COMPACT DEFORMATIONS OF FUCHSIAN GROUPS

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Abstract. A conformal map Φ on the unit disk is called a compact deformation of a Fuchsian group G if Φ has a quasiconformal extension to the plane, h, which conjugates G to a Kleinian group G' and the dilatation of h is compactly supported modulo G. We show that for such deformations $\delta = \dim(\Lambda(G')) = \dim(\Lambda_c(G'))$ (if $\delta \geq 1$) and the image of $\Lambda_e = \Lambda \setminus \Lambda_c$ is contained in a countable union of rectifiable curves and has zero length iff G is divergence type.

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1. Introduction

Suppose G is a non-elementary discrete group of Möbius transformations acting on the hyperbolic ball, $\mathbb{B}^3$. As usual, we let $M = \mathbb{B}^3/G$ be the corresponding hyperbolic orbifold and let $\Lambda \subset S^2 = \partial\mathbb{B}^3$ be the accumulation set of any orbit. Let $\Lambda = \Lambda_c \cup \Lambda_e$ denote the limit points corresponding to geodesics in M which are recurrent (i.e., return to some compact set infinitely often) or escaping (i.e., eventually leave every compact set) in M respectively.

Suppose $\Phi : \mathbb{D} \to \Omega$ is a Riemann mapping. We say that Φ is a *deformation* of G if $\Phi \circ g \circ \Phi^{-1}$ is a Möbius transformation for every $g \in G$. We say Φ is *quasiconformal deformation* if Φ has a quasiconformal extension h to the whole plane so that $h \circ g \circ h^{-1}$ is a Möbius transformation for every $g \in G$. In this case the group $G' = h \circ G \circ h^{-1}$ is called a quasi-Fuchsian group. Finally, we say Φ is a *compact deformation* of G if it is a quasiconformal deformation and the dilatation μ_h of h is compactly supported (modulo G).

Since recurrent geodesics come close to support of μ_h infinitely often, while the escaping ones do not, it is natural to expect that the map h can greatly distort Λ_c but not Λ_e. In fact, we will prove

Theorem 1.1. *Suppose Φ is a compact deformation of a Fuchsian group G to a quasiFuchsian group G'. Then* $\dim(\Lambda_e(G')) = \dim(\Lambda_e(G))$.

Let $\delta(G)$ denote the critical exponent of a Kleinian group G. Clearly, $\dim(\Lambda) = \max(\dim(\Lambda_c), \dim(\Lambda_e))$ and it is known that $\delta = \dim(\Lambda_c)$ for all non-elementary groups (Theorem 1.1, [15]). It is also known that if $\delta(G) = 1$ for a Fuchsian group G, then $\delta(G') \geq 1$ for any deformation G' of G (see e.g., Theorem 0.4 of [28] applied to Λ_c or Section 1 of [11]). Thus

Corollary 1.2. *Suppose Φ is a compact deformation of a Fuchsian group G to a group G' and $\delta(G) = 1$. Then* $\dim(\Lambda(G')) = \delta(G') \geq 1$.

If $\dim(\Lambda_e(G)) = 1$, then Theorem 1.1 says $\dim(\Lambda_e(G')) = 1$ as well. In fact, we can be much more precise.

Theorem 1.3. *Suppose Φ is a compact deformation of a Fuchsian group G. Then $\Phi(\Lambda_e)$ has σ-finite 1-dimensional Hausdorff measure and lies on a countable union* *of rectifiable curves. If, in addition, G is divergence type, then $\Phi(\Lambda_e)$ has zero 1-dimensional measure.*

This is harder than Theorem 1.1 and its proof will take up most of the paper. The techniques of this paper can also be used to show that if Φ is a compact deformation of a divergence type Fuchsian group G, then $\Lambda(G') = \Phi(\mathbb{T})$ is either a circle or $\dim(\Lambda_c(G')) = \delta(G') = \dim(\Lambda(G')) > 1$, thus generalizing the result of Bowen for co-compact groups [18]. However, the first author has recently shown that this holds for all deformations (without the compactness assumption), so we will not include the proof of the special case here. See [10].

Now we would like to describe a little of the philosophy of our approach. The main idea behind Theorem 1.1 is to divide the disk into two types of stopping time regions: those near the support of μ and those at least distance r from it. On a region W far from the support of μ, we will show that Φ is Hölder with exponent close to 1 (depending on r). Moreover, an escaping geodesic eventually stays inside one of these regions. Thus $\Lambda_e(G')$ is a union of Hölder images of pieces of $\Lambda_e(G)$ (with exponent close to 1) and this implies Theorem 1.1.

Theorem 1.3 is an application of the the non-linear L^2 theory for Schwarzian derivatives developed by the authors in [13] and [14]. Also see [26] for a survey of these results. Closely related are the papers by Astala and Zinsmeister [3], [6]. Given a Riemann mapping $\Phi : \mathbb{D} \to \Omega$ we can compute the length of $\partial\Omega$ by

$$I_1 = \int_{\mathbb{T}} |\Phi'(e^{i\theta})| d\theta.$$

The results of [13] and [14] imply that I_1 is bounded below by a multiple of

$$I_2 = \int_{\mathbb{D}} |\Phi'(z)||S(\Phi(z))|^2 \log \frac{1}{|z|}(1-|z|^2)^2 dxdy,$$

where $S(\Phi)$ is the Schwarzian derivative of Φ. For general domains $I_1 \not\simeq I_2$, but $I_1 \simeq I_2 + \mathrm{diam}(\Omega)$ when Ω is a quasidisk. The integral I_2 is where the non-linear Littlewood-Paley theory comes in; had we used the usual theory, our bound for I_1 would have been of the form

$$I_2' = \int_{\mathbb{D}} |\Phi'(z)||\varphi''(z) + \frac{1}{2}\varphi'(z)^2|^2 \log \frac{1}{|z|}(1-|z|^2)^2 dxdy,$$

where $\varphi = \log|\Phi'|$. Since the Schwarzian can be written as $S(\Phi) = \varphi'' - \frac{1}{2}(\varphi')^2$, changing the "+" to a "−" is a non-linear mapping which converts this integral to I_2.

The integral I_2 is preferable to I_2', despite its non-linear nature, since the Schwarzian can be easily estimated in terms of the geometry of Ω and has very nice invariance with respect to composition by Möbius transformations which makes it particularly useful in studying group deformations. Moreover, I_2 is a direct analog of the so called "β-sums" used by the second author to characterize subsets of rectifiable curves in [26].

In this paper we will be more concerned with the related integral

$$I_3 = \int_{\mathbb{D}} |S(\Phi(z))|^2 \log \frac{1}{|z|}(1-|z|^2)^2 dxdy.$$

This represents more localized geometric information than I_2 in the sense that $S(\Phi)(z)$ can be estimated from the geometry of the $\partial\Omega$ near the point $\Phi(z)$ at the scale $\mathrm{dist}(\Phi(z), \partial\Omega)$, whereas estimates of Φ' require information about all larger scales as well. One can also consider I_3 as an analog of the β-sums in [26], but in which each dyadic square is weighted by its harmonic measure in Ω, rather than its diameter. Unfortunately, the convergence of neither I_2 nor I_3 implies the convergence of the other, but, as we shall see below, much useful information can still be derived from I_3. This integral has been studied in detail in by Astala and Zinsmeister (e.g., [2]) as part of their "rectifiable Teichmüller" theory and by the authors (e.g., [14]) for its connections to the geometry of Ω. For example, $I_3 < \infty$ implies $|\Phi'| \in H^p$ for $p < \frac{1}{2}$ and it implies that harmonic measure on Ω satisfies a Lavrentiev type estimate (just as rectifiable curves do). See [14].

Because of the composition properties of the Schwarzian (see Section 4), it turns out that if Φ a deformation Fuchsian group G, then the hyperbolic Schwarzian

$$S_H(\Phi) \equiv |S(\Phi(z))|^2(1-|z|^2)^2,$$

is group invariant (e.g., [20], [31]). Thus we can rewrite I_3 as

$$\int_{\mathbb{D}} S_H(\Phi(z))^2 \log \frac{1}{|z|} dA,$$

where dA denotes hyperbolic area. Since $\log \frac{1}{|z|}$ is the Greens function for the disk, this integral is just the same as

$$\int_R S_H(\Phi(z))^2 G_R(z,w) dA, \tag{1}$$

where $R = \mathbb{D}/G$ is the quotient Riemann surface, $S_H\Phi$ is viewed as a function on R, G_R is the Greens function for R and w is the projection of $0 \in \mathbb{D}$ onto R.

If G is divergence type then $G_R \equiv +\infty$, and so the integral I_3 is infinite (unless $S(\Phi) \equiv 0$). This does not directly imply that $\Phi(\mathbb{T})$ has infinite length, but we can deduce that the integral over the Stolz cone

$$\int_{\Gamma(x)} |S(\Phi)(z)|^2(1-|z|^2)^2 dxdy = \infty,$$

for every conical limit point, and hence almost every point on $\mathbb{T}$ (since G is divergence type). By Theorem 5 of [14], Φ' fails to have non-tangential limits almost everywhere and hence cannot be in H^1. Thus $\Phi(\mathbb{T})$ must have infinite length. This is Pommerenke's result from [30].

Next, suppose G is convergence type, so that G_R is finite. Then (1) might be finite or infinite, depending on Φ. For example, if $S_H(\Phi)$ is bounded away from zero on R then the integral is infinite (since $\int_R G_R dA$ represents the expected time a Brownian motion spends on R, which is infinite). On the other hand, if Φ corresponds to a compact deformation of G we shall show that

$$S_H(\Phi) \leq C\exp(-\rho(z, z_0)), \tag{2}$$

where ρ is hyperbolic distance on R and z_0 is some point of R. Since the hyperbolic area of a ball of radius r Riemann surface can grow at most like e^r, and since G is bounded outside a neighborhood of its pole, we deduce that (1) is always finite for any compact deformation of any convergence type group. In fact, much of this paper is devoted to showing this integral is uniformly bounded, independent of the placement of the pole of Greens function and thus

$$d\mu = |S(\Phi(z))|^2 \log\frac{1}{|z|}(1-|z|^2)^2 dxdy \tag{3}$$

is a Carleson measure on the disk. Proving this requires some estimates on the growth of Greens function for R. The simplest case is when R is "thick", i.e., the injectivity radius at any point is uniformly bounded away from zero. Then the gradient of Greens function is uniformly bounded outside any neighborhood of its pole and so the Greens function has at most linear growth on R. From this and (2) it is easy to deduce a uniform bound for (1), independent of the choice of the pole of Green's function. When R has thin parts, some additional arguments are needed (and only the hyperbolic thin parts cause any serious difficulty).

When $\Omega = \Phi(\mathbb{D})$ is a quasidisk, the fact that (3) is a Carleson measure is equivalent to several different conditions on Φ and Ω including

A: $\varphi = \log|\Phi'|$ is in BMOA,

B: Φ can be extended to a global quasiconformal map whose dilatation μ such that $|\mu(z)|^2(1-|z|)^{-1}dxdy$ is a Carleson measure.

C: For every $z \in \Omega$, there is a uniformly chord-arc subdomain $D_z \subset \Omega$ with $\ell(\partial D_z) \simeq \mathrm{dist}(z, \partial\Omega) \simeq \ell(\partial D_z \cap \partial\Omega)$.

D: Harmonic measures for the two sides of $\partial\Omega$ are A_∞ related.

Thus our methods show that a compact deformation of a Fuchsian group has limit set lying on a curve satisfying all these conditions. These equivalences are proven in [3]. Curves satisfying (A)-(D) are "well approximated" by rectifiable curves (e.g., Corollary 1 of [14]), but need not be rectifiable themselves (e.g., [33], [8], [3]), contrary to a conjecture in [23]. An example of Astala and Zinsmeister in [5] shows that a compact deformation of a convergence group can have dimension > 1, and hence must a curve which is not rectifiable but satisfies (A)-(D). See [27] for further results about such curves.

As noted above, if G is divergence type and Φ is not Möbius, then the integral (1) diverges. However, if $S \subset R$ has a Greens function G_S (e.g., $S = S_r = R \setminus B(x_0, r)$) and Φ corresponds to a compact deformation, then we will show the integral

$$\int_S |S_H(\Phi)(z)|^2 G_S(z,w)\, dA(z) \le A(r) \tag{4}$$

with G_S replacing G_R will converge with an estimate which is uniform for $w \in S$ and tending to 0 as $r \to \infty$ (e.g., Corollary 9.4). Given a $r > 0$ we will construct stopping time regions $W_j \subset \mathbb{D}$ so that W_j projects into some S_r by the quotient map and every escaping geodesic (except possibly for countably many) eventually stays in some W_j. The fact that (4) is small allows us to deduce that $\varphi = \log|\Phi'|$ is in BMO on W_j with small norm (actually we may have to pass to a subregion of W_j to avoid a small set where φ' might be large) and thus that $\Phi(W_j)$ has rectifiable boundary by the John-Nirenberg theorem and A_∞ estimates for harmonic measure on chord-arc curves. Hence $\Phi(\Lambda_e)$ is a subset of a union of rectifiable curves. If G is divergence type then Λ_e has zero Lebesgue measure and the F. and M. Riesz theorem then implies $\Phi(\Lambda_e)$ has zero linear measure. The main technicality is proving (4).

The remainder of the paper is organized as follows.

Section 2: We review definitions and facts concerning chord-arc domains, Carleson measures and BMO.

Section 3: We review background on Fuchsian and Kleinian groups.

Section 4: We recall some facts about the Schwarzian derivative.

Section 5: We prove the Schwarzian derivative of a compact deformation decays exponentially fast away from the support of the dilatation.

Section 6: We show that if the Schwarzian is small on a region then Φ satisfies a Hölder estimate.

Section 7: We prove Theorem 1.1.

Section 8: We prove an estimate on Greens function for a Riemann surface.

Section 9: We show that if Φ is a compact deformation of a convergence type group, then $\varphi = \log|\Phi'|$ is in BMOA.

Section 10: We establish a variant of Greens theorem for chord-arc curves and use it to bound φ in terms of $S(\Phi)$.

Section 11: We prove Theorem 1.3.

2. Carleson measures and BMO

If A, B are quantities that depend on some parameter we write $A \lesssim B$ if the ratio B/A is bounded uniformly independent of the parameter. Similarly for $\gtrsim$. We write $A \simeq B$ if both $A \lesssim B$ and $A \gtrsim B$ hold and say A and B are comparable.

The notation $\text{dist}(z, w)$ or $\text{dist}(z, E)$ will always denote the Euclidean distance between points or sets. We will let ρ denote the hyperbolic metric in hyperbolic space or a Riemann surface R.

If $E \subset \mathbb{R}^d$ we let $|E|_d$ denote the d-dimensional Lebesgue measure of E and use $|E|$ if the dimension is clear from context. Suppose φ is a increasing continuous function from $[0, \infty)$ to itself such that $\varphi(0) = 0$. We define the *Hausdorff content* of a set $E \subset \mathbb{R}^2$ as

$$\mathcal{H}^\varphi_\infty = \inf\{\sum \varphi(r_j) : E \subset \cup_j D(x_j, r_j)\}.$$

If $\varphi(t) = t^\alpha$ we denote $\mathcal{H}^\varphi$ by $\mathcal{H}^\alpha$. The *Hausdorff dimension* of E is

$$\dim(E) = \inf\{\alpha : \mathcal{H}^\alpha_\infty(E) = 0\},$$

and the *Hausdorff measure* of E is

$$\mathcal{H}^\varphi(E) = \lim_{\delta\to 0}[\inf\{\sum \varphi(r_j) : E \subset \cup_j D(x_j, r_j), r_j \le \delta\}].$$

A nth generation *dyadic interval* in $\mathbb{T}$ is an interval of the form $(j2^{-n}\pi, (j+1)2^{-n}\pi]$. The collection of such intervals is denoted $\mathcal{D}_n$. Given a point $x \in \mathbb{T}$, we let $I_n(x)$ denote the unique interval in $\mathcal{D}_n$ which contains x. If J is any bounded interval, the dyadic subintervals, $\mathcal{D}_n(J)$, of J refer to the images of the usual dyadic interval under an affine map from $[0,1]$ to J.

If Ω is a simply connected domain, the harmonic measure for Ω with respect to the point $z_0 \in \Omega$ is the push forward of normalized Lebesgue measure on the circle under any Riemann mapping of the disk to Ω, taking 0 to z_0.

A curve γ is called *chord-arc* with constant M if the shorter arc connecting any two points $x, y \in \gamma$ has arclength bounded by $M|x-y|$. A chord-arc domain W is one whose boundary is a chord-arc curve. A bounded chord-arc domain always contains a point z_W with $\mathrm{dist}(z_W, \partial W) \simeq \mathrm{diam}(W)$. We will call such a point a center of W.

A Carleson square Q in $\mathbb{D}$ is a region of the form

$$Q = Q_I = \{z = re^{i\theta} : \theta \in I, 1 - |I| < r < 1\},$$

for some arc $I \subset \mathbb{T}$. The arc $I = I_Q$ is called the base of Q. For an arc I on the unit circle and $\lambda > 0$ we let λI denote the concentric interval with length $\lambda|I|$. We let $\ell(Q) = |I|$ and

$$T(Q) = \{z = re^{i\theta} : \theta \in I, 1 - |I| < r < 1 - \frac{1}{2}|I|\},$$

denote the top half of Q. We let z_Q denote the center of $T(Q)$. A dyadic Carleson subsquare of Q is a Carleson square whose base is a dyadic subinterval of I_Q. A particular kind of chord-arc domain which we will use are the so called "stopping time" regions. These are of the form $W = Q \setminus \cup_j Q_j$ where $\{Q_j\}$ is a disjoint (except for boundaries) collection of dyadic Carleson subsquares of Q.

A Carleson measure is a positive measure μ on $\mathbb{D}$ such that there is a $C < \infty$ such that $\mu(Q) \le C\ell(Q)$, for every Carleson Q. A convenient test (for our purposes) is the following.

Lemma 2.1. *Suppose f is a positive function on the disk and for every Carleson square Q,*

$$\int_Q f(z) G_Q(z, z_Q) \frac{dxdy}{(1-|z|^2)^2} \leq A < \infty$$

where $G_Q(z,w)$ is the Greens function for Q. Then $d\mu = f(z)(1-|z|)^{-1} dxdy$ is a Carleson measure of norm at most CA.

Proof. We wish to show that there is a constant $M < \infty$ so that

$$\int_Q f(z)(1-|z|^2)^{-1} dxdy \leq M \ell(Q),$$

for every Carleson square Q. Rewriting this, it is equivalent to showing

$$\int_Q \frac{f(z)}{(1-|z|^2)\ell(Q)} dxdy \leq M.$$

It is easy to verify that for $z \in Q$

$$G_{2Q}(z, z_{2Q}) \geq C \frac{1-|z|^2}{\ell(Q)},$$

where $2Q$ denotes the "double" of Q, i.e., the Carleson square whose base is concentric with that of Q, but has twice the length. Thus it suffices to show

$$\int_{2Q} f(z) G_{2Q}(z, z_{2Q}) \frac{dxdy}{(1-|z|^2)^2} \leq M,$$

for every Q. This is true by assumption. □

The space BMO on the unit circle is defined as the measurable functions (modulo constants) such that

$$\|f\|_{\mathrm{BMO}} = \sup_I m_I(f - m_I(f)) < \infty,$$

where the sup is over all subintervals I of $\mathbb{T}$ and

$$m_I(f) = \frac{1}{|I|} \int_I f(x) dx.$$

Holomorphic functions in the Hardy space H^2 on $\mathbb{D}$ with boundary values in BMO will be denoted BMOA.

We will want to use two important facts about BMO. The first is that BMOA can be characterized by a Carleson measure condition, i.e., $\varphi \in$ BMOA iff $|\varphi'(z)|^2(1-|z|^2)dxdy$ is a Carleson measure on $\mathbb{D}$, e.g. Theorem VI.3.4 of [22].

The second fact is the John-Nirenberg theorem ([25] or Theorem VI.2.1 of [22]) which says that if $f \in \mathrm{BMO}$ then there is a constant $C < \infty$, so that

$$|\{x \in I : |f(x) - m_I(f)| \geq \lambda\} \leq C|I| \exp(-C\lambda/\|f\|_{\mathrm{BMO}}),$$

for all $\lambda > 0$. In particular, this implies that there is an $\epsilon_2 > 0$ so that if $\|f\|_{\mathrm{BMO}} \leq \epsilon_2$ then $\int \exp(|f|)dx < \infty$. Thus if F is a conformal map on $\mathbb{D}$ so that $\varphi = \log|F'|$ is in BMOA with small norm, then F maps $\mathbb{T}$ to a rectifiable curve (in fact, to a chord-arc curve). See [29].

If $\Phi : \mathbb{D} \to W$ is conformal and F is a holomorphic function on W so that $F \circ \Phi$ is in BMO of the disk, we say F is in $\mathrm{BMO}(W, d\omega)$ (since it then satisfies the definition of BMO with respect to harmonic measure on ∂W).

Two measures μ and ν are said to be A_∞ related if for every $\epsilon > 0$ there is a $\delta > 0$ so that for every interval I and measurable $E \subset I$, $\mu(E) \leq \delta\mu(I)$ implies $\nu(E) \leq \epsilon\nu(I)$. It is a theorem [19] that this is a equivalence relation and that if μ is A_∞ related to ν then there are constants $C < \infty$ and $\alpha > 0$ so that $\mu(E) \leq C\nu(E)^\alpha$.

Next suppose W is a chord-arc domain and let σ and ω denote arclength and harmonic measure respectively. By a well known theorem of Jerison and Kenig [23], these two measures are A_∞ related. Because ω and σ are A_∞ related the John-Nirenberg theorem implies

$$\begin{aligned}\sigma(\{x \in \partial W : |f(x) - m_I(f)| \geq \lambda\}) &\leq C\omega(|\{x \in \partial W : |f(x) - m_I(f)| \geq \lambda\})^\alpha \\ &\leq C\exp(-C\alpha\lambda/\|f\|_{\mathrm{BMO}}).\end{aligned}$$

Thus $\int \exp(|f|)d\sigma$ will be integrable if the BMO norm of f is small enough, i.e.,

Lemma 2.2. *If W is a M-chord-arc domain and $\varphi = \log|F'| \in \mathrm{BMO}(W, d\omega)$ with small enough norm (depending only on M) then $F(\partial W)$ is rectifiable (indeed, is chord-arc).*

If V is a chord-arc domain and $\Psi : \mathbb{D} \to V$ is a Riemann mapping then the image of any Carleson square is a chord-arc subdomain W of V (see [24]). Thus Lemma 2.1 immediately implies the following.

Corollary 2.3. *Suppose $V \subset \mathbb{D}$ is a chord-arc domain and*

$$\int_W |f'(z)|^2 G(z, z_W)dxdy \leq A$$

for every chord-arc subdomain $W \subset V$. Then $f \in \mathrm{BMO}(V, d\omega)$ with norm $\lesssim A$.

3. Fuchsian and Kleinian groups

A discrete group G of isometries of the hyperbolic metric on $\mathbb{B}^d$ is called *Kleinian group* if $d = 3$ and *Fuchsian* if $d = 2$. Such a group can also be considered as a group of linear fractional transformations on S^2. G is called *elementary* if it contains a finite index Abelian subgroup. In this paper we are only concerned with non-elementary groups. For a non-elementary G, the accumulation of any orbit in $\mathbb{B}^d \cup S^{d-1}$ is a closed set $\Lambda \subset S^{d-1}$ which is independent of the particular orbit. This is the *limit set.* The *conical limit set* Λ_c is the set of points $x \in \Lambda$ for which there is a sequence of orbit points of 0 converging to x within a non-tangential cone in $\mathbb{B}^d$. It is easy to see that x is a conical limit point iff a geodesic ray in $\mathbb{B}$ ending at x projects to a geodesic in $M = \mathbb{B}^d/G$ which returns to some compact set of M infinitely often. We define $\Lambda_e = \Lambda \setminus \Lambda_c$ as the "escaping" part of the limit set.

A Fuchsian group G is called *first kind* if $\Lambda = \mathbb{T}$ and otherwise it is *second kind.* It is called *cocompact* if $R = \mathbb{D}/G$ is compact and *cofinite* if R has finite hyperbolic area.

If $x \in M$ then the *injectivity radius*, $\mathrm{inj}(x)$, is half the length of the shortest loop through x; alternatively it is half the distance between the two closest distinct lifts of x to $\mathbb{B}$. For $d = 2$, the ratio $\mathrm{inj}(x)/\mathrm{inj}(y)$ is at most $C \exp(\rho(x, y))$.

A Fuchsian group is called *divergence type* if

$$\sum_{g \in G} 1 - |g(0)| = \infty,$$

and otherwise it is called *convergence type.* The latter occurs iff $R = \mathbb{D}/G$ has a finite Greens function, which is given by the series

$$G_R(z, w) = \sum_{g \in G} G_{\mathbb{D}}(x, g(y)) = \sum_{g \in G} \log |\frac{x - g(y)}{1 - \bar{x} g(y)}|,$$

where x and y project to z and w respectively.

The *Poincaré exponent* (or *critical exponent*) of the group is

$$\delta(G) = \inf\{s : \sum_G \exp(-s\rho(0, g(0))) < \infty\},$$

where ρ is the hyperbolic metric in $\mathbb{B}^3$. Lemma 2.2 of [17] implies that we can compute δ using any point $z_0 \in \overline{\mathbb{B}^3} \setminus \Lambda$, i.e.,

Lemma 3.1. $\delta(G) = \inf\{s : \sum_G \mathrm{dist}(z_0, \Lambda)^s < \infty\}$, *for any* $z_0 \in \overline{\mathbb{B}^3} \setminus \Lambda$.

A result from [15] says that

Theorem 3.2. *If* G *is a non-elementary, discrete Möbius group on* $\mathbb{B}$ *then* $\delta(G) = \dim(\Lambda_c(G))$.

For Fuchsian groups and geometrically finite Kleinian groups this was previously known, e.g., [34], [35]. The proof given in [15] also shows

Corollary 3.3. *If* G *is any non-elementary, discrete Möbius group,* $x \in R = \mathbb{D}/G$ *and* $\epsilon > 0$ *then there is a* $R = R(\epsilon, x) < \infty$ *such that the set of directions (i.e., unit tangents at* x*) which correspond to geodesic rays starting at* x *which never leave the ball of radius* R *around* x *has dimension* $\geq \delta(G) - \epsilon = \dim(\Lambda_c(G)) - \epsilon$. *In particular, the set of directions at* x *of bounded geodesic rays has dimension exactly* $\delta(G)$.

This is true in all dimensions, but we will only need the case of Fuchsian groups in this paper, and this is due to Fernández and Melián in [21].

The Margulis lemma (e.g., page 134 of [7]) states there is an $\epsilon = \epsilon_M > 0$ independent of G (called the Margulis constant) so that if $y \in \mathbb{B}$, then the group $G_\epsilon(x)$ generated by

$$\{g \in G : \rho(x, g(x)) \leq \epsilon\},$$

is almost-nilpotent and hence elementary. In the case of Fuchsian groups this means the subgroup is cyclic and generated by either a finite order elliptic element or a parabolic or hyperbolic element.

A *thin part* of R is a connected component of $R_{\mathrm{thin}} = \{z \in R : \mathrm{inj}(z) \leq \epsilon_M\}$. On a Riemann surface there are only three types of possible thin parts: parabolic, hyperbolic, and elliptic. The first is conjugate to $\{x + iy : y > 1\}$ modulo $z \to z + 1$. The second is conjugate to $z \to \lambda z$ for λ close to 1. The last is $\mathbb{D}$ modulo $z \to e^{2\pi i/n} z$. In the parabolic and elliptic case the boundary has uniformly bounded diameter and in the hyperbolic case the boundary has two components, each of uniformly bounded diameter, but possibly far apart in the surface. In all three cases the injectivity radius in a thin part P is comparable to $\exp(-\rho(z, \partial P))$ and all three types of thin parts have uniformly bounded areas.

A homeomorphism h of the plane is called H-*quasiconformal* if for every $x \in \mathbb{R}^2$,

$$\liminf_{r \to 0} \frac{\max_{|x-y|=r} |h(x) - h(y)|}{\min_{|x-y|=r} |h(x) - h(y)|} \leq H < \infty.$$

Quasiconformal maps are differentiable almost everywhere and $\mu(w) = h_{\bar{z}}(w)/h_z(w)$ is called the complex dilatation of h. We shall only need two facts about quasiconformal maps. The first is that the map h (with the correct normalization) can be recovered from μ by an explicit formula (described in Section 5). The second is

Lemma 3.4. *Given $\epsilon > 0$ there is a $\delta > 0$ so that if h is a $1 + \delta$ quasiconformal map of the plane and the image of $\mathbb{D}$ is bounded, then then there is a disk D so that $D \subset h(\mathbb{D}) \subset (1 + \epsilon)D$.*

This is a consequence of the well know fact that quasiconformal mappings are also quasisymmetric and that the quasisymmetric constant is close to 1 if the quasiconformal constant is. See [1].

4. Geometric properties of the Schwarzian

In this section we state some basic results concerning the Schwarzian derivative. The Schwarzian derivative of a locally univalent function F is defined by

$$S(F)(z) \;\; = [\tfrac{F''(z)}{F'(z)}]' - \tfrac{1}{2}[\tfrac{F''(z)}{F'(z)}]^2 = [\tfrac{F'''(z)}{F'(z)}] - \tfrac{3}{2}[\tfrac{F''(z)}{F'(z)}]^2.$$

If we write $F' = e^{\varphi}$, then this can be rewritten as

$$S(F)(z) = \varphi'' - \frac{1}{2}(\varphi')^2. \tag{5}$$

Recall that $S(F) \equiv 0$ iff F is a Möbius transformation and that S satisfies the composition law

$$S(F \circ G)(z) = S(F)(G(z))(G'(z))^2 + S(G)(z),$$

(e.g., see Section 8.5 of [20] or Section 1.2 of [31]). In particular, if G is Möbius then

$$S(F \circ G) = S(F) \circ G(G')^2 \qquad \text{and} \qquad S(G \circ F) = S(F).$$

In addition, given an $\epsilon > 0$, hyperbolic disk D and a compact neighborhood K of D, there is a $\delta > 0$ so that $|S(F)| \leq \delta$ on D implies F uniformly approximates a Möbius transformation on K to within ϵ (this is easy to prove by normal families).

The composition laws imply that the *hyperbolic Schwarzian* ,

$$S_H(\Phi)(z) \equiv |S(\Phi)(z)|(1-|z|^2)^2,$$

is invariant under (pre or post) composition with Möbius transformations. Thus if Φ is a deformation of a Fuchsian group G, $S_H(\Phi)$ is constant on the orbits of G. This is what makes the Schwarzian such a powerful tool for studying deformations.

Lemma 4.1. *Suppose F is holomorphic on $\mathbb{D}$. If F is univalent then $\|S_H(\Phi)\|_\infty \leq 6$. Conversely, if $\|S_H(\Phi)\|_\infty \leq 1$ then F is univalent.*

For a proof see Theorem 8.12 in [20] for one direction and the remarks on page 263 for the other. Recall that a simply connected domain is called a κ-quasicircle if it is the image of the disk under a κ-quasiconformal mapping of the plane (see [1]).

Lemma 4.2. *If Ω is a $1+\kappa$ quasicircle, and $\Phi : \mathbb{D} \to \Omega$ is a Riemann mapping then $\|S_h(\Phi)\|_\infty \leq 6\kappa$.*

For a proof, see page 124 of [31] or Corollary 14.7 of [32].

The *Bloch* space, $\mathcal{B}$, on the unit disk is defined to be the collection of analytic functions such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty.$$

Writing $|\nabla_H f(z)| = |f'(z)|(1-|z|^2)$ for the hyperbolic gradient of f we see that this is just the set of analytic functions which are Lipschitz with respect to the hyperbolic metric. Given a subdomain $W \subset \mathbb{D}$ we will say φ is M-Bloch on W (or just Bloch) if $|\varphi'(z)|(1-|z|^2) \leq M$ for all $z \in W$. The following is Proposition 4.1 of [31]):

Lemma 4.3. *Suppose F is holomorphic on $\mathbb{D}$ and $\varphi = \log F'$. If F is univalent then $\|\varphi\|_{\mathcal{B}} \leq 6$. Conversely, if $\|\varphi\|_{\mathcal{B}} \leq 1$ then F is univalent.*

We will we will also need to use the following related results.

Lemma 4.4. *Given $\epsilon > 0$ there is a $\delta > 0$ so that if Ω is a bounded $1+\delta$ quasicircle then there is a choice of Riemann map $\Phi : \mathbb{D} \to \Omega$ so that $\|\varphi\|_{\mathcal{B}} \leq \epsilon$.*

Proof. By Lemma 4.2 the mapping has small Schwarzian. We can use Cauchy estimates to get φ' small at the origin. Then use Lemma 4.6 below to get small φ' on whole disk. □

Lemma 4.5 (Lemma 3.4, [14]). *Given $\epsilon, n > 0$ there exists $C = C(\epsilon) > 0$, $\delta = \delta(\epsilon, n) > 0$ such that if Q is a Carleson square and $|S_H(\Phi)(z)| \leq \delta$ for all $z \in T(Q)$, then there is a geodesic γ such that every point $z \in Q$ such that $1 - |z| \geq 2^{-n}\ell(Q)$ and $|\varphi'(z)| \geq \epsilon/(1-|z|^2)$ lies within hyperbolic distance C of γ.*

Proof. First we check that the lemma is true for $\delta = 0$, i.e., for Möbius transformations. If $\tau(z) = (az+b)/(cz+d)$, with $ad - bc = 1$ and $|d/c| > 1$, (so there is no pole in the disk), then $(\log \tau'(z))' = -2c/(cz+d)$, so $|\varphi'(z)| \geq \epsilon/(1-|z|^2)$ becomes

$$1 - |z|^2 \geq \frac{\epsilon}{2}|z + \frac{d}{c}|.$$

This is clearly contained in a hyperbolic neighborhood of the geodesic containing 0 and $\bar{c}/\bar{d}$. If δ is small enough (depending on n) then Φ is uniformly approximated by a Möbius transformation τ and $(\log \Phi')'$ is approximated to within $\epsilon/(2(1-|z|^2))$ by $(\log \tau')'$ on $Q \cap \{z : |z| \leq 1 - 2^{-n}\ell(Q)\}$. The lemma is true for τ' by the calculation above, hence follows for Φ. $\square$

Lemma 4.6 (Lemma 5.2, [13]). *Given $0 < \epsilon < \frac{1}{4}$ there is a $\delta > 0$ with the following property. Suppose $z_0 = r_0 e^{i\theta} \in \mathbb{D}$ and let $\gamma = [r_0 e^{i\theta}, r_1 e^{i\theta}]$ be a radial line segment with endpoint z_0. Suppose $|\varphi'(z_0)| < \frac{1}{2}\epsilon/(1-|z_0|^2) \leq \epsilon/(1-|z_0|)$ and for all $z \in \gamma$ $|S_H(\Phi)(z)| \leq \delta$. Then for all $z \in \gamma$ $|\nabla_H \varphi(z)| \leq \epsilon$.*

Proof. Suppose $r_0 < r \leq r_1$ is the first point where the lemma fails, i.e., the first point on γ such that $|\varphi'(re^{i\theta})| = \epsilon/(1-r^2)$. Let $r_0 \leq t \leq r$ and note that by (5),

$$\begin{aligned} |\varphi''(te^{i\theta})| &\leq |S(\Phi)(te^{i\theta})| + \frac{1}{2}|\varphi'(te^{i\theta})|^2 \\ &\leq (\delta + \frac{1}{2}\epsilon^2)/(1-t^2)^2 \\ &\leq 4(\delta + \frac{1}{2}\epsilon^2)/(1-t)^2 \\ &< \epsilon/(1-t)^2, \end{aligned}$$

if $\delta < \frac{1}{8}\epsilon$ and $\epsilon \leq \frac{1}{4}$. Integrating, we deduce that

$$|\varphi'(re^{i\theta})| < \epsilon/(1-r)^2 \leq \epsilon/(1-r^2)^2,$$

which is a contradiction, proving the lemma. $\square$

The following is the main application of these results which we will need.

Lemma 4.7. *Given any $0 < \epsilon < \frac{1}{4}$ there is a $\delta > 0$ so that the following holds. Suppose Φ is a conformal map on $\mathbb{D}$ and let $\varphi(z) = \log|\Phi'(z)|$ and suppose W is a stopping time region such that $|S_H(\Phi)(z)| \leq \delta$ on W. Then there are countably many Carleson squares $\{Q_k\}$ so that φ is ϵ-Bloch on each $W_k = Q_k \cap W$ and all radii (except possibly one) which terminate in $\mathbb{T} \cap \partial W$ eventually lie in the closure of some W_k.*

Proof. If Q is a Carleson square and $|\nabla_H \varphi(z)| \leq \epsilon$ on $T(Q)$, then it holds in all of $Q \cap W$ by Lemma 4.6. Thus it either holds on all of W or Lemma 4.5 and iteration imply that $|\nabla_H \varphi(z)| \geq \epsilon$ holds only within some bounded hyperbolic distance C of a single geodesic ray γ. Let U denote this C-neighborhood of γ.

If γ leaves W at some point of $\partial W \setminus \mathbb{T}$ then we simply cover $\mathbb{T} \cap \partial W$ will small enough Carleson squares and we are done.

Finally, we must consider the case when γ remains in W until it hits $\mathbb{T}$ at a single point x_0. Given any point $x \in (\partial W \cap \mathbb{T}) \setminus \{x_0\}$ there is a dyadic Carleson square of size $\simeq |x - x_0|$ containing x and missing U. Taking a maximal collection of such squares proves the lemma. □

5. Exponential decay of the Schwarzian

In this section we will prove

Lemma 5.1. *Suppose $\mu = \mu_h$ is supported in $A_\epsilon = \{z : 1 \leq |z| \leq 1 + \epsilon\}$ and $\|\mu\|_\infty < 1$. Suppose h is a quasiconformal map on $\mathbb{R}^2$ with dilatation μ. Then $|Sh(0)| \leq C\epsilon$, where C depends only on $\|\mu\|_\infty$.*

Proof. We start by recalling the solution of the Beltrami equation, as presented, for example, in Ahlfors' book [1]. Let

$$P(f)(w) = -\frac{1}{\pi} \int f(z)(\frac{1}{z-w} - \frac{1}{z})dxdy,$$

denote the Cauchy transform and let

$$Tf(w) = -\frac{1}{\pi} \int \frac{f(z)}{(z-w)^2} dxdy,$$

be the Beurling transform. Then $h = P(g) + z$ where

$$g = \mu(1 + T\mu + T\mu T\mu + \dots),$$

and the convergence is geometric in L^2 since T is an isometry on L^2 and $\|\mu\|_\infty < 1$. Now h is conformal on $\mathbb{D}$ and for $z \in \mathbb{D}$ we have $h' = e^\varphi$ where we have set $e^\varphi = 1 + Tg(z)$. In particular,

$$|\varphi(z)| \leq C|Tg(z)|,$$

at least if $|Tg(z)| \leq \frac{1}{2}$.

Since $Sf(0) = \varphi''(0) - \frac{1}{2}(\varphi'(0))^2$, the usual Cauchy estimates imply

$$|Sf(0)| \leq C \max_{|z|\leq 1/2} |Tg(z)|.$$

If $|z| \leq 1/2$ and $w \in \operatorname{supp}(\mu)$ then $(z-w)^{-2} \geq 1/4$, and hence $|Tg(z)| \leq 4\|g\|_1$. Thus by Cauchy-Schwarz,

$$|Tg(z)| \leq 4\|g\|_2 \operatorname{area}(\operatorname{supp}(\mu))^{1/2} \leq 4\frac{\|\mu\|_\infty}{1-\|\mu\|_\infty}\|\mu\|_2(4\epsilon)^{1/2} \leq C(\mu)\epsilon.$$

Thus $|Sh(0)| \leq C\epsilon$, as desired. □

We will actually use this in the following form. The proof is immediate from the previous lemma and is left to the reader.

Corollary 5.2. *Suppose Φ is a conformal map on $\mathbb{D}$ with a quasiconformal extension h to the plane with dilatation μ. Then*

$$|S_H\Phi(z)| \leq C\exp(-\rho(z^*, \operatorname{supp}(\mu))),$$

where $z^* = 1/\bar{z}$ and C depends only on $\|\mu\|_\infty$ (and $C \to 0$ as $\|\mu\|_\infty \to 0$).

6. Small Schwarzian implies Hölder

Given $0 < \epsilon \leq 1$ and a Carleson square Q with base I we define

$$Q_\epsilon = \{z \in Q :, 1 - |z| \leq \epsilon|I|\},$$

$$I_\epsilon = \{z \in Q : 1 - |z| = \epsilon|I|\}.$$

Lemma 6.1. *Suppose $\epsilon > 0$ and $K < \infty$ are given. Then there is a $\delta = \delta(\epsilon, K) > 0$ and a $C = C(K,\epsilon) < \infty$ so that the following holds. Suppose $W = Q \setminus \cup_j Q_j \subset \mathbb{D}$ is a stopping time domain and f is a univalent function on $\mathbb{D}$ which has a K-quasiconformal extension to the plane and whose Schwarzian derivative satisfies*

$$|S_H(f)(z)| \leq \delta,$$

for every $z \in W$*. Also assume that* $\operatorname{diam}(f(Q)) \leq K|f'(z_Q)|\operatorname{diam}(Q)$*. Then*

$$\frac{1}{C}|f'(z_Q)|(\frac{1-|z|}{1-|z_Q|})^{\epsilon} \leq |f'(z)| \leq C|f'(z_Q)|(\frac{1-|z_Q|}{1-|z|})^{\epsilon}.$$

Proof. To estimate $|f'|$, we first bound the growth of $\varphi = \log f'$. We wish to show that given $\epsilon > 0$, we can make

$$|\varphi'(z)| \leq \epsilon \log \frac{1}{1-|z|},$$

for all points in $W \cap Q_{\epsilon_0}$, for some $\epsilon_0 > 0$, if δ is chosen small enough. The first step is to show that it is true for all $z \in W \cap I_\epsilon$.

We will prove this by showing it is true for Möbius transformations and then using that fact that if f has small Schwarzian then it is uniformly approximated by some Möbius transformation. In what follows, τ will always denote a Möbius transformation, $\tau(z) = (az+b)/(cz+d)$ with $ad - bc = 1$.

Lemma 6.2. *Given* $K < \infty$ *there is an* $\epsilon_1 > 0$ *so that the following holds. Suppose* f *is a conformal map on the disk which has a* K*-quasiconformal extension to the plane,* $Q \subset \mathbb{D}$ *is a Carleson square with base* I *such that* $\operatorname{diam}(f(I)) \leq \operatorname{diam}(f(\mathbb{T} \setminus I))$*, and* $\tau(z)$ *is a Mobius transformation which approximates* f *to within* $\epsilon_1|f'(z_Q)|$ *on* $Q \setminus Q_{\epsilon_1}$*. Then either* $c = 0$ *or the pole* $p = -d/c$ *of* τ *is at least distance* $\epsilon_1\ell(Q)$ *from* Q*.*

Proof. By composing with linear maps (and replacing $\mathbb{D}$ by a different disk), we may assume $z_Q = 0$, $|f'(z_Q)| = 1$ and $\ell(Q) = 1$. By the Ahlfors three point property (e.g., [1]) the shorter of the arcs $f(I)$ and $f(\mathbb{T} \setminus I)$ has diameter $\leq M$ for some $M = M(K)$ and by assumption this holds for $f(I)$. Standard estimates then imply

$$\operatorname{diam}(f(Q)) \lesssim M. \tag{6}$$

By Koebe's distortion theorem, $|f'|$ is uniformly bounded on $B(0, \frac{1}{10})$ and hence is $|\tau'(z)| = |cz+d|^{-2} = c^2|z-p|^{-2}$. From this is easily follows that c and d are uniformly bounded. If $c = 0$ or p is more than distance $\epsilon_1\ell(Q)$ from Q there is nothing to do, so assume otherwise. Choose a ball B of diameter ϵ_1 in $Q \setminus Q_{\epsilon_1}$ which is distance $\leq 2\epsilon_1$ from p. Then $|\tau'| \gtrsim (\epsilon_1)^{-2}$ on B and hence $\tau(B)$ has diameter $\gtrsim (\epsilon_1)^{-1}$. Thus $\operatorname{diam}(f(Q \setminus Q_{\epsilon_1})) \geq \operatorname{diam}(f(B)) \gtrsim (\epsilon_1)^{-1} - 2\epsilon_1$. This contradicts 6 if ϵ_1 is small enough, so proves the lemma. $\square$

Now we return to the proof of Lemma 6.1. Let $\tau(z)$ be a Möbius transformation and set $\psi(z) = \log \tau'$. A computation shows $|\psi'(z)| = 2/|z + \frac{d}{c}|$. Thus the inequality $|\psi'(z)| \geq \frac{1}{2}\epsilon/(1-|z|)$ becomes

$$(1-|z|) \geq \frac{\epsilon}{4}|z-p|.$$

If Q is Carleson and the point p lies at least distance $\epsilon_1 \ell(Q)$ from Q, then this inequality can only hold for points of Q which satisfy $1-|z| \geq \epsilon_0 \ell(Q)$ (where ϵ_0 depends only ϵ and ϵ_1, hence only on ϵ and K).

So given ϵ and K choose ϵ_1 by Lemma 6.2 and then choose ϵ_0 as above. Now choose $\delta > 0$ so that if $|Sf(z)| \leq \delta$ on W then there is a Möbius transformation τ such that $|f(z) - \tau(z)| \leq \epsilon_1 |f'(z_Q)|$ on $Q \setminus Q_{\epsilon_1}$, By the Cauchy estimates and taking ϵ_1 small enough, we may also assume τ' approximates f' and thus

$$|\varphi'(z)| \leq \epsilon/(1-|z|),$$

for all points $z \in W \cap I_{\epsilon_1}$, if δ is small enough. This completes the first step of the proof of Lemma 6.1.

For the next step we use Lemma 4.6; combined with our previous result, we see that $|\varphi'(z)| \leq \epsilon/(1-|z|)$ for all points in $W \cap Q_{\epsilon_1}$.

Finally, we convert this into an estimate of f'. Using Koebe's $\frac{1}{4}$-theorem, the fact that φ is in the Bloch space and integrating along radial lines, we see

$$|\varphi(z) - \varphi(z_Q)| \leq C + \epsilon \log \frac{1-|z_Q|}{1-|z|},$$

where $z_0 \in T(Q)$ and C is a constant that only depends on K and ϵ. Thus exponentiating gives the desired result. □

We will not need the following result here, but it is used in [9], [12].

Lemma 6.3. *Given $K < \infty$ and $\eta > 0$ there are $C < \infty$ and $r < \infty$ so that the following holds. Suppose f is a conformal map of the disk with a K-quasiconformal extension to the plane with Beltrami coefficient μ. Also suppose that* $\mathrm{diam}(f(\mathbb{D})) \leq K|f'(0)|$. *Suppose $W = Q \setminus \cup_j Q_j$ is a stopping time region and* $\rho(W^*, \mathrm{supp}(\mu)) \geq r$. *Then*

$$\sum_{\mathcal{C}} \mathrm{diam}(f(Q_j))^{1+\eta} \leq C \mathrm{diam}(f(Q))^{1+\eta} \sum_{\mathcal{C}} \frac{\mathrm{diam}(Q_j)}{\mathrm{diam}(Q)} \leq C \mathrm{diam}(f(Q))^{1+\eta}.$$

Proof. By the estimate of the previous lemma,

$$|f'(z)| \leq M|f'(z_Q)|(\frac{1-|z_Q|}{1-|z|})^\epsilon,$$

for $z \in W$. In particular,

$$\begin{aligned} \mathrm{diam}(f(Q_j)) &\simeq |f'(z_j)|\mathrm{diam}(Q_j) \\ &\lesssim |f'(z_Q)|\mathrm{diam}(Q)^\epsilon \mathrm{diam}(Q_j)^{1-\epsilon} \\ &\simeq \mathrm{diam}(f(Q))(\frac{\mathrm{diam}(Q_j)}{\mathrm{diam}(Q)})^{1-\epsilon} \end{aligned}$$

Thus if we choose ϵ so that $1+\eta = (1-\epsilon)^{-1}$, we have

$$\sum_{\mathcal{C}} \mathrm{diam}(f(Q_j))^{1+\eta} \lesssim \mathrm{diam}(f(Q))^{1+\eta} \sum_{\mathcal{C}} \frac{\mathrm{diam}(Q_j)}{\mathrm{diam}(Q)} \lesssim \mathrm{diam}(f(Q))^{1+\eta},$$

as desired. □

7. Proof of Theorem 1.1

Given $r < \infty$ we say a Carleson square is type one if $\rho(T(Q), \mathrm{supp}(\mu)) \leq r$ and type 2 otherwise. If Q is a maximal type 2 square, form a stopping time region $W = Q \setminus \cup_j Q_j$, where $\{Q_j\} \subset Q$ are the maximal type 1 dyadic subsquares of Q. Then

$$\{z : \rho(z, \mathrm{supp}(\mu)) \geq r + 10\} \subset W \subset \{z : \rho(z, \mathrm{supp}(\mu)) \geq r\},$$

so every escaping geodesic eventually remains in a region of this form.

Given any $\epsilon > 0$ we can choose r so large that Lemma 6 holds. Then

$$\frac{1}{1+\epsilon} \dim(\Lambda_e \cap \partial W) \leq \dim(\Phi(\Lambda_e \cap \partial W)) \leq (1+\epsilon) \dim(\Lambda_e \cap \partial W).$$

Taking $\epsilon \to 0$ proves the theorem.

8. The growth of Green's function

For a point z on a Riemann surface R, let $\alpha(z) = 1/\mathrm{area}(B(z,1))$, i.e., it is the reciprocal of the area of a unit ball centered at z. In this section we will prove.

Lemma 8.1. *There is a $C < \infty$ so that the following holds for any convergence type Riemann surface R. Suppose $G_R(z,w)$ is the Greens function of R with pole at w. Then for any $w \in R$, there is a function G on R which satisfies*

$$|\nabla G(z)| \leq C\alpha(z),$$

such that

$$G_R(z,w) \le G(z) + C\alpha(w)\log^+ \rho(z,w)^{-1}.$$

Note that the final term only occurs if $\rho(z,w) < 1$ and that it is integrable (with respect to hyperbolic area on R) with an absolute bound, independent of w. This is because the unit hyperbolic ball around w always has hyperbolic area $\simeq \min(1, \mathrm{inj}(w)^{-1})$.

Proof. The main point is the following estimate from [16] for hyperbolic manifolds. We will use it only in the case $d = 2$.

Lemma 8.2. *Suppose $R = \mathbb{D}/G$ and that Greens function $G(w,z)$ exists on R. Then*

$$\sup_{\rho(z_1,w)=1, \rho(z_2,w)=1} |G(z_1,w) - G(z_2,w)| \le A\alpha(w),$$

and

$$\sup_{z\in M} \sup_{\rho(w,z)\ge 1} |\nabla_z G(z,w)| \le A\alpha(z).$$

In particular, if the injectivity radius is bounded away from zero, then the gradient is uniformly bounded away from the pole.

Now to deduce Lemma 8.1 we write the Greens function for R as a sum over G orbits of the Greens function for the disk, i.e. assume $z, w \in R$ are the projections of points $x, 0 \in \mathbb{D}$ so that

$$G_R(z,w) = \sum_{g\in G} G_{\mathbb{D}}(x, g(0)).$$

Choose a smooth function ψ on $[0,\infty)$ with bounded gradient so that $\psi(t) = 0$ for $t < 1/2$ and $\psi(t) = 1$ for $t > 1$. Then let

$$H_1(x) = \sum_{g\in G} (1 - \psi(\rho(x, g(0)))G_{\mathbb{D}}(x, g(0)),$$

$$H_2(x) = \sum_{g\in G} \psi(\rho(x, g(0))G_{\mathbb{D}}(x, g(0)).$$

Then H_1 and H_2 are G invariant and hence well defined on R and their sum is the Greens function $G_R(z,w)$. We claim that

$$H_1(z) \lesssim \alpha(w)\log^+ \rho(z,w)^{-1}, \tag{7}$$

and

$$|\nabla_H H_2(z)| \lesssim \alpha(z). \tag{8}$$

The first claim is easy since each term in the definition of H_1 satisfies

$$(1-\psi(\rho(x,g(0)))G_{\mathbb{D}}(x,g(0)) \leq C\log^+ \rho(x,G(0))^{-1},$$

and there are at most $\simeq \alpha(w)$ terms in the sum.

To prove (8) we split it in three cases: $\rho(z,w)>1$, $\frac{1}{4}\leq \rho(z,w)\leq 1$ and $\rho(z,w)<\frac{1}{4}$. For $\rho(z,w)>1$, it is true by Lemma 8.2 and the fact that $H_2=G_R$ in this region. For $\frac{1}{4}<\rho(z,w)\leq 1$ we have $H_2=G_R-H_1$. The Greens function has gradient bounded by $C\,\alpha(z)\simeq\alpha(w)$ in this region and H_1 is a sum of $\simeq\alpha(w)$ terms each with bounded gradient. Thus $|\nabla_H H_2|\lesssim \alpha(z)$ in this region as well.

For the final case we lift to the disk, and break the sum defining H_2 into two sums

$$H_2(x)=S_1(x)+S_2(x)=\sum_{g:\rho(0,g(0))<1}+\sum_{g:\rho(0,g(0))\geq 1}.$$

The first has at most $C\alpha(w)$ terms, each with bounded gradient, so it is bounded as desired. The second defines a harmonic function in $\{x:\rho(x,0)<\frac{1}{2}\}$ and hence its gradient on $\{x:\rho(0,x)<\frac{1}{4}\}$ is bounded by its oscillation on $\{x:\frac{1}{4}\leq\rho(x,0)\leq\frac{1}{2}\}$. Here we are using the following fact about harmonic functions: if U is harmonic on $B(0,1)$ then

$$\max_{|z|\leq\frac{1}{2}}|\nabla U(z)|\leq C[\max_{\frac{1}{2}<|z|<1}U(z)-\min_{\frac{1}{2}<|z|<1}U(z)].$$

Since

$$S_2(x)=G_R(x,0)-H_1(x)-S_1(x),$$

and each of the terms on the right has oscillation on the annulus bounded by $C\alpha(w)$ we have completed the proof of the second claim. This completes the proof. □

Lemma 8.3. *Suppose R is Riemann surface and suppose $G(z)$ is a function on R which satisfies $|\nabla G(z)|\leq C\alpha(z)$. Let $z,w\in R$ be two points where the injectivity radius is larger than ϵ_M, the Margulis constant. Then*

$$|G(z)-G(w)|\leq C\exp(\frac{1}{2}\rho(z,w)).$$

Proof. Let γ be a geodesic connecting z to w. Suppose $\gamma_0\subset\gamma$ is a maximal subarc contained in a thin part P. If P is a parabolic thin part, then the endpoints of γ_0 are a uniformly bounded distance apart and so G only changes by a uniformly bounded amount over γ_0. If P is a hyperbolic thin part then the endpoints of γ_0 are either in the same boundary component of P or they are not. In the first case they are only a bounded distance apart and the previous remark holds. Otherwise they are in different boundary components, say distance d apart. The injectivity radius in P is bounded below by $C\exp(-\rho(z,\partial P))$ and hence the gradient of G is bounded above by $C\exp(\rho(z,\partial P))$ and hence the change in G over γ_0 is at most $C\exp(d/2)$.

Since γ can be broken into sections $\{\gamma_j\}$ (each of at least hyperbolic length 1) along which either G has bounded gradient, or changes by at most $C\exp(\ell(\gamma_j)/2)$, the lemma is proven. $\square$

Of course, if R has a positive lower bound on its injectivity radius then the previous result can be greatly improved; away from the pole G changes at most linearly.

9. The Schwarzian integral

In this section we establish the finiteness of the Schwarzian integrals used in the proof of Theorem 1.3. We begin by showing the integral is always finite for compact deformations of convergence groups and end the section by deducing the estimates we need for divergence groups from the proof of the earlier result.

Theorem 9.1. *Suppose G is a convergence type Fuchsian group and $\Phi : \mathbb{D} \to \Omega$ is a compact deformation of G. Then $|S(\Phi)|^2(1-|z|^2)^3 dxdy$ is a Carleson measure (with norm depending on G and μ). Thus Φ satisfies conditions (A)-(D) in the introduction. Moreover, the Carleson norm is close to zero if* $\mathrm{supp}(\mu)$ *is fixed and $||\mu||_\infty$ is small enough.*

By Lemma 2.1 and the fact that $G_Q \lesssim G_{\mathbb{D}}$ (except near the pole z_Q), it suffices to show

$$\int_{\mathbb{D}} |S_H(\Phi)(z)|^2 G_{\mathbb{D}}(z,w)\, dA(z) \le M,$$

for every $w \in \mathbb{D}$, where dA denotes hyperbolic area. By group invariance, this is the same as showing

$$\int_R |S_H(\Phi)(z)|^2 G_R(z,w)\, dA(z) \le M, \tag{9}$$

for every $w \in R = \mathbb{D}/G$. This is just the $t=0$ case of the following more general result which we will need.

Theorem 9.2. *With notation as above, and $z_0 \in R$ we have*

$$\int_{z\in R:\rho(z,z_0)>t} |S_H(\Phi)(z)|^2 G_R(z,w)dA(z) \le Me^{-t/2}, \tag{10}$$

uniformly for all $w \in R$. Moreover, we may take M as close to zero as we wish if the diameter of $\mathrm{supp}(\mu)$ *is fixed and $\|\mu\|_\infty$ is close enough to zero.*

We will only need that the integral goes to zero, so the exponential decay is much stronger than we need (but it might be useful for other applications, e.g., proving differentiability of Φ at certain points on the boundary).

Proof. Fix a point $w \in R$ to be the pole of Greens function and choose a $z_0 \in R$ with $\rho(z_0, \mathrm{supp}(\mu)) \le 1$ and $\rho(z_0, w) \ge 2$. Let $S_t = z \in R : \rho(z, z_0) > t$. By Lemma 8.1,

$$G(z,w) \lesssim G_R(z_0,w) + G(z) + \alpha(w)\log^+ \rho(z,w), \tag{11}$$

where $G(z_0) = 0$ and $|\nabla G(z)| \le C\alpha(z)$, and by Corollary 5.2,

$$|S_H(\Phi)(z)| \le C\exp(-\rho(z,z_0)).$$

Now break the integral in (10) into three terms, corresponding to the three terms in (11). The first and last terms are easy to deal with. The first term is

$$\int_{S_t} |S_H(\Phi)(z)|^2 G_R(z_0,w)dA(z) \le G_R(z_0,w)\int_t^\infty e^{-2r}e^r dr \le CG_R(z_0,w)e^{-t},$$

This is bounded depending on z_0, but independent of w, (as long as $\rho(z_0, w) \ge 1$) since by the symmetry of Greens function and the maximum principle,

$$G_R(z_0,w) = G_R(w,z_0) \le \max_{z:\rho(z,z_0)=1} G(z,z_0).$$

The third term is bounded by

$$\begin{aligned}\int_{S_t} |S_H(\Phi)(z)|^2 \alpha(w)\log^+ \rho(z,w)dA(z)& \\ &\le e^{2(\rho(z_0,w)-1)}\int_{S_t\cap B(w,1)} \alpha(w)\log^+\rho(z,w)dA(z) \\ &\le C\exp(-2\rho(z_0,w)).\end{aligned}$$

Hence we need only deal with the middle term in (11). Thus to prove (10) it suffices to show

$$\int_{S_t} |S_H(\Phi)(z)|^2|G(z)|dA(z) \le Me^{-t/2},$$

for G is any function on R such that $G(z_0) = 0$ and $|\nabla G(z)| \le \alpha(z)$.

To prove this, let $R = R^1 \cup R^2$ denote the thick and thin parts of R (i.e., the points where the injectivity radius is more or less than the Margulis constant ϵ_M). Let S_t^1 and S_t^2 be the thick and thin pieces of S_t. On the thick part we use our estimates

$$|S_H(\Phi)(z)| \leq C\exp(-\rho(z, z_0)),$$
$$G(z) \leq C\exp(\frac{1}{2}\rho(z, z_0)),$$
$$\int_{z:\rho(z,z_0)<r} dA(z) \leq Ce^r,$$

to get

$$\int_{S_t^1} |S_H(\Phi)(z)|^2 G(z) dA(z) \quad \lesssim \quad \int_t^\infty e^{-2r} e^{r/2} e^r \, dr \lesssim e^{-t/2},$$

as desired.

Now we have to bound the integral over the thin parts. For each thin part $P \subset R$ we will show that the integral over P is controlled by the integral over a unit annular neighborhood A_P of its boundary. Since these neighborhoods are in the thick part and are disjoint (if the Margulis constant is chosen small enough), this finishes the proof. Let $P_s = \{z \in P : \rho(z, \partial P) \geq s\}$. We first wish to show that

$$\int_{P_s} |S_H(\Phi)(z)|^2 G(z) dA(z) \leq Ce^{-s} \int_{A_P} |S_H(\Phi)(z)|^2 (G(z)+1) dA(z). \tag{12}$$

If we can do this, then for each thin part whose boundary is distance $d \leq t$ from z_0 we have $S_t \cap P \subset P_{t-d}$ and hence

$$\begin{aligned} \sum_{\partial P \subset S_d} \int_{S_t \cap P} |S_H(\Phi)(z)|^2 (G(z)+1) dA(z) & \\ & \lesssim e^{d-t} \int_{S_d^1} |S_H(\Phi)(z)|^2 (G(z)+1) dA(z) \\ & \lesssim Ce^{d-t}(e^{-d/2} + e^{-d}) \\ & \lesssim e^{-t+d/2} \end{aligned}$$

Now summing over integers all integers d between 0 and t gives

$$\int_{S_t} |S_H(\Phi)(z)|^2 G(z) dA(z) \leq C \sum_{d=0}^{t} e^{-t+d/2} \leq Ce^{-t/2},$$

as desired. Thus it suffices to prove (12) for every thin part P.

First suppose that P is a parabolic or elliptic thin part. For each such thin part, $z \in P$ implies

$$\rho(z, z_0) \geq \mathrm{dist}(z_0, \partial P) + \mathrm{diam}_\rho(\partial P) + \rho(z, \partial P),$$

and thus

$$|S_H(\Phi)(z)| \leq C \exp(-\rho(\partial P, z_0)) \exp(-\rho(z, \partial P)).$$

By the gradient estimate for G,

$$G(z) \leq \max_{z \in A_P} G(z) + C \exp(\rho(z, \partial P),$$

and

$$\int_{z \in P: \rho(z, \partial P) > r} dA(z) \leq Ce^r.$$

Combining these gives

$$\begin{aligned} \int_{P_s} |S_H(\Phi)(z)|^2 G(z) dA(z) &\leq Ce^{-s} (\max_{z \in \partial P} G(z) + 1) \exp(-2\rho(\partial P, z_0) \\ &\leq Ce^{-s} \int_{A_P} |S_H(\Phi)(z)|^2 (G(z) + 1) dA(z) \end{aligned}$$

Now assume P is a hyperbolic thin part. Divide the part into two pieces P_1 and P_2, separated by the geodesic associated to the part P. Then P has two boundary components, which we denote by ∂P_1 and ∂P_2. Let $d = \rho_P(\partial P_1, \partial P_2)$ be the distance between the boundary components through P and let $d_i = \rho(z_0, \partial P_i)$ for $i = 1, 2$. For $z \in P$

$$\rho(z, z_0) \leq \min(d_1 + \rho(z, \partial P_1), d_2 + \rho(z, \partial P_2)),$$

and hence

$$\begin{aligned} |S_H(\Phi)(z)| &\leq C \max(\exp(\rho(-d_1 - \rho(z, \partial P_1)), C \exp(-d_2 - \rho(z, \partial P_2))) \\ &\leq C \exp(-d_1 - \rho(z, \partial P_1)) + C \exp(-d_2 - \rho(z, \partial P_2)) \end{aligned}$$

Thus to prove (10) it is enough to prove an estimate of the form

$$\int_{P_s} \exp(-\rho(z, \partial P_1) G(z) dA(z) \lesssim G(z_1, w) e^{-s}, \tag{13}$$

where $z_1 \in \partial P_1$. On P_1 the function G is bounded by

$$G(z) \leq G(z_1, w) + C \exp(\rho(z, \partial P_1))$$

and on P_2 it is bounded by

$$G(z) \leq G(z_1) + C \exp(d/2).$$

Thus (13) is bounded by

$$C\int_s^{d/2} e^{-2r}(G(z_1,w)+e^r)e^{-r}\,dr$$
$$+C\int_0^{(d/2)-s} e^{-2r-d}(G(z_1,w)+e^{d/2})e^{r-(d/2)}\,dr$$
$$\lesssim G(z_1)(e^{-s}+e^{-d})$$
$$\lesssim G(z_1)e^{-s},$$

since $s \le d$. This completes the proof of the theorem. □

Corollary 9.3. *Suppose G is a Fuchsian group and $\Phi : \mathbb{D} \to \Omega$ is a compact deformation of G. Then if $x \in R$ and $S = R \setminus B(x, r)$ we have*

$$\int_S |S_H(\Phi)(z)|^2 G_S(z,w)\,dA(z) \le Me^{-r/2},$$

for some $M < \infty$ independent of w.

Proof. There are two metrics on S, the restriction of R's hyperbolic metric and the hyperbolic metric that makes S into a complete Riemann surface. First we claim that the Poincare metric for S is comparable to that for R on $S_1 = R \setminus B(x, r + 1)$. To prove this lift S and S_1 to an open sets Ω and Ω_1 in $\mathbb{D}$ by the covering map for R. For $z \in \Omega_1$, $\operatorname{dist}(z, \partial\Omega) \simeq 1 - |z|$ and hence the density for the hyperbolic metric in Ω at z is comparable to the density the hyperbolic metric for $\mathbb{D}$. This proves the claim.

If we choose z_0 in $B(x, r + 1)$ with $\rho(z_0, w) \ge 1$ then it is easy to see $G_S(z_0, w)$ is uniformly bounded. Thus the Greens function G_S satisfies

$$G_S(z,w) \le C + G(z) + \alpha(w)\log^+ \rho(z,w)^{-1},$$

for some G with bounded gradient with respect to the hyperbolic metric on S (and hence with respect to the hyperbolic metric on R). Thus the estimates in the proof of Theorem 9.1 apply here and prove the result. □

Next we want to use the previous corollary to bound integrals on the disk.

Lemma 9.4. *Suppose Φ is a compact deformation of a Fuchsian group G, $V \subset \mathbb{D}$ is a region whose projection into R does not hit the ball $B(x_0, r) \subset R$, Then*

$$\int_V |S_H(\Phi(z)|^2 G_V(z,w)\,dA \lesssim e^{-r/2} \tag{14}$$

uniformly for every $w \in V$ where dA denotes area on the Riemann surface $R = \mathbb{D}/G$, $S_r = R \setminus B(x_0, r)$,

Proof. Write the Greens function G_S for S as a sum over orbits of the Greens function for the planar domain $\Omega = \mathbb{D} \setminus \cup_{g \in G} g(B)$, where B is a lift of the ball $B(x_0, r)$ on R, i.e.,

$$G_S(\Pi(z), \Pi(w)) = \sum_{g \in G} G_\Omega(z, g(w)),$$

where $\Pi : \mathbb{D} \to R$ is the quotient map. Since $V \subset \Omega$, $G_V(z, w) \leq G_\Omega(z, w)$. Hence

$$\begin{aligned}
\int_V |S_H(\Phi)(z)|^2 G_V(z, w)(1 - |z|^2)^{-2} dxdy \\
&\lesssim \int_V \psi(z) |S_H(\Phi)(z)|^2 G_\Omega(z, w) dA(z) \\
&\lesssim \int_{\Pi(V)} |S_H(\Phi)(z)|^2 G_S(w, \Pi(z_V)) dA(w) \\
&\lesssim \int_{S_r} |S_H(\Phi)(z)|^2 G_S(w, \Pi(z_V)) dA(w) \\
&\lesssim e^{-r/2},
\end{aligned}$$

by Corollary 9.3. □

10. A version of Green's theorem

In the last section we proved that

$$\int_{\mathbb{D}} |S(\Phi)(z)|^2 G_{\mathbb{D}}(z, w)(1 - |z|^2)^2 dxdy \tag{15}$$

is uniformly bounded whenever Φ is a compact deformation of a convergence group. By results in [3] or [14] this implies that $\varphi = \log |\Phi'|$ is in BMO. We need two stronger versions this fact.

First, we will want to deduce φ is in BMO on a stopping time domain $W \subset \mathbb{D}$ when we only know

$$\sup_V \int_V |S(\Phi)(z)|^2 G_V(z, z_V)(1 - |z|^2)^2 dxdy, \tag{16}$$

is uniformly bounded for all chord-arc domains $V \subset W$. The difference between (15) and (16) is that $G_{\mathbb{D}}$ has been replace by the Greens function G_V. Since $G_V \leq G_{\mathbb{D}}$ (except near the pole), the second integral is smaller and may be much smaller at some points. Thus we will have reproduce part of the argument from [14], modified to deal with this slightly different integral.

Secondly, for our applications we will need φ in BMO with small norm, and this does not necessarily follow even if the integrand in (15) is Carleson with small norm (just consider a Möbius transformation Φ of the disk to a half-plane; $S(\Phi) \equiv 0$ but φ is not the zero element of BMO). However, we will show that if we assume that φ also has small Bloch norm, then φ has small BMO norm, i.e., we will show

$$\|\varphi\|_{\mathrm{BMO}} \lesssim \|\varphi\|_{\mathcal{B}} + \||S(\Phi)(z)|^2(1-|z|^2)^3 dxdy\|_C. \tag{17}$$

We begin by establishing the following result for chord-arc domains.

Lemma 10.1. *Suppose $V \subset \mathbb{D}$ is a M-chord-arc domain and suppose φ is a Bloch function on $\mathbb{D}$. Then*

$$\int_V |\varphi'(z)|^2 G_V(z, z_V) dxdy$$
$$\lesssim |\varphi'(z_V)| \mathrm{diam}(V)^2 + \int_V |\varphi''(z)|^2 G_V(z, z_V) d_V(z)^2 dxdy,$$

where G_V is the Greens function for V, z_V is a point in V with $\mathrm{dist}(z_V, \partial V) \simeq \mathrm{diam}(V)$ *and the constants in $\lesssim$ depend only on M.*

Proof. The proof closely follows the proof of the related inequality

$$\int_W |F'(z)|^2 d_W(z) dxdy \lesssim |F'(z_W)| d_W(z)^2 + \int_W |F''(z)|^2 d_W(z)^3 dxdy,$$

which is given in [24]. Let $\Psi : \mathbb{D} \to V$ be a Riemann mapping taking 0 to z_V and for $z \in V$, let $\gamma(z)$ be the image under Ψ of the radial segment in $\mathbb{D}$ connecting 0 to $\Psi^{-1}(z)$. If $z_0 \in \gamma(z)$ we let $\gamma(z_0, z) = \gamma(z) \setminus \gamma(z_0)$. Then as in Section 1 of [24], $\gamma(z)$ is a chord-arc curve (with constant depending only on K) and for $0 < \alpha < 1$,

$$\int_{\gamma(z)} d_V(w)^{1-\alpha} ds \simeq d_V(z)^{-\alpha}, \tag{18}$$

and if $z_0 \in \gamma(z)$

$$\int_{\gamma(z, z_0)} d_V(w)^{-\alpha} ds \simeq d_V(z)^{1-\alpha}. \tag{19}$$

By adding a constant to φ' and noting

$$\int_V G_V(z, z_V) dxdy \simeq \mathrm{diam}(V)^2, \tag{20}$$

we may assume that $\varphi'(z_V) = 0$. Fix some $0 < \alpha < 1$ and note that

$$\varphi'(z) = \int_{\gamma(z)} \varphi''(w) ds = \int_{\gamma(z)} \varphi''(w) d_V(w)^{(1+\alpha)/2} d_V(w)^{-(1+\alpha)/2} ds,$$

and hence

$$|\varphi'(z)|^2 \lesssim (\int_{\gamma(z)} |\varphi''(z)|^2 d_V(w)^{1+\alpha} ds) d_V(w)^{-\alpha}.$$

Thus using (18), (19) and the fact that G_V is monotone decreasing on $\gamma(z)$, we deduce

$$\begin{aligned}
\int_{\gamma(z_0)} & |\varphi'(z)|^2 G_V(w, z_V) ds \\
& \lesssim \int_{\gamma(z_0)} G_V(z, z_V) d_V(z)^{-\alpha} [\int_{\gamma(z)} |\varphi''(z)|^2 d_V(w)^{1+\alpha} ds(w)] ds(z) \\
& \lesssim \int_{\gamma(z_0)} |\varphi''(w)|^2 G_V(w, z_V) d_V(w)^{1+\alpha} [\int_{\gamma(z_0, w)} d_V(z)^{-\alpha} ds(z)] ds(w) \\
& \lesssim \int_{\gamma(z_0)} |\varphi''(w)|^2 G_V(w, z_V) d_V(w)^2.
\end{aligned}$$

To finish the proof of Lemma 10.1, we need the following observation,

Lemma 10.2. *With $\gamma(z)$ defined as above, define a positive measure μ on W by*

$$\int g d\mu = \int_{\partial W} \int_{\gamma(z)} g(w) ds(w) ds(z).$$

Then $d\mu$ is boundedly equivalent to $dxdy$ (i.e., $\mu(E) \simeq \mathrm{area}(E)$ for every $E \subset V$)

Proof. We pull back to the disk by the Riemann mapping Ψ and consider what happens on the top half $T(Q)$ of some Carleson square. Let z be some point in $T(Q)$. Then by Koebe distortion the area of $W = \Phi(T(Q))$ is $\simeq |\Phi'(z)|^2 \mathrm{area}(T(Q))$ and the length of each arc $\gamma(w) \cap W$ which hits W is $\simeq |\Phi'(z)| \ell(Q)$. Let I be the base of Q. Then since V is chord-arc $\ell(\Phi(I)) \simeq \mathrm{diam}(W) \simeq |\Phi'(z)| \ell(Q)$. Thus $d\mu$ is boundedly equivalent to $dxdy$ on W, as desired. □

Using this and (18), (19) and (20), we integrate over $\partial\Omega$ and deduce

$$\begin{aligned}
\int_V & |\varphi'(z)|^2 G_V(z, z_V) dxdy \\
& \lesssim |\varphi'(z_V)| \mathrm{diam}(V)^2 + \int_V |\varphi''(z)|^2 G_V(z, z_V) d_V(w)^2 dxdy.
\end{aligned}$$

□

Lemma 10.3. *Let $W \subset \mathbb{D}$ be a chord-arc domain. Then there are constants $\epsilon_1, C < \infty$ such that if Φ is univalent on $\mathbb{D}$, $\varphi(z) = \log|\Phi(z)|$, and φ is ϵ_1-Bloch on W, then*

$$\begin{aligned}\int_W |\varphi'(z)|^2 G_W(z, z_W)dxdy& \\ &\lesssim |\varphi'(z_W)|^2(1-|z_W|^2)^2 + \int_W |S(\Phi)(z)|^2 G_W(z,z_W)dA \\ &\lesssim \|\varphi\|_{\mathcal{B}(W)} + \int_W |S(\Phi)(z)|^2 G_W(z,z_W)dA.\end{aligned}$$

Proof. Apply Lemma 10.1 to W and the fact that

$$|\varphi''|^2 \leq 2|S(\Phi)|^2 + |\varphi'|^4,$$

to get

$$\begin{aligned}\int_W |\varphi'(z)|^2 G_W(z, z_W)dxdy& \\ &\lesssim |\varphi'(w)|^2(1-|z_W|^2)^2 + \int_W |\varphi''(z)|^2 G_W(z,z_W)(1-|z|^2)^2 dxdy \\ &\lesssim |\varphi'(z_W)|^2(1-|z_W|^2)^2 + \int_W |S(\Phi)(z)|^2 G_W(z,z_W)(1-|z|^2)^2 dxdy \\ &\qquad + \epsilon_1^2 \int_W |\varphi'(z)|^2 G_W(z,z_W)dxdy.\end{aligned}$$

The far right term is small compared to the left hand side if ϵ_1 is small enough, and so can be subtracted, and the proof is complete. $\square$

Combining this with our earlier results we obtain the following.

Lemma 10.4. *Suppose Φ is a compact deformation of a Fuchsian group G and $z_0 \in R = \mathbb{D}/G$. There are $\epsilon > 0$ and $r < \infty$ so that if W is a chord-arc subdomain of the disk so that the projection of W into R is at least hyperbolic distance r from z_0 and φ is ϵ-Bloch on W, then $\Phi(W)$ has a rectifiable boundary (in fact, it is a chord-arc curve).*

Proof. This follows immediately from 2.3, 9.4 and 10.3 and the fact that small BMO norm for $\log|\Phi'|$ implies $\Phi(W)$ is rectifiable (see Section 2). $\square$

11. Proof of Theorem 1.3

We start by giving a new proof of a result of Astala and Zinsmeister [4], that a compact deformation with a sufficiently small dilatation gives a rectifiable curve as a limit set.

Lemma 11.1. *Suppose G is a convergence type group and Φ is a compact deformation of G. If $||\mu||_\infty$ is small enough (depending only on the hyperbolic diameter of its support on $R = \mathbb{D}/G$) then $\Phi(\mathbb{T})$ is a chord-arc curve.*

Proof. First, if μ is small, then $\Omega = \Phi(\mathbb{D})$ is a quasi-circle with small constant and so by Lemma 4.4, Φ can be chosen so φ is in Bloch with small norm. Now apply Corollary 2.3, Theorem 9.1 and Lemma 10.3 and the fact that small BMO norm for $\log|\Phi'|$ implies $\Phi(\mathbb{T})$ is chord-arc. ☐

Next we want to prove Theorem 1.3. The proof is essentially the same as the previous result, except that the disk $\mathbb{D}$ must be replaced by a certain stopping time regions.

Lemma 11.2. *Suppose Φ is a compact deformation of a Fuchsian group G. Then $\Phi(\Lambda_e)$ lies on a countable union of rectifiable curves. If, in addition, G is divergence type, then $\Phi(\Lambda_e)$ has zero linear measure.*

Proof. As before let $S_r = R \setminus B(x_0, r)$ and let Ω_r be the lift of S_r to $\mathbb{D}$ (note that S_r and Ω_r need not be connected). Thus $|S_H \Phi| \leq Ce^{-r}$ on all of Ω_r. For any Carleson square Q, let $E_r(Q) \subset I_Q$ be the set of points x so that radial segment in Q ending at x is contained in Ω_r. It suffices to show that each $E_r(Q)$ has h image which is σ finite length, if r is large enough.

Form a stopping time region W by removing from Q the maximal dyadic Carleson subsquares whose top half does not intersect Ω_r. The parent of such a square does hit Ω_r and hence the resulting region W is contained in a hyperbolic C_0-neighborhood of Ω_r. Moreover, W contains all the radial lines which terminate at E_r. Thus $\Phi(E_r) \subset \Phi(\partial W)$.

Since $W \subset \Omega$, the Schwarzian of Φ is small on all of W and so by Lemma 4.7 we may cover all but one point of E_r by stopping times regions where both the Schwarzian integral is small and φ is in Bloch with small norm, with estimates that tend to zero as $r \to \infty$. Fix r large enough so that Lemma 10.4 implies the image boundary is rectifiable and hence E_r is contained in a countable union of rectifiable curves.

If G is divergence type then E_r has zero Lebesgue measure, so its image also has zero 1-dimensional measure by the F. and M. Riesz theorem (e.g., Theorem 6.8 of [31]). This completes the proof. □

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