{1/s} \leq C \sum_{j=N}^{\infty} \left[\frac{N}{|F|} \left(\frac{\gamma_p}{j} \right)^{1/p} \right]^{1/s} \\ &= C(|F|, s) N^{1/s} \sum_{j=N}^{\infty} \frac{1}{j^2} = C(|F|, s) N^{(1/s)-1}. \end{aligned} \tag{14}$$

Since $s > 1$, we may choose N so large that the expression in (14) is at most $\frac{1}{2}$. Then (13) implies that $\sum_{k=1}^{M} \operatorname{diam}_{\lambda^{-1}Y}(\tilde{f}(U_k)) \geq \frac{1}{2}$. Since this holds for all such coverings $U_1, U_2, \ldots$, we conclude that the Hausdorff 1-content of $\tilde{f}(\tilde{F})$ (measured with respect to the nice covering collection $\tilde{f}(\mathcal{C})$) is at least $\frac{1}{2}$. This implies (12) and hence concludes the proof of the proposition.

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C.J. Bishop, Mathematics Department, SUNY at Stony Brook, Stony Brook, NY 11794-3651

E-mail address: bishop@math.sunysb.edu

J.T. Tyson, Mathematics Department, SUNY at Stony Brook, Stony Brook, NY 11794-3651

E-mail address: tyson@math.sunysb.edu