DISTORTION OF DISKS BY CONFORMAL MAPS

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Abstract. We answer a question of Astala, Clop, Mateu, Orobit and Uriarte-Tuero concerning the distortion of a union of disks by a quasiconformal map which is conformal off the disks.

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As part of his celebrated paper [1] on area distortion of quasiconformal mappings, Astala showed that if f is a K-quasiconformal map of the plane which is conformal outside a finite union of disjoint closed disks $\{D_j\}$, then

$$\sum_k \mathrm{diam}(D_j)^2 \sim \sum_k \mathrm{diam}(f(D_k))^2,$$

with a constant that only depends on K (assuming f is normalized so that $f(z) = z + o(1)$ near ∞, i.e., is a principle quasiconformal map). In a sequel paper [2] Astala *et. al.* ask if this is also true if 2 is replaced by some $d \in (0,2)$ (Question 2.4). In this note we shall show that this is false.

Theorem 0.1. *For any $d \in (0,2)$ there is a K so the following holds. Given any $M > 1$ there is a finite collection of disjoint closed disks $\{D_k\}$ and a principle K-quasiconformal map f which is conformal off $\cup_k D_k$ so that*

$$\sum_k \mathrm{diam}(f(D_k))^d \geq M \cdot \sum_k \mathrm{diam}(D_k)^d.$$

Alternatively, one can choose a collection $\{D_k\}$ and a principle K-quasiconformal map g which is conformal off $\cup_k D_k$ so that

$$\sum_k \mathrm{diam}(g(D_k))^d \leq \frac{1}{M} \cdot \sum_k \mathrm{diam}(D_k)^d.$$

The basic idea is to construct nested "rings of disks" so that inside the kth generation rings the maps have derivative $\simeq \lambda^k$ (with $\lambda > 1$ for f and < 1 for g). Moreover, a large proportion of the diameter sums comes from the inner rings, where the distortion is the largest and hence the sums before and after mapping by f or g will differ as much as we wish. The domains will be constructed with the aid of the Koebe's theorem, which states that any finitely connected plane domain can be mapped to to a Koebe domain, i.e., a planar domain so that every boundary component is either a circle or a point. We will also use that simple fact that if the domain is symmetric with respect to rotation by $2\pi/n$ around the origin, then the Koebe domain can be chosen with the same symmetry.

To make this basic idea more precise, we start with a definition. For $n \in \mathbb{N}$, $\epsilon \in (0,1)$ and $0 < r < s \leq 1$, let

$$S(n, \epsilon, r, s,) = \{z \in \mathbb{D} : r \leq |z| \leq s, |\arg(z^n)| \leq (1-\epsilon)\pi\}.$$

This set looks like a annulus centered at the origin with n radial channels of width ϵ/n cut through it. Let $\Omega_n = \mathbb{C} \setminus S(n, \frac{1}{2}, \frac{3}{4}, 1)$. Then Ω is a planar domain with n boundary components and is invariant under a rotation of $2\pi/n$ around the origin. By the Koebe circle theorem, there is a conformal map of Ω_n to a domain W_n bounded by n circles. Applying a Möbius transformation we map assume both 0 and ∞ are fixed by this mapping, and that the bounding circles are all equal size and centered at the nth roots of unity on the unit circle. Since the disks are disjoint, the have radius at most $O(1/n)$. (In fact, the radii must be comparable to $1/n$ for otherwise there would be annuli with large conformal modulus separating them in W_n and hence separating the components of $\partial\Omega_n$, which is clearly not true. However, we do not need the lower bound for what follows).

By a normal families argument, we see that as $n \to \infty$, the map $f_n : W_n \to \Omega_n$ tends to the identity on $|z| > 2$ and to $z \to \frac{3}{4}z$ on $|z| < \frac{1}{2}$. Moreover, for each n, f_n has a K_n-quasiconformal extension to the plane (although $K_n \to \infty$ as $n \to \infty$). Using a partition of unity near the circles $|z| = \frac{1}{2}$ and $|z| = 2$, we can modify f_n to get a quasiconformal map f of the plane which (1) maps W_n to Ω_n, (2) is the identity on $|z| \geq 2$ and (3) equals $z \to \frac{4}{3}z$ on $|z| \leq \frac{1}{2}$.

The map f has two parts to its Beltrami dilatation. On the disks W_n^c, the dilatation is large and the same as for f_n. On W_n, it is as small as we wish (by taking n large enough) and is supported in two annuli (which we may take to be $\{\frac{1}{2} \leq |z| \leq \frac{3}{8}\}$ and $\{\frac{5}{4} \leq |z| \leq 2\}$).

Now choose a collection of "first generation" disks $\mathcal{D}_1 = \{D_j\}$ so that the double of each disk is inside $D(0, \frac{1}{2})$, and

$$\sum_j \operatorname{diam}(D_j)^d = 2^d = \operatorname{diam}(D(0,1))^2. \tag{0.1}$$

We can certainly do this by choosing M disks of radius $r = 2 \cdot M^{-1/d}$, which will all fit into $D(0, \frac{1}{2})$ if M is large enough (since $d < 2$).

To form the second generation disks, take a conformal linear map from $D_0 = D(0,1)$ to D_j and take the image of each disk in $\mathcal{D}_1$. Continue for N generations. Let $\mathcal{C}_k$ be the collection formed by replacing each disk D in $\mathcal{D}_k$ by the image of ∂W_n

under a conformal,linear map sending the unit disk to D. Then clearly,

$$\sum_{k=1}^{N}\sum_{D\in\mathcal{C}_k} \mathrm{diam}(D)^d \simeq n\cdot n^{-d}\sum_{k=1}^{N}\sum_{D\in\mathcal{D}_k} \mathrm{diam}(D)^d = n^{1-d}N\mathrm{diam}(D_0)^d. \tag{0.2}$$

Note that if we choose the first generation disks to all be the same size r, then kth generation disks will all be size r^k.

Using f_n we can build a K_n quasiconformal map f of the plane as follows. Outside the doubles first generation disks we let $f = f_n$. Inside the double $2D$ of a disk $D \in \mathcal{D}_1$, but outside the doubles of the second generation disks, we let $f = L_1 \circ f_n \circ L_2$ where L_2 is the linear, conformal map from $2D$ to $D(0,2)$ and L_1 is the linear map which sends $\partial D(0,2)$ to $f(\partial 2D)$ (which is already define). The linear maps can be chosen to make this continuous and hence quasiconformal with the same constant as f_n. Continue in the obvious way over all generations.

The resulting map expands disks in $\mathcal{D}_k$ by a factor of $(\frac{4}{3})^k$ and hence

$$\sum_{k=1}^{N}\sum_{D\in\mathcal{C}_k} \mathrm{diam}(f(D))^d = \sum_{k=1}^{N} n(\frac{4}{3})^k\mathrm{diam}(D_0)^d \geq Cn(\frac{4}{3})^N. \tag{0.3}$$

For fixed n and $N \to \infty$, this sum is much larger than the one above. However this does not quite prove the theorem, since f is not conformal on W_n, but this is not difficult to fix.

Let h be a normalized solution of the Beltrami equation where the data is the dilatation of f^{-1} restricted to Ω_n. Then $h \circ f$ is conformal on W_n. From our remarks above, the L^∞ norm of the dilatation of f on W_n may be chosen as small as we wish by taking n large enough. Choose it so small that

$$\frac{1}{C}\mathrm{diam}(E)^{1+\epsilon} \leq \mathrm{diam}(h(E)) \leq C\mathrm{diam}(E)^{1-\epsilon}.$$

where ϵ is so small that

$$s^{1+\epsilon} > \frac{7}{8}s,$$

for $s \geq r$, where r is the size of the disks in $\mathcal{D}_1$. Thus the choice of d determines the choice of r in the definition of $\mathcal{D}_1$, which determines the choice of ϵ here, which determines the choice of n, which fixes the size of $K = K_n$ in Theorem 0.1.

For $D \in \mathcal{D}_k$, we have $\operatorname{diam}(f(D)) \simeq (\frac{4}{3})^k \operatorname{diam}(D) \geq \operatorname{diam}(D)$, and hence

$$\begin{aligned}
\operatorname{diam}(h(f(D)) &\geq \frac{1}{C}(\operatorname{diam}(f(D))^{1+\epsilon} \\
&\geq \frac{1}{C}((\frac{4}{3})^k r^k)^{1+\epsilon} \\
&\geq \frac{1}{C}(\frac{4}{3}\cdot\frac{7}{8})^k \operatorname{diam}(D) \\
&\geq \frac{1}{C}(\frac{7}{6})^k \operatorname{diam}(D).
\end{aligned}$$

Thus the conformal map $h \circ f$ expands disks in $\mathcal{D}_k$ be a factor of at least $(\frac{7}{6})^k$. Hence the estimate in (0.3) holds with the $\frac{4}{3}$ replaced by $\frac{7}{6} > 1$, which is still sufficient to prove the first statement of Theorem 0.1.

To prove the second part, we use the same argument, but start with a Koebe domain and a map which shrinks, rather than expands, near the center of the unit disk. Consider the domain

$$\Omega = \mathbb{C} \setminus \cup_{k=1}^n S(n, \frac{1}{2}, 1 - \frac{2k}{n^2}, 1 - \frac{2k+1}{n^2}).$$

Note that $D(0, 1 - \frac{3}{n}) \subset \Omega \subset D(0,1)$ and that the n^2 boundary components are arranged in n rings concentric with the origin. We apply Koebe's theorem to this domain and get a image whose boundary has n^2 disks around the origin, arranged in groups of n on each of the radials segments corresponding to an nth root of unity. We claim that every boundary disk has radius $\sim \frac{1}{n}$. To prove this, consider two boundary components of Ω_n which occupy adjacent positions in the same ring. It is easy to see the extremal distance between them in Ω is ~ 1 and hence the same is true of the corresponding disks in $\tilde{\Omega}$. By conformal invariance, the extremal distance between the two boundary circles in Ω is greater or equal to their extremal distance in the plane, and their diameters are bounded above by $2\pi/n$ (since their centers are at most this far apart). Thus each disk has diameter comparable to $\frac{1}{n}$, or else the extremal distance between them would be large.

We have normalized the Koebe domain so that the outermost ring of disks is centered along $|z| = 1$. Thus on each radial segment, the Euclidean distance between the innermost and outermost disk is about $n \cdot \frac{1}{n} \simeq 1$. Then the conformal map from $\tilde{\Omega}$ to is K_n-quasiconformal, converges to the identity on $|z| \geq 2$ and converges to $z \to \lambda z$ for some $\lambda < 1$ on some disk $|z| \leq \lambda/2$.

We can now proceed as before, constructing generations of disks $\mathcal{D}_k$ and $\mathcal{C}_k$ which satisfy (0.2). The images, however, satisfy

$$\sum_{k=1}^{N} \sum_{D \in \mathcal{C}_k} \operatorname{diam}(f(D))^d = \sum_{k=1}^{N} n \cdot \lambda^k \cdot \operatorname{diam}(D_0)^d = O(n). \tag{0.4}$$

We then correct f with a quasiconformal map of small norm, h, so that $\operatorname{diam}(h(f(D)) \leq O(\tau^k)$ for some $\lambda < \tau < 1$. Proceeding as before proves the second part of Theorem 0.1.

In [2] the authors also ask the following question (Conjecture 2.3): Suppose $d \in (0,2)$, E is compact and h is a K-quasiconformal principle map which is conformal off E, then $\mathcal{M}^d(E) \simeq \mathcal{M}^d(h(E))$, with constants depending only on K.

This conjecture is crucial to understanding the absolute continuity of Hausdorff measures under quasiconformal mappings and would have been implied by an positive answer to Question 2.4 of [2] (the one we have just disproved). However, our examples do not seem to lead to a counterexample of this weaker conjecture (as far as I can tell).

References

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[2] K. Astala, A. Clop, J. Mateu, J. Orobity, and I. Uriarte-Tuero. Distortionof Hausdorff measures and improved Painlevé removability for quasiregular mappings. 2006.

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