A TRANSCENDENTAL JULIA SET OF DIMENSION 1

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Abstract. We construct a non-polynomial entire function whose Julia set has finite 1-dimensional spherical measure, and hence Hausdorff dimension 1. In 1975, Baker proved the dimension of such a Julia set must be at least 1, but whether this minimum could be attained has remained open until now. Our example also has packing dimension 1, and is the first transcendental Julia set known to have packing dimension strictly less than 2. It is also the first example with a multiply connected wandering domain where the dynamics can be completely described.

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1. Statement of results

Suppose f is an entire function (holomorphic on the whole complex plane, $\mathbb{C}$). The Fatou set $\mathcal{F}(f)$ is the union of all open disks on which the iterates $f, f^2, f^3, \dots$ of f form a normal family and the Julia set $\mathcal{J}(f)$ is the complement of this set. The function f is called transcendental if it is not a polynomial. We will define 1-dimensional Hausdorff measure in Section 3; spherical Hausdorff measure refers to using the spherical metric in its definition, rather than the usual Euclidean metric.

Theorem 1.1. *There is a transcendental entire function f so that $\mathcal{J}(f)$ has finite 1-dimensional spherical Hausdorff measure.*

In 1975 Baker [4] proved that if f is transcendental, then the Fatou set has no unbounded, multiply connected components, and hence the Julia set cannot be totally disconnected. Therefore it must contain a non-trivial continuum, and hence has Hausdorff dimension at least 1. Thus the Julia set given by Theorem 1.1 must have Hausdorff dimension equal to 1, the first example of transcendental Julia set with this property. This answers a well known question, e.g., see page 307 of [39], Question 8.5 of [42] or Question 2 of [49].

In 1981 Misiurewicz [34] proved that the Julia set of $f(z) = \exp(z)$ is the whole plane (see [33], [43] for alternate proofs of this). McMullen [31] gave transcendental examples where the Julia set is not the whole plane, but still has dimension 2 (even positive area) and Stallard [47], [48] has shown that the Hausdorff dimension of a transcendental Julia set can attain every value in the interval $(1, 2]$. Together with these results, Theorem 1.1 implies:

Corollary 1.2. *The set of possible Hausdorff dimensions of a transcendental Julia set is exactly the closed interval* $[1, 2]$.

The corresponding question for packing dimension is still open. In all previous examples where the packing dimension is known, it is equal to 2, but we shall prove the example in Theorem 1.1 has packing dimension 1. Although it is reasonable to conjecture that all packing dimensions in the closed interval $[1, 2]$ are possible for a transcendental Julia set, only the two integer values $\{1, 2\}$ are currently proven to occur. See Section 3 for the definition of packing dimension and its relation to Hausdorff dimension.

As mentioned above, the Julia set of $\exp(z)$ is the whole plane, and so this function is as chaotic as possible, although its definition and geometry are perhaps the simplest possible for a transcendental entire function. In contrast, the example of Theorem 1.1 is among the dynamically "least chaotic" transcendental entire functions that can exist, although it definition is significantly more complicated than $\exp(z)$. Thus in transcendental dynamics it is easy to build highly chaotic functions (Julia sets with large dimension), but hard to build less chaotic examples (Julia sets with dimension close to or equal to 1). This is the reverse of the situation in other areas of conformal dynamics, where building examples with large dimension is challenging. Thus we can think of Theorem 1.1 as the transcendental analogue of the construction of polynomial Julia sets of dimension two by Shishikura [44], or to the construction of positive area Julia sets by Buff and Chéritat [22] (also see [3]). It is also analogous to the construction of finitely generated Kleinian limit sets of dimension 2 that are not the whole Riemann sphere. These are the geometrically infinite examples (they do not have a finite sided fundamental polyhedron); such groups were first shown to exist indirectly [13], [26], [30], then examples were constructed explicitly [27], [32], and finally all such groups were proven to have limit sets of dimension two [20], [50]. Unlike the case of polynomial Julia sets, finitely generated Kleinian limit sets with positive area do not exist [1], [23].

Stallard's examples, mentioned above, lie in the Eremenko-Lyubich class; these are transcendental entire functions with a bounded singular set, i.e., a bounded set of critical values and finite asymptotic values. More recently, the author and Simon Albrecht [19] showed that one can find examples with Hausdorff dimension close to 1 even in the smaller Speiser class (transcendental entire functions with a finite singular set). Stallard [46] proved that Julia sets of Eremenko-Lyubich functions have Hausdorff dimension strictly larger than 1, and Stallard and Rippon [40] proved their packing dimension is always equal to 2. Thus the example in Theorem 1.1 can't lie in the Eremenko-Lyubich class.

Baker's theorem actually implies that any transcendental Julia set either contains an unbounded continuum or a sequence of continua with diameters tending to infinity (the latter will be true in our example). Thus our example has infinite, but locally finite, 1-dimensional planar Hausdorff measure (finite 1-dimensional spherical measure implies every bounded subset has finite planar measure)

We shall prove even more: $\mathcal{J}(f)$ is a rectifiable set in the sense of geometric measure theory: except for a set of 1-dimensional measure zero, it can be covered by a countable union of C^1 curves (in our case, the exception set even has small Hausdorff dimension). On the other hand, we will also prove that for our example, $\mathcal{J}(f)$ is not a subset of a rectifiable curve on the Riemann sphere; indeed, $\mathcal{J}(f) \cap D(x, r)$ does not lie on a rectifiable curve for any $x \in \mathcal{J}(f)$ and $r > 0$. Whether this holds for every transcendental Julia set is open.

We will prove that, for our example, each connected component of the Fatou set is an infinitely connected bounded open set whose boundary consists of countably many C^1 curves that accumulate only on the single boundary component that separates this Fatou component from ∞. Baker [6] had shown that infinitely connected Fatou components could exist, but ours is the first example where the geometry of the components and the dynamics of f have been completely described; previously it had been thought that such examples might be too pathological to admit a concise description, as is given in this paper. Markus Baumgartner [8] has used our methods to describe the topology of multiply connected Fatou components in other examples, including Baker's original example.

Given an entire function f, the escaping set of f is defined as

$$I(f) = \{z : f^n(z) \to \infty\}.$$

As with polynomials, $\mathcal{J}(f) = \partial I(f)$ for any entire function. This is due to Eremenko [24] in the transcendental case; the hard part is to show $I(f) \neq \emptyset$. The escaping set plays a more interesting role in transcendental dynamics because it can be subdivided according to rates of escape (for a polynomial, all escaping points escape at the same rate). For example, the "fast escaping set", $A(f)$, is defined as follows. Fix a large S_0 and inductively define

$$S_{n+1} = \max_{|z|=S_n} |f(z)|.$$

It easy to see this gives an upper bound for $|f^n(z)|$ if $|z| \leq S_0$. The fast escaping set

$$A(f) = \{z : \text{ there is a } k \geq 0 \text{ so that } |f^{n+k}(z)| \geq S_n \text{ for all } n \geq 0\}.$$

are the points that almost achieve the upper bound. The fast escaping set was introduced by Bergweiler and Hinkkanen in [11], and has come to play a crucial role in transcendental dynamics. By a theorem of Baker [5], multiply connected components of the Fatou set lie in the escaping set, but a stronger result of Rippon and Stallard [41] implies that the closure of each Fatou component is in $A(f)$. Thus in our example, $A(f) \cap \mathcal{J}(f)$ contains Jordan curves and so has dimension 1. On the other hand, we will prove that $I(f) \setminus A(f)$ is non-empty, but has Hausdorff dimension zero. Moreover, we will show that $\dim(\mathbb{C} \setminus A(f))$ may be taken as small as we wish. Thus for the function f given by the proof of Theorem 1.1, all points in the plane iterate to infinity at the same rate, except for an exceptional set of small Hausdorff dimension. In this sense, our example behaves like a polynomial whose Julia set is a Cantor set of small dimension, i.e., it has "simple dynamics" (however, our example also has unbounded orbits that are not escaping, which a polynomial can't have).

We summarize the discussion above (and other results we will prove) as follows.

Theorem 1.3. *There is a transcendental entire function f so that*

(1) $\mathcal{H}^1(\mathcal{J}(f) \cap D(x,r)) = O(r)$ *for every disk $D(x,r)$ in the plane, i.e.,* 1*-dimensional planar Hausdorff measure on $\mathcal{J}(f)$ is upper Ahlfors regular.*
(2) *Every Fatou component Ω is a bounded, infinitely connected domain whose boundary consists of a countable number of C^1 curves, and the accumulation set of these curves is the outer component of $\partial\Omega$ (the unique boundary component that separates Ω from ∞).*
(3) *The fast escaping set, $A(f)$, is the union of the closures of all the Fatou components. Thus, $A(f) \cap \mathcal{J}(f)$ is the union of boundaries of the Fatou components.*
(4) *Given any $\alpha > 0$, f may be chosen so* $\dim(\mathbb{C} \setminus A(f)) < \alpha$.
(5) $\dim(I(f) \setminus A(f)) = 0$.
(6) *$\mathcal{J}(f)$ has packing dimension* 1*, but $D(x,r) \cap \mathcal{J}(f)$ has infinite* 1*-dimensional packing measure for every $x \in \mathcal{J}(f)$ and $r > 0$.*
(7) *Given any function $\psi(t)$ so that* $\lim_{t\to\infty} \psi(t)t^{-n} = \infty$ *for every n, f may be chosen so that $|f(z)| = o(\psi(|z|))$ as $|z| \to \infty$, i.e., we can choose f to grow more slowly than any given super-polynomial function.*

By (1), the linear measure of $\mathcal{J}_0 = \mathcal{J}(f) \cap D(0,1)$ is $O(1)$, and the linear measure of $\mathcal{J}_n = \mathcal{J}(f) \cap \{z : 2^{n-1} \leq |z| \leq 2^n\}$ is at most $O(2^n)$ for $n \geq 1$. Thus the spherical linear measure of $\mathcal{J}_n$ is $O(2^{-n})$ for $n \geq 0$, so summing over n gives Theorem 1.1. Linear measure is not lower regular on $\mathcal{J}(f)$ because $\mathcal{J}(f)$ is not uniformly perfect; the Fatou set contains round annuli of arbitrarily large modulus that surround points of the Julia set (this is true whenever there are multiply connected Fatou components, see e.g., [12]).

The conclusion that $\dim(\mathbb{C} \backslash A(f)) < \alpha$ can't be improved to dimension zero because the set of points with bounded orbits always has positive Hausdorff dimension (e.g., see the proof of Corollary 2.11 in [37], or [45] which contains essentially the same argument). Examples of entire functions where points with bounded orbits have small Hausdorff dimension were also given by Bergweiler [10].

The final conclusion implies our example has order of growth zero, i.e.,

$$\rho(f) \equiv \limsup_{z \to \infty} \frac{\log \log |f(z)|}{\log |z|} = 0.$$

Moreover, we can choose it to be as "close to" polynomial growth as we wish.

In this paper we will often use the "big-Oh" and "little-oh" notation. If A_n and B_n are two quantities that depend on a parameter n, then $A_n = O(B_n)$ means that $\sup_n A_n/B_n < \infty$ and $A_n = o(B_n)$ means that $\lim_n A_n/B_n = 0$.

The idea for the construction originates in a series of papers [14], [15], [16], [17], [18], which construct entire functions by a method I call quasiconformal folding, although this method does not explicitly appear here. The chain of papers started with a conversation with Alex Eremenko during his visit to Stony Brook in March 2011, and was encouraged by further exchanges with both Eremenko and Lasse Rempe-Gillen. I thank Lasse Rempe for a detailed reading of the first draft of this paper and for his numerous helpful comments and suggestions. I also thank David Sixsmith for his many helpful comments on a later draft, and for pointing out an error in the proof, and to Jack Burkart for finding some mistakes. Many thanks to Walter Bergweiler and Markus Baumgartner for pointing out several errors in an earlier version and suggesting simplifications that significantly improved the paper. Extra thanks to Markus for his many detailed comments and corrections to later versions as well. An anonymous referee's report on an earlier version of this paper provided numerous corrections and questions that greatly improved the manuscript, and I am extremely grateful for the effort that went into that report. A second referee's report also provided valuable feedback and suggestions that further improved the paper.

2. Overview of the proof

This section gives an overview of the main definitions and steps of the proof.

We will define a family of entire functions depending on a positive integer parameter N, real parameters $\lambda, R > 1$ and an infinite subset $S \subset \mathbb{N} = \{1, 2, \dots\}$. Each f is defined as an infinite product of the form

$$f(z) = F_0(z) \cdot \prod_{k=1}^{\infty} F_k(z),$$

where F_0 is the Nth iterate of the quadratic polynomial $p_\lambda(z) = \lambda(2z^2 - 1)$, $F_k(z)$ is the constant function 1 if $k \notin S$, and

$$F_k(z) = \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right)$$

if $k \in S$. The sequences $\{n_k\}$, $\{R_k\}$ increase rapidly to ∞ and will be defined inductively in terms of the parameters $\{N, \lambda, R, S\}$. After giving the details of these definitions, we will quickly verify that the infinite product converges uniformly on compact subsets of the plane and prove that the resulting function has order of growth zero. The set S is only used to verify part (7) of Theorem 1.3; the reader may set $S = \mathbb{N}$ in order to simplify the proof and still get a function f with $\rho(f) = 0$ and satisfying conditions (1)-(6) in Theorem 1.3.

The zeros of F_k, $k \in S$ are evenly spaced on a circle of radius

$$r_k = R_k(1 + \frac{\log 2}{n_k} + O(n_k^{-2})).$$

This symmetric placement of the zeros is what allows us to estimate f precisely and leads to the differentiability of the boundaries of the Fatou components.

To understand the dynamics of f, define, for $k \geq 1$, the annuli

$$A_k = \{z : \frac{1}{4}R_k \leq |z| \leq 4R_k\},$$

$$B_k = \{z : 4R_k < |z| < \frac{1}{4}R_{k+1}\}.$$

Note that the A_k have bounded moduli, but the moduli of the B_k get larger and larger because $R_{k+1}/R_k \nearrow \infty$ rapidly (e.g. we will prove $R_{k+1} \geq 2R_k^2$). The main step of the proof is to prove the inclusions

$$f(B_k) \subset B_{k+1}, \quad \text{and} \quad A_{k+1} \subset f(A_k) \subset D_{k+2}, \quad k \geq 1, \tag{2.1}$$

where $D_k = \{z : |z| < R_k/4\}$ is the bounded complementary component of A_k. The first inclusion implies the annuli $\{B_k\}$ iterate uniformly to ∞ and hence they are in the Fatou set. Thus the Julia set is contained in $D_1 \bigcup \cup_{k\geq 1} A_k$.

Inside D_1, $f \approx F_0$, since all the other factors are close to 1 here. If λ is large, then the polynomial $p_\lambda(z) = \lambda(2z^2 - 1)$ has a Julia set that is a Cantor set with small dimension. The same is true for F_0, since this function is just an iterate of p_λ. Since f approximates F_0 in D_1, f restricted D_1 will also have an invariant set E of small dimension and points not in E will eventually iterate out of D_1. Let $\widetilde{E}$ be the subset of $\mathcal{J}(f)$ that eventually iterates into E. Then $X = \mathcal{J}(f) \setminus \widetilde{E}$, are the points whose orbits are in $\cup_k A_k$ infinitely often.

Points that map under f into $\mathcal{J}(f) \cap (D \setminus E)$ eventually re-enter A_1, and thus they actually land in some preimage of A_1. For non-positive indices we set

$$A_{-k} = f^{-k-1}(A_1) \cap D_1.$$

With this notation, the orbit of a point in X stays inside $A = \cup_{k=-\infty}^{\infty} A_k$ forever and we can associate to each such orbit an itinerary as follows. For each $z \in X$ we define a sequence of integers $k(z,n), n \geq 0$ by the condition $f^n(z) \in A_{k(z,n)}$. These sequences must always satisfy

$$k(z, n+1) \leq k(z,n) + 1 \text{ if } k(z,n) \geq 1 \tag{2.2}$$

because $f(A_k) \cap A_j = \emptyset$ for $j > k+1$ (by the third inclusion in 2.1), and

$$k(z, n+1) = k(z,n) + 1 \text{ if } k(z,n) \leq 0 \tag{2.3}$$

because $f(A_k) = A_{k+1}$ by definition for $k \leq 0$.

Any integer sequence is either eventually strictly increasing or it is not, so every sequence satisfying (2.2) and (2.3) must satisfy exactly one of the following two conditions:

(1) $k(z, n+1) \leq k(z,n)$ infinitely often,
(2) $k(z, n+1) = k(z,n) + 1$ for all large enough n.

These two conditions define subsets $Y, Z \subset X$ respectively. The set Z is the fast escaping part of the Julia set and it consists exactly of the closed C^1 curves that are the boundary components of Fatou components and so it has dimension 1. See Figure 1. The set Y contains points that do not escape as fast as possible; orbits in Y can either escape, remain bounded or oscillate (the "bungee set" in the terminology of [35]). We will show that the Hausdorff dimension of Y can be taken as close to zero as we wish by an appropriate choice of the parameters defining f, and that $Y \cap I(f) = I(f) \setminus A(f)$ has Hausdorff dimension zero.

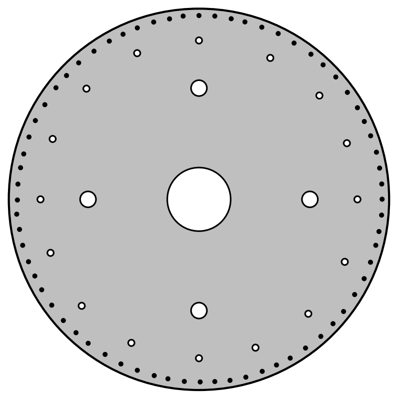

FIGURE 1. A model of a Fatou component. There is a C^1 outer boundary curve that separates the Fatou component from ∞ and this curve is the accumulation set of other boundary curves; these are grouped into levels which lie on curves roughly parallel to the outer boundary. The figure is not to scale; the levels of boundary curves lie in an annulus A_k of bounded modulus, but the component contains an annulus B_{k-1} of huge modulus. The single innermost boundary component is sometimes replaced by a bounded number of curves.

3. Minkowski, Hausdorff and packing dimensions

In this section we recall the definition of Minkowski, packing and Hausdorff dimensions, and some basic properties that we will need. For further details see [21].

For a bounded set $K \subset \mathbb{R}^n$, let $N(K, \epsilon)$ denote the minimal number of sets of diameter at most ϵ needed to cover K. We define the upper Minkowski dimension as

$$\overline{\mathrm{Mdim}}(K) = \limsup_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon},$$

and the lower Minkowski dimension

$$\underline{\mathrm{Mdim}}(K) = \liminf_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon}.$$

If the two values agree, the common value is simply called the Minkowski dimension of K and denoted by $\mathrm{Mdim}(K)$. Note that the upper Minkowski dimension of a set and its closure agree. If a connected set K has $\overline{\mathrm{Mdim}}(K) = 1$, then it is easy to check that $\mathrm{Mdim}(K)$ exists and equals 1.

There is an alternate formulation of the upper Minkowski dimension in terms of dyadic cubes and Whitney covers that is convenient to use in many cases, including in the current paper. For $n \in \mathbb{Z}$, we let $\mathcal{D}_n$ denote the collection of nth generation closed dyadic intervals

$$Q = [j2^{-n}, (j+1)2^{-n}],$$

and let $\mathcal{D}$ be the union of $\mathcal{D}_n$ over all integers n. A dyadic cube in $\mathbb{R}^d$ is any product of dyadic intervals that all have the same length. The side length of such a square is denoted $\ell(Q) = 2^{-n}$ and its diameter is denoted $|Q| = \sqrt{d}\ell(Q)$. Each dyadic cube is contained in a unique dyadic cube $Q^\uparrow$ with $|Q^\uparrow| = 2|Q|$; we call $Q^\uparrow$ the parent of Q. The *nested property* of dyadic cubes says that any two dyadic cubes either have disjoint interiors or one is contained inside the other.

Suppose $\Omega \subset \mathbb{R}^d$ is open. Every point of Ω is contained in a dyadic cube such that $Q \subseteq \Omega$ and $|Q| \leq \mathrm{dist}(Q, \partial\Omega)$. Thus every point is contained in a maximal such cube. By maximality, we have $\mathrm{dist}(Q^\uparrow, \partial\Omega) \leq |Q^\uparrow|$ and therefore $\mathrm{dist}(Q, \partial\Omega) \leq |Q^\uparrow| + |Q| = 3|Q|$. Thus the collection of such cubes forms a *Whitney decomposition*, i.e., a collection of cubes $\{Q_j\}$ in Ω that are disjoint except along their boundaries, whose union covers Ω and that satisfy

$$\frac{1}{\lambda}\mathrm{dist}(Q_j, \partial\Omega) \leq |Q_j| \leq \lambda \mathrm{dist}(Q_j, \partial\Omega),$$

for some finite λ.

For any compact set $K \subset \mathbb{R}^d$ we can define an exponent of convergence

$$\alpha = \alpha(K) = \inf\left\{\alpha : \sum_{Q \in \mathcal{W}} |Q|^\alpha < \infty\right\}, \tag{3.1}$$

where the sum is taken over all cubes in some Whitney decomposition $\mathcal{W}$ of $\Omega = \mathbb{R}^d \setminus K$ that are within distance 1 of K (we have to drop the "far away" cubes or the series might not converge). It is easy to check that α is independent of the choice of Whitney decomposition. The following is Lemma 2.6.1 of [21].

Lemma 3.1. *For any compact set K in $\mathbb{R}^d$, we have $\alpha(K) \leq \overline{\mathrm{Mdim}}(K)$. If K also has zero Lebesgue measure then $\alpha(K) = \overline{\mathrm{Mdim}}(K)$.*

Given any set K in a metric space X, we define the α-dimensional Hausdorff content as

$$\mathcal{H}^\alpha_\infty(K) = \inf\left\{\sum_i \mathrm{diam}(U_i)^\alpha : K \subset \bigcup_i U_i\right\},$$

where $\{U_i\}$ is a countable cover of K by any sets. The Hausdorff dimension of K is defined to be

$$\dim(K) = \inf\{\alpha : \mathcal{H}^\alpha_\infty(K) = 0\}.$$

More generally we define

$$\mathcal{H}^\alpha_\epsilon(K) = \inf\left\{\sum_i \mathrm{diam}(U_i)^\alpha : K \subset \bigcup_i U_i, \mathrm{diam}(U_i) < \epsilon\right\},$$

where each U_i is now required to have diameter less than ϵ. The α-dimensional Hausdorff measure of K is defined as

$$\mathcal{H}^\alpha(K) = \lim_{\epsilon\to 0} \mathcal{H}^\alpha_\epsilon(K).$$

If the metric space X is the Euclidean plane, then we refer to $\mathcal{H}^\alpha$ as α-dimensional planar measure; if X is the Riemann sphere, then we call it α-dimensional spherical measure. The statement of Theorem 1.1 uses the latter measure. More precisely, the theorem is equivalent to saying there is a $C < \infty$, so that for any $\epsilon > 0$, $\mathcal{J}(f)$ can be covered by disks $\{D(x_k, r_k)\}$ of radius at most ϵ so that

$$\sum_k \frac{r_k}{1+|x_k|^2} \leq C.$$

If we admit only open sets in the covers of K, then the value of $\mathcal{H}^\alpha_\epsilon(K)$ does not change. This is also true if we use only closed sets or use only convex sets. Using only balls might increase $\mathcal{H}^\alpha_\epsilon$ by at most a factor of 2^α, since any set K is contained in a ball of at most twice the diameter. Still, the values for which $\mathcal{H}^\alpha(K) = 0$ are the same whether we allow covers by arbitrary sets or only covers by balls.

A standard result (e.g. Proposition 1.2.6 of [21]) says that

$$\mathcal{H}^\alpha(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^\alpha_\infty(E) = 0$$

and therefore

$$\begin{aligned}\dim E &= \inf\{\alpha\colon \mathcal{H}^\alpha(E)=0\} = \inf\{\alpha\colon \mathcal{H}^\alpha(E)<\infty\}\\ &= \sup\{\alpha\colon \mathcal{H}^\alpha(E)>0\} = \sup\{\alpha\colon \mathcal{H}^\alpha(E)=\infty\}\,.\end{aligned}$$

The following relationship to Minkowski dimension is clear

$$\dim(K) \le \underline{\mathrm{Mdim}}(K) \le \overline{\mathrm{Mdim}}(K). \tag{3.2}$$

For any set E in a metric space, define the α-dimensional packing pre-measure by

$$\widetilde{\mathcal{P}}^\alpha(E) = \lim_{\epsilon\downarrow 0}\left(\sup\sum_{j=1}^{\infty}(2r_j)^\alpha\right),$$

where the supremum is over all collections of disjoint open balls $\{B(x_j,r_j)\}_{j=1}^\infty$ with centers in E and radii $r_j<\epsilon$. This pre-measure is finitely sub-additive, but not countably sub-additive. Define the packing measure in dimension α:

$$\mathcal{P}^\alpha(E) = \inf\left\{\sum_{i=1}^{\infty}\widetilde{\mathcal{P}}^\alpha(E_i) : E\subset\bigcup_{i=1}^{\infty}E_i\right\}. \tag{3.3}$$

The pre-measure of a set and its closure are the same, so we may assume the sets $\{E_i\}$ in the definition are closed. It is easy to check that $\mathcal{P}^\alpha$ is a metric outer measure, hence all Borel sets are $\mathcal{P}^\alpha$-measurable. Finally, define the packing dimension of E:

$$\mathrm{Pdim}(E) = \inf\{\alpha : \mathcal{P}^\alpha(E)=0\}\,. \tag{3.4}$$

However, it is usually more convenient to compute packing dimension using the following result (Proposition 2.7.1 of [21]):

Lemma 3.2. *The packing dimension of any set A in a metric space may be expressed in terms of upper Minkowski dimensions:*

$$\mathrm{Pdim}(A) = \inf\left\{\sup_{j\ge 1}\ \overline{\mathrm{Mdim}}(A_j)\ :\ A\subset\bigcup_{j=1}^{\infty}A_j\right\}, \tag{3.5}$$

where the infimum is over all countable covers of A. Since the upper Minkowski dimension of a set and its closure are the same, we can assume all the sets $\{A_j\}$ above are closed.

From this result it is immediate that

$$\dim(E) \le \mathrm{Pdim}(E) \le \overline{\mathrm{Mdim}}(E). \tag{3.6}$$

We will use Lemma 3.1 in Section 20 to prove that bounded pieces of the the Julia set in Theorem 1.1 have Minkowski dimension 1, and hence the whole Julia set has packing dimension 1 by Lemma 3.2.

4. A Cantor repeller

In this section we give an example of a polynomial whose Julia set has small Hausdorff dimension. This will be used later to give the first term, F_0, of the infinite product defining our entire function.

Lemma 4.1. *Let $p_\lambda(z) = \lambda(2z^2 - 1)$ where $\lambda \geq 1$. The Julia set of p_λ is a Cantor subset of $[-1, 1]$ whose upper Minkowski dimension tends to zero as $\lambda \nearrow \infty$.*

Proof. The iterates of $T_2(z) = 2z^2 - 1$ are fairly easy to understand. The map $z \mapsto \frac{1}{2}(z + \frac{1}{z})$ maps $\mathbb{D}^* = \{z : |z| > 1\}$ to $U = \mathbb{C} \setminus [-1, 1]$ and conjugates the action of z^2 on $\mathbb{D}^*$ to the action of T_2 on U. Thus the Julia set for T_2 is the segment $[-1, 1]$ and points off this segment are iterated towards ∞. Note that 0 is mapped to -1.

If we replace T_2 by $p_\lambda = \lambda \cdot T_2$ with $\lambda > 1$, then some interval around 0 is mapped outside $[-1, 1]$ and then iterates to ∞. Thus the Julia set of the new map is a Cantor set of Hausdorff dimension < 1. The Julia set is contained in two intervals $I, -I$ whose endpoints maps to ± 1; a little arithmetic shows that

$$I = [\sqrt{\frac{1}{2} - \frac{1}{2\lambda}}, \sqrt{\frac{1}{2} + \frac{1}{2\lambda}}].$$

Then $|p'_\lambda(z)| = 4\lambda|z|$ is minimized over I at the left endpoint, so

$$|p'_\lambda| \geq 4\lambda\sqrt{\frac{1}{2} - \frac{1}{2\lambda}} \geq 2\lambda, \tag{4.1}$$

on I if $\lambda \geq 2$. Thus the Julia set of p_λ is covered by 2^n n-generation pre-images of these intervals and each has size at most $(2\lambda)^{-n}$, and so the upper Minkowski dimension of the Julia set is at most $\frac{\log 2}{\log 2 + \log \lambda}$. This tends to zero as λ increases. □

The level lines of the iterates $\{|p_\lambda^n| = 2\}$ form nested loops around the Julia set and their length decreases exponentially with n. See Figure 2. A more precise statement that we shall need later is (the proof is easy and left to the reader):

Lemma 4.2. *Suppose p_λ is as above. For $r \geq 2$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, let γ_n be a connected component of $\{|p_\lambda^n| = r\}$. There is a constant C_λ, so that* $\mathrm{diam}(p_\lambda(\gamma_n)) \geq C_\lambda \mathrm{diam}(\gamma_n)$ *and C_λ may be taken as large as we wish by taking λ large enough.*

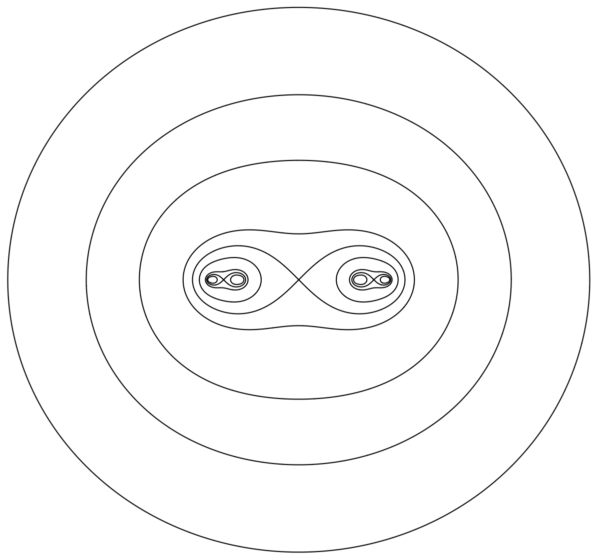

FIGURE 2. The level lines of iterates of $p = 2 \cdot T_2$. This shows inverse images of three circles of radius $\frac{1}{2}r, r, 2r$ under p, p^2, p^3. The value $r = 14$ was chosen so that $p^{-2}(|z| = r)$ would contain the critical point at 0.

If we holomorphically perturb p_λ by multiplying by an entire function that is close to 1 on a disk $D = D(0, \frac{1}{2}R)$, then the new function f is a polynomial-like map of degree 2 and pre-images of the unit circle will be close to the pre-images under p_λ. Thus there is a Cantor set E of small dimension that is invariant under f and so that any point outside E eventually iterates outside D. The inverse images of ∂D are finite unions of closed curves whose total length decays exponentially.

Next we record a calculation that we will need in the next section.

Lemma 4.3. *Let p_λ^n denote the nth iterate of p_λ. For $n \geq 1$, $|(p_\lambda^n)''(0)| \geq (4\lambda)^n$.*

Proof. For $n = 1$ an easy calculation shows $(p_\lambda^n)''(z) \equiv 4\lambda$. The other cases we prove inductively. Let $g_n = p_\lambda^n$, the nth iterate of p_λ. The $g_1(z) = \lambda(2z^2 - 1)$ and

$g_1'(z) = 4\lambda z$. By two applications of the chain rule,

$$\begin{aligned} g_n''(0) &= (g(g_{n-1}))''(0) \\ &= (g'(g_{n-1})g_{n-1}')'(0) \\ &= g''(g_{n-1}(0))(g_{n-1}'(0))^2 + g'(g_{n-1}(0))g_{n-1}''(0) \end{aligned}$$

By the chain rule $g_n'(0) = 0$ for $n \geq 1$, so the left term in the last line above is zero. Moreover, is easy to check that $|g_n(0)| \geq 1$ for $n \geq 1$, so using the induction hypothesis, we get

$$|g_n''(0)| = |g'(g_{n-1}(0))g_{n-1}''(0)| = 4\lambda|g_{n-1}(0))|(4\lambda)^{n-1} \geq (4\lambda)^n. \quad \square$$

5. The definition of f

Rather than produce a single entire function, we will define a collection of entire functions defined by related infinite products. Each element of the family will depend on the choice of a natural number N, two positive real numbers λ and R and an infinite subset S of the natural numbers $\mathbb{N} = \{1, 2, \dots\}$; we will assume $\{1\} \in S$. We also assume throughout that $\min(\lambda, R) \geq 4$, although several additional lower bounds will be imposed later. For a first reading, it might be simplest to take $S = \mathbb{N}$; this choice suffices to prove parts (1)-(6) of Theorem 1.3 and gives a function with order of growth zero. Sparser sets are only needed to verify part (7) of that theorem.

Let $F_0(z) = p_\lambda^N(z)$, which we defined earlier as the Nth iterate of $p_\lambda(z) = \lambda(2z^2-1)$. Then F_0 has degree $m = 2^N$ with leading coefficient $(2\lambda)^{m-1}$. If $R > 0$ is large enough, then $|z| \geq R$ implies

$$\frac{1}{2} \leq \left| \frac{F_0(z)}{(2\lambda)^{m-1}z^m} \right| \leq \frac{3}{2}, \text{ if } |z| \geq R. \tag{5.1}$$

Assume R is large enough that this holds.

Suppose $S \subset \mathbb{N} = \{1, 2, 3 \dots\}$ is an infinite set and that $1 \in S$. For $k \in \mathbb{N}$, let $S_k = S \cap \{1, \dots, k\}$ and let $\#(S_k)$ denote the number of elements in S_k. Note that $\#(S_1) = 1$ since we assumed $1 \in S$ and that $\#(S_k) \leq k$.

Next define

$$n_k = \begin{cases} 2^{N+\#(S_k)-1} & \text{if } k \in S, \\ 0 & \text{if } k \notin S. \end{cases}$$

Note that $n_1 = 2^N$, and for $k \in S$ we have $n_k \geq 2^N \geq 2$. In the special case $S = \mathbb{N}$, we have $n_k = 2^{N+k-1}$ for all $k \geq 1$.

We define an increasing sequence of real numbers $\{R_k\}_{k=1}^{\infty}$ and two sequences of polynomials $\{f_k\}_{k=0}^{\infty}$ and $\{F_k\}_{k=1}^{\infty}$ inductively as follows. Set $f_0 = F_0$, where F_0 is as defined above. Set

$$R_1 = 2R \tag{5.2}$$

where R is as above. Assuming $k \geq 1$ and R_k, f_{k-1} and F_{k-1} have already been defined, set $F_k(z) = 1$ if $k \notin S$ and for $k \in S$ set

$$F_k(z) = 1 - \frac{1}{2}(\frac{z}{R_k})^{n_k}. \tag{5.3}$$

For all $k \geq 1$ we set

$$f_k(z) = f_{k-1}(z) \cdot F_k(z) = \prod_{j=0}^{k} F_j(z), \tag{5.4}$$

$$R_{k+1} = M(f_k, 2R_k) = \max\{|f_k(z)| : |z| = 2R_k\}. \tag{5.5}$$

Since F_k is only non-trivial when $k \in S$, the product defining f_k could be taken just over the indices in the set S, i.e.,

$$f_k(z) = F_0(z) \cdot \prod_{j \in S_k} F_j(z), \tag{5.6}$$

and sometimes it will be convenient to view it this way.

Lemma 5.1. *$R_k \nearrow \infty$.*

Proof. By the product rule,

$$f_k''(0) = \sum_j F_j''(0) \prod_{k \neq j} F_k(0) + \sum_j \sum_{n \neq j} F_j'(0) F_n'(0) \prod_{k \neq j,n} F_k(0).$$

Since for $k \geq 1$ we have $F_k(0) = 1$ and $F_k'(0) = F_k''(0) = 0$, it is then easy to deduce that $|f_k''(0)| = |F_0''(0)| \geq (4\lambda)^N > 4\lambda$ by Lemma 4.3. By the Cauchy estimates

$$4\lambda \leq |f_k''(0)| \leq \frac{2M(f_k, r)}{r^2},$$

for any $r > 0$. Taking $r = 2R_k$, and using $\lambda \geq 2$ gives

$$R_{k+1} \geq \frac{1}{2}(2R_k)^2 4\lambda \geq 8R_k^2\lambda \geq 16R_k^2. \tag{5.7}$$

Since $R_1 \geq R > 1$, this implies $\{R_k\}$ increases to ∞ at least exponentially. $\square$

A simple induction using (5.7) shows:

$$R_{k+1} \geq 4^{2^k} R^{2^k}, k \geq 1. \tag{5.8}$$

This is the first of several lower bounds on the growth of $\{R_k\}$ that we will give.

Lemma 5.2. *The infinite product*

$$f(z) = \prod_{k=0}^{\infty} F_k(z) = F_0(z) \cdot \prod_{k \in S} F_k(z) = \lim_{k \to \infty} f_k(z), \tag{5.9}$$

converges uniformly on compact sets of $\mathbb{C}$.

Proof. Fix $s > 0$ and choose j so that $R_j > 2s$. Then for $|z| \leq s$ and $k \geq j$, either $F_k \equiv 1$ (if $k \notin S$) or

$$F_k(z) \quad = \quad 1 - \frac{1}{2}(\frac{z}{R_k})^{n_k} = 1 + O(2^{-n_k}) = 1 + O(2^{-2^k}).$$

Hence the infinite product converges uniformly on $\{|z| \leq s\}$. □

In particular

$$f(z) = \lim_{k \to \infty} f_k(z) = \prod_{k=0}^{\infty} F_k(z) \tag{5.10}$$

defines an entire function. The rest of the paper is devoted to proving this function satisfies the conclusions of Theorems 1.1 and 1.3.

6. Some product estimates

Here we record some simple consequences of the definitions in the previous section that we will use later to approximate our infinite product by finite ones.

Let

$$m_k = \max(n_1, \ldots, n_k) = 2^{N+\#(S_k)-1}. \tag{6.1}$$

This is clearly increasing and $m_k = n_k$ if $k \in S$ More generally, $m_k = n_j$ where j is the last element of S_k. (When $k \in S$, m_k and n_k can be used interchangeable; we will most often use n_k, but not universally.) In the special case $S = \mathbb{N}$, then $m_k = n_k$ for all $k \geq 1$, which is one reason the proof in this case is simpler.

Note that

$$m_k \leq 2m_{k-1} \text{ and } m_k \geq 2^N \tag{6.2}$$

for all k. Furthermore,

$$\begin{aligned} \deg(f_k) &= \sum_{j=0}^{k} \deg(F_j) = 2^N + \sum_{j\in S_k} n_j \\ &= 2^N\left(1+\sum_{j\in S_k} 2^{\#(S_j)-1}\right) = 2^N\left(1+\sum_{n=0}^{\#(S_k)-1} 2^n\right) \\ &= 2^N(1+(2^{\#(S_k)}-1) = 2^{N+\#(S_k)} \\ &= 2m_k \end{aligned}$$

and this equals $2\deg(F_k)$ if $k \in S$. Hence

$$\deg(f_k) = 2\deg(F_k) \quad \text{if } k \in S. \tag{6.3}$$

Also for future reference, we note that we have shown

$$2^N + \sum_{j\in S_k} n_j = 2m_k. \tag{6.4}$$

If $k \in S$, this becomes

$$2^N + \sum_{j\in S_{k-1}} n_j = 2n_{k-1} = n_k. \tag{6.5}$$

We will use the following consequences of (5.7) later in the paper

Lemma 6.1. *Suppose $\{R_k\}$ is as above and $m \geq 1$. Then*

$$\prod_{j=1}^{k-1}\left(1+\frac{R_j}{R_k}\right) = 1+O\left(\frac{R_{k-1}}{R_k}\right) = 1+O(R_k^{-1/2}) \tag{6.6}$$

$$\prod_{j=1}^{k-1}\left(1+\left(\frac{R_j}{R_k}\right)^m\right) = 1+O(R_k^{-m/2}), \tag{6.7}$$

$$\prod_{j=k+1}^{\infty}\left(1+\frac{R_k}{R_j}\right) = 1+O\left(\frac{R_k}{R_{k+1}}\right) = 1+O(R_k^{-1}) \tag{6.8}$$

$$\prod_{j=k+1}^{\infty} F_j(z) = 1+O(R_k^{-1}), \qquad |z| \leq 4R_k. \tag{6.9}$$

Proof. We prove (6.7) since it contains (6.6) as a special case. Note that (5.7) implies that that $R_{k-1} \leq \sqrt{R_k/8}$ and $R_j \leq R_{j+1}/2$, and hence that for $j = 1, \dots, k$,

$$\begin{aligned}
\prod_{j=1}^{k-1}\left(1+\left(\frac{R_j}{R_k}\right)^m\right) &= \exp\left(\log\prod_{j=1}^{k-1}\left(1+\left(\frac{R_j}{R_k}\right)^m\right)\right) = \exp\left(\sum_{j=1}^{k}\log\left(1+\left(\frac{R_j}{R_k}\right)^m\right)\right) \\
&\leq \exp\left(\left(\frac{\sqrt{R_k}}{2\sqrt{2}R_k}\right)^m\left(1+\frac{1}{2}+\frac{1}{4}+\dots\right)^m\right) \leq \exp\left(\left(\frac{1}{\sqrt{2R_k}}\right)^m\right) \\
&\leq 1 + R_k^{m/2}
\end{aligned}$$

where the last line uses $e^x \leq 1 + 2x$ if $0 \leq x \leq 1$.

The proof of (6.8) is very similar:

$$\begin{aligned}
\prod_{j=k+1}^{\infty}(1+\frac{R_k}{R_j}) &= \exp\left(\sum_{j=k+1}^{\infty}\log\left(1+\frac{R_k}{R_j}\right)\right) \leq \exp\left(\sum_{j=k+1}^{\infty}\frac{R_k}{R_j}\right) \\
&\leq \exp\left(\frac{1}{R_k}\left(1+\frac{1}{2}+\frac{1}{4}+\dots\right)\right) \leq \exp\left(\frac{2}{R_k}\right) \\
&\leq 1+\frac{4}{R_k}
\end{aligned}$$

Finally we consider (6.9). Assume $|z| \leq 4R_k$. Then

$$\begin{aligned}
\prod_{j=k+1}^{\infty}F_j(z) &= \prod_{j=k+1}^{\infty}\left(1-\frac{1}{2}\left(\frac{z}{R_j}\right)^{n_j}\right) = \exp\left(\sum_{j=k+1}^{\infty}\log\left(1-\frac{1}{2}\left(\frac{z}{R_j}\right)^{n_j}\right)\right) \\
&\leq \exp\left(\sum_{j=k+1}^{\infty}\frac{1}{2}\left(\frac{4R_k}{R_j}\right)^{n_j}\right) \leq \exp\left(\sum_{j=k+1}^{\infty}2\frac{R_k}{R_j}\right) \\
&\leq \exp\left(2R_k\left(\frac{1}{R_k^2}+\frac{1}{R_k^4}+\dots\right)\right) \leq \exp\left(\frac{2}{R_k}\left(1+\frac{1}{R_k^2}+\dots\right)\right) \\
&\leq \exp\left(\frac{4}{R_k}\right) \leq 1+\frac{8}{R_k}. \quad \square
\end{aligned}$$

7. The order of growth

Lemma 7.1. *The function f defined by (5.9) has order of growth* 0.

Proof. We need an upper bound for the $\{R_k\}$. Assume $|z| = 2R_k$. From (5.1) we deduce that if $|z| \geq R$, then

$$|F_0(z)| \leq \frac{3}{2}(2\lambda)^{2^N-1}|z|^{2^N}.$$

Equation (5.3) implies that if $|z| = 2R_k$, and $j \in S_k$, then since $R \geq 1$,

$$|F_j(z)| \leq 1 + \frac{1}{2}\left|\frac{z}{R_j}\right|^{n_j} \leq 1 + \frac{1}{2}\left|\frac{2R_k}{R_j}\right|^{n_j} \leq 1 + \frac{1}{2}|2R_k|^{n_j} \leq (2R_k)^{n_j}.$$

for any $k \geq 1$. Thus for $|z| = R_k$,

$$\begin{aligned} |f_k(z)| &\leq |F_0(z)| \cdot \prod_{j \in S_k} |F_j(z)| \\ &\leq \frac{3}{2}(2\lambda)^{2^N-1}|2R_k|^{2^N} \cdot (2R_k)^{\sum_{j\in S_k} n_j} \\ &\leq \frac{3}{2}(2\lambda)^{2^N-1} \cdot (2R_k)^{2^N+\sum_{j\in S_k} n_j} \\ &\leq \frac{3}{2}(2\lambda)^{2^N-1} \cdot (2R_k)^{2m_k} \end{aligned}$$

If we let $C = \frac{3}{2}(2\lambda)^{2^N-1}$ and use the fact that $m_k = 2^{N+\#(S_k)-1} \leq 2^{N+k}$, then this becomes $R_{k+1} \leq C(2R_k)^{2^{N+k+1}}$. Iterating this inequality gives

$$\begin{aligned} R_{k+2} &\leq C \cdot (2R_{k+1})^{2^{N+k+2}} \\ &\leq C \cdot (2C \cdot (2R_k)^{2^{N+k+1}})^{2^{N+k+2}} \\ &\leq C \cdot (2C)^{2^{N+k+2}} \cdot (2R_k)^{2^{2N+2k+3}} \end{aligned}$$

Hence if $2R_k \leq |z| \leq 2R_{k+1}$, then

$$\frac{\log\log|f(z)|}{\log|z|} \leq \frac{\log\log R_{k+2}}{\log 2R_k} \leq \frac{\log\log\left(C \cdot (2C)^{2^{N+k+2}} \cdot (2R_k)^{2^{2N+2k+3}}\right)}{\log 2R_k}$$

Using $\log(x+y) \leq \log x + \log y$, for $x, y \geq 2$, twice implies that this is bounded by

$$\frac{\log\log C}{\log R_k} + \frac{\log\log(2C)^{2^{N+k+2}}}{\log R_k} + \frac{\log\log(2R_k)^{2^{2N+2k+3}}}{\log R_k} \tag{7.1}$$

The first term in (7.1) clearly tends to 0. Using (5.8) we see that the second term is bounded by

$$\begin{aligned} &\leq \frac{\log\left(2^{N+k+2}\log 2C\right)}{\log R^{2^k}} \\ &\leq \frac{(N+k+2)\log 2 + \log\log 2C}{2^k \log R} \end{aligned}$$

and hence it also tends to zero. The third term in (7.1) is bounded by

$$\begin{aligned} &\le \frac{\log\left[2^{2N+2k+3}\log 2R_k\right]}{\log R_k} \\ &\le \frac{(2N+2k+3)\log 2 + \log\log 2R_k}{\log R_k} \\ &\le \frac{(2N+2k+3)\log 2}{2^k \log R} + \frac{\log\log 2R_k}{\log R_k}. \end{aligned}$$

These both tend to zero as k tends to ∞, hence the order of f is zero. □

Note that f grows like a polynomial of degree $2m_k$ for $|z| \le R_{k+1}$, so choosing m_k to grow slowly, we can make f grow more slowly than any super-polynomial function. We can make m_k grow as slowly as we wish, by taking $S \subset \mathbb{N}$ sufficiently sparse. This proves part (7) of Theorem 1.3.

8. The growth of $\{R_k\}$

As explained in Section 2, the Julia set of f roughly consists of points whose orbits land near near one of the circles of radius R_k, $k \in S$, infinitely often. To make this a small set, we want landing near a circle to be a rare event, which means that we want the circles to be rather sparse. In other words, if $R_k \nearrow \infty$ rapidly, then we expect the Julia set to be small. We saw in equation (5.7) that $R_{k+1} \ge 16R_k^2$. However, in our estimates of Hausdorff dimension, we will need something stronger than this.

Lemma 8.1. *For $k \ge 1$,*

$$R_{k+1} \ge 2^{m_k} R_k^{2^{N-1}+m_{k-1}} \ge 2^{m_k} R_k^{2^{N-1}+m_k/2} \ge 2^N R_k^{2^N}.$$

Proof. The second and third inequalities are immediate from (6.2), so we only need to prove the first inequality. For any $k \in S_k$ it follows from (5.3) that

$$\frac{1}{2}\left(\frac{|z|}{R_k}\right)^{n_k} - 1 \le |F_k(z)| \le \frac{1}{2}\left(\frac{|z|}{R_k}\right)^{n_k} + 1. \tag{8.1}$$

For $k \notin S$, $F_k \equiv 1$, so using the left side of (8.1), and (5.1), we get for any k

$$\begin{aligned} R_{k+1} &= \max_{|z|=2R_k} |f_k(z)| \\ &\geq \max_{|z|=2R_k} |F_0(z)| \cdot \prod_{j\in S_k} \min_{|z|=2R_k} |F_k(z)| \\ &\geq 2^{2^N}\frac{1}{4}\lambda^{2^N-1}R_k^{2^N} \cdot \prod_{j\in S_k}\left(\frac{1}{2}\left(\frac{2R_k}{R_j}\right)^{n_j} - 1\right) \end{aligned}$$

Next we will use $\lambda \geq 4$, (5.7) and (6.4)

$$\begin{aligned} &\geq 2^{2^N}R_k^{2^N} \cdot \prod_{j\in S_k}\left(2^{n_j-1}R_k^{n_j/2} - 1\right) \\ &\geq 2^{2^N}R_k^{2^N} \cdot \prod_{j\in S_k} 2^{n_j-2} \cdot \prod_{j\in S_{k-1}} R_k^{n_j/2} \\ &\geq 2^{2^N+\sum_{j\in S_k}(n_j-2)} \cdot R_k^{2^{N-1}+(2^N+\sum_{j\in S_{k-1}} n_j)/2} \\ &\geq 2^{2m_k-2\#(S_k)} \cdot R_k^{2^{N-1}+m_{k-1}} \\ &\geq 2^{m_k} \cdot R_k^{2^{N-1}+m_k/2} \end{aligned}$$

where, in the last line, we have used $m_k \leq 2m_{k-1}$ and $2\cdot\#(S_k) \leq 2^{\#(S_k)} \leq m_k$. $\square$

Corollary 8.2. *If $R \geq 8, N \geq 2$, then $4R_{k+1}4^{2m_k} \leq \frac{1}{4}R_{k+2}$.*

Proof. By Lemma 8.1 we have

$$\begin{aligned} \frac{1}{4}R_{k+2} &\geq \frac{1}{4}2^{m_k+1}R_{k+1}^{(2^{N-1}+m_k)/2} \geq \frac{1}{4}2^{m_k}R_{k+1}R_{k+1}^{m_k/2} \\ &\geq \frac{1}{4}2^{m_k}R_{k+1}(2R_k)^{m_k} \geq \frac{1}{4}2^{m_k}R_{k+1}2^{2^N}\cdot 8^{m_k} \\ &\geq 4R_{k+1}4^{2m_k}. \quad \square \end{aligned}$$

Corollary 8.3. *If $k \geq 1$, $R_{k+1} \geq (2R)^{2^{kN}}$.*

Proof. The case $k = 0$ is (5.2). By Lemma 8.1,

$$R_{k+1} \geq (R_k)^{2^N} \geq \left((2R)^{2^{(k-1)N}}\right)^{2^N} = (2R)^{2^{kN}}. \quad \square$$

Corollary 8.4. *Suppose $1 \leq s \leq k$. Then $R_k \geq 2^{(k-s)m_s}R_s$.*

Proof. Use Lemma 8.1, $m_j \geq 2$, and the fact that $\{m_j\}$ is non-decreasing to deduce that for $j \geq s$ we have

$$R_{j+1} \geq 2^{m_j} R_j^{m_j/2} \geq 2^{m_s} R_j.$$

The corollary is now obvious. $\square$

Lemma 8.5. *Let $M_k = m_1 \dots m_k$. For any $\alpha > 0$ and any $R > 1$,*

$$\sum_{k=1}^{\infty} 2^k M_k R_k^{-\alpha} < \infty. \tag{8.2}$$

For fixed α, the sum can be made as small as we wish by taking R sufficiently large.

Proof. Let $a_k = 2^k M_k R_k^{-\alpha}$ be the kth term of the series. We will use the ratio test for convergence. By Lemma 8.1 and the fact that $m_k \leq 2m_{k-1}$, we have

$$\frac{a_k}{a_{k-1}} = \frac{2M_k R_k^{-\alpha}}{M_{k-1} R_{k-1}^{-\alpha}} \leq \frac{2m_k R_{k-1}^{-(2^N + m_{k-1})\alpha/2}}{R_{k-1}^{-\alpha}} \leq 2m_k (R_{k-1}^{\alpha/2})^{-m_{k-1}} \leq 2m_k (R^{\alpha/4})^{-m_k}$$

Since $\lim_{n\to\infty} nx^{-n} = 0$ for any $x > 1$, we see that the ratio test is satisfied and the sum converges for any $R > 1$.

Clearly $\sum_k a_k \leq 2\sum_n nx^n$ where $x = R^{-\alpha/2}$. Using the differentiated geometric series, the latter sum is bounded by $2x/(1-x)^2$, so tends to zero as $R \nearrow \infty$. $\square$

9. The geometry of T_2

Let $T_2(z) = 2z^2 - 1$ be the degree two Chebyshev polynomial. The terms $\{F_k\}$ of our infinite product can be written in terms of $T_2(z^m)$. In this section we recall some facts about the geometry of T_2 that will be useful to us in understanding F_k.

Let $z_2 = -1/\sqrt{2}$ be the left root of T_2 and let $w_2 = 0$ be its critical point. Let Ω_2 be the component of $\{z : |T_2(z)| < 1\}$ that contains z_2. See Figure 3. Let $r_2 = \text{dist}(z_2, -1) = 1 - 2^{-1/2}$ and $\tilde{r}_2 = \text{dist}(z_2, w_2) = 1/\sqrt{2}$ and

$$D_2 = D(z_2, r_2) = D(-\frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}), \qquad \widetilde{D}_2 = D(z_2, \tilde{r}_2) = D(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

Lemma 9.1. *$|T_2| \geq 1$ on $\partial \widetilde{D}_2$ and ≤ 1 on ∂D_2. Thus $D_2 \subset \Omega_2 \subset \widetilde{D}_2$.*

Proof. Each point of $\partial \widetilde{D}_2$ is the same distance from z_2, and w_2 is the closest point of $\partial \widetilde{D}_2$ to the other zero of T_2. Hence the product of the distances from any point on

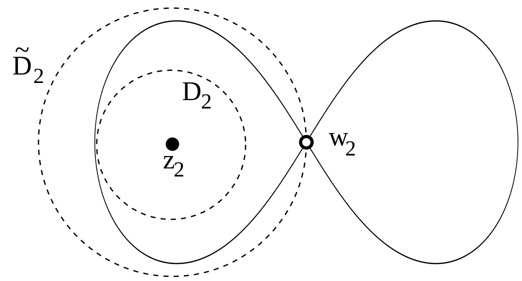

FIGURE 3. The black dot is z_2 and the white is w_2. Ω_2 is the left lobe of the curve and is trapped between D_2 and $\widetilde{D}_2$.

this circle to both zeros of T_2 is minimized at $w_2 = 0$. Thus $|T_2|$ (which is proportional to this product) takes its minimal value over $\widetilde{D}_2$ at w_2, giving the first inequality.

Because $T_2'(z) = 2z$,

$$\frac{d}{dr}|T_2|(z_2 + re^{i\theta}) \leq |T_2'(z_2 + re^{i\theta})| \leq |T_2'(z_2 - r)| = \frac{d}{dr}|T_2|(z_2 - r),$$

where we have used $T_2'(z) = 2z$ in the second step. Integrating gives

$$|T_2(z_2 + re^{i\theta})| \leq |T_2(z_2 - r)|.$$

Taking $r = r_2$ gives the second inequality. □

Let

$$H_m(z) = -T_2(\tilde{r}_2 z^m + z_2) = z^m(2 - z^m).$$

Note that

$$H_m'(z) = mz^{m-1}(2 - z^m) + z^m(-mz^{m-1}) = 2mz^{m-1}(1 - z^m),$$

and this shows all the non-zero critical points are on the unit circle. The complement of the level line $\gamma_m = \{z : |H_m(z)| = 1\}$ is an open set $\Omega_m = \mathbb{C} \setminus \gamma_m$ that has $m+2$ connected components, as illustrated in Figure 4. This includes a central component Ω_m^0 (the one containing 0), an unbounded component Ω_m^∞ and m other bounded components that each have one critical point on their boundary. We call these the "petals" of Ω_m and denote their union by Ω_m^p.

H_m is a m-to-1 branched covering map from Ω_m^0 to $\mathbb{D}$ with a single critical point at the origin. H_m is conformal from the interior of each petal to $\mathbb{D}$.

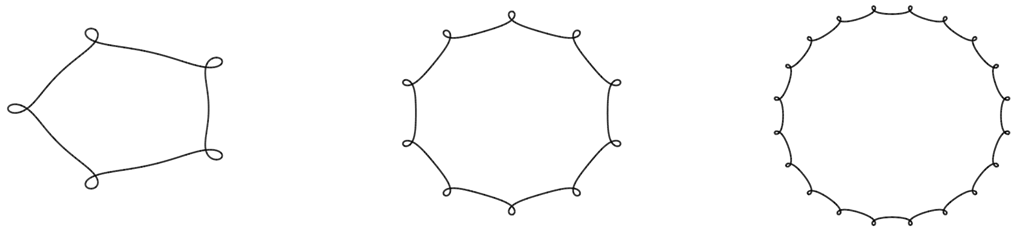

FIGURE 4. Level sets of the form $\{z : |T_2(z^m)| = 1\}$, for $m = 5, 10, 20$.

Lemma 9.1 says that

$$\partial\Omega_1^0 \subset \{z : \frac{r_2}{\tilde{r}_2} \leq |z| \leq 1\},$$

and so taking roots,

$$\partial\Omega_m^0 \subset \{z : (\frac{r_2}{\tilde{r}_2})^{1/m} \leq |z| \leq 1\}.$$

Since $r_2/\tilde{r}_2 = \sqrt{2} - 1$ an easy computation says we can take

$$\partial\Omega_m^0 \subset \{z : 1 - \frac{a}{m} \leq |z| \leq 1\},$$

where $a = -\log(\sqrt{2} - 1) \approx .881374 \leq 1$. Thus

Lemma 9.2. $\{z : |z| < 1 - \frac{1}{m}\} \subset \Omega_m^0 \subset \mathbb{D}$.

For the unbounded component we have

Lemma 9.3. $\{z : |z| > 1 - \frac{1}{m}\} \supset \Omega_m^\infty \supset \{z : |z| > 1 + \frac{2}{m}\}$ *if* $m \geq 2$.

Proof. Since Ω^0 and Ω^∞ are disjoint, the first inclusion is immediate from the first inclusion of Lemma 9.2. For the second inclusion we note that $\{|T_2| = 1\}$ is contained in a ball of radius $1 + \tilde{r}_2$ around z_2 and hence γ_1 is contained in a ball of radius $(1 + \tilde{r}_2)/\tilde{r}_2 = 1 + \tilde{r}_2^{-1}$ around 0. Thus

$$\partial\Omega_m^\infty \subset \{z : |z| \leq (1 + \sqrt{2})^{1/m}\}.$$

A simple calculation shows shows

$$(1 + \sqrt{2})^{1/m} \leq 1 + \frac{\sqrt{2}}{m \log(1 + \sqrt{2})} \approx 1 + \frac{1.60}{m} \leq 1 + \frac{2}{m}$$

and this proves the lemma. $\square$

10. f LOOKS LIKE H_{m_k} ON A_k

As discussed in Section 2, we define families of disjoint annuli

$$A_k = \{z : \frac{1}{4}R_k \leq |z| \leq 4R_k\}, \qquad B_k = \{z : 4R_k \leq |z| \leq \frac{1}{4}R_{k+1}\},$$

and one of the key steps in the proof of Theorem 1.1 is to show that $A_{k+1} \subset f(A_k)$. Both A_k and A_{k+1} are round annuli of the same modulus, and A_{k+1} contains the circle of radius R_{k+1} that, by the definition of R_{k+1}, hits the image of the circle of radius $2R_k$ that is contained in A_k. Thus it would be enough to show that f looks like a power function on A_k and that the image of A_k is a "roundish" annulus of large moduli. This is not true, because if $k \in S$, then f has zeros in A_k near the circle of radius R_k. However, this idea does work if we restrict to a sub-annulus of A_k. We set

$$V_k = \{z : \frac{3}{2}R_k \leq |z| \leq \frac{5}{2}R_k\}, \qquad U_k = \{z : \frac{5}{4}R_k \leq |z| \leq 3R_{k+1}\}.$$

Note that $V_k \subset U_k \subset A_k$. In this section we give estimates for f on A_k and U_k that we can use to prove the desired inclusions in a later section. We introduce the two annuli U_k, V_k so we can use the Cauchy estimates to turn bounds for f on U_k into estimates for f' on V_k.

Lemma 10.1. *For $k \in S$,*

$$F_k(z) = \frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k} H_{n_k}\left(\frac{z}{R_k}\right).$$

Proof. This is just arithmetic:

$$\begin{aligned}\frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k} H_{n_k}(\frac{z}{R_k}) &= \frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k}\left(\frac{z}{R_k}\right)^{n_k}\left(2 - (\frac{z}{R_k})^{n_k}\right) \\ &= \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right) = F_k(z). \quad \square\end{aligned}$$

We can break the infinite product defining f into three parts: F_k, the finite product over $j < k$ and the infinite product over $j > k$. On A_k, the infinite product approximates 1 and the finite product approximates $C_k z^{n_k}$ for some constant C_k. Thus the last lemma implies that f is very close to a multiple of H_{n_k} on A_k, and this leads to very precise estimates of f. This is the reason we defined F_k as we did.

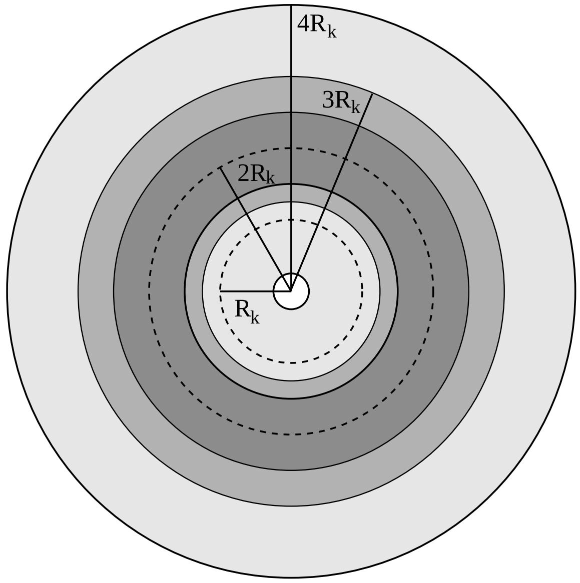

FIGURE 5. The nested annuli A_k (largest, light gray), U_k (medium) and V_k (smallest, darkest). The latter two are neighborhoods of $\{|z| = 2R_k\}$ but are separated from $\{|z| = R_k\}$.

Lemma 10.2. *For $k \in S$ and $z \in A_k$ we have*

$$f(z) = C_k H_{n_k}(\frac{z}{R_k}) \left(1 + O(R_k^{-1})\right) \tag{10.1}$$

where

$$C_k = (-1)^{\#S_{k-1}-1} 2^{-\#(S_{k-1})} (2\lambda)^{2^N-1} R_k^{n_k} \cdot \prod_{j \in S_{k-1}} R_j^{-n_j}. \tag{10.2}$$

Proof. Write

$$f(z) \quad = \quad F_0(z) \cdot \prod_{j=S_{k-1}} F_j(z) \cdot F_k(z) \cdot \prod_{j \in S \setminus S_k} F_j(z).$$

The second product is easy to handle using (6.9) and we get

$$\prod_{j \in S \setminus S_k} F_j(z) = 1 + O(R_k^{-1}), z \in A_k.$$

For the remaining terms, we use Lemma 10.1 (recall that $k \in S_k$ by assumption) and Equation (6.5)

$$\begin{aligned} F_0(z) \cdot \prod_{j \in S_k} F_j(z) &= z^{-2^N} F_0(z) \cdot \left(\prod_{j \in S_{k-1}} z^{-n_j} F_j(z) \right) \cdot z^{n_k} F_k(z) \\ &= z^{-2^N} F_0(z) \cdot \left(\prod_{j \in S_{k-1}} z^{-n_j} F_j(z) \right) \cdot \frac{1}{2} H_{n_k}(\frac{z}{R_k}) \cdot R_k^{n_k}. \end{aligned}$$

Since $\deg(F_0) = 2^N = n_1$ and $\deg(F_j) = n_j$, every term except the last one tends to a constant as $|z| \to \infty$ (and that constant is the coefficient of the highest degree term of the polynomial). In particular, the first term is

$$z^{-n_1} F_0(z) = (2\lambda)^{2^N - 1} \left(1 + O(R_k^{-1})\right),$$

for $z \in A_k$. The terms of the product over $j \in S_{k-1}$ are

$$z^{-n_j} F_j(z) \quad = \quad z^{-n_j} \cdot \left(1 - \frac{1}{2}\left(\frac{z}{R_j}\right)^{n_j}\right) = -\frac{1}{2} R_j^{-n_j} \cdot \left(1 - 2\left(\frac{R_j}{z}\right)^{n_j}\right).$$

If $z \in A_k$, then this becomes

$$= \quad -\frac{1}{2} R_j^{-n_j} \cdot \left(1 + O(\frac{R_j}{R_k}\right)^{n_j})).$$

Note that $n_k \leq 2^{N+k} \leq 2^N R^{2^k/2} \leq R_k^{1/2}$, so (6.7) applies to give

$$\prod_{j \in S_{k-1}} \left(1 - 2\left(\frac{R_j}{R_k}\right)^{n_j}\right) = 1 + O(R_k^{-1}).$$

This completes the proof. $\square$

Corollary 10.3. *For $k \notin S$ and $z \in A_k$ we have*

$$f(z) = C_k z^{2n_k}(1 + O(R_k^{-1}))$$

where $C_k = C_j$ where j is the last element of S_k (note that $j < k$ since $k \notin S_k$).

Proof. The proof mimics the previous proof, except that now the last non-trivial term of the product over S_k occurs for some $j \in S$ with $j < k$. By (6.4) we have $1 = z^{-2^N - \sum_{i \in S_j} n_i} \cdot z^{2n_j}$, so if we multiply the product $F_0 \cdot \prod_{i \in S_k} F_i$ by these two terms, the negative powers of z in the first term are paired with the non-trivial terms in the product, leaving the z^{2n_j} term to appear alone. Making this change, and estimating the products as before, proves the corollary. $\square$

Lemma 10.4. *For $k \geq 1$, $|C_{k+1}| \geq |C_k| \geq 1$.*

Proof. Consider the first inequality. For $k+1 \not\in S$, we have equality by definition (see Corollary 10.3). For $k+1 \in S$, using the definition and $n_{k+1} \geq 2m_k$ gives

$$|C_{k+1}/C_k| \geq \frac{1}{2} R_{k+1}^{n_{k+1}} \cdot R_k^{-m_k} \geq \frac{1}{2} R_{k+1}^{m_k} \cdot (R_{k+1}/R_k)^{m_k} \geq 1.$$

The second inequality follows by induction and a computation that $|C_1| > 1$. $\square$

If $j \in S$ is the nth element before k, then $n_j = n_k/2^n$ and $R_j \leq R_k^{2^{-n}}$. Thus

$$\prod_{s \in S_{k-1}} R_j^{n_j} \leq \prod_{n=1}^{k} R_k^{n_k/4^n} = R_k^{n_k \sum_{n=1}^{k} 4^{-n}} = R_k^{n_k/2}.$$

Putting this into (10.2) gives

$$|C_k| \geq (2\lambda)^{2^N - 1} R_k^{m_k/2} \geq 8R_k, \quad k \in S. \tag{10.3}$$

If $k \not\in S$, then we get

$$|C_k| \geq 8R_j, \tag{10.4}$$

where j is the last element of S_k.

Lemma 10.5. *For $k \geq 1, k \in S$ and $\frac{5}{4}R_k \leq |z| \leq 4R_k$ we have*

$$f(z) = C_k \cdot z^{2n_k} \cdot \left(1 + O\left(\frac{4}{5}\right)^{m_k}\right) \cdot \left(1 + O(R_k^{-1})\right).$$

Proof. Since $k \in S$, in (10.1) we can replace H_{n_k} with an approximation by a power function as follows:

$$\begin{aligned} H_{n_k}(\frac{z}{R_k}) &= -R_k^{-2n_k} z^{2n_k} \left(1 - 2\left(\frac{R_k}{z}\right)^{n_k}\right) \\ &= -R_k^{-2n_k} z^{2n_k} \left(1 + O\left(2\frac{R_k}{\frac{5}{4}R_k}\right)^{n_k}\right) \\ &= -R_k^{-2n_k} z^{2n_k} \left(1 + O\left(\left(\frac{4}{5}\right)^{n_k}\right)\right). \quad \square \end{aligned}$$

Corollary 10.3 and Lemma 10.5 both apply to the circle $\{|z| = 2R_k\}$, so for any $k \in \mathbb{N}$ we have the following estimate that we shall use repeatedly:

$$\frac{1}{2} \leq \frac{R_{k+1}}{|C_k|(2R_k)^{2m_k}} \leq 2. \tag{10.5}$$

Lemma 10.6. *For $k \geq 1, k \in S$ and $\frac{1}{4}R_k \leq |z| \leq \frac{4}{5}R_k$ we have*

$$f(z) = 2C_k \cdot R_k^{-n_k} \cdot z^{n_k} \cdot \left(1 + O\left(\frac{4}{5}\right)^{m_k}\right) \cdot \left(1 + O(R_k^{-1})\right).$$

In particular, this holds for $z \in U_k^*$.

Proof. Since $k \in S$, then in (10.1) we use

$$\begin{aligned} H_{n_k}(\frac{z}{R_k}) &= R_k^{-n_k} z^{n_k} \left(2 - \left(\frac{z}{R_k}\right)^{n_k}\right) \\ &= 2R_k^{-n_k} z^{n_k} \left(1 + O\left(\left(\frac{4}{5}\right)^{n_k}\right)\right) \quad \square \end{aligned}$$

Thus on U_k we have $f(z) = C_k z^{2m_k}(1 + h_k(z))$, where h_k is holomorphic on U_k and

$$|h_k(z)| = O\left(\left(\frac{4}{5}\right)^{m_k} + R_k^{-1}\right), \tag{10.6}$$

on U_k; this bound is clearly summable over k. Moreover, we can assume the sum is small, say less than $1/2$ if $m_1 = 2^N$ and R are large. We make this assumption from this point on.

For future use we let

$$\epsilon_k = C \cdot \left(\left(\frac{3}{4}\right)^{m_k} + R_k^{-1}\right), \tag{10.7}$$

where the constant is chosen so that $|h_k(z)| \leq \epsilon_k$ on U_k. Also note that $\sum_{k=1}^{\infty} \epsilon_k$ is as small as we wish if we choose the parameters defining f to all be large enough.

Corollary 10.7. *For $k \geq 1$, f' is non-zero on V_k.*

Proof. By the Cauchy estimate, for $z \in V_k$

$$|h_k(z)| \leq \frac{\epsilon_k}{\operatorname{dist}(V_k, U_k^c)} = \frac{4\epsilon_k}{R_k}.$$

Thus

$$\begin{aligned} f'(z) &= C_k \left(z^{2m_k}(1 + h_k(z))\right)' \\ &= C_k \left(2m_k z^{2m_k - 1}(1 + h_k(z)) + z^{2m_k} h_k'(z)\right) \\ &= C_k z^{2m_k - 1} \left(2m_k(1 + h_k(z)) + z h_k'(z)\right) \end{aligned}$$

Since $|z| \simeq R_k$ for $z \in V_k$,

$$|f'(z)| = |C_k| \cdot |z|^{2m_k - 1} \left(2m_k + O(\epsilon_k m_k) + O\left(|z| \frac{\epsilon_k}{|R_k|}\right)\right) \geq |C_k| \cdot |z|^{2m_k - 1} m_k. \quad \square$$

11. The first annulus estimate: $A_{k+1} \subset f(A_k)$

Finally, we start proving the inclusions (2.1), which control the dynamics of f. Recall that for $k = 1, \dots$ we set

$$A_k = \{z : \frac{1}{4}R_k \leq |z| \leq 4R_k\}, \quad B_k = \{z : 4R_k \leq |z| \leq \frac{1}{4}R_{k+1}\},$$

$$V_k = \{z : \frac{3}{2}R_k \leq |z| \leq \frac{5}{2}R_k\},$$

Suppose $A = \{z : a \leq |z| \leq b\}$. Let $\partial_i A = \{z : |z| = a\}$ denote the "inner" boundary of A, and let $\partial_o A = \{z : |z| = b\}$ denote the "outer" boundary. Clearly $\partial A = \partial_i A \cup \partial_o A$. Throughout this section and the next we will use the following elementary facts about analytic functions on annuli.

Lemma 11.1. *Suppose g is a holomorphic on an annulus $W = \{a < |z| < b\}$ and continuous up to the boundary. Let $U = \{c < |z| < d\}$.*
(1) Assume $|g(z)| \leq c$ on $\partial_i W$ and $|g(z)| \geq d$ on $\partial_o W$. Then $U \subset g(W)$.
(2) Suppose g has no zeros in W and that $|g(\partial W)| \subset g(\overline{U})$. Then $g(W) \subset \overline{U}$.

Proof. To prove (1), note that $g(W)$ must contain points in U since it is connected and contains points in both complementary components of U. If $g(W)$ omits a point w of U, then there is a point in U that is on the boundary of $g(W)$. This point is in $\overline{g(W)}$, but it is not in $g(W)$, since holomorphic maps are open maps, and it is not in $g(\partial W)$ by our boundary assumptions. But $\overline{g(W)} = g(\overline{W}) = g(\partial W \cup W)$ since g is continuous and $\overline{W}$ is compact. This is a contradiction, so $U \subset g(W)$.

To prove (2), note that by the maximum principle $|g| \leq d$ on W. If g has no zeros, then we can apply the maximum principle to $1/g$ to deduce $|1/g| \leq 1/c$ on W or $|g| \geq c$, as desired. (Note that if g is non-constant then we can improve the conclusion to $g(W) \subset \overline{U}$, since g is an open mapping in this case.) $\square$

Lemma 11.2. *Assume $N \geq 3$. For $k \geq 1$, $A_{k+1} \subset f(V_k) \subset f(A_k)$. Moreover, the inner boundary of V_k maps into B_k and its outer boundary maps into B_{k+1}.*

Proof. The second inclusion is trivial, so we only prove the first. By Corollary 10.3, Lemma 10.5, and (10.5) we know that if $|z| = \frac{3}{2}R_k$, then

$$\begin{aligned} |f(z)| &\leq C_k|z^{2m_k}|(1+\epsilon_k) \\ &\leq C_k\left(\frac{3}{4}\right)^{2m_k}(2R_k)^{2m_k}\cdot(1+\epsilon_k) \\ &\leq \left(\frac{3}{4}\right)^{2m_k}2\cdot R_{k+1}\cdot(1+\epsilon_k) \\ &\leq 4\cdot\left(\frac{3}{4}\right)^{2m_k}\cdot R_{k+1} \\ &\leq \frac{1}{4}\cdot R_{k+1}. \end{aligned}$$

In the other direction, by Lemma 10.4 we have $|C_k| leq 1$, so

$$|f(z)| \geq C_k|z^{2m_k}|(1-\epsilon_k) \geq \frac{1}{2}\left(\frac{3}{2}\right)^{2m_k}R_k^{2m_k} \geq 4R_k.$$

Thus $f(\partial_i V_k) \subset B_k$. Similarly, if $|z| = \frac{5}{2}R_k$, then

$$|f(z)| \geq \left(\frac{5}{2}\right)^{2m_k}\frac{1}{2}\cdot R_{k+1}\cdot(1-\epsilon_k) > 4\cdot R_{k+1},$$

and

$$|f(z)| \leq 2\cdot C_k\cdot\left(\frac{5}{4}\right)^{2m_k}\cdot(2R_k)^{2m_k} \leq 4\cdot 2^{2m_k}\cdot R_{k+2} \leq \frac{1}{4}R_{k+2},$$

where the last inequality holds by Corollary 8.2. Thus $f(\partial_o V)k) \subset B_{k+1}$. Hence $f(V_k)$ contains A_{k+1} by part (1) of Lemma 11.1, and the images of the boundaries of V_k satisfy the desired inclusions. □

12. The second annulus estimate: $f(B_k) \subset B_{k+1}$

Recall that $\partial_i A_k = \{z : |z| = R_k/4\}$ and $\partial_o A_k = \{z : |z| = 4R_k\}$ denote the inner and outer boundary circles of A_k.

Lemma 12.1. *If $R \geq 8$ and $k \in S$, then $f(\partial_o A_k) \subset B_{k+1}$ and $f(\partial_i A_k) \subset B_k$.*

Proof. Suppose $j \geq 1$. By Lemma 10.5 and Corollary 10.4

$$\min\{|f(z)| : |z| = 4R_k\} \geq \frac{1}{2}|C_k|\cdot|4R_k|^{2m_k} \geq 4R_{k+1}.$$

On the other hand, using $|C_k| \leq |C_{k+1}|$ (Lemma 10.4), (10.5), and Corollary 8.2 we get:

$$\begin{aligned}\max\{|f(z)| : |z| = 4R_k\} &\leq 2|C_k| \cdot |4R_k|^{2m_k} \\ &\leq 2|C_{k+1}| \cdot |2R_k|^{2m_k} 2^{2m_k} \\ &\leq 4R_{k+1} 2^{2m_k} \\ &\leq \frac{1}{4} R_{k+2}\end{aligned}$$

Thus $f(\partial_o A_k) \subset B_{k+1}$.

The inner boundary is similar, but easier. Using Lemma 10.6 and (10.3)

$$\min\{|f(z)| : |z| = \frac{1}{4}R_k\} \geq \frac{1}{2}|C_k| \cdot \left(\frac{R_k}{4}\right)^{2n_k} \geq \frac{1}{2}|C_k| \geq 4R_k.$$

For the other direction, the maximum principle says

$$\max\{|f(z)| : |z| = \frac{1}{4}R_k\} \leq \max\{|f(z)| : |z| = 2R_k\} = R_{k+1}$$

which is only a factor of 4 larger than we want. However, using Lemma 10.2 and assuming R is large enough that the error term there is at most 2, we get

$$\begin{aligned}\frac{\max\{|f(z)| : |z| = \frac{1}{4}R_k\}}{\max\{|f(z)| : |z| = 2R_k\}} &\leq 2 \cdot \frac{\max\{|H_{n_k}(z/R_k)| : |z| = \frac{1}{4}R_k\}}{\min\{|H_{n_k}(z/R_k)| : |z| = 2R_k\}} \\ &\leq 2 \cdot \frac{(R_k/4)^{n_k} + 2}{(2R_k)^{n_k} - 2} \\ &\leq 4 \cdot 8^{-n_k} \frac{1 + R_k^{-n_k}/2}{1 - R_k^{-n_k}/2} \\ &\leq \frac{1}{4},\end{aligned}$$

if R and N are large enough, say $R \geq 8$ and $N \geq 3$. This proves $f(\partial_i A_k) \subset B_k$. $\square$

Lemma 12.2. *If $R \geq 8$ and $k \geq 1, k \notin S$ then $f(A_k) \subset B_k \cup A_{k+1} \cup B_{k+1}$.*

Proof. The proof that

$$\max\{|f(z)| : z \in A_k\} \leq \frac{1}{4} R_{k+2},$$

is the same as in Lemma 12.1. The argument for the inner boundary of A_k is only slightly different. If $|z| = \frac{1}{4}R_k$ and $k \notin S$ then Lemma 10.3 gives

$$\begin{aligned} \min\{|f(z)| : |z| = \frac{1}{4}R_k\} &\geq \frac{1}{2}|C_k| \cdot (\frac{1}{4}R_k)^{2m_k} \\ &\geq 4R_k \cdot (\frac{1}{4}R_k)^{2m_k} \\ &\geq 4R_k \end{aligned}$$

Since $k \notin S$, a has no zeros in A_k, so applying part (2) of Lemma 11.1 completes the proof. □

Corollary 12.3. *For $k \geq 1$, $f(B_k) \subset B_{k+1}$.*

Proof. By Lemmas 12.1 (for $k \in S$) and 12.2 (for $k \notin S$) both the inner and outer boundaries of B_j map into B_{j+1}. Since f has no zeros in B_k, the corollary follows from part (2) of Lemma 11.1. □

Corollary 12.4. *For $k \geq 1$, B_k is in the Fatou set of f.*

Proof. B_k maps into B_{k+1} and hence iterates uniformly to infinity. Thus the iterates of f form a normal family on B_k. □

13. The Julia set in A_k

Recall that Ω_m^p denotes the "petals" of Ω_m, i.e., the m components of $|H_m(z)| < 1$ other than the central component Ω_m^0 that contains the origin. See Section 9 and particularly Figure 4.

Lemma 13.1. *If $k \in S$, $\mathcal{J}(f) \cap A_k \subset V_k \cup (R_k \cdot \Omega_{n_k}^p)$.*

Proof. We will break the complement of $V_k \cup (R_k \cdot \Omega_{n_k}^p)$ in A_k into three pieces and verify that each of them is in the Fatou set.

First, there is the annulus that lies between V_k and the outer boundary of A_k. The outer boundary of A_k maps into B_{k+1} by the proof of Lemma 12.1 and the outer boundary of V_k maps into B_{k+1} by Lemma 11.2. Since f has no zeros between these circles, this region maps in B_{k+1} by part (2) of Lemma 11.1 and so is in the Fatou set by Corollary 12.4.

Second, consider the region between the inner boundary of A_k and the boundary of $R_k \cdot \Omega_{m_k}^0$. From the proof of Lemma 12.1 we know the inner boundary of A_k maps into B_k. Since the inner boundary of V_k also maps into B_k we can use the maximum principle to deduce the boundary of $R_k \cdot \Omega^0_{m_k}$ also maps into B_k. Since f has no zeros in this region, the minimum and maximum principles imply it maps into B_k.

Finally, there is the region between the union of petals, $R_k \cdot \Omega^p_{m_k}$ and the inner boundary of V_k. Fix $0 < \delta < 1$ and note that this region is a subset of

$$T_k = \{z : \frac{1}{4}R_k \le |z| \le \frac{3}{2}R_k, H(z/R_k) > \delta\}.$$

To show $f(T_k) \subset B_k$, we need to prove the two inequalities

$$\max\{f(z) : z \in T_k\} \le \frac{1}{4}R_{k+1}, \qquad \min\{f(z) : z \in T_k\} \ge 4R_k.$$

The first follows from the maximum principle and the fact that the inner boundary of V_k maps into B_k. The second follows from

$$\begin{aligned}
\min\{|f(z)| : z \in T_k\} &\ge \frac{1}{2}|C_k| \cdot H(z/R_k) \ge \frac{\delta}{2}|C_k| \\
&\ge \delta 2^{-\#(S_k)-1} R_k^{n_k} \prod_{j \in S_{k-1}} R_j^{-n_j} \\
&\ge \delta 2^{-\#(S_k)-1} R_k^{n_k} \prod_{j \in S_{k-1}} R_{k-1}^{-n_j} \\
&\ge \delta 2^{-\#(S_k)-1} R_k^{n_k} \cdot R_{k-1}^{-\sum_{j \in S_{k-1}} n_j}
\end{aligned}$$

By (6.4) this becomes (recall $2m_{k-1} = n_k$ for $k \in S$)

$$\ge \quad \delta 2^{-\#(S_k)-1} R_k^{n_k} \cdot R_{k-1}^{-2m_{k-1}+2^N} \ge \delta 2^{-\#(S_k)-1} R_k^4 \left(\frac{R_k^{n_k-4}}{R_{k-1}^{n_k}} \right) R_{k-1}^{2^N}.$$

We now use Lemma 8.1

$$\ge \quad \delta 2^{-\#(S_k)-1} \cdot R_k^4 \cdot R_{k-1}^{(2^{N-1}+m_{k-1}/2)(n_k-4)-n_k+2^N}$$

If $N \ge 4$ then $n_k \ge 16$, so this becomes

$$\ge \quad \delta 2^{-\#(S_k)-1} R_k^4 \cdot R_{k-1}^{96+6m_{k-1}-n_k+16}$$

and again we use $m_{k-1} = m_k/2 = n_k/2$ (since $k \in S$) to get

$$\begin{aligned} &\geq \delta 2^{-\#(S_k)-1} R_k^4 \cdot R_{k-1}^{128} \cdot R_{k-1}^{2n_k} \\ &\geq \delta R_k^4 \cdot R_{k-1}^{n_k}. \end{aligned}$$

This is larger than $4R_k$ as long as

$$\delta \geq R_k^{-3} \cdot R_{k-1}^{-n_k}. \quad \square \tag{13.1}$$

Recall that H_m is a conformal map of each petal in Ω_m^p to the unit disk. Thus the part of the petal were $|H_m(z)| \leq \delta$ has diameter similar to δ times the diameter of the petal. Since the components of $R_k \cdot \Omega_{m_k}^p$ (the "petals") each have diameter $\simeq R_k/m_k$, the part of the Julia set contained in each petal has diameter at most $(R_k/m_k) \cdot R_k^{-3}/R_{k-1}^{-n_k} \ll R_k^{-2}/m_k^2$ by (13.1). (And we have not been careful; the actual size is much smaller). Thus the boundary components of the Fatou component containing B_k will be tiny compared to the component itself, except for the inner and outer boundary components (those separating the component from 0 and ∞ respectively).

14. Critical points are in the Fatou set

An entire function is called hyperbolic if the set of singular values (critical values and finite asymptotic values) is bounded and all such points iterate to attracting cycles. See, e.g., [38]. These conditions allow one to iterate a small neighborhood of a Julia set point until it grows to about unit size, without introducing much distortion.

Our function f has an unbounded set of critical values, so it is not hyperbolic, but we claim that all the critical points of f are in the Fatou set. In particular, the distance from the critical set to the Julia set is strictly positive and critical points in A_k will be distance about R_k/m_k from the Julia set, i.e., the distance to the Julia set grows almost linearly with distance from the origin. This weaker version of hyperbolicity will be enough to show that small components of the Fatou set can be iterated under f until they become large, also with only small distortion.

Lemma 14.1. *For $k \geq 1$, $k \notin S$, f has no critical points in A_k.*

Proof. By Lemma 10.3 we know that $f(z) = C_k z^{2m_k}(1 + h(z))$ where h is a holomorphic function on a $R_k/2$ neighborhood of A_k such that $|h_k| = O(1/R_k)$. By the

Cauchy estimate, $|h_k'| = O(1/R_k^2)$ on A_k. Differentiating and using $|z| \simeq R_k$ for $z \in A_k$, we get

$$\begin{aligned} f'(z) &= C_k 2m_k z^{2m_k-1}\left(1+O\left(\frac{1}{R_k}\right)\right) + C_k z^{2m_k} O\left(\frac{1}{R_k^2}\right) \\ &= C_k z^{2m_k-1}\left(2m_k + O\left(\frac{m_k}{R_k}\right) + O\left(\frac{z}{R_k^2}\right)\right) \\ &= C_k z^{2m_k-1}\left(2m_k + O\left(\frac{m_k}{R_k}\right) + O\left(\frac{1}{R_k}\right)\right). \end{aligned}$$

Since $R_k \gg m_k$, this shows f' is never zero on A_k. □

Lemma 14.2. *Suppose $k \in S$. Any critical point z of f in A_k satisfies $f(z) \in B_k$, and hence it is in the Fatou set.*

Proof. By Lemma 13.1, we need only check critical points that lie either in V_k or in the union of petals $R_k \cdot \Omega^p_{n_k}$. There are no critical points in the former by Lemma 10.7. Thus we are reduced to checking if there are any critical points $z \in R_k \cdot \Omega^p_{m_k}$ with $H(z/R_k) < 1/2$. We claim that there are none.

Lemma 10.2 shows that for $z \in A_k$,

$$f(z) = C_k H_{n_k}(\frac{z}{R_k})(1+h(z)),$$

where h is holomorphic on A_k and $|h(z)| = O(R_k^{-1})$ on A_k. Therefore by Cauchy's estimate $|h'(z)| = O(R_k^{-2})$ in the petals (which are distance $\simeq R_k$ from ∂A_k). By the product rule,

$$f'(z) = C_k H'_{n_k}(\frac{z}{R_k})\frac{1}{R_k}(1+h(z)) + C_k H_{n_k}(\frac{z}{R_k})h'(z).$$

So at a critical point of f we must have

$$\begin{aligned} H'_{n_k}(\frac{z}{R_k}) &= \frac{R_k H_{n_k}(z/R_k)h'(z)}{1+h(z)} \\ &= \frac{R_k H_{n_k}(z/R_k)O(R_k^{-2})}{1+O(R_k^{-1})} \\ &= H_{n_k}(z/R_k)O(R_k^{-1}) \end{aligned}$$

Using that

$$\max_{|w|\le 2}|H_{n_k}(w)| \le 2^{n_k}(2+2^{n_k}) \le 2\cdot 2^{2n_k} = 2^{1+2n_k},$$

we get

$$H'_{n_k}(\frac{z}{R_k}) = O\left(\frac{2^{1+2n_k}}{R_k}\right).$$

From this we want to deduce that H must be large at the critical point of f.

A simple calculation shows that

$$1 - H_m(z) = 1 - z^m(2 - z^m) = (1 - z^m)^2 = \left(\frac{H'_m(z)}{2mz^{m-1}}\right)^2.$$

Therefore, at a critical point of f

$$\begin{aligned}
|1 - H_{n_k}(z/R_k)| &= \left|\frac{H'_{n_k}(z/R_k)}{2n_k z^{n_k-1}}\right|^2 \\
&\leq \left|\frac{1}{2n_k} H'_{n_k}(z/R_k)(2/R_k)^{n_k-1}\right|^2. \\
&\leq \frac{1}{4n_k^2} 2^{2+6n_k} R_k^{-2(n_k-1)} \\
&\leq n_k^{-2} 2^{8n_k} R_k^{-2n_k+2}
\end{aligned}$$

Assume $R \geq 128 = 2^7$. By Corollary 8.3 we get $R_k \geq (2R)^{2^{kN}} \geq 2R \geq 2^8$, so

$$\begin{aligned}
|1 - H_{n_k}(z/R_k)| &\leq n_k^{-2} 2^{8n_k} (2^8)^{(-2(n_k-1)} \\
&\leq n_k^{-2} 2^{8n_k - 16(n_k-1)} \leq n_k^{-2} 2^{-8(n_k-2)} \\
&\leq n_k^{-2} \leq \frac{1}{4}
\end{aligned}$$

since $n_k \geq 2$ (since $N \geq 1$). Hence $|H(z/R_k)| \geq 3/4$ at such a critical point. □

Next we address the critical points near the origin.

Lemma 14.3. *We can choose R in the definition of f so that any critical points of f in $\{|z| < R\}$ are in the Fatou set. Moreover, such an R may be as large as we wish.*

Proof. The idea of the proof is simple: we show that the critical points of f iterate to infinity so quickly that we can choose R (and hence R_1) so that the annulus A_1 fits "in between" the orbits of these points. Thus the critical orbits land in Fatou set, and hence the critical points were in the Fatou set to begin with. We will first describe the critical orbits for the polynomial F_0, and then use the fact that f approximates F_0 as closely as we wish if R is large enough.

Since F_0 is the Nth iteration of the degree two polynomial $p_\lambda(z) = \lambda(2z^2 - 1)$, it has $2^N - 1$ critical points consisting of 0, its two preimages under p_λ, the four pre-images of those two points, going back to the 2^{N-1} pre-images of 0 under p_λ^{N-1}. The critical values of F_0 are the 2^N-1 images of the critical points under F_0. These are the same as the first N iterates of 0 under p_λ. The F_0-images of the critical values are the $(N+1)$st through $(2N)$th iterates of 0. In particular, the post-critical set does not have any points in the

$$K = \{z : 2|p_\lambda^N(0)| \le |z| \le \frac{1}{2}|p_\lambda^{N+1}(0)|\}.$$

In fact, the post-critical set has distance at least 1 from this set (actually, much larger distance).

Now iterate K under F_0 until it surrounds the disk $\{|z| < 2^N R^{2^N}\}$ and then choose R_1 so that $A = \{z : R_1/8 \le |z| \le 8R_1\}$ is contained in this iterate of K (we can do this because the moduli of the images of K increase to infinity under iteration, hence the images contain round annuli of arbitrarily large modulus). With this choice, the critical values of F_0, iterated by F_0, miss the annulus A. In fact, if we iterate any point that is within distance 1 of a critical point, it also misses the annulus A. We claim that if the parameter R is large enough, then the iterates the critical values of f by f do not hit $A_1 = \{z : R_1/4 \le |z| \le 4R_1\} \subset A$, assuming that we stop iterating when the orbit leaves $\{|z| \le 4R_1\}$.

To prove the claim, choose R so large that $K \subset \{|z| < R\}$. On $\{|z| < R\}$, the function f is a small perturbation of F_0; we have

$$f(z) = F_0(z)\left(1 + O\left(\frac{1}{\sqrt{R_1}}\right)\right),$$

by (6.9). However the number of iterates under f needed to go from size $\simeq \lambda$ to $\simeq R_1$ is only $O(\log\log R_1)$. So if we iterate a point by f until it has absolute value at least $4R_1$, and iterate the same point under F_0, the ratio between the absolute values of corresponding points is at most

$$(1 + O(R_1^{-1/2}))^{O(\log\log R_1)} = 1 + O(R_1^{-1/2}\log\log R_1) \le 2,$$

if R_1 (hence if R) is large enough. Since the iterates under F_0 miss A, the iterates under f will miss $A_1 = \{z : \frac{1}{4}R_1 \le |z| \le 4R_1\}$, as desired. $\square$

15. Negative indices

If the orbit of z visits $\cup_{k=1}^{\infty} A_k$ infinitely often, there may still be iterates that land near the origin and then iterate several steps in $D_1 = \{z : |z| < R_1/4\}$ before reaching A_1. It will be convenient to think of these intermediate steps as still being in some A_k, so in this section we define A_k for non-positive indices.

Define $A_0 = \{z \in D_1 : f(z) \in A_1\}$. More generally, for $k > 0$, let A_{-k} be the set of points so that $\{z, f(z), \dots, f^k(z)\} \subset D_1$ and $f^{k+1}(z) \in A_1$. Thus A_{-k} maps to A_1 in $k+1$ steps without ever leaving D_1. Alternatively, these are the points in this disk that iterate to A_1 in $k+1$ steps and don't visit A_1 on any earlier iteration.

We make analogous definitions of V_k and U_k for $k \leq 0$.

For $k \leq 0$, each V_k is a union of topological annuli that surround the Cantor set E (see Section 4), and each component of V_k, $k \leq 0$ is mapped by f to a component of V_{k+1}. From the discussion in Section 14, we know that the $2^N - 1$ critical points of f that are in $D(0, R)$ are is the same Fatou component. This means there is an integer T so that V_{-k} surrounds all these points for $k = 0, 1, \dots T$ and does not surround any critical points for $k > T$. Hence there is one connected component of V_{-k} for $k = 0, \dots, T$ and that f acts as a 2^N-to-1 covering map from V_{-k} to V_{1-k} for these indices. The set V_{-T-1} has 2^N distinct connected components that are each mapped conformally to V_{-T} by f. Similarly, V_{-T-j} has 2^{jN} connected components, and groups of 2^N are each mapped conformally to some component of V_{-T-j+1}.

Since f is conformal on U_k, the map $f : V_{-k} \mapsto V_{-k+1}$ actually extends to be conformal of a uniformly larger annulus, so by the distortion theorem for conformal maps (see e.g., Section I.4 of [25]) it has bounded distortion independent of the component and independent of k.

Previously we have defined m_k for $k \geq 0$. We now define $m_k = 2^N$ for $-T \leq k \leq 0$, and $m_k = 1$ for $k < -T$. Note that the covering map $f : A_k \to A_{k+1}$, $k \leq 0$ has degree m_k. Let $M_0 = 2^{NT} = \prod_{k \leq 0} m_k$. Then M_0 bounds the number of pre-images of a single point $z \in V_1$ that will be found in any connected component of V_k, $k \leq 0$.

Corollary 15.1. *Let $A = \cup_{k=-\infty}^{\infty} A_k$. Then $f^{-1}(A) \subset A$.*

Proof. This follows from the definitions and Lemma 12.3. □

16. Partitioning the Julia set

We now have shown that if z is in the Julia set, then either the orbit of z eventually lands on the Cantor repeller E (the set of such z we denote by $\widetilde{E}$), or it stays in $A = \cup_{k=-\infty}^{\infty} A_k$ forever (these points we denote by X). By construction, E has small Hausdorff dimension, and hence so does $\widetilde{E}$. Thus to prove Theorem 1.3 we need only consider $X = \mathcal{J}(f) \setminus \widetilde{E}$. We will split X into two pieces. We first need to observe a simple rule about how orbits can behave:

Lemma 16.1. *Any connected component W of $f^{-1}(A_k)$ is contained in A_j for some $j \geq k-1$. If $j \geq k$, then $j \in S$.*

Proof. If $j \leq 0$ then $f(A_j) = A_{j+1}$. If $j \geq 1$ and $j \notin S$ then $f(A_j) \subset B_j \cup A_{j+1} \cup B_{j+1}$ by Lemma 12.2. Thus in either case, A_{j+1} is the only set of the form A_k hit by $f(A_j)$. Thus if $f(A_j)$ hits A_k for some $k < j+1$, we must have $j \in S$. □

The cases $W \subset A_{k-1}$ and $W \subset A_j$, $j \geq k$ in the previous lemma are quite different. We call these type I and type II respectively. See Figure 6.

Recall that Ω_m, Ω_m^0 and Ω_m^∞ are defined in Section 9.

Lemma 16.2. *If $k \in S$, then the only components of $f^{-1}(A_j)$, $j \leq k$, that are inside A_k are inside the petals of $R_k \cdot \Omega_{m_k}$. (If $k \notin S$, there are no such pre-images.)*

Proof. Since $H_{m_k} \geq 1$ on $\Omega_{m_k}^\infty$ and $C_k \geq 8R_k$, there are no pre-images in $(R_k \cdot \Omega_{m_k}^\infty)$. On $\partial(R_k \cdot \Omega_{m_k}^0)$, f is bigger than $4R_k$ by Lemma 10.2. Also, $\partial A_k \cap (R_k \cdot \Omega_{m_k}^0) \subset \partial B_{k-1}$ and hence is mapped into B_k by f. Thus $|f| \geq 4R_k$ on both boundary components of $A_k \cap (R_k \cdot \Omega_{m_k}^0)$. Since f has no zeros in $A_k \cap (R_k \cdot \Omega_{m_k}^0)$, the minimum principle holds on this region and the lemma follows. □

If the orbit of z stays in $A = \cup_{k=-\infty}^{\infty} A_k$ forever, we associate to z the sequence of integers $k(z,n), n \geq 0$, such that $f^n(z) \in A_{k(z,n)}$. By the discussion above,

$$k(z, n+1) \leq k(z,n) + 1 \text{ if } k(z,n) \in S,$$

$$k(z, n+1) = k(z,n) + 1 \text{ if } k(z,n) \notin S.$$

Obviously any numerical sequence is either eventually strictly increasing or it is not. We denote the points corresponding to each type of sequence by Z and Y respectively. The dimensions of these two sets are given by:

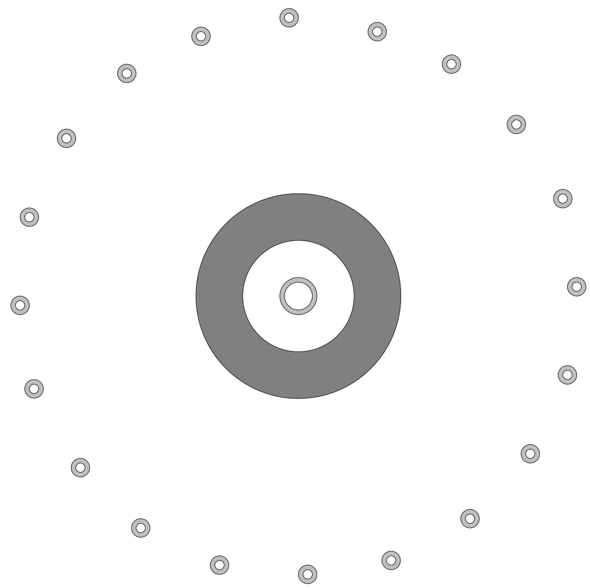

FIGURE 6. The darker ring is A_k. Inside, is the single type I pre-image $W \subset A_{k-1}$ were f acts as $2m_{k-1}$-to-1 covering map. Outside, there are infinitely many rings of type II pre-images in A_j, $j > k, j \in S$ (only one such ring is drawn) were f acts conformally. The picture is not to scale; the inner pre-image should be much smaller and thinner; the outer pre-images should be much further out and smaller (but their conformal moduli are the same as that of A_k).

Lemma 16.3. *Let $Y \subset X$ be the set of points z such that $k(z, n+1) \leq k(z, n)$ infinitely often. Suppose $\alpha > 0$. If λ, R and N are large enough, then* $\dim(Y) \leq \alpha$.

Lemma 16.4. *Let $Z \subset X$ be the set of z so that $k(z, n+1) = k(z, n) + 1$ for all sufficiently large n. Then Z is a union of C^1 closed Jordan curves. Moreover, Z has locally finite 1-measure.*

The set Z equals $\mathcal{J}(f) \cap A(f)$ (the fast escaping points of the Julia set). The set Y corresponds to everything that is not fast escaping and does not eventually land on E. The escaping, but not fast escaping, set corresponds to the subset of Y where the indices tend to infinity but are not eventually strictly monotonic.

17. Proof of Lemma 16.3

Since $Y \subset A = \cup_k A_k$, and Y is invariant under f, it suffices to show $\dim(Y \cap A_m) \leq \alpha$ for any m, say $m = 1$. We will do this by building nested coverings of $Y \cap A_1$ using certain components W_k^n of $f^{-n}(A_k)$ that lie inside A_1. We allow $n = 0$ where f^0 is the identity map; hence the single element $W_1^0 = A_1$ is our first covering.

We form generations of nested covers of $Y \cap A_m$ by the following replacement procedure. Suppose $W_k^n \subset f^{-n}(A_k)$ is an element of the current cover. If $z \in W_k^n$, then the definition of Y says that eventually $f^{n+q}(z) \in A_j$ with $j < k+q$. At this point we stop and cover z by a component of the form W_j^{n+q}. Thus $Y \cap W_k^n$ can be covered by components of the form W_j^{n+q} where $q \geq 1$ and $j \leq k+q-1$. We replace W_k^n by all components that arise in this way. Since every $z \in Y \cap W_k^n$ is in such a component, we get another covering.

Instead of considering all $j \leq k+q-1$, it suffices to take just $j = k+q-1$ if we cover Y using the topological disks $\widehat{W}_k^n$ obtained by "filling in" the hole of W_k^n (this is also called the "polynomial hull" of W_k^n). Every component with $j < k+q-1$ is contained in such a hole and W_{k+q-1}^{n+q} and $\widehat{W}_{k+q-1}^{n+q}$ have the same diameter. Thus using these does not change the sum in the definition of Hausdorff measure and dimension. Note that using the filled-in components requires us to also consider the $q = 0$ case: $Y \cap (\widehat{W}_k^n \setminus W_k^n)$ is covered by components of the form $\widehat{W}_{k-1}^n$ in the next generation.

Thus to prove $\dim(Y \cap A_m) \leq \alpha$, it suffices to show the α-sum of the diameters tends to zero as we repeatedly refine the covers. In fact, we claim it tends to zero geometrically fast (we will use this stronger estimate later in Sections 18 and 20). We write the α-sum as two sums corresponding to $q = 0$ and $q > 0$:

$$\sum_{W_{k-1}^n \subset \widehat{W}_k^n} \operatorname{diam}(W_{k-1}^n)^\alpha \leq \frac{1}{4} \operatorname{diam}(W_k^n)^\alpha \tag{17.1}$$

$$\sum_{q \geq 1} \sum_{W_{k+q-1}^{n+q} \subset W_k^n} \operatorname{diam}(W_{k+q-1}^{n+q})^\alpha \leq \frac{1}{4} \operatorname{diam}(W_k^n)^\alpha \tag{17.2}$$

Proof of (17.1). This is the easy case. If $k \geq 1$, then W_{k-1}^n has just one component in $\widehat{W}_k^n$ and its diameter is $O(R_1^{-1}) \cdot \operatorname{diam}(W_k^n)$, so its contribution is small. For $k \leq 0$ there a bounded number (depending on N) of connected components of W_{k-1}^n inside W_k^n, and by Lemma 4.2 each such component has small diameter compared to W_k^n if λ is large enough. In either case we can make the left side of (17.1) small compared to the right side of (17.1). □

Proof of (17.2). By definition

$$\begin{aligned} W^{n+q}_{k+q-1} \subset W^n_k &\subset A_m, \\ f^n(W^{n+q}_{k+q-1}) \subset f^n(W^n_k) &= A_k, \\ f^{n+1}(W^{n+q}_{k+q-1}) &\subset A_{k+1}, \\ f^{n+2}(W^{n+q}_{k+q-1}) &\subset A_{k+2}, \\ &\vdots \quad \vdots \\ f^{n+q-1}(W^{n+q}_{k+q-1}) &\subset A_{k+q-1}, \\ f^{n+q}(W^{n+q}_{k+q-1}) &\subset A_{k+q-1}, \end{aligned}$$

The point is that, by our choice of q, the annuli on the right increase monotonely until the last one, which is repeated. The first $q-1$ maps are restrictions of the covering maps $A_{k+i-1} \supset f^{-1}(A_{k+i}) \to A_{k+i}$ (these were called case I earlier) and the final map is the restriction of a petal map (case II). The ith covering map for $i = 1, \dots q-1$ is

(1) $2m_{k+i-1}$-to-1, if $k+i-1 \geq 1$,
(2) 2^N-to-1, if $-T \leq k+i-1 \leq 0$,
(3) 1-to-1, if $k \leq -T$.

Hence the number of possible new components when we replace a single component W^n_k by multiple components of the form W^{n+q}_j is less than

$$2^{NT} 2^q m_k \cdot m_{k+1} \cdots m_{k+q-2} \leq 2^{NT} 2^q M_{k+q-2}.$$

The size of a single pre-image is determined by the final petal map:

$$\operatorname{diam}(W^{n+q}_{k+q-1}) \lesssim \frac{R_{k+q-1}}{R_{k+q}} \operatorname{diam}(W^n_k) \lesssim \frac{\operatorname{diam}(W^n_k)}{R_{k+q-1}},$$

since $R_{k+q} \geq R^2_{k+q-1}$ by (5.7). Thus for each q, the total contribution to (17.2) is

$$O\left(2^{NT} 2^q M_{k+q-2} \left(\frac{\operatorname{diam}(W^n_k)}{R_{k+q-1}}\right)^{\alpha}\right) = O\left(2^{NT} 2^q M_{k+q-2} R^{-\alpha}_{k+q-1}\right) \cdot \operatorname{diam}(W^n_k)^{\alpha}.$$

By Lemma 8.5 the sum of the terms inside the "big-Oh" term over q converges, and is as small as we wish if R is large enough. □

This completes the proof of Lemma 16.3 and gives part (4) of Theorem 1.3. Before proceeding with the proof of Lemma 16.4, we pause to prove part (5) of Theorem 1.3. Recall that $I(f)$ denotes the escaping set of f and $A(f)$ the fast escaping set.

Corollary 17.1. *For our example,* $\dim(I(f) \cap Y) = \dim(I(f) \setminus A(f)) = 0$.

Proof. During the proofs of (17.1) and (17.2), we needed to take R large enough, depending on α. However, if we only cover points whose orbits visit A_j for $j \geq J$, then the same claims are true if R_J is large enough. Thus inequalities (17.1) and (17.2) are true for any $\alpha > 0$, if we take J large enough, depending on α. Thus the set of points whose orbits eventually only visit A_j for $j \geq J$ has Hausdorff dimension that tends to 0 as $J \to \infty$ and hence the escaping points in Y have dimension 0. $\square$

18. Proof of Lemma 16.4

The pre-image $W \subset V_k$ of the round annulus V_{k+1} under a power function would be another round annulus. Since f is a small perturbation of a power function, W will be a small perturbation of a round annulus. We make this precise with the estimate:

Lemma 18.1. *Suppose h is a holomorphic function on $A = \{z : 1 < |z| < 4\}$ and suppose that $|h|$ is bounded by ϵ on A. Let $H(z) = (1 + h(z))z^m$. For any fixed θ, the segment $S(\theta) = \{re^{i\theta} : \frac{3}{2} \leq r \leq \frac{5}{2}\}$ is mapped by H to a curve that makes angle at most $O(\epsilon/m)$ with any radial ray it meets.*

Proof. We want to look at the image of a circle under H and show that this image at a point $H(z)$ is close to perpendicular to the ray from the origin to $H(z)$. The angle between the ray and the image is $\arg(zH'(z)/H(z))$, so we want to show this is small. By the Cauchy estimate, h' is bounded by $O(\epsilon)$ on $\{z : \frac{3}{2} \leq |z| \leq \frac{5}{2}\}$, so

$$\begin{aligned} z\frac{H'(z)}{H(z)} &= z\frac{h'(z)z^m + (1+h(z))mz^{m-1}}{(1+h(z))z^m} \\ &= \frac{zh'(z)}{1+h(z)} + \frac{zmz^{m-1}}{z^m} \\ &= O(\epsilon) + m. \end{aligned}$$

Hence $\arg\left(z\frac{H'(z)}{H(z)}\right) = O(\epsilon/m)$. $\square$

We can deduce that $W \subset V_k$ is a topological annulus whose width is approximately $R_k/2m_k$ and whose boundary components are smooth curves that are ϵ_k-close to circles (ϵ_k was defined in (10.7)).

For $k \geq 1$, consider

$$\Gamma_{k,n} = \{z \in A_k : f^j(z) \in A_{k+j}, j = 1, \dots, n\}.$$

These are nested topological annuli whose widths decrease to zero uniformly in j. Each $\Gamma_{k,n}$ has a foliation by closed analytic curves (including its boundary curves) that go around $\Gamma_{k,n}$ once, obtained by pulling back circles in A_{k+n}. By Lemma 18.1 the curves for $\Gamma_{k,n+1}$ make angle at most $O(\epsilon_k)$ with the curves for $\Gamma_{k,n}$. This is a summable estimate, so we deduce that nested annuli $\Gamma_{k,n}$ limit on a C^1 Jordan curve Γ_k that makes at most angle $O(\sum_k \epsilon_k)$ with the circular arcs foliating V_k. Therefore Γ_k has finite length and this length is comparable to its diameter.

This deals with sequences that are strictly monotone and start at a value ≥ 1. Every eventually strictly monotone sequence has a finite initial sequence that is followed by an infinite strictly monotone sequence. The corresponding component of Z maps onto one of the components discussed above by a map that is conformal on a large neighborhood of the component and hence has bounded distortion with uniform estimates. Thus every component of Z is a C^1 curve with uniform estimates.

To finish the proof of the lemma, we have to show the sums of the lengths of all the components of Z in a bounded region of the plane is finite. By the last sentence of the previous paragraph it suffices to sum the diameters of the components. Since each component of Z is associated by containment to a unique set of the form W_k^n (notation as in the proof of Lemma 16.3), the sum of the diameters over components of Z is dominated by the sum of diameters over sets of the form W_k^n. This is just the $\alpha = 1$ case of (17.1) and (17.2). To prove Lemma 16.3 we only needed the sum over each cover to tend to zero, but here we use the fact that the cover sums decay exponentially fast to see that the sum over all generations is finite.

This completes the proof of Lemma 16.4 and implies parts (1), (2) and (3) of Theorem 1.3. The remaining sections prove part (6). This requires a more careful geometric description of the shape of the Fatou components, which we shall give in the next section.

19. The shape of the Fatou components

In this section we justify Figure 1 as the "shape" of general Fatou components and describe this shape a little more precisely.

For $k \geq 1$, consider the component Ω_k that contains the inner boundary component of A_k. This component has an outer boundary curve γ_k that separates it from ∞

and is contained in V_k, and this is a C^1 closed Jordan curve that approximates the circle $\{|z| = 2R_k\}$.

The component Ω_k has an inner boundary curve that is also the outer boundary γ_{k-1} of Ω_{k-1} if $k \geq 2$. The other boundary components of Ω_k are curves that we can group into "levels" that lie on approximately circular curves that limit onto the outer boundary γ_k. The levels are indexed by values of $j \in S$ such that $j \geq k$. Components of level $j \geq k$ of Ω_k are mapped onto the curve γ_j after $1 + j - k$ iterations of f.

If $k \in S$, then first level consists of m_k components, one in each petal of $R_k \cdot \Omega^0_{m_k}$. These boundary components lie inside the components W^1_k used to cover the set Y in Section 16.3. The next level consists of $(2m_k) \cdots (2m_j)$ components and lie in the pre-images under f^{j-k} of the petals of $R_j \cdot \Omega^0_{m_j}$ where j is the next element of S after k. These correspond to the components W^{n+q}_{k+q-1} in Section 16.3 where $q = j - k + 1$. Higher levels lie in similar pre-images corresponding to later elements of S. The level corresponding to $j \in S$, $j > k$ consists of $m_j \cdot 2^d$ components that lie approximately distance $R_k 2^{-d}$ from the outer boundary, where $d = 2(m_k + \cdots + m_{j-1})$ (so 2^d is the degree of f as a covering map from $f^{k-j}(V_j)$ to V_j). Adjacent boundary components in level j are about distance $R_k m_j^{-1} 2^{-d}$ apart.

For $k \geq 1$, $k \in S$, Ω_k contains m_k critical points; the critical points of H_{m_k} are the points where the petals join $\partial(R_k \cdot \Omega^0_{m_k})$, so the critical points of f are perturbations of these. If $k \in S$, the map f acts as a $2m_k$-to-1 branched cover from Ω_k to Ω_{k+1}, with the outer boundary mapping to the outer boundary (as a $2m_k$-to-1 map), the inner boundary maps to the inner boundary (as a m_k-to-1 map), the first layer boundary components also map to the inner boundary (1-to-1 maps on each component) and higher level components map 1-to-1 to components one level lower (so the second level in Ω_k becomes the first level in Ω_{k+1}). The map on each boundary component in a given level is 1-to-1, but $2m_k$ different components will map to the same boundary component of Ω_{k+1}.

If $k \geq 1$, but $k \notin S$, then the picture is the same except that there is no first level corresponding to petals of $R_k \cdot \Omega^0_{m_k}$, but there are levels corresponding to each $j \in S$, $j > k$ as described above. All other components of the Fatou set eventually map onto one of these by iterating f. For $k \leq 0$, the Fatou components Ω_k are defined as inverse images of Ω_{k+1} under f. For $-T \leq k \leq 0$ this is a 2^N-to-1 covering map and for $k < T$ it is conformal (1-to-1). All the critical points of f are in the components Ω_k, $k = -T$ and $k \geq 1$, so every other component of the Fatou set is a conformal image of one of these and hence has the same geometry as Ω_k, for some $k \geq -T$, up to bounded distortion. When $k \notin S$, the action of f on Ω_k is $2m_k$-to-1 on both the outer and inner boundaries, and each component in a level of Ω_k maps 1-to-1 to a boundary component in the same level of Ω_{k+1} (but $2m_k$ components map onto the same image component).

20. Packing dimension equals one

A theorem of Rippon and Stallard [40] says that packing dimension agrees with the local upper Minkowski dimension for Julia sets of entire functions. The local upper Minkowski dimension is the upper Minkowski dimension of the Julia set intersected with a neighborhood of any point in the Julia set; Rippon and Stallard show this is constant (except possibly at one point). They also show that the packing dimension is 2 for any function in the Eremenko-Lyubich class of transcendental entire functions with bounded singular sets. Other conditions involving the growth of f that imply packing dimension 2 are given in Bergweiler's paper [9]. Interestingly, one of his conditions is that all large circles centered at zero map to curves that deviate from circles by a fixed amount; the exact opposite of the property that was crucial in this paper (most large circles centered at zero are mapped very close to circles).

In addition to the standard results about dimension reviewed in Section 3 we will also need a couple of other easy results. Recall that the Whitney decomposition of an open set was defined in Section 3.

Lemma 20.1. *Suppose a bounded open set Ω contains disjoint open subsets $\{\Omega_j\}$ so that $\Omega \setminus \cup_j \Omega_j$ has zero area. Then for any $1 \leq s \leq 2$,*

$$\sum_{Q \in \mathcal{W}(\Omega)} \mathrm{diam}(Q)^s \leq \sum_j \sum_{Q \in \mathcal{W}(\Omega_j)} \mathrm{diam}(Q)^s.$$

Proof. By rescaling, we can assume Ω has diameter at most 1. By the nested property of dyadic squares, each square $Q' \in \cup_j \mathcal{W}(\Omega_j)$ is contained in some square $Q \in \mathcal{W}(\Omega)$ and almost every (area measure) point of Q is covered by the squares Q' it contains.

Thus

$$\sum_{Q\in\mathcal{W}(\Omega)} \operatorname{diam}(Q)^s = \sum_{Q\in\mathcal{W}(\Omega)} \operatorname{diam}(Q)^{s-2}\cdot 2\cdot \operatorname{area}(Q)$$
$$= \sum_{Q\in\mathcal{W}(\Omega)} \operatorname{diam}(Q)^{s-2}\left(\sum_j \sum_{Q'\in\mathcal{W}(\Omega_j),Q'\subset Q} 2\cdot\operatorname{area}(Q')\right).$$

If $Q'\subset Q$ then $\operatorname{diam}(Q')\leq \operatorname{diam}(Q)\leq 1$. Since $s-2\leq 0$, this means $\operatorname{diam}(Q)^{s-2}\leq \operatorname{diam}(Q')^{s-2}$. Using this and rearranging the sums gives

$$= \sum_j \sum_{Q'\in\mathcal{W}(\Omega_j)}\left(\sum_{Q\in\mathcal{W}(\Omega),Q'\subset Q} \operatorname{diam}(Q)^{s-2}\cdot\operatorname{diam}(Q')^2\right)$$
$$\leq \sum_j \sum_{Q'\in\mathcal{W}(\Omega_j)}\left(\sum_{Q\in\mathcal{W}(\Omega),Q'\subset Q} \operatorname{diam}(Q')^{s-2}\cdot\operatorname{diam}(Q')^2\right)$$
$$= \sum_j \sum_{Q'\in\mathcal{W}(\Omega_j)} \operatorname{diam}(Q')^s.$$

The last equality holds because for each Q' there is only one Q that contains it. $\square$

Lemma 20.2. *If $f:\Omega_1\to\Omega_2$ is biLipschitz, then for any $0<s\leq 2$,*

$$\sum_{Q\in\mathcal{W}(\Omega_1)} \operatorname{diam}(Q)^s \simeq \sum_{Q\in\mathcal{W}(\Omega_2)} \operatorname{diam}(Q)^s.$$

Proof. For any $Q\in\mathcal{W}(\Omega_1)$, the image $f(Q)$ can be covered by $O(1)$ elements of $\mathcal{W}(\Omega_2)$, so the left hand side is bounded by a multiple of the right hand side. The argument reverses, giving the lemma. $\square$

Theorem 20.3. *For our example,* $\operatorname{Pdim}(\mathcal{J}(f))=1$.

Proof. By Lemma 3.2 we can bound the packing dimension by bounding the upper Minkowski dimension of bounded pieces of the Julia set. We estimate the upper Minkowski dimension using Lemma 3.1.

Consider the Fatou component Ω_k that contains B_{k-1}. Its outer boundary γ_k is in A_k and let $\mathcal{U}_k$ be the collection of all Fatou components contained inside γ_k. We must show that for any $s>1$

$$\sum \operatorname{diam}(Q_j)^s < \infty$$

where the sum is taken over all the Whitney squares for all these components. We break this sum into sums over the Whitney squares in each Fatou components. Then we use two estimates. First,

$$\sum_{\Omega \in \mathcal{U}_k} \mathrm{diam}(\Omega)^s < \infty \tag{20.1}$$

where the sum is over all components Ω of the Fatou set that are contained in a bounded set. This was already done in the proof of Lemma 16.4 for $s = 1$, and this implies the result for $s > 1$.

Second, we claim that for any $s > 1$ there is a $C_s < \infty$ so that

$$\sum_j \mathrm{diam}(Q)^s \leq C_s \mathrm{diam}(\Omega)^s, \tag{20.2}$$

uniformly for every component Ω of the Fatou set. The sum is over the Whitney squares contain in a single component Ω, but the constant C_s must be independent of the component Ω.

This follows from our knowledge of the geometry of the Fatou components, as described in Sections 18 and 19. The boundary components of a given level all lie on a Lipschitz graph in polar coordinates for the correct choice of center. If we remove this curve from the Fatou component Ω, we cut it into topological annuli, each of which is biLipschitz equivalent to a round annulus of the form $A_j = \{z : r < |z| < (1+\delta_j)r\}$ and where the δ_j decrease rapidly with the level and $r \simeq \mathrm{diam}(\Omega)$. A direct computation for a single round annuli shows the s-Whitney sum is

$$O\left(\frac{1}{s} \cdot \delta_j^{s-1} \cdot r^s\right).$$

and for a fixed r. Since the δ's go to zero rapidly (more than exponentially fast), this is summable over j and is bounded by

$$O\left(\frac{1}{s} \cdot r^s\right) = O(\mathrm{diam}(\Omega)^s).$$

Lemmas 20.1 and 20.2 then apply to prove (20.2). □

Theorem 20.4. *The 1-dimensional packing measure of the example constructed in this paper is infinite. In fact, the measure of $\mathcal{J}(f) \cap D(x,r)$ is infinite for any $x \in \mathcal{J}(f)$ and $r > 0$.*

Proof. Consider one Fatou component, say the one with its outer boundary in V_1. As described in Section 19, the jth layer layer of boundary components have diameter at most $D_j = R_1 2^{-d_j}/m_j$ and have distance at least $R_1 2^{-d_j} = m_j D_j$ from the outer boundary (and from other layers of boundary components). Thus we can put disjoint disks of radius $\simeq D_j$ around each component of the jth layer, and the sum of diameters is comparable to R_1. Since we can do this for every layer, the total sum of diameters is infinite. Thus the 1-dimensional packing pre-measure of the boundary of the Fatou component is infinite.

If we take any neighborhood of any point in the Julia set, it contains a Fatou component that iterates to the one described above with bounded distortion, and hence the boundary of that component also has infinite packing pre-measure. If we take a bounded piece of the Julia set, say the part J_1 inside A_1, the packing measure is computed from the pre-measure by taking countable coverings. Since the pre-measure of a set and its closure are the same, we can assume the sets in the countable union are closed. By Baire's theorem, if J is a countable union of closed sets, one of them must have interior relative to J_1, and hence contains the Julia set intersected with some disk centered on the Julia set. By the argument above, this piece contains the entire boundary of some Fatou component that has infinite 1-dimensional packing pre-measure. Thus J_1 must have infinite 1-dimensional packing measure. $\square$

This proves part (6) of Theorem 1.3 and thus completes the proof of that theorem.

Corollary 20.5. *The set $\mathcal{J}(f) \cap D(x,r)$ is not contained in a curve of finite length, for any $x \in \mathcal{J}(f)$ and $r > 0$.*

Proof. Any subset of a rectifiable curve has finite 1-dimensional packing measure. $\square$

21. Remarks and questions

(1) There seem to be no known examples of transcendental entire functions whose Julia sets have packing dimension strictly between 1 and 2, but it may be possible to build such examples using our construction by replacing F_0 by a polynomial whose Julia set has Hausdorff and packing dimension bigger than 1 (but verifying this may be difficult). If we vary F_0 in the construction and leave the rest alone, does the dimension of $\mathcal{J}(f)$ vary continuously? Figure 7 shows the known pairs of Hausdorff

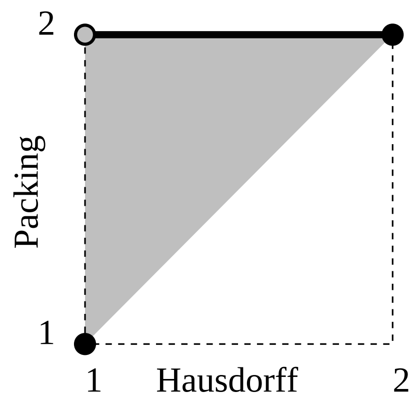

FIGURE 7. Since packing dimension is an upper bound for Hausdorff dimension, the shaded region illustrates the possible values for the pairs $(\dim(\mathcal{J}), \mathrm{Pdim}(\mathcal{J}))$ for a transcendental entire function. Black represents known examples: the dot at $(2, 2)$ is e^z and McMullen's examples, the black line $\{(t, 2) : 1 < t < 2\}$ are Stallard's examples and the dot at the bottom is the example in this paper.

and packing dimensions that can occur for transcendental entire functions. Which of the remaining possibilities actually occur? A few interesting special cases are:

- Can we have $1 < \dim(\mathcal{J}) = \mathrm{Pdim}(\mathcal{J}) < 2$?
- Can we have $\dim(\mathcal{J}) = 1, \mathrm{Pdim}(\mathcal{J}) = 2$?
- Does every packing dimension between 1 and 2 occur?

(2) Bergweiler and Zheng [12] have shown that multiply connected Fatou components can sometimes have uniformly perfect boundaries, even though the Julia set itself is never uniformly perfect if there are multiply connected Fatou components. Our example does not have this property. Can we have $\dim(\mathcal{J}(f)) = 1$ for an example with multiply connected Fatou components whose boundaries are uniformly perfect?

(3) A stronger version of the last question is to ask if there is an example with $\dim(\mathcal{J}(f)) = 1$ where the Fatou components are finitely connected? Can we take the boundary components to be rectifiable? Finitely connected Fatou components can exist by examples of Masashi and Shishikura [29], and such components may make it easier to understand the rectifiability of the Julia set. Is there a transcendental Julia set of dimension 1 where the Fatou components are all simply connected?

(4) We noted at the end of Section 5 that our examples all have order zero, and can be taken to grow as slowly as we wish by taking S sparse enough. Can we build examples of positive or infinite order? This may be possible by replacing the use of the degree two Chebyshev polynomial in this paper by higher degree Chebyshev polynomials or generalized Chebyshev polynomials. Can we use such constructions to show the conditions in Bergweiler's paper [9] implying $\mathrm{Pdim}(\mathcal{J}) = 2$ are sharp?

(5) The boundaries of the Fatou components in our example are at least C^1 curves. Can this be improved to C^2? C^∞? Analytic? Smooth curves can occur as the boundaries of Fatou components of rational maps, (see [2] and its references for the history of such results), and it would be interesting to see if they also occur in this context.

(6) The connected components of the Julia set constructed in this paper are all either points or continua of Hausdorff dimension one. This also occurs for many other transcendental functions, such as certain exponential functions where the connected components of the Julia set are all curves that tend to infinity (e.g., see [28], [7]). However, the situation for polynomials is open. If a polynomial Julia set is connected, then it is either a generalized circle/segment or has Hausdorff dimension strictly greater than 1 (this follows from work of Zdunik [51] and Przytycki [36]). Is this also true of the non-trivial connected components when the Julia set is disconnected? In other words, if $\mathcal{J}(p)$ is disconnected, is every connected component either a point or a set of Hausdorff dimension strictly greater than 1?

(7) Is there a transcendental Julia set which is a subset of rectifiable curve on the Riemann sphere? Corollary 20.5 says this is not true for our example. If such an example exists, it would be the ultimate in "smallness" for a transcendental Julia set.

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