Equi-triangulation of polygons

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Abstract. We prove that any two polygons with the same area can be triangulated using the same set of triangles. This strengthens the 200-year-old Wallace–Bolyai–Gerwien theorem that equal area polygons have an equi-dissection.

Keywords: dissections, triangulations, Wallace-Bolyai-Gerwien theorem, dynamical system

1. INTRODUCTION. A simple polygon P is a Jordan curve in the plane consisting of a finite number of line segments. A polygonal region Ω is a compact set in the plane whose boundary $P = \partial\Omega$ is a simple polygon, and so that Ω is the closure of its interior.. A dissection of Ω is a finite collection of polygonal sub-regions that have disjoint interiors and whose union is Ω. We say that dissections of two polygonal regions are equivalent if there is a 1-to-1 correspondence between their pieces so that corresponding pieces are images of each other under a translation and rotation (an orientation preserving isometry). If two polygonal regions are dissection equivalent, i.e., if they have an equi-dissection, then clearly they must have the same area, and the Wallace-Bolyai-Gerwien (WBG for brevity) theorem says the converse is also true: any two polygonal regions with equal areas have an equi-dissection. See Figure 1 for an example.

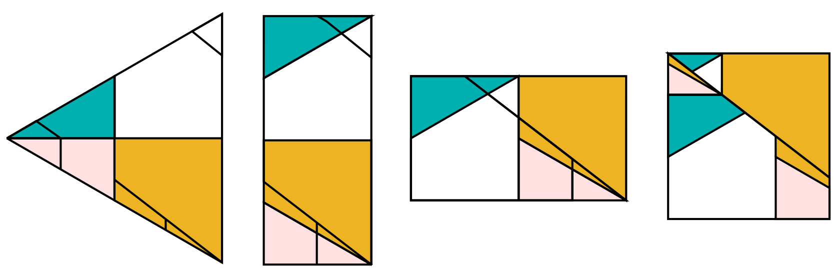

Figure 1. Equivalent dissections of a triangle and a square (and two intermediate rectangles showing how the pieces move).

A triangular dissection is one where all the pieces are triangles. A triangular dissection is called a triangulation if the simplex condition holds, i.e., any two pieces are either disjoint or they intersect in a common vertex or a common edge. See Figure 2. A dissection that is not a triangulation contains at least one exceptional point, i.e., a vertex of one piece that is an interior edge point of another piece. The dissection in Figure 2 has seven exceptional points. Although triangulations are much more restrictive than dissections, we will show that an analog of the Wallace-Bolyai-Gerwien theorem still holds for them.

Theorem 1. *Any two polygonal regions with the same area have triangulations that use the same set of triangles (up to rotation and translation).*

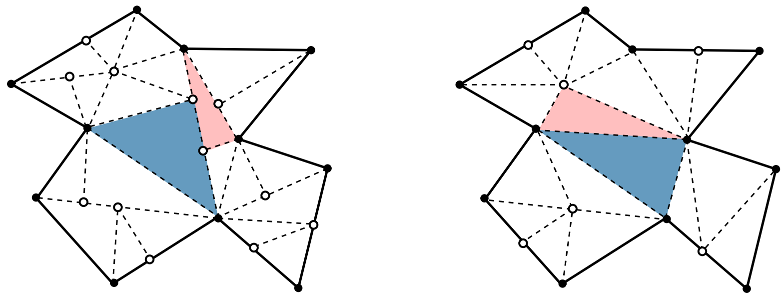

Figure 2. A dissection (left) and a triangulation (right) of a polygon.

We call this an equi-triangulation of the domains. Despite a large literature on optimal meshing and triangulation, the question of whether two equal area polygons always have an equi-triangulation seems not to have been previously considered.

Triangulations are generally more common than dissections in applications in topology, numerical analysis and computer science. This is because triangulations have a number of useful properties compared with triangular dissections, e.g., they remain triangulations under small perturbations of the vertices, functions on the vertices can be extended linearly to the edges and faces, and the dual graph has degree at most three, which is very useful in a variety of data structures and algorithms.

The basic idea behind Theorem 1 is to convert an equi-dissection of two polygonal regions into a simple dynamical system $\Phi : X \to X$, where the space X, to be defined precisely in Section 4, will consist of copies of the edges of the dissection pieces plus two other points. We will show that we can refine (see Section 2) the equi-dissection to an equi-triangulation if and only if the orbits of the exceptional points are finite. Simple examples show that these orbits can be infinite in general (see Section 7), but we will prove that any equi-dissection can be modified to give another equi-dissection of the same regions so that in the new dynamical system all exceptional points have finite orbits.

2. REFINEMENTS. One dissection of a polygonal region Ω is a refinement of another dissection of Ω if each piece of the second dissection is a union of pieces from the first dissection. It is well known that any polygonal region Ω can be triangulated without adding new vertices (e.g., Theorem 3.1 of [4]). By triangulating each piece of a dissection (include any exceptional points on the boundary of a piece as vertices of that piece), we observe that any dissection has a refinement that is a triangulation. See Figure 3. We will prove Theorem 1 by showing that given any equi-dissection of two polygons there is a refinement that gives an equi-triangulation of the same polygons.

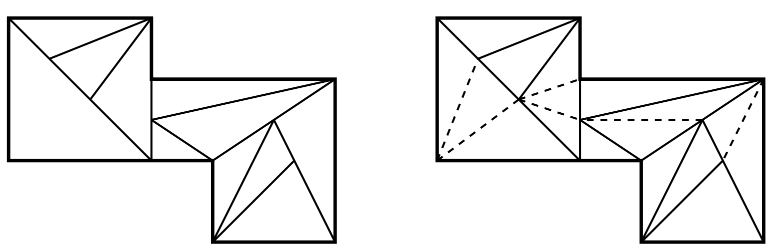

Figure 3. A triangular dissection (left) and a refinement that is a triangulation (right).

Given two dissections of the same polygonal region Ω, we can form a third dissection that is a refinement of both, by taking all intersections of pairs of pieces, one from each dissection. This is called their common refinement. Figure 4 shows the common refinement of two dissections of a square.

Lemma 2. *Dissection equivalence is transitive.*

Proof. Suppose $\Omega_1, \Omega_2, \Omega_3$ are polygonal regions so that Ω_1 and Ω_2 have an equi-dissection, and similarly for Ω_2 and Ω_3. Thus we have two dissections of Ω_2 and their common refinement gives an equi-dissection of Ω_1 and Ω_3. See Figure 4. ■

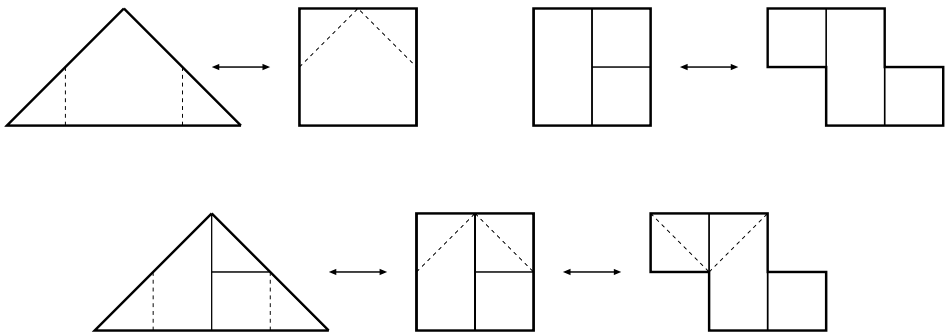

Figure 4. Dissection equivalence is transitive. The common refinement of the two dissections of the square in the top row can be rearranged into either of the other two shapes (bottom row).

3. THE WALLACE–BOLYAI–GERWIEN THEOREM. According to Ian Stewart [1] the Wallace-Bolyai-Gerwien theorem seems to have been proven by William Wallace around 1808, and independently by Paul Gerwien in 1833, in response to a question of Wolfgang Bolyai (father of the hyperbolic geometry Janos Bolyai). Eduardo Giovannini [2] credits John Lowry with a solution in 1814, in response to a question of Wallace, and cites independent proofs by Bolyai (somewhat sketchy) in 1831 and by Gerwien (very detailed) in 1833. See also the historical remarks in [3]. The proof of the theorem is elementary and well known, but we sketch it here for the convenience of the reader.

Proof of the WBG theorem. As noted in Section 2, any polygonal region Ω can be triangulated, and we claim that any triangle of area a is dissection equivalent to a 1-by-a rectangle. If so, then stacking the rectangles corresponding to the elements of a triangulation of Ω, shows that Ω is dissection equivalent to 1-by-A rectangle, where A is the area of Ω. By Lemma 2, the WBG theorem follows from the claim.

The proof that a triangle of area a is dissection equivalent to a 1-by-a rectangle is given in Figure 5 by a series of pictures (one also needs a little high school geometry involving similar triangles). The topmost picture shows that any triangle is dissection equivalent to some rectangle. Every triangle has at least two angles strictly less than 90° (since all three angles sum to 180°), and we have chosen the horizontal edge of the triangle in Figure 5 to be an edge between two such angles. This ensures the opposite vertex projects vertically into this edge, as shown. The middle of Figure 5 shows that any rectangle is dissection equivalent to another rectangle with one side twice as long and the other side half as long. Repeating this process, we can obtain a rectangle with one side length $s \in [1, 2]$, and the other side length $t \in [a/2, a]$. The three-piece dissection at the bottom of Figure 5 shows this is equivalent to a $1 \times a$ rectangle, as desired. ■

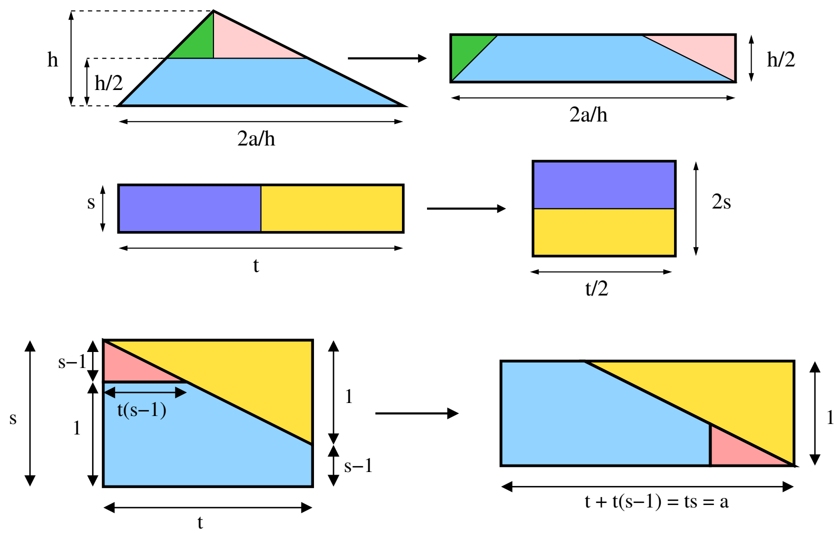

Figure 5. (1) A triangle is always dissection equivalent to some rectangle. (2) Any rectangle is equivalent to another rectangle where one side is twice as long and the other side is half as long. (3) A rectangle with side length $1 < s \leq 2$, is equivalent to one with side length exactly 1.

4. FLIPS AND SWAPS. In this section, given an equi-dissection of two polygonal regions, we will define a space X and a mapping $\Phi : X \to X$. Informally, the space X will consist of all the edges of all the pieces in both dissections, plus two additional special points. Points that belong to the boundaries of exactly two adjacent pieces of the same dissection are considered as two separate points of our space X, and the "flip map" will exchange them. The flip map will send other points (e.g., exceptional points and boundary points) to one of the two special points in X. The "swap map" will take points in each dissection and map them to the corresponding points in the other dissection. Composing the flip and swap maps will give a single map $\Phi : X \to X$, and this is the dynamical system we will study in the remainder of this paper.

As noted earlier, a dissection of Ω is a finite collection of closed polygonal sub-regions with disjoint interiors and whose union is all of Ω. The boundaries of the pieces can overlap, but to define the two maps discussed above, we would like to think of these overlapping boundaries as distinct. We do this by considering the dissection pieces as isometric images of polygons that are actually disjoint. This could lead to some ambiguity about what a "piece" of a dissection is, but we will refer to the images as "pieces" (the sub-regions of Ω) and to the corresponding disjoint polygons as the "components". An equi-dissection will consist of a single set of components and two sets of isometries mapping them to the pieces of the two dissections. The rest of this section makes these ideas precise.

Suppose $\mathcal{P} = \{P_k\}_1^N$ is a finite collection of disjoint simple polygons. We let R_k denote the compact region bounded by P_k, and set $R = \cup_1^N R_k$. A map $F : R \to \Omega$ is a dissection of Ω with **components** $\{R_k\}_1^N$ and **pieces** $\{Q_k\}_1^N = \{F(R_k)\}_1^N$ if

1. F restricted to each R_k is an orientation preserving planar isometry,
2. the images $Q_k = F(R_k)$, $k = 1, \ldots, N$ have pairwise disjoint interiors,
3. and $\cup_1^N Q_k = \Omega$.

Let V_k denote the vertex set of P_k. For each k, the points of $E_k = P_k \setminus V_k$ are called the edge points of P_k (or the interior edge points, to be very precise). The connected components of E_k are the (open) edges of P_k. Define $E = \cup_1^N E_k$ to be the (disjoint) union of all the edge points. A segment $I \subset E$ is called a boundary segment if $F(I) \subset \partial\Omega$. We define boundary points similarly. In a dissection $F : R \to \Omega$, an interior edge point $x \in E$ must be one of three possible types (see Figure 6).

1. Regular: $F(x)$ an interior edge point of exactly two different dissection pieces.
2. Exceptional: $F(x)$ an interior edge point of one piece and the vertex of one or more other pieces.
3. Boundary: $F(x)$ is an interior point of a boundary edge of Ω.

A triangular dissection is a triangulation if and only if no exceptional points occur.

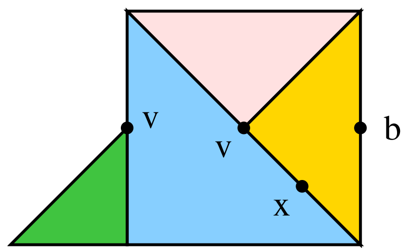

Figure 6. The three types of edge points: x is regular, v's are exceptional, and b is a boundary point.

We say two polygonal regions, Ω_1 and Ω_2, have an equi-dissection (or are dissection equivalent) if there are dissection maps $F_1 : R \to \Omega_1$ and $F_2 : R \to \Omega_2$ using the same collection of components. The pair of maps F_1 and F_2 is then an equidissection of the polygonal regions Ω_1 and Ω_2. Since F_1 and F_2, restricted to each component of R, preserves orientation, these restrictions are compositions of rotations and translations. Even if we allowed reflections in the definition, dissection equivalent regions would still have the same area, and hence they would be dissection equivalent in the more restrictive "no-reflections" sense, by the WGB theorem (the proof in Section 3 only uses orientation preserving maps). Thus allowing reflections gives no greater generality.

Suppose $F_1 : R \to \Omega_1$ and $F_2 : R \to \Omega_2$ form an equi-dissection of Ω_1 and Ω_2. Let $X = E_1 \sqcup E_2 \cup \{v\} \cup \{b\}$ consist of the disjoint union of two copies of E, plus two special points: v (for vertex) and b (for boundary). We can restrict F_1 to E_1 to define a map $F_1 : E_1 \to \Omega_1$, and similarly for $F_2 : E_2 \to \Omega_2$.

The **swap** map $\sigma : X \to X$ simply exchanges E_1 and E_2; each edge is mapped, via the identity, to the corresponding edge in the other copy. We extend σ to the points v and b as the identity: $\sigma(v) = v$ and $\sigma(b) = b$. Clearly σ^2 is just the identity, so σ is an involution. (Here we let $\sigma^2 = \sigma \circ \sigma$ and inductively define $\sigma^n = \sigma \circ \sigma^{n-1}$ to be the nth iterate of σ.)

Next we define the **flip** map $\phi : X \to X$. Each copy of E in X will be mapped into the union of itself and the two special points v and b. Fix $j \in \{1, 2\}$. To define $\phi(x)$ for an edge point $x \in E_j$, we consider three cases.

1. If x is regular, we set $\phi(x) = y$ if x and y belong to different components of E_j but $F_j(x) = F_j(y)$. Thus $x \neq y = \phi(x)$ are distinct in X but they have the same image in Ω_j. Clearly $\phi^2(x) = \phi(y) = x$.
2. If x is exceptional, we set $\phi(x) = v$.
3. If x is a boundary point, we set $\phi(x) = b$.

As with σ, define $\phi(v) = v$ and $\phi(b) = b$. Let $EV = \phi^{-1}(v) \setminus \{v\} \subset X$ denote the set of exceptional points. For brevity, we use "exceptional orbit" to mean "the forward orbit of an exceptional point", i.e., $\{\Phi(v), \Phi^2(v), \ldots\}$ for some $v \in EV$.

Unlike σ, the flip map ϕ sends intervals of regular points from each copy of $E \subset X$ into the same copy, and it reverses their orientations. Like σ, the dynamics of ϕ by itself are simple, since each regular point x satisfies $x = \phi^2(x)$, and every other point satisfies either $\phi^2(x) = \phi(x) = v$ or $\phi^2(x) = \phi(x) = b$.

However, things become more interesting when we combine the flip map with the swap map. Define $\Phi = \phi \circ \sigma : X \to X$. Then v and b are fixed by Φ and every other point of X has at most one preimage. If a forward orbit consists only of regular points, then it either has infinitely many distinct points or it is periodic (it can not be strictly pre-periodic since regular points have unique preimages). We record this observation as a lemma for future use.

Lemma 3. *Every forward orbit of a point either*

1. *is an infinite sequence of distinct regular values,*
2. *is a periodic orbit of regular points,*
3. *or eventually lands on either v or b*

In particular, an exceptional orbit is finite if and only if it eventually lands on v or b.

If an orbit lands on v or b, then it stays there forever. Such an orbit is called "eventually fixed", and this is the only kind of strictly pre-periodic orbit that can occur in the dynamical systems we are considering. (Such points are also known as "absorbing states" or "cemetery states".) Since σ preserves orientations of edges and ϕ reverses them, Φ also reverses orientation. For $j = 1, 2$, if $\Phi^n(E_j) \cap E_j \neq \emptyset$ then n is even, so if an iterate of an interval under Φ intersects itself, then the orientations must be the same. We will use this observation in the proof of Lemma 6.

5. SOME EXAMPLES OF FINITE ORBITS. Next, we examine a few simple examples to illustrate how Φ acts, and to show the various types of finite orbits. While looking at specific examples, we will also make an observation that applies in general.

Figure 7 shows two regions, Ω_1 and Ω_2, equi-dissected by four triangles. In the left-hand dissection, there is one exceptional point, labeled 1. An arrow is included to show which component the point belongs to. (An exceptional point is only an interior edge point of one component, so the arrow is not really needed, but it does make the diagram easier to read, especially after iterating a few times.)

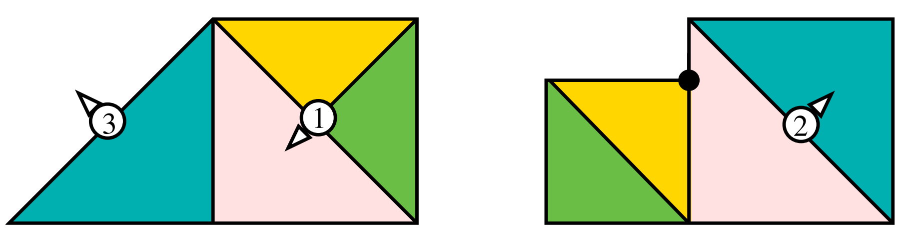

Figure 7. Φ maps the exceptional point 1 to 2, and then to 3 (which the special fixed point b).

The swap map σ moves a point from one dissection to the corresponding point in the other dissection, and then the flip map ϕ reverses the direction of the arrow. Thus $\Phi(1) = 2$. The next application of Φ sends 2 to the point labeled 3, with an arrow pointed outside of Ω_1; this is the point $b \in X$. The exceptional point on the right side of Figure 7 (the black dot) also iterates to b in two steps.

Figures 8 and 9 both show two rectangles equi-dissected into six triangles. Figure 8 shows a periodic orbit of a regular point. Figure 9 shows an exceptional orbit: $1 \to 2 \to 3 = v$. The second exceptional point also iterates to v. In general (but informally), if an exceptional point x iterates to an exceptional point y, then y iterates to x. More precisely, since $\phi^2 = \sigma^2 = \mathrm{Id}$ on regular points, the following holds.

Lemma 4. *If $n \geq 0$ and if x and $y = \sigma(\Phi^n(x))$ are both exceptional points, then $\sigma(\Phi^n(y)) = x$. Thus $\Phi^{n+1}(x) = \phi(\sigma(\Phi^n(x))) = \phi(y) = v = \Phi^{n+1}(y)$.*

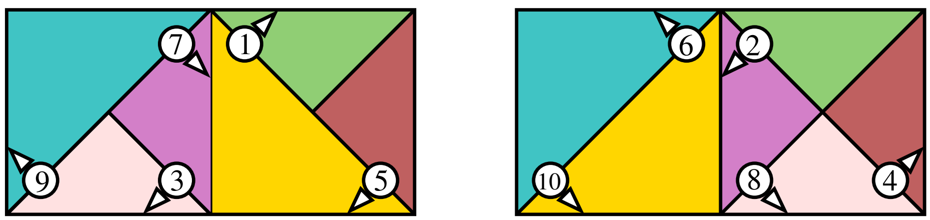

Figure 8. A periodic orbit of regular points of length 10.

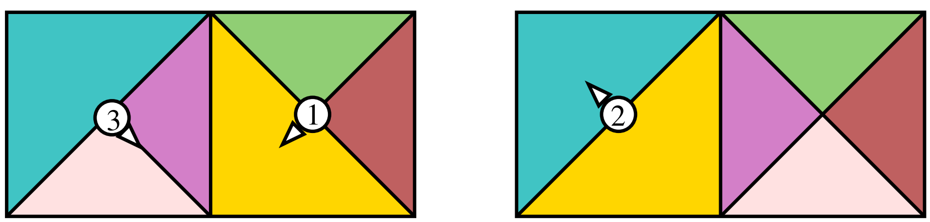

Figure 9. There are two exceptional points. The point marked 1 is first mapped to the regular point 2, and then to 3, which is the other exceptional point. Thus the orbit ends at $v \in X$. Similarly, the orbit of the exceptional point 3 also ends at v.

Figure 10 gives another equi-dissection (this time into rectangles), where there are two exceptional points. In this case, the left-hand exceptional point iterates to the right-hand one in eleven steps (and vice versa).We have chosen all edge lengths to be rational, and the exceptional orbits can never hit the boundary, so they must eventually reach v. Similar examples show that every interval exchange map on $[0, 1]$ occurs within the class of equi-dissections we are considering. There is an extensive theory of such maps, e.g., [5], but we make no use of it here.

6. FINITE ORBITS GIVE TRIANGULATIONS. Suppose Ω_1 and Ω_2 are polygonal regions that have a triangular equi-dissection. If the dissection of Ω_1 is not a triangulation, then there must be some exceptional vertices and, as noted in Section 2, we can refine the dissection to a triangulation by triangulating each component (including any exceptional vertices on the component's boundary). However, when we map these new triangles forward by the swap map, we might create new exceptional vertices in the dissection of Ω_2. Again, we can refine the dissection to get a triangulation of Ω_2, but new exceptional points might appear in Ω_1 when we apply the swap map again. We will never obtain an equi-triangulation unless this "ping-pong" process eventually stops creating new exceptional points.

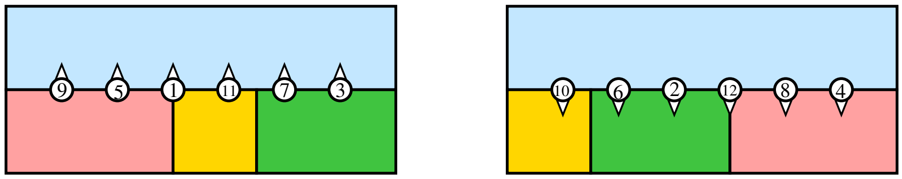

Figure 10. An interval exchange map as a equi-dissection. In this example, all the edge length ratios were chosen to be rational, and each of the two exceptional points iterates to the other one.

Lemma 5. *Suppose we have an equi-dissection of polygonal regions Ω_1 and Ω_2. Then every exceptional orbit is finite if and only if there is a equi-triangulation of Ω_1 and Ω_2 that refines the given equi-dissection.*

Proof. Suppose the exceptional orbits are all finite. Take all these orbit points as vertices of the polygons in $\mathcal{P}$ and triangulate using exactly these vertices. Transferring these triangulation to the dissections, we get triangulations of the dissection pieces so that there are no exceptional vertices. Thus we have an equi-triangulation. Conversely, if the equi-dissection has a refinement that is an equi-triangulation, then the orbits of the exceptional points must lie among the vertices of the triangulation. This is a finite set, and hence every exceptional orbit if finite. ∎

We claim that all the exceptional orbits are finite if and only if the set of exceptional points is contained in a finite Φ-invariant set Y, i.e., $EV \subset Y \subset X$ and $\Phi(Y) \subset Y$. To see this, note that if all the exceptional orbits are finite, then their union is such a set Y. The converse is obvious. Note that it suffices to check that $\sigma(Y) \subset Y$ and $\phi(Y) \subset Y$; this formulation is how we will utilize Lemma 5.

7. HOW TO END AN INFINITE ORBIT. Next, we will study a specific example with two infinite exceptional orbits, and we will modify the given dissections so that all exceptional orbits become finite. In Sections 8 and 9, we will use this construction to prove Theorem 1. The basic step will be to introduce two new triangular components for each infinite orbit, and to replace some existing components by sub-regions where we have removed triangular subsets to "make room" for the new components.

Figure 11 shows an equi-dissection of two rectangles $F_1 : R \to \Omega_1$ and $F_2 : R \to \Omega_2$ where $R = R_1 \cup R_2 \cup R_3$ has three components: a small triangle, a large triangle and a pentagon. Assume that the hypotenuse of the larger triangle has length 1, and that the hypotenuse of the smaller triangle has length $x < 1$. If we identify the long hypotenuse with the interval $[0, 1]$, then the two exceptional points correspond to x and $1 - x$. The nth iterate of the first point is $nx \mod 1$ for $n \in \mathbb{N} = \{1, 2, \dots\}$, and the iterates of the second point are $-(n+1)x \mod 1$ for $n \in \mathbb{N}$. If x is irrational, then both these orbits are infinite. (Irrational rotations of the circle have infinite orbits: if $nx = mx \mod 1$ for some $m > n$, then $x = k/(m-n)$ for some integer k, and so x must be rational. Indeed, such orbits are dense and even satisfy a stronger property called Weyl equidistribution, but we will not need this fact.)

We will refine the given equi-dissection to obtain a new equi-dissection of the same two rectangles in which all the exceptional orbits are finite. The process is described in Figures 11 to 14. (These show the regions Ω_1, Ω_2, but by mentally separating the pieces we can also think of these pictures as representing the space X.) Figure 11 shows the first nine iterates of the exceptional point labeled "1". Note that iterate 9 lies on a segment I whose endpoints are earlier iterates (1 and 3). Trapping exceptional orbit points between nearby earlier iterates is a key feature of our method.

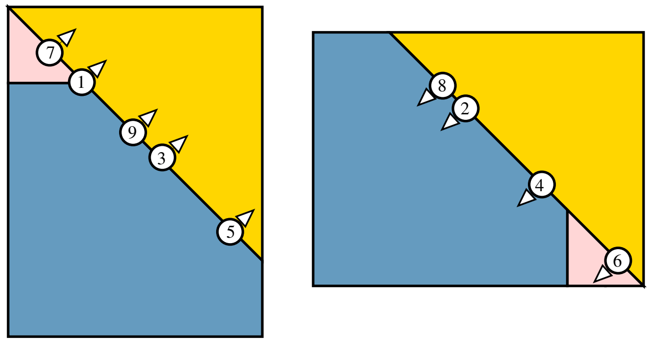

Figure 11. Suppose the hypotenuse of the larger triangle has length 1, and the hypotenuse of the smaller triangle has length $x < 1$. If x is irrational, the orbits of both exceptional points are infinite. Here we show the first nine iterates of an exceptional point (the point labeled "1").

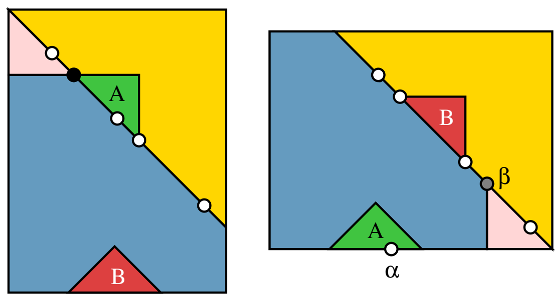

Figure 12. This is the same as Figure 11, except that we have removed the numbers and arrows to simplify the picture, and we have added two new triangular components; each is a right triangle with hypotenuse length X. The images (under F_1 and F_2) of the first new component are labeled A and the images of the second are labeled B. The large triangular component R_2 and the pentagon R_3 have been replaced by sub-components R_2', R_3' that have six and eight sides respectively.

We start modifying the equi-dissection by adding two new triangular components denoted A and B. See Figure 12, where the pieces $F_1(A)$ and $F_2(A)$ are labeled "A" and the two images of B are labeled "B". In the figure we show these components as both being right triangles, but the exact shapes are not important. However, the two new components must have the same shape as each other, and they must each have a side J of length x. We define F_1 on A so that J maps to the segment between 1 and 3 in Ω_1, and so that $F_1(A)$ lies inside the large triangle, $F_1(R_2)$. This is always possible if A is shallow enough. Note that R_2 is the component containing the point 9.

We modify the component R_2, obtaining R_2', by removing a copy of A from the indicated location. We then define F_2 on B so that $F_2(B)$ "fills the gap" in $F_2(R_2')$, and the side of length x in B maps to the segment between 2 and 4. (This is why A and B must have the same shape.)

Next, we define F_2 on component A, so that J maps into a boundary edge of $F_2(R_3)$, and so that $F_2(A)$ is contained in that piece. The choice of R_3 here is arbitrary, as is the location inside R_3 (except that $F_2(J)$ must be on the boundary of Ω_2). As with R_2, we remove a triangle from R_3 to get R_3', in order to "make room" for A. The image of point 9 under swapping is labeled α in Figure 12.

The choice of $F_2(A)$ forces the definition of $F_1(B)$, since both images have to fit into the same triangular region cut out of R_3. The maps F_1 and F_2 are restricted from

R_2, R_3 to R_2', R_3' in the obvious way. The original three components have become five components: three triangles (R_1, A, B) and two polygons $(R_2', R_3',)$ with six and eight sides respectively.

Next we terminate the orbit of the second exceptional point, labeled β. This is easier than the previous step, because β is already located between two suitable iterates, 4 and 6, of the first exceptional point. As shown in Figure 13, we add two more triangular components, labeled C and D, with the same shape as each other (but again the precise shape is unimportant and they can differ from A and). Both C and D are chosen to have a side I of length x, and we define $F_2(C)$ so that I maps to the segment between 4 and 6 in Ω_2, with $F_2(C)$ lying inside $F_2(R_2')$ (i.e., R_2' is the modified triangular component containing β). As before, we modify R_2' to make room for C, and call the result R_2''. Side I of D must be mapped by F_1 to the segment between 3 and 5. The image of $F_1(C)$ is arbitrary, except that side I of C must land in a boundary segment of Ω_1 and $F_1(C)$ is contained in some existing component (we choose R_3', but this is arbitrary). We then define a new component $R_3'' \subset R_3'$ by removing a copy of C. Then the definition of F_2 on D is forced. The image of β under the swap map is labeled γ. The final equi-dissection has seven components: five are triangles (R_1, A, B, C, D) and two are polygons (R_2'', R_3''). that have eight and eleven edges respectively.

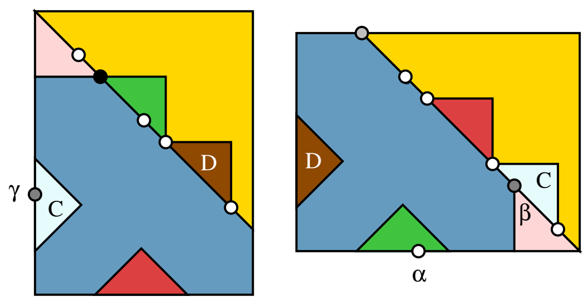

Figure 13. We can also terminate the second exceptional orbit. The second exceptional point is already inside the segment between iterates 4 and 6 of the first exceptional point. We now have seven components replacing the original three.

Finally, we must check that all exceptional orbits are finite. Define sets $Y_o \subset \Omega_1$ and $Y_e \subset \Omega_2$ as follows ("o" for odd and "e" for even): Y_o is the set of points labeled $1, 3, 5, 7, 9$, together with the point γ, and Y_e corresponds to $2, 4, 6, 8$ along with α and β. Let $Y = F_1^{-1}(Y_o) \cup F_2^{-1}(Y_e) \cup \{v, b\} \subset X$. Since F_1 and F_2 are finite-to-one maps, Y is a finite set. Indeed, each point can have at most two preimages. For example, the point 1 has zero preimages since it is now a vertex of every piece it belongs to. The point 2 is exceptional for the new equi-dissection and so it has only one preimage, but 7 has two preimages since it is still regular for the new equi-dissection.

When we add a triangle component T in the construction above, one side I of T either has both endpoints in Y, or both endpoints on a boundary edge. In the latter case, the endpoints of I are boundary points, thus are not exceptional. The vertex of T opposite I is a vertex of both new components that it belongs to, so does not create any exceptional points either. Therefore any new exceptional points are in Y. Moreover, $\phi(Y) \subset Y$ since the flip map only exchanges points with the same image under F_1 or F_2 and Y contains all preimages of a point if it contains any preimage. Also, $\sigma(Y) \subset Y$ since β swaps with γ, α swaps with 9, and preimages of other points with numerical labels only swap with other such points. Thus $EV \subset Y$ and $\Phi(Y) \subset Y$, and hence all exceptional Φ-orbits are finite by the remark at the end of Section 6.

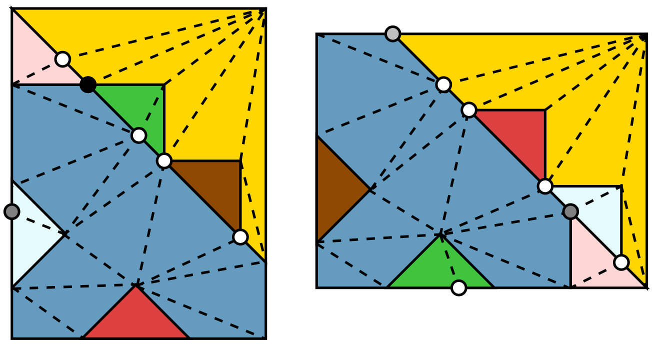

Figure 14. Here we triangulate the dissection pieces from Figure 13. Note each dissection piece is triangulated in exactly the same way in both dissections. Thus we have obtained an equi-triangulation that refines the original equi-dissection.

By Lemma 5, the new equi-dissection has a refinement that is an equi-triangulation of the two domains we started with. Figure 14 shows one such equi-triangulation. How we triangulate the polygonal components R_2'' and R_3'' is not important (as long as no vertices are added to their boundaries).

8. INFINITE ORBITS RECUR. In the previous section, we saw that an infinite orbit of an exceptional point w can be terminated by modifying the dissection so that some iterate of Φ maps w onto the point $b \in X$. This required two things to happen: (1) the exceptional orbit had to land inside some interval I whose endpoints were themselves earlier orbit points of exceptional vertices, and (2) I must be short enough to be mapped into some boundary segment of the other dissection. As noted in Section 7, the placement of the boundary triangles is arbitrary, except for having to lie inside existing dissection components and needing to be pairwise disjoint. Doing this for all exceptional orbits requires that the total length of all the segments I used is so small that the corresponding segments all fit disjointly on the boundaries of the dissected regions. The following lemma says this is always possible.

Lemma 6. *Suppose Ω_1, Ω_2 are polygonal regions with an equi-dissection, and let $\Phi : X \to X$ be the associated dynamical system described in Section 4. Suppose that EV is the set of exceptional vertices, that $v_1 \in EV$ is an exceptional point, and that this point has an infinite orbit consisting of distinct points $v_1, v_2, v_3, \ldots$. Let $EV_k = EV \cup \{v_2, \ldots, v_k\}$. Then given any $\epsilon > 0$, there is a positive integer n and an interval I on the boundary of some dissection piece so that*

1. $v_n \in I$
2. *I has length $< \epsilon$,*
3. *the endpoints of I are in EV_{n-1}.*

Proof. The basic idea is that an infinite sequence inside a finite set of bounded intervals must get close to itself eventually; a consequence of the pigeonhole principle. The only difficulty is that we need to find an orbit point v_n that is sufficiently close to points of EV_{n-1} on both sides on v_n. We will assume this fails and derive a contradiction.

To start, note that there must be a pair of indices, $m < n$, so that v_m and v_n are on the same edge of the same dissection component, and are within ϵ of each other; otherwise some edge contains infinitely many orbit points all ϵ apart, which is impossible.

Let J be the closed segment connecting v_m and v_n, oriented to go from v_m to v_n. Replacing v_m by another point of EV_{n-1}, if necessary, we may assume v_m is the only point of EV_{n-1} in J. Now iterate J forward one step to get an interval J' of the same length with endpoints v_{m+1} and v_{n+1}. J' is not on the boundary of Ω_1 or Ω_2, for then v_n would be on the boundary too, giving a finite orbit, contrary to our assumption.

If J' contains an exceptional vertex w, then w must be an interior point of J', since the orbit $\{v_k\}$ never hits an exceptional vertex other than v_1 (if it did, by Lemma 4 the orbit would be finite). We then replace J' by the sub-segment between w and v_{n+1}. We then iterate this new interval forward. At each stage, we either get an image interval of the same length which can then be iterated again, or we encounter another exceptional vertex as an interior point.

We claim we never encounter the same exceptional vertex twice, so this "segment shortening" step only happens finitely often. If we ever encounter an exceptional vertex w twice, say at v_{n+k} and at v_{n+j} with $j > k$, then v_{n+j} is strictly closer to w than v_{n+k} was, since the interval being iterated has length at most $|w - v_{n+k}|$ after encountering w the first time. If v_{n+j} is between w and v_{n+k}, then our orbit has landed inside an interval of length $< \epsilon$ with endpoints in EV_{n+j-1}. But we assumed this never happens. Thus w must separate v_{n+j} and v_{n+k}. But then the orientation of the image of J has changed, which is not possible by our remark at the end of Section 4. This proves the claim that each exceptional point is encountered at most once.

Thus the iterated interval is only shortened finitely often, and eventually the lengths stay fixed forever. Hence there are two such iterates that overlap. The overlap cannot be the entire interval, for then the endpoints agree and we would get a period orbit, which is impossible for exceptional points. Thus both intervals have some length $\delta > 0$ and one is equal to the other shifted by some $\eta < \delta$. If the iterates are M steps apart in the orbit, then taking another M steps gives a third interval that is a η-shift of the second interval and in the same direction, i.e., it is 2η-shift of the first interval. But since the forward iterates never hit the boundary or an exceptional point, this process can be continued forever. Thus there is an edge of the equi-dissection that contains infinitely many η-shifts of a fixed point, a clear contradiction, proving the lemma. ■

9. PROOF OF THEOREM 1.

Proof. Suppose we are given an equi-dissection of polygonal regions Ω_1 and Ω_2. Each has at least one boundary edge and so we can choose $\delta > 0$ so that each dissection has a piece that contains a square with side length δ, so that one side of this square lies along the boundary of the region being dissected. Let M be the number of exceptional vertices in the equi-dissection and choose a positive $\epsilon < \delta/4M$.

If there are exceptional vertices with infinite orbits, choose one, say v, and for the ϵ chosen above, select an integer n_v so that its orbit $v = w_1, \dots, w_{n_v}$ satisfies the conclusion of Lemma 6. We modify the equi-dissection by inserting two triangular components that terminate the orbit of v by mapping w_{n_v} to a boundary point, as described in Section 7. The boundary images of these new components may be placed on adjacent intervals inside the δ-boxes chosen above. As in Section 7, the new orbit of v will be finite and no new infinite orbits are created (although new exceptional points with finite orbits may be created). If any exceptional points with infinite orbits remain, we repeat the construction, and continue to do so until no infinite exceptional orbits remain. Then we are done by Lemma 5. ■

10. QUESTIONS AND REMARKS. Dissections and equi-dissections have been the source of many problems in recreational mathematics, often involving the minimal number of pieces needed to decompose a polygon into shapes of a certain type, or needed to equi-decompose two given polygons. Numerous references related to dissections and equi-dissections using the least number of pieces are given in [6].

Can we bound the number of triangles needed in our argument in terms of the number and side lengths of the given dissection pieces? Is computing the minimum number of polygons needed in an equi-dissection of two polygonal regions NP-hard? Triangular dissections? Equi-triangulations? Upper and lower bounds for the equi-dissection problem have been given in terms of the geometry of the polygons, and in various special cases, e.g., by Alfred Tarski [11].

In a hinged dissection, pieces are "connected" at vertices and can only move by rotations around these vertices that avoid self-intersection of the pieces, e.g., see [12] and Figure 15. Any pair of equal area polygons has a hinged equi-dissection by [13], but do they always have a hinged equi-triangulation?

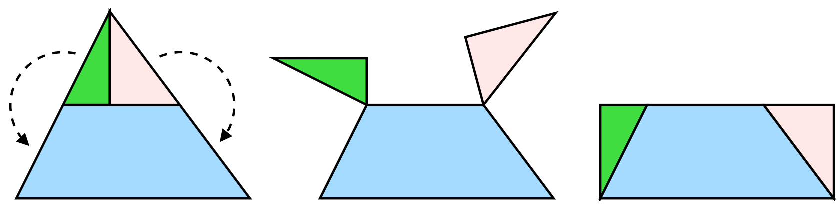

Figure 15. This equi-dissection of a triangle and a rectangle is hinged. It becomes a hinged equi-triangulation by triangulating the trapezoid piece appropriately. What happens in general?

Yuri Burago and Victor Zalgaller [7] proved that every polygon has an acute triangulation (all angles $< 90°$). Even more, every dissection of a polygon has a refinement that is an acute triangulation. See [8], [9], and [10]. Thus any two equal area polygons have an equi-dissection in which every element is an acute triangle. Using the result in this paper (and other arguments) one can prove they also have an acute equi-triangulation.

We ended infinite orbits by mapping some exceptional points to the boundary. What if there is no boundary? Do two closed polygonal surfaces of the same area have an equi-triangulation? Two such surfaces have an equi-dissection; the proof of the Wallace–Bolyai–Gerwien theorem still applies. Can we modify our construction so that pairs of exceptional orbits collide with each other? Is the number of exceptional points in a equi-dissection of two closed surfaces always even?

Two sets are called equi-decomposable if they can each be written as a finite union of disjoint subsets, where the sub-pieces of one decomposition can be rotated and translated to give the pieces of the other decomposition. The Banach-Tarski theorem [14] says that two planar polygonal regions are equi-decomposable if and only if they are equi-dissectable if and only if they have equal area. This fails in higher dimensions: the famous Banach-Tarski paradox says that a unit cube in $\mathbb{R}^3$ is equi-decomposable with the union of two disjoint unit cubes. This decomposition depends on the Axiom of Choice and the pieces are "non-constructible" in a precise sense. See [15], [16].

The 3-dimensional version of the Wallace–Bolyai–Gerwien theorem is better known as Hilbert's third problem: do any two polyhedra of equal volume have equi-dissections into polyhedral pieces? In 1900, Max Dehn [17] showed this is false, and that equi-dissectable polyhedra must have the same value of what we now call the "Dehn invariant", a certain tensor defined using the edge lengths and dihedral angles (angles between adjacent faces). For example, the unit cube and a regular tetrahedron of volume 1 have different values of the Dehn invariant, so they do not have an equi-dissection. That equal volumes and equal Dehn invariants are sufficient for a equi-dissection to exist was proven in 1965 by Jean-Pierre Sydler. See [19], [18]. In

[20], Johan Dupont and Han Sah show that Sydler's intricate geometric argument can be replaced by a purely algebraic proof involving the homology of $\mathrm{SO}(3, \mathbb{R})$.

By Sydler's theorem, any two polyhedra with the same volume and same Dehn invariant have an equi-dissection into tetrahedra. Do they have an equi-triangulation into tetrahedra, i.e., so that the simplex condition holds (adjacent pieces intersect in vertices, edges or faces)? Is there an acute equi-triangulation (all angles between faces are $< \pi/2$)? Finding an acute triangulation for a single polyhedron is already a difficult task. Even for a cube in $\mathbb{R}^3$, acute triangulations were only constructed fairly recently: [21] uses 2715 tetrahedra via a conceptual argument, and [22] uses 1370 found by a computer-assisted construction.

It is well known that Hilbert's third problem was the first on his list to be solved, but it is perhaps less well known that it was solved even before he stated it. According to [3], the problem was well known in the mathematical community, and in 1882 Władysław Kretkowski offered a prize of 500 French francs (about $10,000 today) for its solution, and this was awarded by the Polish Academy of Arts and Sciences to a 28-year-old math teacher, Ludwig Birkenmajer. Although unpublished, his manuscript was found by the authors of [3] in the Scientific Library of the Polish Academy of Sciences. In 1881 Birkenmajer became a *Privatdozent* at Jagiellonian University, and from 1897 until his death in 1929, he held the Chair of the History of Exact Sciences there. Birkenmajer was active in number theory, geophysics, astronomy, and the history of science, but never seems to have claimed credit for solving Hilbert's problem before Dehn. The prize sponsor, Kretkowski, published more than twenty mathematical papers, but as a wealthy nobleman, he made his greatest impact by supporting other mathematicians with prizes and scholarships during his life, and bequeathing his entire estate for the development of mathematics, on his death in 1910. See [3] for more details and references.

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