Math. Nachr. (1999),

Local Spectra and Index of Singular Integral Operators with Piecewise Continuous Coefficients on Composed Curves

By C.J. Bishop of Stony Brook, A. Böttcher of Chemnitz,
Yu.I. Karlovich of Chemnitz, I. Spitkovsky of Williamsburg

"The spectrum gives an operator a personality."

L.N. Trefethen [60]

(Received January 21, 1997)

Abstract. We establish a symbol calculus for deciding whether singular integral operators with piecewise continuous coefficients are Fredholm on the Lebesgue space $L^p(\Gamma, w)$ where $1 < p < \infty$, Γ is a composed Carleson curve, and w is a Muckenhoupt weight in the class $A_p(\Gamma)$. We also provide index formulas for the operators in the closed algebra of singular integral operators with piecewise continuous matrix-valued coefficients. Our main theorem is based upon three pillars: on the identification of the local spectrum of the Cauchy singular integral operator at the endpoints of simple Carleson arcs, on an appropriate "N projections theorem", and on results of geometric function theory pertaining to the problem of extending Carleson curves and Muckenhoupt weights.

1. Introduction

In 1960, Harold Widom discovered that the spectrum of the Cauchy singular integral operator on the space $L^p(0,\infty)$ is a circular arc joining the points -1 and 1 whose shape depends on p and which degenerates to the line segment $[-1,1]$ in case $p = 2$. About ten years later, I. Gohberg and N. Krupnik had accomplished their Fredholm theory of singular integral operators with piecewise continuous coefficients on L^p spaces over composed Lyapunov curves with power weights. Only in recent time, some phenomena caused by general Carleson curves and general Muckenhoupt

1991 *Mathematics Subject Classification.* Primary 47B35; Secondary 16R20, 30C20, 30D55, 30E20, 42A50, 45E05, 46J15, 47A53, 47C05, 47D30.

Keywords and phrases. Ahlfors-David curve, Carleson condition, Muckenhoupt condition, singular integral operator, spectral theory, index, idempotent, geometric function theory

weights were understood, and it became clear that Carleson curves and Muckenhoupt weights effect a surprising and beautiful metamorphosis of the circular arcs prevailing the Widom/Gohberg/Krupnik theory.

Some kind of symbol calculus for deciding whether a singular integral operator with piecewise continuous coefficients is Fredholm on $L^p(\Gamma, w)$ $(w \in A_p(\Gamma))$ is presently available in two cases: if Γ is a composed Lyapunov curve [35] or if Γ is a Carleson Jordan curve [8]. Curves composed by arcs with oscillating tangents were considered in [48], [49], [11] under the assumption that w is a so-called power weight. In this paper we provide a symbol calculus for the case of an arbitrary composed Carleson curve. We remark that, in a sense, composed Carleson curves constitute the maximal class of curves for which a symbol calculus with matrices of finite order is possible.

The paper is organized as follows. Sections 2 and 3 are devoted to the preliminaries. In Section 2 we introduce composed Carleson curves and Muckenhoupt weights. In Section 3 we discuss some basic properties of singular integral operators and define the notion of the local spectrum of such operators. The main results are stated in Section 4. Several special cases of the main theorem as well as pieces of its history are illuminated in Section 5. The main results are proved in Sections 6 to 10.

In Section 6 we prove that Carleson stars can be extended to Carleson flowers and that Muckenhoupt weights on a Carleson star can be extended to Muckenhoupt weights on the corresponding Carleson flower. In Section 7 we apply an appropriate N projections theorem to local algebras of Carleson flowers, which reduces the invertibility problem in these algebras to the problem of studying the invertibility of certain matrix functions living on certain sets. The latter sets are called leaves. Some preliminary information about the structure of the leaves is obtained in Section 8. In Section 9 we connect the local algebras of an arbitrary composed Carleson curve with the local algebras of Carleson flowers. The precise shape of the leaves is then determined in Section 10.

Once the theorems of Section 4 are available, many results on singular integral operators can be extended to the most general setting by using standard techniques. In Section 11 we supplement the results of Section 4 by showing that neither the consideration of systems of singular integral equations nor relaxation of the question about Fredholmness to the one about semi-Fredholmness yield qualitatively new phenomena. In Section 12 we establish index formulas which do not only extend known index formulas to the case of Carleson curves with Muckenhoupt weights but also differ in their form from previous index formulas. Section 13 is dedicated to "Coburn's lemma", which says that certain operators, including the operator $S_\Gamma - \lambda I$, are invertible if and only if they are Fredholm of index zero. The spectrum of the Cauchy singular integral operator S_Γ over an arbitrary composed Carleson curve with an arbitrary Muckenhoupt weight is finally determined in Section 14.

2. The Cauchy singular integral

An oriented rectifiable curve in the complex plane is called a *simple arc* if it is homeomorphic to a line segment. The union of finitely many simple arcs each pair of which have at most endpoints in common is referred to as a *composed curve.*

Let Γ be a composed curve and equip Γ with Lebesgue length measure $|d\tau|$. A measurable function $w:\Gamma\to[0,\infty]$ is called a *weight* if $w^{-1}(0)$ and $w^{-1}(\infty)$ have measure zero. We denote by $L^p(\Gamma)$ $(1\le p\le\infty)$ the usual Lebesgue spaces of Γ and by $L^p(\Gamma,w)$ $(1<p<\infty)$ the weighted Lebesgue spaces with the norms

$$\|f\|_{p,w}:=\left(\int_\Gamma |f(\tau)|^p w(\tau)^p\,|d\tau|\right)^{1/p}.$$

We write $A_p(\Gamma)$ $(1<p<\infty)$ for the class of all weights w on Γ such that $w\in L^p(\Gamma), w^{-1}\in L^q(\Gamma)$, and

$$\sup_{t\in\Gamma}\sup_{\varepsilon>0}\frac{1}{\varepsilon}\left(\int_{\Gamma(t,\varepsilon)} w(\tau)^p\,|d\tau|\right)^{1/p}\left(\int_{\Gamma(t,\varepsilon)} w(\tau)^{-q}\,|d\tau|\right)^{1/q}<\infty, \tag{2.1}$$

where here and in what follows

$$\Gamma(t,\varepsilon):=\{\tau\in\Gamma:|\tau-t|<\varepsilon\}\ \text{ and }\ 1/p+1/q=1.$$

The weights in $A_p(\Gamma)$ are usually called *Muckenhoupt weights.*

Not every composed curve Γ may carry a Muckenhoupt weight. Indeed, if (2.1) holds then Hölder's inequality gives

$$\sup_{t\in\Gamma}\sup_{\varepsilon>0}\frac{1}{\varepsilon}|\Gamma(t,\varepsilon)|<\infty \tag{2.2}$$

where $|\Gamma(t,\varepsilon)|$ is the Lebesgue length measure of the portion $\Gamma(t,\varepsilon)$. On the other hand, if (2.2) is valid then constant weights obviously belong to $A_p(\Gamma)$ for all $p\in(1,\infty)$. Thus, $A_p(\Gamma)\ne\emptyset$ if and only if (2.2) is true. Composed curves Γ satisfying (2.2) are usually referred to as *Carleson curves* (Ahlfors-David curves, Ahlfors regular curves, David regular curves, or simply as regular curves).

Now let $1<p<\infty$, let Γ be a composed curve, and let w be a weight on Γ. For $f\in L^1(\Gamma)$ and $t\in\Gamma$, consider the limit

$$\lim_{\varepsilon\to 0}\frac{1}{\pi i}\int_{\Gamma\setminus\Gamma(t,\varepsilon)}\frac{f(\tau)}{\tau-t}\,d\tau. \tag{2.3}$$

If the limit (2.3) exists, it is denoted by $(S_\Gamma f)(t)$. Suppose

(i) for each $f\in L^1(\Gamma)\cap L^p(\Gamma,w)$ the limits exist a.e. on Γ and define a function $S_\Gamma f\in L^p(\Gamma,w)$;

(ii) $L^1(\Gamma)\cap L^p(\Gamma,w)$ is dense in $L^p(\Gamma,w)$;

(iii) there is a constant $B=B(\Gamma,p,w)\in(0,\infty)$ such that $\|S_\Gamma f\|_{p,w}\le B\|f\|_{p,w}$ for all
$f\in L^1(\Gamma)\cap L^p(\Gamma,w)$.

In that case there is a unique extension of the map

$$L^1(\Gamma)\cap L^p(\Gamma,w)\to L^p(\Gamma,w),\ f\mapsto S_\Gamma f$$

to a bounded linear operator on $L^p(\Gamma, w)$. This operator is called the *Cauchy singular integral operator* on $L^p(\Gamma, w)$ and is denoted by S_Γ.

Theorem 2.1. *Let $1 < p < \infty$, let Γ be a composed curve, and let w be a weight on Γ. Then* (i), (ii), (iii) *hold if and only if $w \in A_p(\Gamma)$.*

In other words, the Cauchy singular integral S_Γ is a well-defined and bounded operator on $L^p(\Gamma, w)$ if and only if Γ is a Carleson curve and w is a Muckenhoupt weight in $A_p(\Gamma)$. Note that if $w \in A_p(\Gamma)$, then automatically $L^p(\Gamma, w) \subset L^1(\Gamma)$ by Hölder's inequality, so that (i) and (ii) hold with $L^1(\Gamma) \cap L^p(\Gamma, w)$ replaced by $L^p(\Gamma, w)$ in this case.

Under various hypotheses concerning Γ and w, the boundedness of S_Γ on $L^p(\Gamma, w)$ had been studied for a long time by many mathematicians, including Hardy, Littlewood, M. Riesz, Mikhlin, Babenko, Khvedelidze, Helson, Szegö, Widom, Forelli, Danilyuk, Shelepov, Calderón, and many others. Theorem 2.1 as it is quoted here was established by Hunt, Muckenhoupt, and Wheeden [38] in the case where Γ is the complex unit circle $\mathbf{T}$. Paatashvili and Khuskivadze [45] showed that if S_Γ is bounded on $L^2(\Gamma)$, then Γ must be a Carleson curve. Finally, David [19], [20] proved that for S_Γ to be bounded on $L^2(\Gamma)$ it is necessary and sufficient that Γ be Carleson. After David's result, a theorem like Theorem 2.1 was stated by many people (see, e.g., [23], [33], [35], [6]).

To compute $S_\Gamma f$ one may employ the formula

$$(S_\Gamma f)(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau) - f(t)}{\tau - t}\, d\tau + \lim_{\varepsilon \to 0} \frac{f(t)}{\pi i} \int_{\Gamma \setminus \Gamma(t,\varepsilon)} \frac{d\tau}{\tau - t}$$

in which the first integral does not contain a singularity provided f is sufficiently nice, say a rational function. In this connection the following result proves useful.

Theorem 2.2. *Let Γ be a composed (rectifiable) curve, $1 < p < \infty$, and $w \in L^p(\Gamma)$. Then the set $R(\Gamma)$ of all rational functions without poles on Γ is dense in $L^p(\Gamma, w)$.*

This theorem is well known, but we have not found an explicit reference. For the reader's convenience, we will give a proof.

Proof. Let first $\eta \subset \Gamma$ be a simple rectifiable arc. Without loss of generality assume $|\eta| = 2\pi$, put $I := [0, 2\pi]$, and denote by $\mu : I \to \eta$ any homeomorphism such that $|\mu'(x)| = 1$ for almost all $x \in I$. Put $\rho := (w|\eta) \circ \mu$. Since

$$\int_\Gamma |f(\tau)|^p w(\tau)^p |d\tau| = \int_I |f(\mu(x))|^p w(\mu(x))^p dx = \int_I |f(\mu(x))|^p \rho(x)^p dx,$$

the map $f \mapsto f \circ \mu$ is an isometric isomorphism of $L^p(\eta, w|\eta)$ onto $L^p(I, \rho)$. In particular, if $w|\eta \in L^p(\eta)$ then $\rho \in L^p(I)$.

We claim that $C(I)$ is dense in $L^p(I, \rho)$. To see this, assume $C(I)$ is not dense in $L^p(I, \rho)$. Then there is a nonzero function $h \in (L^p(I, \rho))^* = L^q(I, \rho^{-1})$ such that

$$\int_I h(x)\varphi(x) dx = 0 \text{ for all } \varphi \in C(I). \tag{2.4}$$

Hölder's inequality gives

$$\int_I |h(x)|dx \le \left(\int_I |h(x)|^q \rho(x)^{-q} dx\right)^{1/q} \left(\int_I \rho(x)^p dx\right)^{1/p},$$

and since $h \in L^q(I, \rho^{-1})$ and $\rho \in L^p(I)$, it follows that $h \in L^1(I)$. Taking $\varphi(x) = e^{inx}$ with $n \in \mathbf{Z}$ in (2.4), we see that all Fourier coefficients of h vanish, which implies that $h = 0$ a.e. on I. This contradiction proves that $C(I)$ is dense in $L^p(I, \rho)$. Let $\chi_\varepsilon \in C(I)$ be any function satisfying

$$0 \le \chi_\varepsilon \le 1, \ \ \chi_\varepsilon(0) = \chi_\varepsilon(2\pi) = 0, \ \ \chi_\varepsilon = 1 \ \text{ on } \ [\varepsilon, 2\pi - \varepsilon].$$

Then for $\varphi \in C(I)$,

$$\int_I |\varphi(x) - \chi_\varepsilon(x)\varphi(x)|^p \rho(x)^p dx \le \|\varphi\|_\infty^p \int_{[0,\varepsilon]\cup[2\pi-\varepsilon,2\pi]} \rho(x)^p dx = o(1) \ \text{ as } \ \varepsilon \to 0$$

because $\rho^p \in L^1(I)$. We therefore arrive at the conclusion that $C_0(I)$, the functions in $C(I)$ vanishing at the endpoints of I, is also dense in $L^p(I, \rho)$.

Thus, given $f \in L^p(\eta, w|\eta)$ we can find functions $\varphi_n \in C_0(I)$ such that $f \circ \mu - \varphi_n \to 0$ in $L^p(I, \rho)$, implying that $f - \varphi_n \circ \mu^{-1} \to 0$ in $L^p(\eta, w|\eta)$. Consequently, the functions in $C(\eta)$ vanishing at the endpoints of η are dense in $L^p(\eta, w|\eta)$.

We may write $\Gamma = \eta_1 \cup \ldots \cup \eta_m$ where $\eta_1, \ldots, \eta_m$ are simple arcs each pair of which have at most endpoints in common. Put

$$\begin{aligned} C_0(\eta_j) &:= \{f \in C(\Gamma) : f|\eta_j \in C(\eta_j), \ \ f|\Gamma \setminus \eta_j = 0\}, \\ C_0(\eta_1) + \ldots + C_0(\eta_m) &:= \{f_1 + \ldots + f_m : f_j \in C_0(\eta_j)\}. \end{aligned}$$

From what was shown in the preceding paragraph we infer that $C_0(\eta_1) + \ldots + C_0(\eta_m)$ is dense in $L^p(\Gamma, w)$. Clearly, $C_0(\eta_1) + \ldots + C_0(\eta_m) \subset C(\Gamma)$, and since, by Mergelyan's theorem, $R(\Gamma)$ is (uniformly) dense in $C(\Gamma)$, it follows that $R(\Gamma)$ is dense in the space $L^p(\Gamma, w)$. □

3. Singular integral operators

Let Γ be a composed Carleson curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. Obviously, for every function $a \in L^\infty(\Gamma)$ the multiplication operator $f \mapsto af$ is bounded on $L^p(\Gamma, w)$. This operator is traditionally denoted by aI, and in case it follows another operator, say B, one abbreviates aIB to aB. Let $\mathcal{B}(\Gamma) := \mathcal{B}(L^p(\Gamma, w))$ denote the Banach algebra of all bounded linear operators on $L^p(\Gamma, w)$. In the notation $\mathcal{B}(\Gamma)$ and in similar notations we will introduce in the following, we suppress the dependence on p and w; we rather think of Γ as an object which is a curve and carries a number p and a weight w. Given a closed subalgebra E of $L^\infty(\Gamma)$, we denote by alg (S_Γ, E) the smallest closed subalgebra of $\mathcal{B}(\Gamma)$ containing the operator S_Γ and all multiplication operators aI with $a \in E$. The operators in alg (S_Γ, E) are called *singular integral operators with coefficients in E*.

An operator $A \in \mathcal{B}(\Gamma)$ is said to be *Fredholm* if it is invertible modulo compact operators. Thus, if we denote by $\mathcal{K}(\Gamma) := \mathcal{K}(L^p(\Gamma, w)) \subset \mathcal{B}(\Gamma)$ the ideal of all compact operators, then an operator $A \in \mathcal{B}(\Gamma)$ is Fredholm if and only if the coset $\pi(A) := A + \mathcal{K}(\Gamma)$ is invertible in the quotient algebra $\mathcal{B}(\Gamma)/\mathcal{K}(\Gamma)$. The *essential spectrum* $\mathrm{sp}_{\mathrm{ess}}(A)$ of an operator $A \in \mathcal{B}(\Gamma)$ is the set

$$\mathrm{sp}_{\mathrm{ess}}(A) := \{\lambda \in \mathbf{C} : A - \lambda I \text{ is not Fredholm on } L^p(\Gamma, w)\}.$$

Clearly, the algebra $\mathrm{alg}\,(S_\Gamma, E)$ is the more complicated the larger the coefficient algebra E is. One might expect that the simplest case is the case where E is the algebra $C(\Gamma)$ of all continuous functions on Γ. However, the restriction to continuous functions is unnatural. In fact, the simplest and at the same time natural class of coefficients is the algebra $PC(\Gamma)$ of all *piecewise continuous functions* on Γ. Here $PC(\Gamma)$ stands for the smallest closed subalgebra of $L^\infty(\Gamma)$ containing all characteristic functions of simple subarcs of Γ. For example, if $\Gamma = \eta_1 \cup \ldots \cup \eta_m$ where $\eta_1, \ldots, \eta_m$ are simple arcs, if $a \in L^\infty(\Gamma)$ and if $a|\eta_1, \ldots, a|\eta_m$ are continuous functions, then $a \in PC(\Gamma)$ (but, in general, $a \notin C(\Gamma)$). That $PC(\Gamma)$ is more natural than $C(\Gamma)$ but not more complicated than $C(\Gamma)$ can be seen as follows. Let Δ be a proper subset of Γ and suppose Δ itself is also a composed curve. Denote the characteristic function of Δ by χ_Δ. If a is any function in $C(\Delta)$, then the operator aS_Δ is Fredholm on $L^p(\Delta, w|\Delta)$ if and only if the operator $\chi_\Delta a S_\Gamma \chi_\Delta I + \chi_{\Gamma\setminus\Delta} I$ is Fredholm on $L^p(\Gamma, w)$. Thus, considering operators in $\mathrm{alg}\,(S_\Delta, C(\Delta))$ naturally results in studying operators in $\mathrm{alg}\,(S_\Gamma, PC(\Gamma))$.

Our purpose is to establish Fredholm criteria and thus to give descriptions of the essential spectrum for operators A in $\mathrm{alg}(S_\Gamma, PC(\Gamma))$. The essential spectrum $\mathrm{sp}_{\mathrm{ess}}(A)$ may be a rather intricate set, which leads to the problem of explaining why certain parts of the essential spectrum appear. This problem is tackled by identifying the local spectra of A.

The following theorem is the *local principle* of G.R. Allan [3] and R.G. Douglas [21] (see also [14] for a proof and for a discussion).

Theorem 3.1. *Let B be a (complex) Banach algebra with identity element e and let Z be a closed subalgebra of B such that $e \in Z$ and*

$$bz = zb \text{ for all } z \in Z \text{ and all } b \in B. \tag{3.1}$$

Denote the maximal ideal space of Z by Ω, and for $\omega \in \Omega$, let $\mathcal{J}_\omega$ refer to the smallest closed two-sided ideal of B containing the ideal ω. Then an element $b \in B$ is invertible in B if and only if $b + \mathcal{J}_\omega$ is invertible in $B/\mathcal{J}_\omega$ for every $\omega \in \Omega$.

A subalgebra Z of B satisfying (3.1) is usually said to be a *central subalgebra*. The algebra $B/\mathcal{J}_\omega$ is referred to as the *local algebra of B at $\omega \in \Omega$*, and the spectrum of $b + \mathcal{J}_\omega$ is called the *local spectrum of b at $\omega \in \Omega$*. We may write the conclusion of Theorem 3.1 also in the form

$$\mathrm{sp}_B\, b = \bigcup_{\omega \in \Omega} \mathrm{sp}_{B/\mathcal{J}_\omega}(b + \mathcal{J}_\omega),$$

where $\mathrm{sp}_A\, a := \{\lambda \in \mathbf{C} : a - \lambda e \text{ is not invertible in } A\}$. Sometimes, if the algebra A is clear from the context, we will abbreviate $\mathrm{sp}_A\, a$ simply to $\mathrm{sp}\, a$.

Now let Γ be a composed Carleson curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. We are interested in Fredholmness and thus in the invertibility of cosets $\pi(A) := A + \mathcal{K}(\Gamma)$ in the quotient algebra $\pi(\mathcal{B}(\Gamma)) := \mathcal{B}(\Gamma)/\mathcal{K}(\Gamma)$. Put

$$
\begin{aligned}
(3.2) \qquad Z(\Gamma) &:= \{\pi(cI) : c \in C(\Gamma)\},\\
\Lambda(\Gamma) &:= \{A \in \mathcal{B}(\Gamma) : \pi(A)\pi(cI) = \pi(cI)\pi(A) \text{ for all } c \in C(\Gamma)\}.
\end{aligned}
$$

Thus, $\Lambda(\Gamma)$ is the collection of all operators in $\mathcal{B}(\Gamma)$ which commute modulo compact operators with multiplications by continuous functions. Following [56], we refer to the operators in $\Lambda(\Gamma)$ as *operators of local type.* It is easily seen that $\Lambda(\Gamma)$ is a closed subalgebra of $\mathcal{B}(\Gamma)$ and that an operator $A \in \Lambda(\Gamma)$ is Fredholm if and only if $\pi(A)$ is invertible in the quotient algebra $\pi(\Lambda(\Gamma)) := \Lambda(\Gamma)/\mathcal{K}(\Gamma)$. Consequently, in order to decide whether operators in $\Lambda(\Gamma)$ are Fredholm, we may have recourse to Theorem 3.1 with $B = \pi(\Lambda(\Gamma))$ and $Z = Z(\Gamma)$ as in (3.2). The maximal ideal space of $Z(\Gamma)$ may be identified with Γ: the maximal ideal corresponding to $t \in \Gamma$ is the set

$$
(3.3) \qquad \{\pi(cI) : c \in C(\Gamma),\ c(t) = 0\}.
$$

For $t \in \Gamma$, let $\mathcal{J}_t(\Gamma)$ be the smallest closed two-sided ideal of $\pi(\Lambda(\Gamma))$ containing the set (3.3) and define

$$
\mathcal{B}_t(\Gamma) := \pi(\Lambda(\Gamma))/\mathcal{J}_t(\Gamma), \quad \pi_t(A) := \pi(A) + \mathcal{J}_t(\Gamma).
$$

Theorem 3.1 now gives the following.

Proposition 3.2. *An operator A of local type is Fredholm on $L^p(\Gamma, w)$ if and only if $\pi_t(A)$ is invertible in $\mathcal{B}_t(\Gamma)$ for every $t \in \Gamma$. Equivalently, if A is of local type, then*

$$
(3.4) \qquad \mathrm{sp}_{\mathrm{ess}}\,(A) = \bigcup_{t\in\Gamma} \mathrm{sp}\ \pi_t(A). \quad \square
$$

We call $\mathcal{B}_t(\Gamma)$ the *local algebra of* Γ (and of p and w) *at* t and $\mathrm{sp}\ \pi_t(A)$ the *local spectrum of* A *at* t. In (3.4), the essential spectrum of an operator is represented as the union of the local spectra of the operator. If $\varphi \in C(\Gamma)$ and $\varphi(t) = 1$, then

$$
\pi(\varphi I) - \pi(I) = \pi((\varphi - 1)I) \in \mathcal{J}_t(\Gamma)
$$

and hence, $\pi_t(\varphi I) = \pi_t(I)$. It follows that

$$
(3.5) \qquad \pi_t(A) = \pi_t(\varphi I)\pi_t(A)\pi_t(\varphi I) = \pi_t(\varphi A \varphi I)
$$

for every operator $A \in \Lambda(\Gamma)$. In particular, A and $\varphi A \varphi I$ have the same local spectrum at a point t. Since $\varphi A \varphi I$ does not depend on the properties of Γ and w outside the support $\mathrm{supp}\,\varphi$ of φ and because $\mathrm{supp}\,\varphi$ may be chosen to be contained in any open neighborhood of t, it is clear that $\mathrm{sp}\pi_t(A)$ depends in fact only on the "local properties" of Γ and w at t.

Proposition 3.3. *The operators in* $\mathrm{alg}\,(S_\Gamma, L^\infty(\Gamma))$ *are of local type.*

P r o o f. Multiplication operators are obviously of local type. Since $\Lambda(\Gamma)$ is a closed algebra, it remains to show that S_Γ is of local type, i.e. that $\varphi S_\Gamma - S_\Gamma \varphi I$ is compact on $L^p(\Gamma, w)$ for every $\varphi \in C(\Gamma)$. If $\varphi \in R(\Gamma)$, then

$$\Big((\varphi S_\Gamma - S_\Gamma \varphi I)f\Big)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t) - \varphi(\tau)}{t - \tau} f(\tau)\, d\tau \quad (t \in \Gamma),$$

and on decomposing φ into partial fractions it is readily seen that $\varphi S_\Gamma - S_\Gamma \varphi I$ is a finite-rank operator. Since $R(\Gamma)$ is uniformly dense in $C(\Gamma)$ by Mergelyan's theorem, we arrive at the assertion. □

4. Main results

Let Γ be a composed Carleson curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. We first associate a certain compact subset $\mathcal{L}_t(\Gamma) := \mathcal{L}_t(\Gamma, p, w)$ of the complex plane with each point $t \in \Gamma$, which is called the leaf of the triple p, Γ, w at t, and then we construct a Banach algebra homomorphism

$$\mathrm{Sym}_{\,t,x} : \mathrm{alg}\,(S_\Gamma, PC(\Gamma)) \to \mathbf{C}^{N(t) \times N(t)}$$

for each $x \in \mathcal{L}_t(\Gamma)$ such that

$$\mathrm{sp}\, \pi_t(A) = \bigcup_{x \in \mathcal{L}_t(\Gamma)} \mathrm{sp}\, \mathrm{Sym}_{\,t,x}(A)$$

for every $A \in \mathrm{alg}\,(S_\Gamma, PC(\Gamma))$. Here $\mathbf{C}^{N(t) \times N(t)}$ stands for the Banach algebra of all $N(t) \times N(t)$ matrices and $\mathrm{sp}\,\mathrm{Sym}_{\,t,x}(A)$ is the spectrum (= set of eigenvalues) of the matrix $\mathrm{Sym}_{\,t,x}(A)$. We also emphasize that by a Banach algebra homomorphism we always understand a continuous (= bounded) algebra homomorphism.

Suppose first that η is a simple Carleson arc, $1 < p < \infty$, and $w \in A_p(\eta)$. Let t be an endpoint of η and denote by η^+ the arc η with the orientation in which t is the starting point. Put $S_\eta^+ := S_{\eta^+}$ and $P_\eta^+ := (I + S_\eta^+)/2$. We define the *leaf* $\mathcal{L}_t(\eta)$ *of the simple Carleson arc* η *at its endpoint* t as the local spectrum of P_η^+ at t:

$$\mathcal{L}_t(\eta) := \mathrm{sp}\, \pi_t(P_\eta^+). \tag{4.1}$$

Notice that $\mathcal{L}_t(\eta)$ does not depend on the orientation of η. We remark that if both η and w are "nice" at the point t, then $\mathcal{L}_t(\eta)$ is a certain circular arc between the points 0 and 1. The points 0 and 1 always belong to $\mathcal{L}_t(\eta)$. More about the shape of leaves will be said in the next section.

Theorem 4.1. *Let η_1 and η_2 be simple Carleson arcs which have a common endpoint t but no other common points. Let further $1 < p < \infty$, $w_1 \in A_p(\eta_1)$, $w_2 \in A_p(\eta_2)$. If $\eta_1 \cup \eta_2$ is a Carleson curve and the weight w given by $w|\eta_1 = w_1$ and $w|\eta_2 = w_2$ belongs to $A_p(\eta_1 \cup \eta_2)$, then*

$$\mathcal{L}_t(\eta_1) = \mathcal{L}_t(\eta_2).$$

In the case where both η_1 and η_2 are simple Lyapunov arcs, a result like the previous theorem is implicitly already in [35].

A composed Carleson curve γ comprised of a finite number $N \geq 2$ of simple Carleson arcs $\eta_1, \dots, \eta_N$ which have exactly one point in common is called a *Carleson star* and the common point is referred to as the *center* of the star. Let $\gamma = \eta_1 \cup \dots \cup \eta_N$ be a Carleson star with the center t, let $1 < p < \infty$, and let $w \in A_p(\gamma)$. Theorem 1.1 implies that $w|\eta_i \cup \eta_j \in A_p(\eta_i \cup \eta_j)$. From Theorem 4.1 we so infer that

$$\mathcal{L}_t(\eta_1) = \dots = \mathcal{L}_t(\eta_N). \tag{4.2}$$

We denote the set (4.2) by $\mathcal{L}_t(\gamma) := \mathcal{L}_t(\gamma, p, w)$ and call it the *leaf of the Carleson star γ at its center γ*. Notice again that $\mathcal{L}_t(\gamma)$ is independent of the orientations of the arcs γ is comprised of.

Now let Γ be a general composed Carleson curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. Pick a point $t \in \Gamma$ and denote by $\Gamma_0(t, \varepsilon)$ the connected component of the portion $\Gamma(t, \varepsilon)$ containing t. If $\varepsilon > 0$ is sufficiently small, then $\Gamma_0(t, \varepsilon)$ is a simple Carleson arc or a Carleson star and hence, we may define $\mathcal{L}_t(\Gamma_0(t, \varepsilon))$. From (4.1), (3.5), and Proposition 3.3 we infer that $\mathcal{L}_t(\Gamma_0(t, \varepsilon))$ does not depend on ε. We denote $\mathcal{L}_t(\Gamma_0(t, \varepsilon))$ by $\mathcal{L}_t(\Gamma) := \mathcal{L}_t(\Gamma, p, w)$ and refer to it as the *leaf of the composed Carleson curve at the point t*.

If $\varepsilon > 0$ is small enough, then the number of connected components of $\Gamma_0(t, \varepsilon) \setminus \{t\}$ is independent of ε. This number is denoted by $N(t)$ and called the *multiplicity* of t. Clearly, there is only a finite number of points $t \in \Gamma$ with $N(t) \neq 2$. Let

$$\Gamma_0(t, \varepsilon) = \eta_1 \cup \dots \cup \eta_{N(t)} \tag{4.3}$$

where $\eta_1, \dots, \eta_{N(t)}$ are simple arcs. Each of these arcs has an orientation. We put $\varepsilon_j(t) = 1$ if η_j is an arc starting at t and we let $\varepsilon_j(t) = -1$ if η_j is an arc terminating at t. If $a \in PC(\Gamma)$, then the limits

$$a_j(t) := \lim_{\substack{\tau \in \eta_j, \\ \tau \to t}} a_j(\tau) \quad (j = 1, \dots, N(t)) \tag{4.4}$$

exist and are finite.

We are now in a position to state our main result.

Theorem 4.2. *Suppose Γ is a composed Carleson curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. For $t \in \Gamma$, put $E(t) := \operatorname{diag}\Big(\varepsilon_1(t), \dots, \varepsilon_{N(t)}(t)\Big)$, and given $(t, x) \in \Gamma \times \mathbf{C}$, define the map*

$$\operatorname{Sym}_{t,x} : \{S_\Gamma\} \cup \{aI : a \in PC(\Gamma)\} \to \mathbf{C}^{N(t) \times N(t)}$$

as follows. For $x \notin \{0, 1\}$, let

$$\operatorname{Sym}_{t,x}(S_\Gamma) := E(t) \begin{pmatrix} 2x-1 & 2x-2 & 2x-2 & \dots & 2x-2 \\ 2x & 2x-1 & 2x-2 & \dots & 2x-2 \\ 2x & 2x & 2x-1 & \dots & 2x-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2x & 2x & 2x & \dots & 2x-1 \end{pmatrix}, \tag{4.5}$$

for $x \in \{0,1\}$, *let*

(4.6) $$\mathrm{Sym}_{\,t,x}(S_\Gamma) := (2x-1)E(t),$$

and for all x, *let*

(4.7) $$\mathrm{Sym}_{\,t,x}(aI) := \mathrm{diag}\,\Big(a_1(t),\ldots,a_{N(t)}(t)\Big).$$

(a) *For each* x *in the leaf* $\mathcal{L}_t(\Gamma)$ *the map* $\mathrm{Sym}_{\,t,x}$ *extends to a Banach algebra homomorphism*

$$\mathrm{Sym}_{\,t,x} : \mathrm{alg}\,(S_\Gamma, PC(\Gamma)) \to \mathbf{C}^{N(t)\times N(t)}.$$

(b) *If* $A \in \mathrm{alg}\,(S_\Gamma, PC(\Gamma))$, *then*

$$\mathrm{sp}\ \pi_t(A) = \bigcup_{x\in\mathcal{L}_t(\Gamma)} \mathrm{sp}\ \mathrm{Sym}_{\,t,x}(A).$$

(c) *An operator* $A \in \mathrm{alg}\,(S_\Gamma, PC(\Gamma))$ *is Fredholm on* $L^p(\Gamma,w)$ *if and only if* $\mathrm{Sym}_{\,t,x}(A)$ *is an invertible matrix for all* (t,x) *in the leaf bundle*

$$\mathcal{M} := \mathcal{M}(\Gamma,p,w) := \bigcup_{t\in\Gamma}\Big(\{t\}\times\mathcal{L}_t(\Gamma)\Big).$$

In other terms,

$$\mathrm{sp}_{\mathrm{ess}}\, A = \bigcup_{(t,x)\in\mathcal{M}} \mathrm{sp}\ \mathrm{Sym}_{\,t,x}(A) = \bigcup_{t\in\Gamma}\ \bigcup_{x\in\mathcal{L}_t(\Gamma)} \mathrm{sp}\ \mathrm{Sym}_{\,t,x}(A).$$

Clearly, part (c) of this theorem follows from part (b) and Propositions 3.2 and 3.3. Let us consider three simple examples.

We call a curve a *Carleson Jordan curve* if it is a Carleson curve homeomorphic to a circle and if it is oriented counter-clockwise. If Γ is a Carleson Jordan curve and $t\in\Gamma$, then Theorem 4.2. holds with

(4.8) $$\mathrm{Sym}_{\,t,x}(S_\Gamma) = \begin{pmatrix} 2x-1 & 2x-2 \\ -2x & 1-2x \end{pmatrix} \quad (x\notin\{0,1\}),$$

(4.9) $$\mathrm{Sym}_{\,t,0}(S_\Gamma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathrm{Sym}_{\,t,1}(S_\Gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(4.10) $$\mathrm{Sym}_{\,t,x}(aI) = \begin{pmatrix} a(t+0) & 0 \\ 0 & a(t-0) \end{pmatrix}.$$

The eigenvalues of the matrices (4.8), (4.9) are -1 and 1. Hence $\mathrm{sp}\,\pi_t(S_\Gamma) = \{-1,1\}$ for all $t\in\Gamma$ and thus, $\mathrm{sp}_{\mathrm{ess}}\,(S_\Gamma) = \{-1,1\}$ and $\mathrm{sp}_{\mathrm{ess}}\,(P_\Gamma) = \{0,1\}$, where $P_\Gamma := (I+S_\Gamma)/2$.

Secondly, suppose $\Gamma=\eta$ is a simple Carleson arc starting at t_0 and terminating at t_1. We then have Theorem 4.2 with

$$\mathrm{Sym}_{\,t_0,x}(S_\Gamma) = 2x-1,\quad \mathrm{Sym}_{\,t_0,x}(aI) = a(t_0+0),$$
$$\mathrm{Sym}_{\,t_1,x}(S_\Gamma) = 1-2x,\quad \mathrm{Sym}_{\,t_1,x}(aI) = a(t_1-0),$$

and with $\mathrm{Sym}_{t,x}(S_\Gamma)$ and $\mathrm{Sym}_{t,x}(aI)$ as in (4.8), (4.9), (4.10) for $t \in \eta \setminus \{t_0, t_1\}$. In particular,

$$\begin{aligned} \mathrm{sp}\, \pi_{t_0}(S_\eta) &= 2\mathcal{L}_{t_0}(\eta) - 1, \quad \mathrm{sp}\, \pi_{t_0}(P_\eta) = \mathcal{L}_{t_0}(\eta), \\ \mathrm{sp}\, \pi_{t_1}(S_\eta) &= 1 - 2\mathcal{L}_{t_1}(\eta), \quad \mathrm{sp}\, \pi_{t_1}(P_\eta) = 1 - \mathcal{L}_{t_1}(\eta). \end{aligned}$$

Thirdly, assume $\Gamma = \eta_1 \cup \eta_2$ is a Carleson star centered at t_0 and η_1 and η_2 are outgoing arcs terminating at t_1 and t_2, respectively. Then Theorem 4.2 is true with

$$\mathrm{Sym}_{t_0,x}(S_\Gamma) = \begin{pmatrix} 2x-1 & 2x-2 \\ 2x & 2x-1 \end{pmatrix} \quad (x \notin \{0,1\}), \tag{4.11}$$

$$\mathrm{Sym}_{t_0,0}(S_\Gamma) = \mathrm{diag}\,(-1,-1), \;\; \mathrm{Sym}_{t_0,1}(S_\Gamma) = \mathrm{diag}\,(1,1),$$
$$\mathrm{Sym}_{t_1,x}(S_\Gamma) = 1 - 2x, \;\; \mathrm{Sym}_{t_2,x}(S_\Gamma) = 1 - 2x,$$

with $\mathrm{Sym}_{t,x}(S_\Gamma)$ as in (4.8), (4.9) for $t \in \Gamma \setminus \{t_0, t_1, t_2\}$ and with $\mathrm{Sym}_{t,x}(aI)$ given by (4.10) for $t \in \Gamma \setminus \{t_0, t_1, t_2\}$ and by

$$\begin{pmatrix} \lim\limits_{t \to t_0, t \in \eta_1} a(t) & 0 \\ 0 & \lim\limits_{t \to t_0, t \in \eta_2} a(t) \end{pmatrix}, \quad a(t_1 - 0), \quad a(t_2 - 0)$$

for $t = t_0$, $t = t_1$, $t = t_2$, respectively. The eigenvalues of the matrix (4.11) are $2x - 1 \pm 2i\sqrt{x(1-x)}$. Thus,

$$\mathrm{sp}\, \pi_{t_0}(S_\Gamma) = \left\{ 2x - 1 \pm 2i\sqrt{x(1-x)} : x \in \mathcal{L}_{t_0}(\Gamma) \right\}$$

where $\mathcal{L}_{t_0}(\Gamma) := \mathcal{L}_{t_0}(\eta_1) = \mathcal{L}_{t_0}(\eta_2)$.

Theorem 4.2 does everything provided we know the leaves $\mathcal{L}_t(\Gamma)$ $(t \in \Gamma)$ or, equivalently, the local spectra of the operators P_η^+ at the endpoints of all simple arcs η contained in Γ. We now proceed to the computation of the leaves.

Suppose first that Γ is a Carleson Jordan curve and fix a point $t \in \Gamma$. Given a weight $\psi : \Gamma \to [0, \infty]$ such that $\log \psi \in L^1(\Gamma)$, we construct a new function $V_t^0 \psi : (0, \infty) \to [0, \infty]$, the V_t^0 *transform* of ψ (this terminology proved useful in [8], [9]) as follows: for $\varepsilon > 0$, put

$$M_{t,\psi}(\varepsilon) := \exp\left(\frac{1}{|\Gamma(t,\varepsilon)|} \int_{\Gamma(t,\varepsilon)} \log \psi(\tau)\, |d\tau| \right)$$

and then define

$$(V_t^0 \psi)(\xi) := \begin{cases} \limsup\limits_{R \to 0} M_{t,\psi}(\xi R)/M_{t,\psi}(R) & \text{for } \xi \in (0,1], \\ \limsup\limits_{R \to 0} M_{t,\psi}(R)/M_{t,\psi}(\xi^{-1} R) & \text{for } \xi \in [1, \infty). \end{cases}$$

We call

$$\alpha(V_t^0 \psi) := \limsup_{\xi \to 0} \frac{\log (V_t^0 \psi)(\xi)}{\log \xi}, \quad \beta(V_t^0 \psi) := \limsup_{\xi \to \infty} \frac{\log (V_t^0 \psi)(\xi)}{\log \xi}$$

the *lower* and *upper indices* of $V_t^0\psi$, respectively.

For $\tau \in \Gamma$, we have $\tau - t = |\tau - t|e^{i\arg(\tau - t)}$, and $\arg(\tau - t)$ may be chosen as a continuous function of $\tau \in \Gamma \setminus \{t\}$. Taking into account that Γ is a Carleson curve, one can show that the function $\tau \mapsto \arg(\tau - t)$ belongs to $L^1(\Gamma)$ (see [8]). It is also well known that $\log W \in L^1(\Gamma)$ whenever $W \in A_p(\Gamma)$ (again see [8]). Hence, if we put

$$\varrho_t(\tau) := e^{-\arg(\tau - t)} \quad \text{for} \quad \tau \in \Gamma \setminus \{t\},$$

then $\log(\varrho_t^x W) = x \log \varrho_t + \log W \in L^1(\Gamma)$ for each $x \in \mathbf{R}$ and we may carry out the above construction with $\psi := \varrho_t^x W$ to obtain

$$\alpha_t(x) := \alpha(V_t^0(\varrho_t^x W)), \quad \beta_t(x) := \beta(V_t^0(\varrho_t^x W)). \tag{4.12}$$

In [8], [9] it is shown that

$$-\infty < \alpha_t(x) \le \beta_t(x) < +\infty \quad \text{for all} \quad x \in \mathbf{R}, \tag{4.13}$$

$$-1/p < \alpha_t(0) \le \beta_t(0) < 1/q, \tag{4.14}$$

$$\alpha_t \text{ is a concave function and } \beta_t \text{ is a convex function.} \tag{4.15}$$

The latter property implies that α_t and β_t are continuous functions on $\mathbf{R}$. We call α_t and β_t the *indicator functions* of the pair Γ, W at t.

Given two continuous functions $\alpha, \beta : \mathbf{R} \to \mathbf{R}$ such that $\alpha(x) \le \beta(x)$ for all $x \in \mathbf{R}$, we put

$$Y(p; \alpha, \beta) := \left\{ z = x + iy \in \mathbf{C} : \frac{1}{p} + \alpha(x) \le y \le \frac{1}{p} + \beta(x) \right\} \tag{4.16}$$

and

$$\mathcal{L}^0(p; \alpha, \beta) := \left\{ \frac{e^{2\pi z}}{e^{2\pi z} - 1} : z \in Y(p; \alpha, \beta) \right\} \cup \{0, 1\}. \tag{4.17}$$

We refer to $\mathcal{L}^0(p; \alpha, \beta)$ as the leaf of the triple p, α, β. If α_t, β_t are the indicator functions of Γ, W at t, then $\mathcal{L}^0(p; \alpha_t, \beta_t)$ is called the *indicator leaf* of the triple Γ, p, W at the point t. Sometimes, in case p and W are known from the context, we will simply speak of $\mathcal{L}^0(p; \alpha_t, \beta_t)$ as the indicator leaf of the (Carleson and Jordan !) curve Γ at the point t.

Of course, the above construction is not easy. However, in the next section we will see that the indicator functions can be given by pretty nice formulas in many interesting cases. At the present moment we remark that solely (4.13) and the continuity of the indicator functions imply that $\mathcal{L}^0(p; \alpha_t, \beta_t)$ is a compact connected set which does not separate the plane and which contains the points 0 and 1.

The following theorem identifies the leaves $\mathcal{L}_t(\eta)$ we are interested in in terms of indicator leaves.

Theorem 4.3. *Let η be a simple Carleson arc, $1 < p < \infty$, and $w \in A_p(\eta)$.*

(a) *There exist a Carleson Jordan curve Γ and a weight $W \in A_p(\Gamma)$ such that $\eta \subset \Gamma$ and $W|\eta = w$.*

(b) *Let* Γ *be any Carleson Jordan curve and* $W \in A_p(\Gamma)$ *be any weight such that* $\eta \subset \Gamma$ *and* $W|\eta = w$. *For* $t \in \Gamma$, *let* $\mathcal{L}^0(p; \alpha_t, \beta_t)$ *be the indicator leaf of* Γ, p, W *at* t. *Then*

$$\mathcal{L}_t(\eta) = \mathcal{L}^0(p; \alpha_t, \beta_t) \tag{4.18}$$

at each endpoint t *of* η.

Thus, in order to find $\mathcal{L}_t(\eta)$, we first have to extend η and w to a Carleson Jordan curve Γ and a weight $W \in A_p(\Gamma)$ (which is possible by Theorem 4.3(a)) and then we have to compute the indicator leaf $\mathcal{L}^0(p; \alpha_t, \beta_t)$ (which is independent of the extension due to Theorem 4.3(b)).

The following theorem describes the local spectrum of the Cauchy singular integral operator as the union of certain leaves. Suppose t has the multiplicity $N := N(t)$ and that in (4.3) we have N_o outgoing and N_i incoming arcs. The difference $\varepsilon := N_o - N_i$ is called the *valency* of t. Of course, $\varepsilon = \varepsilon_1(t) + \ldots + \varepsilon_N(t)$. For a continuous function $f : \mathbf{R} \to \mathbf{R}$, define $T_n f : \mathbf{R} \to \mathbf{R}$ by $(T_n f)(x) = f(nx)$.

Theorem 4.4. *Let* Γ *be a composed Carleson curve,* $1 < p < \infty$, *and* $w \in A_p(\Gamma)$. *Let* $t \in \Gamma$ *and let* α_t, β_t *be any functions such that* $\mathcal{L}_t(\Gamma) = \mathcal{L}^0(p; \alpha_t, \beta_t)$. *Given integers* $n > 0$ *and* $k \geq 0$, *put*

$$\mathcal{L}^0_{k,n} := \mathcal{L}^0\left(p; \frac{1}{n}\left(\frac{1}{p} - \frac{n}{p} + k + T_n\alpha_t\right), \frac{1}{n}\left(\frac{1}{p} - \frac{n}{p} + k + T_n\beta_t\right)\right).$$

Then, $\operatorname{sp} \pi_t(S_\Gamma)$ *equals*

$$\{-1, 1\} \quad \textit{if } \varepsilon = 0, \tag{4.19}$$

$$\bigcup_{k=0}^{\varepsilon-1} (2\mathcal{L}^0_{k,\varepsilon} - 1) \quad \textit{if } \varepsilon > 0, \tag{4.20}$$

$$\bigcup_{k=0}^{|\varepsilon|-1} (1 - 2\mathcal{L}^0_{k,|\varepsilon|}) \quad \textit{if } \varepsilon < 0. \tag{4.21}$$

We remark that, curiously, a result of this type was first explicitly stated only in the recent papers [11] and [50].

5. Comments and history

Theorem 4.2 stands at the end of a long chain of results. This theorem associates a family of matrix functions with every operator in alg $(S_\Gamma, PC(\Gamma))$ in terms of which one can decide whether the operator is Fredholm or not. These matrix functions have a special structure and live on certain sets, the leaves. The history of Theorem 4.2 is the history of understanding the shape of the leaves on the one hand and the history of finding the right structure of the matrix functions on the other.

5.1. Genesis of leaves

The theory of singular integral operators with continuous coefficients was essentially disposed of with I. Gohberg's 1952 paper [28]. B.V. Khvedelidze was the first to investigate Fredholm properties of singular integral equations with piecewise continuous coefficients on L^p spaces with power weights [40] (also see his survey [41]). In particular, he established sufficient conditions for such operators to be Fredholm. In 1960, Shamir [54] and Widom [61] studied singular integral operators on the line and on the half-line. Widom's paper [61] is the earliest reference we know in which the term "circular arc" explicitly appeared in connection with the spectral theory of singular integral operators. Also in the sixties, Simonenko created his local principle [56] and the method of generalized Wiener-Hopf factorization [55], [57] and thus laid the foundation of the today's edifice of singular integral equations. In the late sixties, Gohberg and Krupnik turned to operators with discontinuous symbols and within only a few years elaborated a Fredholm theory for algebras of singular integral operators with piecewise continuous coefficients on composed Lyapunov curves with power weights; we here confine ourselves to refering to their book [33] and their three papers [30], [31], [32].

A simple arc η is called a *Lyapunov arc* if the derivative φ' of its natural parametrization $\varphi : [0, l] \to \mathbf{C}$ satisfies a Hölder condition,

$$|\varphi'(t) - \varphi'(\tau)| \leq C|t - \tau|^\lambda \text{ for all } t, \tau \in \eta$$

with $C > 0$ and $\lambda \in (0, 1)$. Two Lyapunov arcs η_1 and η_2 are said to be compatible if either $\eta_1 \cap \eta_2 = \emptyset$ or η_1 and η_2 have exactly one endpoint in common, at which they do not form a cusp. A *composed Lyapunov curve* is a finite union of pairwise compatible Lyapunov arcs. It is easily seen that composed Lyapunov curves are composed Carleson curves. A *power weight* is a weight w of the form

$$w(\tau) = \prod_{j=1}^{m} |\tau - t_j|^{\lambda_j} \quad (\tau \in \Gamma) \tag{5.1}$$

where $t_1, \ldots, t_m$ are distinct points on Γ and $\lambda_1, \ldots, \lambda_m$ are real numbers. It is well known that the weight (5.1) belongs to $A_p(\Gamma)$ if and only if $-1/p < \lambda_j < 1/q$ for all j; in the case of general Carleson curves this was perhaps first proved by Danilov [18] (his proof is reproduced in [6]).

Let now η be a simple Lyapunov arc, let w be the power weight (5.1) with $\Gamma = \eta$, and let t be an endpoint of η. Gohberg and Krupnik proved that then $\operatorname{sp} \pi_t(P_\eta^+)$ is a certain circular arc. In the language of Section 4, this means that the leaf $\mathcal{L}_t(\eta)$ is a circular arc. In terms of Theorem 4.3, this circular arc can be described as the indicator leaf corresponding to the indicator functions

$$\alpha_t(x) = \beta_t(x) = \lambda_t \tag{5.2}$$

where $\lambda_t := 0$ if $t \notin \{t_1 \ldots, t_m\}$ and $\lambda_t := \lambda_j$ if $t = t_j$. Clearly, $Y(p; \alpha_t, \beta_t)$ is the horizontal line $\{z \in \mathbf{C} : \operatorname{Im} z = 1/p + \lambda_t\}$. The map $z \mapsto e^{2\pi z}$ maps $Y(p; \alpha_t, \beta_t)$ to a ray starting at the origin and making the angle $2\pi(1/p + \lambda_t) \in (0, 2\pi)$ with the positive real axis. Finally, the Möbius transform $\zeta \mapsto \zeta/(\zeta-1)$ maps this ray to a circular arc between 0 and 1 which degenerates to the segment $[0,1]$ in case $1/p+\lambda_t = 1/2$. This circular arc is $\mathcal{L}_t(\eta)$.

In 1990, after a pause of about 15 years, the next step was done by one of the authors [58], who determined $\operatorname{sp}\pi_t(P_\eta^+)$ and thus the leaf $\mathcal{L}_t(\eta)$ in case η is a simple Lyapunov arc and w is an arbitrary weight in $A_p(\eta)$. Using Theorem 4.3, the leaf $\mathcal{L}_t(\eta)$ can be identified as the indicator leaf induced by the indicator functions

$$\alpha_t(x) = \mu_t \quad \text{and} \quad \beta_t(x) = \nu_t,$$

where μ_t and ν_t are certain real numbers measuring the "powerlikeness" of the weight w at t. We always have $-1/p < \mu_t \le \nu_t < 1/q$. Thus, $Y(p;\alpha_t,\beta_t)$ is now the stripe $\{z \in \mathbf{C} : 1/p+\mu_t \le \operatorname{Im} z \le 1/p+\nu_t\}$ and from what was said in the preceding paragraph we infer that $\mathcal{L}_t(\eta)$ is a *horn*, i.e. the closed set between two circular arcs joining 0 and 1.

In [1] (also see [2]), Aizenshtat, Litvinchuk, and one of the authors were led to the investigation of singular integral operators on Carleson curves very close to logarithmic spirals and stated a Fredholm criterion for such operators. Only recently, two of the authors turned attention to general Carleson curves and were able to describe $\operatorname{sp}\pi_t(P_\eta^+)$ for simple Carleson arcs η and weights $w \in A_p(\eta)$ provided there is a Carleson Jordan curve Γ and a weight $W \in A_p(\Gamma)$ such that $\eta \subset \Gamma$ and $W|\eta = w$ (see [6], [7], [8], [9]). By virtue of Theorem 4.3(a), we therefore know $\operatorname{sp}\pi_t(P_\eta^+)$ and thus $\mathcal{L}_t(\eta)$ for arbitrary simple Carleson arcs η and arbitrary weights $w \in A_p(\eta)$. We remark that the concept of indicator functions and the description of $\mathcal{L}_t(\eta)$ in terms of indicator leaves as in Theorem 4.3(b) were introduced only in [8].

We now examine some examples of leaves. Suppose η is a simple Carleson arc, t is an endpoint of η, and $w \in A_p(\eta)$. Let Γ be a Carleson Jordan curve containing η and let W be a weight in $A_p(\Gamma)$ extending w. Theorem 4.3(b) tells us that $\mathcal{L}_t(\eta) = \mathcal{L}^0(p;\alpha_t,\beta_t)$, where α_t, β_t are the indicator functions of Γ, W at t.

Recall first the definition (4.12) of the indicator functions. We call the two numbers

$$\delta_t^- := \alpha(V_t^0 \varrho_t), \quad \delta_t^+ := \beta(V_t^0 \varrho_t)$$

the *spirality indices* of the curve Γ at the point t. Obviously, δ_t^- and δ_t^+ have nothing to do with a weight on Γ and are therefore "intrinsic" characteristics of the curve. The curve Γ is said to be *spiralic* at t if $\delta_t^- = \delta_t^+ =: \delta_t$. Let, for example,

$$\eta = \Big\{ r\exp\Big(ih(\log(-\log r))(-\log r) \Big) : r \in (0,1) \Big\} \cup \{0,1\}$$

where h is a twice continuously differentiable function and h, h', h'' are bounded on $\mathbf{R}$. One can show that η is a Carleson arc and that the spirality indices at $t=0$ of any extension Γ of η are

$$\delta_t^- = \liminf_{x\to+\infty}(h(x)+h'(x)), \quad \delta_t^+ = \limsup_{x\to+\infty}(h(x)+h'(x)).$$

If $h(x) = \delta + \varepsilon \sin x$, then

$$\delta_t^- = \delta - |\varepsilon|\sqrt{2} \quad \text{and} \quad \delta_t^+ = \delta + |\varepsilon|\sqrt{2}.$$

In case $h(x) = \delta$, the arc η is a logarithmic spiral and $\delta_t^- = \delta_t^+ = \delta_t = \delta$. Piecewise C^1 curves and, in particular, Lyapunov curves are always spiralic with $\delta_t = 0$.

We need still another pair of indices. We put

$$\mu_t := \alpha_t(0) := \alpha(V_t^0 W), \quad \nu_t := \beta_t(0) := \beta(V_t^0 W) \tag{5.3}$$

and call μ_t and ν_t the *indices of powerlikeness* of the weight W at t. Note that in (5.3) the curve Γ does not explicitly appear, although, of course, the weight W lives on Γ, by virtue of which Γ enters (5.3) implicitly. We say that W is *powerlike* at t if $\mu_t = \nu_t$. To have a concrete example, suppose $\eta := [0, 1]$ and

$$w(r) := \exp\Big(g(\log(-\log r))\log r\Big) \text{ for } r \in (0,1)$$

where g is a twice continuously differentiable function on $\mathbf{R}$ for which g, g', g'' are bounded on $\mathbf{R}$. One can show that $w \in A_p(\eta)$ if and only if $-1/p < \mu_t \le \nu_t < 1/q$ where

$$\mu_t := \liminf_{x\to+\infty}\,(g(x) + g'(x)), \quad \nu_t := \limsup_{x\to+\infty}\,(g(x) + g'(x))$$

and that the indices of powerlikeness at $t = 0$ of any extension W of w are just μ_t and ν_t. Letting $g(x) = \lambda + \varepsilon \sin x$, we get

$$\mu_t = \lambda - |\varepsilon|\sqrt{2} \quad \text{and} \quad \nu_t = \lambda + |\varepsilon|\sqrt{2}.$$

If $g(x) = \lambda$, then w is a power weight and $\mu_t = \nu_t = \lambda$.

The results of the next two paragraphs were established in [7] and [8] (also see [9]).

If Γ is spiralic at t and W is an arbitrary weight in $A_p(\Gamma)$, then the indicator functions are

$$\alpha_t(x) = \mu_t + \delta_t x, \quad \beta_t(x) = \nu_t + \delta_t x. \tag{5.4}$$

Thus, $Y(p; \alpha_t, \beta_t)$ is the stripe

$$\Big\{z = x + iy \in \mathbf{C} : 1/p + \mu_t + \delta_t x \le y \le 1/p + \nu_t + \delta_t x\Big\}$$

(which degenerates to a line in case $\mu_t = \nu_t$). The map $z \mapsto e^{2\pi z}$ maps the line $\{z = x + iy \in \mathbf{C} : y = \lambda + \delta_t x\}$ to a logarithmic spiral, and the Möbius transform $\zeta \mapsto \zeta/(\zeta - 1)$ maps this logarithmic spiral to a logarithmic double spiral wriggling out of 0 and scrolling up at 1. Thus, $\mathcal{L}^0(p; \alpha_t, \beta_t)$ is the closed set between two such logarithmic double spirals. In [6], such sets were called *spiralic horns.*

Now suppose Γ is an arbitrary Carleson Jordan curve but $W \in A_p(\Gamma)$ is powerlike at t. Then the indicator functions are given by

$$\alpha_t(x) = \lambda_t + \min\Big\{\delta_t^- x, \delta_t^+ x\Big\}, \quad \beta_t(x) = \lambda_t + \max\Big\{\delta_t^- x, \delta_t^+ x\Big\} \tag{5.5}$$

where $\lambda_t := \mu_t = \nu_t$. The set $Y(p; \alpha_t, \beta_t)$ is now the double sector between two lines of slopes δ_t^- and δ_t^+. In [7], the corresponding leaves $\mathcal{L}^0(p; \alpha_t, \beta_t)$ were called skew spiralic horns, in the terminology of [8], these leaves are logarithmic leaves with a separating point.

In summary, we know the leaves $\mathcal{L}_t(\eta) = \mathcal{L}^0(p; \alpha_t, \beta_t)$ in case Γ is spiralic at t or W is powerlike at t. In these two cases, $\mathcal{L}_t(\eta)$ is always bounded by pieces of at most two logarithmic double spirals. In the case where Γ is not spiralic at t and W is not powerlike at t, the situation is more delicate. Taking into account (5.4) and (5.5) one might guess that

$$\alpha_t(x) = \mu_t + \min\left\{\delta_t^- x, \delta_t^+ x\right\}, \quad \beta_t(x) = \nu_t + \max\left\{\delta_t^- x, \delta_t^+ x\right\},$$

but one of the big surprises of the story is that this guess is wrong ! It turns out that in the general case some kind of interference between the oscillation of the curve and the oscillation of the weight may result in a complete demolition of spirality: it may happen that no piece of the boundary of $\mathcal{L}_t(\eta)$ is a piece of some logarithmic double spiral. The leaves that may occur in the general case are completely described in [8]. This paper and [11] also contain pictures of several leaves.

5.2. Genesis of symbol matrices

Here now are a few remarks on the other aspect of Theorem 4.2, that is, on the structure of the symbol matrices $\mathrm{Sym}_{t,x}(A)$. It was once more Gohberg and Krupnik [30], [31], [32], [33] who first realized the possibility of a symbol calculus for algebras of singular integral operators and who also first understood the structure of the symbol matrices. They considered composed Lyapunov curves and power weights. In this case their theorem does the same as Theorem 4.2, but it is more complicated.

For the sake of simplicity, let us suppose that Γ is a Lyapunov Jordan curve. Gohberg and Krupnik define their symbol on the cylinder $\Gamma \times [0,1]$ and put all information about the leaf $\mathcal{L}_t(\eta)$ into the function $x \mapsto \mathrm{Sym}_{t,x}(A)$. This works well if the leaves themselves are curves (e.g. circular arcs), but it is clear that difficulties arise as soon as the leaves become heavy sets. Instead of this, we define the symbol on the leaf bundle $\mathcal{M}$ and hence, for each A, the map $x \mapsto \mathrm{Sym}_{t,x}(A)$ is one and the same function. Redefining the Gohberg/Krupnik symbol to a symbol on $\mathcal{M}$ gives

$$\mathrm{Sym}_{t,x}^{GK}(aI) = \begin{pmatrix} a(t+0) & 0 \\ 0 & a(t-0) \end{pmatrix}, \tag{5.6}$$

$$\mathrm{Sym}_{t,x}^{GK}(P_\Gamma) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}, \tag{5.7}$$

whereas Theorem 4.2 yields

$$\mathrm{Sym}_{t,x}(aI) = \begin{pmatrix} a(t+0) & 0 \\ 0 & a(t-0) \end{pmatrix}, \quad \mathrm{Sym}_{t,x}(P_\Gamma) = \begin{pmatrix} x & x-1 \\ -x & 1-x \end{pmatrix}$$

for $x \notin \{0,1\}$. The enigma's resolution is in fact simple [50]: letting

$$D_x := \mathrm{diag}\left(\sqrt[4]{x/(1-x)},\ -\sqrt[4]{(1-x)/x}\right)$$

we have

$$\mathrm{Sym}_{t,x}^{GK}(A) = D_x \mathrm{Sym}_{t,x}(A) D_x^{-1}.$$

Thus, the differences between the Gohberg/Krupnik symbol and the symbol delivered by Theorem 4.2 are minor ones if Γ is a Jordan curve. However, the differences become serious when passing to composed curves.

In 1978, Douglas [22] studied the algebra of singular integral operators on L^2 over the complex unit circle $\mathbf{T}$. His approach was as follows. The algebra $\operatorname{alg}(S_{\mathbf{T}}, PC(\mathbf{T}))$ is generated by $P_{\mathbf{T}}$ and $\{aI : a \in PC(\mathbf{T})\}$. Since $P_{\mathbf{T}}$ is a projection, the element $P := \pi_t(P_{\mathbf{T}})$ is an idempotent element of the local algebra $\mathcal{B}_t(\mathbf{T})$, i.e. $P^2 = P$. Every function $a \in PC(\mathbf{T})$ may be written in the form

$$a(\tau) = a(t+0)\chi_t(\tau) + a(t-0)(1-\chi_t(\tau)) + \varphi(\tau)$$

where χ_t is the characteristic function of some subarc of $\mathbf{T}$ starting at t and φ is continuous in a neighborhood of t and vanishes at t. Hence,

$$\begin{aligned} \pi_t(aI) &= a(t+0)\pi_t(\chi_t I) + a(t-0)(\pi_t(I) - \pi_t(\chi_t I)) \\ &= a(t+0)p + a(t-0)(e-p), \end{aligned}$$

where $e := \pi_t(I)$ and $p := \pi_t(\chi_t I)$. The element p is also an idempotent. Thus, the smallest closed subalgebra $\mathcal{A}_t(\mathbf{T})$ of $\mathcal{B}_t(\mathbf{T})$ containing the set $\{\pi_t(A) : A \in \operatorname{alg}(P_{\mathbf{T}}, PC(\mathbf{T}))\}$ is generated by the identity and two idempotents ("projections"). Douglas then employed a two projections theorem, which gave him a homomorphism

$$\operatorname{Sym}_t^D : \mathcal{A}_t(\mathbf{T}) \to C^{2\times 2}(\mathcal{L}_t)$$

such that e, p, P are sent to the matrix functions

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix},$$

respectively, and such that $a \in \mathcal{A}_t(\mathbf{T})$ is invertible in $\mathcal{B}_t(\mathbf{T})$ if and only if $\operatorname{Sym}_t^D(a)$ is invertible in $C^{2\times 2}(\mathcal{L}_t)$. The set $\mathcal{L}_t$ was the spectrum of $pPp + (e-p)(e-P)(e-p)$ and thus, in principle, the spectrum of $\pi_t(P_\eta^+)$, where η is the support of χ_t. In the case at hand, $\mathcal{L}_t = [0,1]$. Thus, in the end Douglas got exactly the matrices (5.6) and (5.7) for $(\operatorname{Sym}_t^D \pi_t(aI))(x)$ and $(\operatorname{Sym}_t^D \pi_t(P_{\mathbf{T}}))(x)$.

The local algebras $\mathcal{A}_t(\mathbf{T})$ introduced by Douglas were C^*–algebras, and the two projections theorem one had at that time worked just for C^*–algebras. Ten years later, in 1988, Roch and Silbermann [51] proved a two projections theorem for Banach algebras and derived the Gohberg/Krupnik symbol calculus on Lyapunov Jordan curves by Douglas' elegant method.

If Γ is a composed Lyapunov curve, then the analogues $\mathcal{A}_t(\Gamma)$ of the local algebras $\mathcal{A}_t(\mathbf{T})$ are in general not generated by two idempotents. These algebras were studied by Duduchava, Krupnik, Nyaga, Costabel, Plamenevski, Senichkin, and others. Roch and Silbermann [52] (also see [36]) used techniques of locally straightening the curve Γ at t to a star comprised of $N(t)$ rays and then employed Mellin convolutions techniques to obtain a theorem like Theorem 4.2. In [11] it was shown that the latter approach is also applicable to curves that may locally "straightened" to a star of $N(t)$ logarithmic spirals. Notice in this connection that besides the unit circle $\mathbf{T}$, the real half-line $G_0 := \{r > 0\}$ and the logarithmic spirals $G_\delta := \{r^{1+i\delta} : r > 0\}$ $(\delta \neq 0)$ are the only non-trivial Lie subgroups of the multiplication group $\mathbf{C} \setminus \{0\}$. The methods of [52] and [11] nicely work for power weights and have only recently been extended by Rabinovich and two of the authors [12], [13] to large classes of general Muckenhoupt weights.

Immediately after the paper [58], which showed that circular arcs may metamorphose into horns, people started investigating algebras of singular integral operators over composed Lyapunov curves with general Muckenhoupt weights. However, this time one encountered a new obstacle. Suppose Γ is a Lyapunov Jordan curve and $A \in \text{alg}\,(S_\Gamma, PC(\Gamma))$. The two projections theorem tells us that the spectrum of $\pi_t(A)$ in the local algebra $\mathcal{A}_t(\Gamma)$ is a horn. But to get $\text{sp}_{\text{ess}}\,(A)$, we need the spectrum of $\pi_t(A)$ in the local algebra $\mathcal{B}_t(\Gamma) \supset \mathcal{A}_t(\Gamma)$. If $\mathcal{B}_t(\Gamma)$ and $\mathcal{A}_t(\Gamma)$ are C^*–algebras (which was the case in Douglas' situation) or if the horn degenerates to a circular arc, then the spectra of $\pi_t(A)$ in $\mathcal{A}_t(\Gamma)$ and $\mathcal{B}_t(\Gamma)$ coincide. Since in general a horn is a heavy set, i.e. a set that may arise from another set by "filling in holes", it is not clear whether the spectrum of $\pi_t(A)$ in $\mathcal{B}_t(\Gamma)$ is a horn if this spectrum is a horn in $\mathcal{A}_t(\Gamma)$. These obstacles were overcome by Finck, Roch, Silbermann [26] and Gohberg, Krupnik [34], who elaborated the appropriate techniques for studying the inverse closedness of $\mathcal{A}_t(\Gamma)$ in $\mathcal{B}_t(\Gamma)$ and thus for constructing a symbol calculus for $\text{alg}\,(S_\Gamma, PC(\Gamma))$ on $L^p(\Gamma, w)$ in case Γ is a Lyapunov Jordan curve and w is a weight in $A_p(\Gamma)$. This symbol calculus was extended to Carleson Jordan curves in [8]. The case of a composed Lyapunov curve Γ and a weight $w \in A_p(\Gamma)$ was finally disposed of in [35] by having recourse to the original approach of [30] and [32].

The difference between the methods of [26] and [35] motivated the search for an N projections theorem. Such a theorem was found in [50], and we will cite and make essential use of this theorem in Section 7. The paper [50] contains results close to Theorem 4.2. Nevertheless, Theorem 4.2 cannot be straightforwardly deduced from [50]; it still requires some results on the extension of Carleson curves and Muckenhoupt weights and on the behavior of leaves under such extensions. In a sense, almost all that will follow in the rest of this paper is devoted to proving and subsequently employing such results.

6. Extending Carleson stars and Muckenhoupt weights

In Section 4, we introduced the notion of a Carleson star $\gamma = \eta_1 \cup \ldots \cup \eta_N$ $(N \geq 2)$ centered at a point t. We now also regard a simple Carleson arc $\gamma = \eta$ as a Carleson star whose center is one of the endpoints of η.

A composed curve F is said to be a *Carleson flower* if it is a Carleson Jordan curve or if it is the union of a finite number $N \geq 2$ of Carleson Jordan curves with exactly one common point. Thus, a Carleson flower F may be written as $F = \Gamma_1 \cup \ldots \cup \Gamma_N$ where $\Gamma_1, \ldots, \Gamma_N$ are Carleson Jordan curves and $\Gamma_i \cap \Gamma_j = \{t\}$ for $i \neq j$. If $N \geq 2$, then the common point t is referred to as the *center* of the flower. Each point of a Jordan curve may be its center.

We always think of Jordan curves as being oriented counter-clockwise. Accordingly, a flower F consists of a finite number of counter-clockwise oriented Jordan curves. A star γ is comprised of a finite number of oriented simple arcs. When writing $\gamma \subset F$ we always mean that the simple arcs of γ have the orientation inherited from F.

The behavior of the Cauchy singular integral operator on Carleson flowers is much better than on Carleson stars. The point is that if F is a Carleson flower, then $S_F^2 = I$ and hence $P_F := (I + S_F)/2$ is a projection. This is not true for the Cauchy singular integral operator on a Carleson star. However, if γ is a Carleson star and $F \supset \gamma$ is a Carleson flower, then many properties of S_γ are equivalent to the corresponding properties of the operator $\chi_\gamma S_F \chi_\gamma I + \chi_{F\setminus\gamma} I$, which belongs to alg $(S_F, PC(F))$. We are thus led to the problem of extending Carleson stars γ to Carleson flowers F and of extending weights $w \in A_p(\gamma)$ to weights $W \in A_p(F)$. The first of these two problems may be quickly disposed of by having recourse to the following result.

Theorem 6.1. *Suppose Ω is a simply connected domain in $\mathbf{C}\cup\{\infty\}$ and its boundary $\partial\Omega$ is a Carleson curve. Let $\varphi : \mathbf{D} \to \Omega$ be a Riemann map of the unit disk $\mathbf{D}$ onto Ω. If σ is a Carleson curve in $\mathbf{D}$, then $\varphi(\sigma)$ is a Carleson curve in Ω.*

This result is a special case of Proposition 4 in Zinsmeister's paper [62]. The proof is based on the Hayman-Wu theorem (for which there are now a variety of proofs [37], [24], [27], [44]). Note that in [62] the term "Carleson domain" does not mean a domain bounded by a Carleson curve (although such a domain is an example of one), but something more general.

Theorem 6.2. *If γ is a Carleson star, then there exists a Carleson flower F such that $\gamma \subset F$.*

Proof. Suppose γ is made up of the simple arcs $\eta_1, \ldots, \eta_N$ with the common endpoint t and the other endpoint of η_i being y_i $(i = 1, \ldots, N)$. Put $\Omega := \mathbf{C} \setminus \gamma$ and let $\varphi : \mathbf{D} \to \Omega$ be a Riemann map. Because $\partial\Omega = \gamma$ is rectifiable, it is certainly locally connected, and hence, φ extends continuously to $\mathbf{D} \cup \mathbf{T}$. Each point $y_j \in \{y_1, \ldots, y_N\}$ has exactly one preimage $z_j \in \mathbf{T}$, and the center t has exactly N preimages $t_1, \ldots, t_N$ on the circle $\mathbf{T}$ which separate the points $z_1, \ldots, z_N$. We assume everything is labeled so that as we traverse $\mathbf{T}$ in the clockwise direction, we encounter the points in the order $t_1, z_1, t_2, z_2, \ldots, t_N, z_N$. Put $t_{N+1} := t_1$.

For $j \in \{1, \ldots, N\}$, let σ_j be the hyperbolic geodesic connecting z_j to t_{j+1} if η_j is directed away from t, and let σ_j stand for the hyperbolic geodesic connecting t_j to z_j in case η_j is an incoming arc. Then $\sigma := \sigma_1 \cup \ldots \cup \sigma_N$ is clearly a Carleson curve and we may assume that no point of σ is mapped to ∞ (otherwise replace φ by $\varphi \circ \psi$ with an appropriate Möbius transform $\psi : \mathbf{D} \to \mathbf{D}$). Thus, by Zinsmeister's Theorem 6.1, $\varphi(\sigma) \cup \gamma$ is a Carleson flower containing γ. $\square$

We should note that in the preceding proof, the use of hyperbolic geodesics was completely irrelevant and was only made for concreteness. Any choice of N simple Carleson arcs connecting the desired points without crossing each other would have worked just as well. Moreover, every possible completion of γ to a Carleson flower arises in this way. In fact, for any simply connected domain $\Omega \subset \mathbf{C} \cup \{\infty\}$ whose boundary is a Carleson curve, any Riemann map φ of $\mathbf{D}$ onto Ω, and any Carleson curve Γ, the preimage $\varphi^{-1}(\Gamma \cap \Omega)$ is always a Carleson curve in the disk. If $\Gamma \subset \Omega$, this was established in [25], the general case is in [4].

We now turn to the problem of extending Muckenhoupt weights from Carleson stars to Muckenhoupt weights on Carleson flowers. The following construction also yields a self-contained proof of Theorem 6.2, that is, a proof which does not make use of Theorem 6.1.

The *hyperbolic metric* in the complex unit disk $\mathbf{D}$ and the upper complex half-plane $\mathbf{C}_+ := \{z \in \mathbf{C} : \mathrm{Im}\, z > 0\}$ is given by

$$|dz|_H := \frac{2}{1-|z|^2}|dz| \quad \text{and} \quad |dz|_H := \frac{1}{\mathrm{Im}\, z}|dz|,$$

respectively. In what follows we denote by $|\Gamma|$ and $\mathrm{dist}\,(z, \Gamma)$ the Euclidean length of Γ and the Euclidean distance between z and Γ, while $|\Gamma|_H$ stands for the hyperbolic length of Γ.

Let $\Omega \subset \mathbf{C} \cup \{\infty\}$ be a simply connected domain with boundary $\partial\Omega$ and let $\varphi : \mathbf{D} \to \Omega$ be a Riemann map. *Koebe's distortion theorem* (see, e.g., [46, p. 22]) says that for all $z \in \mathbf{D}$,

$$\frac{1}{4}(1-|z|^2)|\varphi'(z)| \le \mathrm{dist}\,(\varphi(z), \partial\Omega) \le (1-|z|^2)|\varphi'(z)|. \tag{6.1}$$

Furthermore, if $z, z_0 \in \mathbf{D}$ and the hyperbolic distance between z and z_0 is $h(z, z_0)$, then

$$\frac{e^{2h(z,z_0)}-1}{4e^{2h(z,z_0)}} \le \frac{|\varphi(z)-\varphi(z_0)|}{\mathrm{dist}\,(\varphi(z_0), \partial\Omega)} \le e^{2h(z,z_0)} - 1 \tag{6.2}$$

(see, e.g., [27, Lemma 3.1]). In the following we frequently write $A_\tau \simeq B_\tau$ to indicate that the ratio A_τ/B_τ is bounded and bounded away from zero as τ ranges over some set. In particular, (6.1) implies that $|\varphi'(z)|(1-|z|^2) \simeq \mathrm{dist}\,(\varphi(z), \partial\Omega)$.

Suppose now that $\partial\Omega$ is rectifiable and equip $\partial\Omega$ with Lebesgue length measure. Given a measurable subset $E \subset \partial\Omega$ and a point $z \in \Omega$, we denote by $\omega(z, E, \Omega)$ the *harmonic measure of* E *at* z:

$$\omega(z, E, \Omega) := |\varphi_z^{-1}(E)|/(2\pi)$$

where $\varphi_z : \mathbf{D} \to \Omega$ is a Riemann map taking 0 to z. Equivalently, $\omega(z, E, \Omega)$ may be defined as the value at z of the harmonic extension of the characteristic function of E into Ω. Finally, we say that $\{\eta_n\}_{n\in\mathbf{Z}}$ is a *partition of the simple arc* η if each η_n itself is a simple arc, if each pair η_j, η_k $(j \ne k)$ have at most an endpoint in common, and if $\eta = \bigcup_{n\in\mathbf{Z}} \eta_n$.

Lemma 6.3. *Let $\sigma \subset \mathbf{D}$ be a (maximal) hyperbolic geodesic and let $\tau \subset \mathbf{T}$ be anyone of the two arcs of $\mathbf{T}$ between the endpoints of σ. There exist partitions $\{I_n\}_{n\in\mathbf{Z}}$ of σ and $\{J_n\}_{n\in\mathbf{Z}}$ of τ such that if we denote the endpoints of I_n by z_n and z_{n+1}, then*

$$|I_n|_H = 1 \ \textit{ for all } n, \tag{6.3}$$

$$|J_n| \simeq \mathrm{dist}\,(z_n, J_n) \simeq 1 - |z_n|, \tag{6.4}$$

$$\omega(z_n, J_n, \mathbf{D}) > 4/(10\pi). \tag{6.5}$$

Proof. Without loss of generality assume that -1 is an endpoint of σ. Denote by $\mathbf{C}_+$ the upper half-plane and consider the Riemann maps

$$\varphi : \mathbf{D} \to \mathbf{C}_+, \quad z \mapsto i(1-z)/(1+z),$$
$$\psi : \mathbf{C}_+ \to \mathbf{D}, \quad w \mapsto (i-w)/(i+w).$$

Clearly, $\varphi(\sigma)$ is a half-line of the form $\{w \in \mathbf{C}_+ : \operatorname{Re} w = a\}$. For $n \in \mathbf{Z}$, put

$$w_n := a + ie^n, \quad z_n := \psi(w_n),$$

and let $I_n \subset \sigma$ be the arc with the endpoints z_n and z_{n+1}. Further, set

$$v_n := a + \varepsilon e^n, \quad t_n := \psi(v_n),$$

where $\varepsilon = 1$ if ψ maps $\{x \in \mathbf{R} : x > a\}$ to τ and $\varepsilon = -1$ if ψ maps $\{x \in \mathbf{R} : x < a\}$ to τ. Denote by $J_n \subset \mathbf{T}$ the arc whose endpoints are t_n and t_{n+1}.

It is easily verified that the line segment $[w_n, w_{n+1}]$ has hyperbolic length 1. Since hyperbolic length is a conformal invariant, we so obtain (6.3). A straightforward computation gives

$$|J_n| \simeq |t_{n+1} - t_n| \simeq e^{-|n|}, \quad 1 - |z_n| \simeq e^{-|n|}, \quad \operatorname{dist}(z_n, J_n) \simeq |z_n - t_n| \simeq e^{-|n|},$$

which proves (6.4). It is well known that $\pi\omega(z_n, J_n, \mathbf{D})$ coincides with the angle at which the line segment $[v_n, v_{n+1}]$ is seen at w_n. Thus,

$$\omega(z_n, J_n, \mathbf{D}) = \frac{1}{\pi}\Big(\arctan e - \frac{\pi}{4}\Big) > \frac{4}{10\pi}.$$

$\square$

Throughout what follows in this section, let γ be a Carleson star, let $\Omega := \mathbf{C} \setminus \gamma$, and let $\varphi : \mathbf{D} \to \Omega$ be a Riemann map. Choose the hyperbolic geodesics σ_j as in the proof of Theorem 6.2 and suppose that none of the arcs $\varphi(\sigma_j)$ contains the point at infinity.

Lemma 6.4. *Let $\{z_n\}$ and $\{J_n\}$ be as in Lemma 6.3. Then there exist a constant $M < \infty$ and measurable sets $F_n \subset J_n$ such that*

(6.6) $$|F_n| \ge (3/4)|J_n|,$$

(6.7) $$\omega(z_n, F_n, \mathbf{D}) \ge 3/(10\pi),$$

(6.8) $$|\varphi(t) - \varphi(z_n)| \le M|\varphi'(z_n)|(1 - |z_n|^2) \text{ for all } t \in F_n.$$

Proof. Theorem 10.8 on p. 311 of [46] says that there exists a function $K : (0,\infty) \to (0,\infty)$ with the following property: if $h : \mathbf{D} \to \mathbf{C}$ is a univalent (analytic) function,

$h(0) = 0$, $h'(0) = 1$, and if $\varepsilon > 0$ is any number, then there is a measurable subset $E \subset \mathbf{T}$ such that $|E| < \varepsilon$ and

$$\int_0^1 |h'(r\tau)|dr \le K(\varepsilon)$$

for all $\tau \in \mathbf{T} \setminus E$.

Define $\rho_n : \mathbf{D} \to \mathbf{D}$ by $\rho_n(z) := (z + z_n)/(1 + \overline{z_n} z)$. Then ρ_n is a Möbius transform of $\mathbf{D}$ onto itself mapping 0 to z_n. Further, let $\sigma_n : \mathbf{C} \to \mathbf{C}$ be the linear map given by

$$\sigma_n(z) := \Big(z - \varphi(z_n)\Big) \Big/ \Big(\varphi'(z_n)(1 - |z_n|^2\Big).$$

The composition $h_n := \sigma_n \circ \varphi \circ \rho_n$ is univalent on $\mathbf{D}$, $h_n(0) = 0$, and $h_n'(0) = 1$. From what was said in the preceding paragraph we deduce that there are $E_n \subset \mathbf{T}$ such that $|E_n| < \varepsilon$ and

$$|h_n(\tau)| = |h_n(\tau) - h_n(0)| \le \int_0^1 |h_n'(r\tau)|dr \le K(\varepsilon)$$

for all $\tau \in \mathbf{T} \setminus E_n$. Consequently,

$$|\varphi(\rho_n(\tau)) - \varphi(z_n)| = |h_n(\tau)||\varphi'(z_n)|(1 - |z_n|^2) \le K(\varepsilon)|\varphi'(z_n)|(1 - |z_n|^2) \tag{6.9}$$

whenever $\tau \notin E_n$. Put $F_n := J_n \setminus \rho_n(E_n)$. From (6.9) we infer that

$$|\varphi(t) - \varphi(z_n)| \le K(\varepsilon)|\varphi'(z_n)|(1 - |z_n|^2) \;\text{ for }\; t \in F_n. \tag{6.10}$$

We have

$$\omega(z_n, \rho_n(E_n), \mathbf{D}) = \frac{1}{2\pi}|\rho_n^{-1}(\rho_n(E_n))| = \frac{1}{2\pi}|E_n| < \frac{\varepsilon}{2\pi},$$

and since $\omega(z_n, J_n, \mathbf{D}) > 4/(10\pi)$ by (6.5), it follows that $\omega(z_n, F_n, \mathbf{D}) > 3/(10\pi)$ whenever $\varepsilon \le 2/10$. If $t \in J_n$, then $|t - z_n| \le C(1 - |z_n|^2)$ with some C independent of n by virtue of (6.4). Thus, for $t \in J_n$,

$$\Big|(\rho_n^{-1})'(t)\Big| = \frac{1 - |z_n|^2}{|1 - \overline{z_n} t|^2} = \frac{1 - |z_n|^2}{|t - z_n|^2} \ge \frac{1}{C^2}\frac{1}{1 - |z_n|^2},$$

whence

$$\begin{aligned} \varepsilon > |E_n| &= \int_{\rho_n(E_n)} |(\rho_n^{-1})'(t)||dt| \\ &\ge \int_{\rho_n(E_n)\cap J_n} |(\rho_n^{-1})'(t)||dt| \ge \frac{1}{C^2}\frac{1}{1 - |z_n|^2}|\rho_n(E_n) \cap J_n|. \end{aligned}$$

Combining the latter inequality and (6.4), we get

$$|\rho_n(E_n) \cap J_n| \le C^2\varepsilon(1 - |z_n|^2) \le D\varepsilon|J_n|$$

with some constant D independent of ε and n. Thus, $|F_n| \ge (1 - D\varepsilon)|J_n| \ge (3/4)|J_n|$ if only $\varepsilon \le 1/(4D)$. This completes the proof of (6.6) and (6.7). The estimate (6.10) gives (6.8) with $M = K(\varepsilon_0)$ where $\varepsilon_0 \le \min\{2/10, 1/(4D)\}$. $\square$

Given z_n, J_n, F_n $(n \in \mathbf{Z})$, we put

$$A_n := \varphi(F_n), \quad \Gamma_n := \varphi(I_n), \quad x_n := \varphi(z_n), \quad r_n := \operatorname{dist}(x_n, \gamma).$$

We let $D(x,r)$ stand for the disk $\{z \in \mathbf{C} : |z-x| < r\}$. Our next objective is to prove Lemma 6.7. For this purpose, we need two more auxiliary results.

Lemma 6.5. *There exists a function $H : (1,\infty) \to [0,1)$ such that $H(s) \to 0$ as $s \to \infty$ and such that if Ω' is any simply connected open subset of $\mathbf{C}$ with rectifiable boundary, $x \in \Omega'$ and $r = \operatorname{dist}(x, \partial\Omega')$, then for all $s > 1$,*

$$\omega(x, \partial\Omega' \setminus D(x,sr), \Omega') \le H(s)$$

Proof. Let $K : (0,\infty) \to (0,\infty)$ be the function introduced in the beginning of the proof of Lemma 6.4. We may assume that K is monotonically decreasing and that $K(\varepsilon) \to +\infty$ as $\varepsilon \to 0$ (otherwise replace K by $K^*(\varepsilon) := (1/\varepsilon) + \sup\{K(\eta) : \eta \ge \varepsilon\}$). Denote by $\varphi : \mathbf{D} \to \Omega'$ a Riemann map such that $\varphi(0) = x$ and $\varphi'(0) > 0$. By Koebe's distortion theorem, $(1/4)\varphi'(0) \le r \le \varphi'(0)$. Put $f(z) := (\varphi(z) - x)/\varphi'(0)$. Since f is univalent, $f(0) = 0$, $f'(0) = 1$, it follows from the definition of K that there is a measurable subset $E \subset \mathbf{T}$ such that $|E| < \varepsilon$ and

$$|f(t) - f(0)| \le \int_0^1 |f'(rt)|dr \le K(\varepsilon)$$

whenever $t \in \mathbf{T} \setminus E$. Thus,

$$|\varphi(t) - x| \le K(\varepsilon)\varphi'(0) \le 4K(\varepsilon)r \tag{6.11}$$

for all $t \in \mathbf{T} \setminus E$.

If $s > 4K(\varepsilon)$ and $|\varphi(t) - x| \ge sr$, then (6.11) implies that $t \in E$. Consequently,

$$\omega(x, \partial\Omega' \setminus D(x,sr), \Omega') = \frac{1}{2\pi}\left|\varphi^{-1}(\partial\Omega' \setminus D(x,sr))\right| \le \frac{1}{2\pi}|E| < \frac{\varepsilon}{2\pi}. \tag{6.12}$$

For $s > 1$, define $H(s) := \inf\left\{\varepsilon/(2\pi) : 4K(\varepsilon) < s\right\}$. With this definition, we may rewrite (6.12) in the form $\omega(x, \partial\Omega' \setminus D(x,sr), \Omega') \le H(s)$. It remains to prove that $H(s) \to 0$ as $s \to \infty$. The function H is monotonically decreasing because if $s_1 < s_2$ then

$$2\pi H(s_1) = \inf\{\varepsilon : 4K(\varepsilon) < s_1\} \ge \inf\{\varepsilon : 4K(\varepsilon) < s_2\} = 2\pi H(s_2).$$

If H would not go to zero, there would exist a $\delta > 0$ and an $s_0 \in (1,\infty)$ such that

$$\delta \le \inf\{\varepsilon : 4K(\varepsilon) < s\} \text{ for all } s > s_0.$$

However, this is impossible since $\bigcup_{s>s_0}\{\varepsilon : 4K(\varepsilon) < s\} = (0,\infty)$. □

Lemma 6.6. *Let Ω' be a simply connected open subset of $\mathbf{C}$ with rectifiable boundary, let $x \in \Omega'$, and let E be a measurable subset of $\partial\Omega'$. If $r = \operatorname{dist}(x, \partial\Omega')$, then*

$$\omega(x, E, \Omega') \le C\frac{\log(|\partial\Omega'|/r)}{|\log(|E|/r)| + 1} \tag{6.13}$$

with some universal constant $C < \infty$.

The estimate (6.13) is known as *Lavrentiev's inequality.* A proof of Lemma 6.6 (which actually deserves the title "theorem") is in [47, pp. 125-127] and also in [5].

Lemma 6.7. *There are constants $M_1, M_2 < \infty$ such that*

$$A_n \subset \gamma \cap D(x_n, M_1 r_n) \text{ for all } n \in \mathbf{Z}, \tag{6.14}$$

$$|\Gamma_n| \le M_2 |A_n| \text{ for all } n \in \mathbf{Z}, \tag{6.15}$$

$$|A_n| \simeq r_n. \tag{6.16}$$

Proof. From (6.8) and (6.1) we infer that if $t \in F_n$, then

$$|\varphi(t) - \varphi(z_n)| \le M|\varphi'(z_n)|(1 - |z_n|^2) \le 4M \operatorname{dist}(x_n, \gamma) = 4Mr_n,$$

whence $A_n \subset D(x_n, 4Mr_n)$. Thus, we get (6.14) with $M_1 = 4M$.

Since γ is a Carleson curve, we have $|\gamma \cap D(x, r)| \le C_\gamma r$ for all $x \in \gamma$. We claim that $|\gamma \cap D(x, r)| \le 2C_\gamma r$ for all $x \in \mathbf{C}$. This is clear if $\gamma \cap D(x, r) = \emptyset$. If there is a point $t \in \gamma \cap D(x, r)$, then $|\gamma \cap D(x, r)| \le |\gamma \cap D(t, 2r)| \le 2C_\gamma r$, which gives our claim. Using (6.14) we therefore get

$$|A_n| \le |\gamma \cap D(x_n, M_1 r_n)| \le 2M_1 C_\gamma r_n. \tag{6.17}$$

We now show that $|A_n| \ge cr_n$ with some positive constant c independent of n. Let H be the function of Lemma 6.5 and choose $N_0 > 0$ so that $N_0 M > 1$ and $H(N_0 M_1) < 2/(10\pi)$.

Suppose first that r_n is smaller than $(\operatorname{diam} \gamma)/(2N_0 M_1)$. Then the boundary of the disk $D(x_n, N_0 M_1 r_n)$ necessarily hits γ. Hence, the connected component Ω_n' of $\Omega \cap D(x_n, N_0 M_1 r_n)$ containing x_n is a (bounded) simply connected domain. We claim that

$$\omega(x_n, A_n, \Omega_n') \ge \omega(x_n, A_n, \Omega) - \omega(x_n, \Sigma_n, \Omega_n') \tag{6.18}$$

where $\Sigma_n := \partial\Omega_n' \setminus \gamma$. By the maximum principle for harmonic functions, (6.18) will follow as soon as we have shown that

$$\omega^*(\zeta, A_n, \Omega_n') \ge \omega^*(\zeta, A_n, \Omega) - \omega^*(\zeta, \Sigma_n, \Omega_n') \tag{6.19}$$

for almost all $\zeta \in \partial\Omega_n'$, where $\omega^*(\zeta, E, G)$ stands for the boundary value of $\omega(z, E, G)$ at $\zeta \in \partial G$. If $\zeta \in A_n$, then (6.19) reads $1 \ge 1 - \omega^*(\zeta, \Sigma_n, \Omega_n')$, which is certainly true. For $\zeta \in \partial\Omega_n' \cap (\gamma \setminus A_n)$, inequality (6.19) is $0 \ge 0 - \omega^*(\zeta, \Sigma_n, \Omega_n')$, which is also true. Finally, if $\zeta \in \partial\Omega_n' \setminus \gamma$, then (6.19) requires that $0 \ge \omega^*(\zeta, A_n, \Omega) - 1$, which is again satisfied. This proves our claim (6.18).

Since the points of Σ_n have distance at least $N_0 M_1 r_n$ to x_n, we deduce from Lemma 6.5 that

$$\omega(x_n, \Sigma_n, \Omega_n') \le H(N_0 M_1) < 2/(10\pi). \tag{6.20}$$

From (6.7) and the conformal invariance of harmonic measure we see that $\omega(x_n, A_n, \Omega)$ is at least $3/(10\pi)$. This together with (6.18) and (6.20) shows that

$$\omega(x_n, A_n, \Omega_n') \ge 1/(10\pi). \tag{6.21}$$

Since γ is a Carleson curve, we have

$$|\partial\Omega_n'| \le |\Sigma_n| + |\gamma \cap D(x_n, N_0 M_1 r_n)| \le (2\pi + 2C_\gamma)N_0 M_1 r_n. \tag{6.22}$$

Combining (6.21) and (6.22) with Lemma 6.6 we obtain

$$\frac{1}{10\pi} \le C\, \frac{\log\big((2\pi + 2C_\gamma)N_0 M_1\big)}{\big|\log(|A_n|/r_n)\big| + 1},$$

whence

$$\frac{|A_n|}{r_n} \ge e\left((2\pi + 2C_\gamma)N_0 M_1\right)^{-10\pi C} =: c_1. \tag{6.23}$$

Clearly, there are only finitely many n for which $r_n \ge (\operatorname{diam}\gamma)/(2N_0 M_1)$. For these n we have $|A_n|/r_n \ge c_2 > 0$ with some constant c_2 independent of n. This along with (6.23) gives the desired estimate $|A_n| \ge c r_n$ with $c := \min\{c_1, c_2\}$. Taking into account (6.17), we arrive at (6.16).

We finally prove (6.15). If $x \in \Gamma_n$, then $x = \varphi(z)$ with $z \in I_n$. By (6.3), the hyperbolic distance between z and z_n is at most 1. So (6.2) implies that

$$|\varphi(z) - \varphi(z_n)| \le (e^2 - 1)\operatorname{dist}(\varphi(z_n), \gamma) = (e^2 - 1) r_n,$$

and because

$$\operatorname{dist}(\varphi(z), \gamma) \le |\varphi(z) - \varphi(z_n)| + \operatorname{dist}(\varphi(z_n), \gamma) = |\varphi(z) - \varphi(z_n)| + r_n,$$

we arrive at the estimate

$$\operatorname{dist}(\varphi(z), \gamma) \le e^2 r_n \ \text{ for all } \ z \in I_n. \tag{6.24}$$

By virtue of (6.1),

$$|\Gamma_n| = \int_{\Gamma_n} |dw| = \int_{I_n} |\varphi'(z)||dz| \simeq \int_{I_n} \frac{\operatorname{dist}(\varphi(z), \gamma)}{1 - |z|^2}|dz|,$$

and hence, due to (6.24) and (6.3),

$$|\Gamma_n| \le c_0 e^2 r_n \int_{I_n} \frac{2|dz|}{1 - |z|^2} = c_0 e^2 r_n \tag{6.25}$$

with some $c_0 < \infty$ independent of n. Combining (6.16) and (6.25) we get (6.15). $\square$

Now extend γ to a flower as in the proof of Theorem 6.2. If γ is comprised of N simple Carleson arcs, $\gamma = \eta_1 \cup \ldots \cup \eta_N$, we have N subarcs $\tau_1, \ldots, \tau_N \subset \mathbf{T}$ and N hyperbolic geodesics $\sigma_1, \ldots, \sigma_N \subset \mathbf{D}$. For each $j \in \{1, \ldots, N\}$, we construct the sets $I_n^{(j)}, J_n^{(j)}, \Gamma_n^{(j)}, A_n^{(j)}$ as above. Put

$$\{\Gamma_n\}_{n\in\mathbf{Z}} := \bigcup_{m\in\mathbf{Z}} \bigcup_{j=1}^{N} \{\Gamma_m^{(j)}\}, \quad \{A_n\}_{n\in\mathbf{Z}} := \bigcup_{m\in\mathbf{Z}} \bigcup_{j=1}^{N} \{A_m^{(j)}\}$$

and suppose the enumeration is so that Γ_n is associated with A_n: if $\Gamma_n = \Gamma_m^{(j)}$, then $A_n = \varphi(F_m^{(j)})$. Further, for $\Gamma_n = \Gamma_m^{(j)}$ let $x_n := \varphi(z_m^{(j)})$ and $r_n := \operatorname{dist}(x_n, \gamma)$. Thus,

$$\gamma \supset \bigcup_{n \in \mathbf{Z}} A_n, \quad F \setminus \gamma = \bigcup_{n \in \mathbf{Z}} \Gamma_n.$$

For each $n \in \mathbf{Z}$, set

$$w_n := \left(\frac{1}{|A_n|} \int_{A_n} w(t)^p |dt| \right)^{1/p}$$

and then define a weight $\tilde{w}$ on $F \setminus \gamma$ by $\tilde{w}(x) := w_n$ for $x \in \Gamma_n$. Finally, let

$$W := w\chi_\gamma + \tilde{w}\chi_{F\setminus\gamma}.$$

Obviously, W is an extension of w from γ to a weight on F. We are left with showing that $W \in A_p(F)$.

Lemma 6.8. *Let $x \in F \setminus \gamma$ and let $r \le (1/10) \operatorname{dist}(x, \gamma)$. Then at most $2N$ of the arcs Γ_n hit $D(x, r)$ and there are constants $0 < B_1 \le B_2 < \infty$ independent of x and r such that*

(6.26) $\quad B_1 \le w_n/w_k \le B_2$ *whenever* $\Gamma_n \cap D(x,r) \ne \emptyset$ *and* $\Gamma_k \cap D(x,r) \ne \emptyset$.

Proof. We have $x = \varphi(z_0)$ with some z_0 in the hyperbolic geodesic $\sigma \in \{\sigma_1, \ldots, \sigma_N\}$. If z is any point in $\mathbf{D}$ such that $\varphi(z) \in D(x, r)$, then

$$|\varphi(z) - \varphi(z_0)| < r \le (1/10) \operatorname{dist}(\varphi(z_0), \gamma),$$

while (6.2) implies that

$$\frac{|\varphi(z) - \varphi(z_0)|}{\operatorname{dist}(\varphi(z_0), \gamma)} \ge \frac{e^{2h(z,z_0)} - 1}{4e^{2h(z,z_0)}}.$$

Therefore,

$$\frac{e^{2h(z,z_0)} - 1}{4e^{2h(z,z_0)}} \le \frac{1}{10},$$

whence

$$h(z, z_0) \le \frac{1}{2} \log \frac{10}{6} < \frac{1}{2} \log e = \frac{1}{2}.$$

Since the arcs $I_n \subset \sigma_1 \cup \ldots \cup \sigma_N$ all have hyperbolic length 1, it follows that at most $2N$ arcs $\Gamma_n = \varphi(I_n)$ can hit $D(x, r)$.

Now suppose $\Gamma_n \cap D(x, r) \ne \emptyset$. We claim that

$$r_n \simeq \operatorname{dist}(x, \gamma). \tag{6.27}$$

From (6.1) we know that

$$r_n \simeq |\varphi'(z_n)|(1 - |z_n|), \quad \operatorname{dist}(x, \gamma) \simeq |\varphi'(z_0)|(1 - |z_0|). \tag{6.28}$$

Divide the (maximal) geodesics through the points z_n and z_0 into arcs $\tilde{I_m}$ as in the proof of Lemma 6.3. The arc I_n has the endpoints z_n and z_{n+1}, and at least one point of this arc lies in the hyperbolic disk $\{z : h(z, z_0) < 1/2\}$. As $h(z_n, z_{n+1}) = 1$, it follows that $h(z_0, z_n) < 3/2$. This shows that the points z_n and z_0 belong to arcs $\tilde{I_m}$ and $\tilde{I_k}$ with $|m-k| \le 2$. Consequently,

$$1 - |z_n| \simeq e^{-|m|} \simeq e^{-|k|} \simeq 1 - |z_0|$$

(also see the proof of Lemma 6.3). Due to (6.28), the relation (6.27) will follow as soon as we have shown that $|\varphi'(z_n)| \simeq |\varphi'(z_0)|$.

Theorem 9.4 on page 269 of [46] with $h(z) := (\varphi(z) - \varphi(0))/\varphi'(0)$ implies that $\log \varphi'(z)$ is a Bloch function, which means that

$$\left| \frac{\mathrm{d}}{\mathrm{d}z} \log \varphi'(z) \right| \le \frac{2C}{1-|z|^2} \text{ for all } z \in \mathbf{D}$$

with some constant $C < \infty$. It results that

$$\begin{aligned} \Big| \log|\varphi'(z_n)| - \log|\varphi'(z_0)| \Big| &\le \Big| \log\varphi'(z_n) - \log\varphi'(z_0) \Big| \\ = \left| \int_{z_0}^{z_n} \frac{\mathrm{d}}{\mathrm{d}z} \log\varphi'(z) dz \right| &\le C \int_{z_0}^{z_n} \frac{2|dz|}{1-|z|^2} \\ = C \int_{z_0}^{z_n} |dz|_H = C h(z_0, z_n) &< (3/2)C =: C_0, \end{aligned}$$

whence $e^{-C_0} \le |\varphi'(z_n)/\varphi'(z_0)| \le e^{C_0}$, which completes the proof of (6.27).

To prove (6.26), we will have recourse to the so-called A_∞ condition. Two non-negative Borel measures ν and μ on $\mathbf{C}$ are said to be comparable if there are numbers $a, b \in (0,1)$ so that if Δ is a disk and $E \subset \Delta$ is Borel, then

$$\nu(E) < a\nu(\Delta) \Longrightarrow \mu(E) < b\mu(\Delta).$$

Lemma 5 of [17] states that this is an equivalence relation on measures. Moreover, it states that if ν and μ are comparable, then there exist constants $0 < c < \infty$ and $\delta > 0$ independent of E and Δ such that

$$c \left(\frac{\nu(E)}{\nu(\Delta)} \right)^{\delta} \le \frac{\mu(E)}{\mu(\Delta)} \le 1 \tag{6.29}$$

for all Borel subsets $E \subset \Delta$. Define ν and μ by

$$\nu(E) := |\gamma \cap E|, \quad \mu(E) := \int_{\gamma\cap E} w(t)^p |dt|.$$

Since $w \in A_p(\gamma)$, we may deduce from [59, Theorem 18 on p. 9] that ν and μ are comparable. Let A_n be the subset of γ associated with Γ_n. By (6.16) and (6.27), there exist constants $0 < M_3 \le M_4 < \infty$ such that

$$M_3 r_x \le |A_n| \le M_4 r_x, \quad M_3 r_x \le r_n \le M_4 r_x \tag{6.30}$$

where $r_x := \operatorname{dist}(x, \gamma)$. By (6.14),

$$A_n \subset \gamma \cap D(x_n, M_1 r_n) \subset \gamma \cap D(x, |x_n - x| + M_1 r_n),$$

and we have $|x_n - x| < r + |\Gamma_n|$, $r \le r_x/10$. From (6.15) and (6.30) we see that $|\Gamma_n| \le M_2|A_n| \le M_2 M_4 r_x$. Thus,

$$A_n \subset \gamma \cap D(x, M_5 r_x)$$

with $M_5 := 1/10 + M_2M_4 + M_1M_4$. Since

$$\nu(A_n) = |A_n| \ge M_3 r_x \text{ and } \nu(D(x, M_5 r_x)) = |\gamma \cap D(x, M_5 r_x)| \le 2C_\gamma M_5 r_x,$$

we obtain from (6.29) that

$$c\left(\frac{M_3}{2C_\gamma M_5}\right)^\delta \le c\left(\frac{\nu(A_n)}{\nu(D(x, M_5 r_x))}\right)^\delta \le \frac{\mu(A_n)}{\mu(D(x, M_5 r_x))} \le 1.$$

Hence, $\mu(A_n) \simeq \mu(D(x, M_5 r_x))$ for all n such that $\Gamma_n \cap D(x, r) \ne \emptyset$, while $|A_n| \simeq r_x$ for these n by virtue of (6.30). This implies that

$$\mu(A_n) \simeq \mu(A_k), \;\; |A_n| \simeq |A_k| \text{ if } \Gamma_n \cap D(x,r) \ne \emptyset \text{ and } \Gamma_k \cap D(x,r) \ne \emptyset,$$

whence

$$w_n := \left(\frac{\mu(A_n)}{|A_n|}\right)^{1/p} \simeq \left(\frac{\mu(A_k)}{|A_k|}\right)^{1/p} =: w_k.$$

□

Theorem 6.9. *Let γ be a Carleson star, $1 < p < \infty$, and $w \in A_p(\gamma)$. Then there exist a Carleson flower F and a weight $W \in A_p(F)$ such that $\gamma \subset F$ and $W|\gamma = w$.*

Proof. We have already constructed F and W. It remains to show that $W \in A_p(F)$, which automatically implies that F is Carleson.

Suppose first that $x \in F \setminus \gamma$ and that $r \le (1/10) \operatorname{dist}(x, \gamma)$. Then the point $z_0 := \varphi^{-1}(x)$ lies on some geodesic $\sigma_j \in \{\sigma_1, \dots, \sigma_N\}$. In the same way we proved (6.27), one can show that if $\varphi(z) \in D(x, r)$, then

$$|\varphi'(z)|(1 - |z|^2) \simeq \operatorname{dist}(x, \gamma) =: r_x. \tag{6.31}$$

Moreover, for such z we have, by (6.2),

$$\frac{r}{r_x} > \frac{|\varphi(z) - \varphi(z_0)|}{\operatorname{dist}(\varphi(z_0), \gamma)} \ge \frac{e^{2h(z,z_0)} - 1}{4e^{2h(z,z_0)}},$$

whence

$$h(z, z_0) < \frac{1}{2}\log\frac{r_x}{r_x - 4r} = \frac{1}{2}\log\left(1 + \frac{4r}{r_x - 4r}\right) < \frac{1}{2}\frac{4r}{r_x - 4r} < \frac{2r}{(1–4/10)r_x} = \frac{10}{3}\frac{r}{r_x}.$$

Thus, if $\varphi(z) \in D(x,r)$, then z belongs to the hyperbolic disk $D_H := \{z : h(z,z_0) < 10r/(3r_x)\}$. This observation together with (6.31) shows that

$$|\varphi(\sigma) \cap D(x,r)| \le \int_{\sigma\cap D_H} |\varphi'(z)|\,|dz| = \int_{\sigma\cap D_H} |\varphi'(z)|(1-|z|^2)\frac{|dz|}{1-|z|^2}$$
$$\le M_6 r_x \int_{\sigma\cap D_H} |dz|_H \le M_6 r_x \frac{20}{3}\frac{r}{r_x} = \frac{20}{3} M_6 r$$

with some $M_6 < \infty$ independent of z and x. As we have N geodesics, we get

$$|F \cap D(x,r)| \le (20/3) M_6 N r =: M_7 r.$$

By Lemma 6.8, the weight W is piecewise constant on $F \cap D(x,r)$ and the ratio of the different values of the weight on $F \cap D(x,r)$ lies between B_1 and B_2. Thus,

$$\left(\frac{1}{r}\int_{F\cap D(x,r)} W(t)^p |dt|\right)^{1/p} = \left(\frac{1}{r}\sum_n \int_{\Gamma_n \cap D(x,r) \ne \emptyset} w_n^p |dt|\right)^{1/p} \tag{6.32}$$
$$\le \left(\frac{1}{r} B_2^p w_{n_0}^p |F \cap D(x,r)|\right)^{1/p} \le M_7^{1/p} B_2 w_{n_0}$$

and similarly,

$$\left(\frac{1}{r}\int_{F\cap D(x,r)} W(t)^{-q} |dt|\right)^{1/q} \le M_7^{1/q} B_1^{-1} w_{n_0}^{-1}. \tag{6.33}$$

Clearly, the product of (6.32) and (6.33) is uniformly bounded.

Now assume that either $x \in \gamma$ or that $x \in F \setminus \gamma$ and $r > (1/10)\,\mathrm{dist}\,(x,\gamma)$. Obviously, it suffices to consider only the case where $(F \setminus \gamma) \cap D(x,r) \ne \emptyset$. Let $\{\Gamma_n\}$ be the arcs which hit $D(x,r)$ and let $\{A_n\}$ be the corresponding subsets of γ. Pick anyone of the arcs Γ_n. There is a $z \in I_n$ such that $\varphi(z) \in D(x,r)$. If $z \in I_n$ then, by (6.2) and by the definition of z_n, x_n, r_n,

$$\frac{e^{2h(z,z_n)}-1}{4e^2} \le \frac{|\varphi(z)-x_n|}{r_n} \le e^{2h(z,z_n)}-1$$

and

$$\frac{e^{2h(z,z_n)}-1}{4e^2} \le \frac{|\varphi(z)-x_n|}{\mathrm{dist}\,(\varphi(z),\gamma)} \le e^{2h(z,z_n)}-1,$$

whence

$$\frac{r_n}{4e^2} \le \mathrm{dist}\,(\varphi(z),\gamma) \le 4e^2 r_n. \tag{6.34}$$

Since $A_n \subset D(x_n, M_1 r_n)$ due to (6.14) and $|\varphi(z) - x_n| \le |\Gamma_n| \le M_8 r_n$ by virtue of (6.15) and (6.16), it follows that

$$A_n \subset D(\varphi(z), (M_1+M_8)r_n) =: D(\varphi(z), M_9 r_n).$$

Because $|\varphi(z) - x| < r$, we so see that $A_n \subset D(x, r + M_9 r_n)$. From (6.34) and the assumption that $r > (1/10)\,\mathrm{dist}\,(x,\gamma)$ we infer that

$$r_n \le 4e^2 \mathrm{dist}\,(\varphi(z),\gamma) \le 4e^2(r + \mathrm{dist}\,(x,\gamma)) \le 44e^2 r.$$

Consequently, $A_n \subset D(x, M_{10}r)$ with $M_{10} := 1 + 44e^2 M_9$. Put

$$(F \setminus \gamma)(x,r) := (F \setminus \gamma) \cap D(x,r), \quad \gamma(x,r) := \gamma \cap D(x,r).$$

We will prove that

$$\int_{(F\setminus\gamma)(x,r)} \tilde{w}(t)^p |dt| \le M_2 \int_{\gamma(x,M_{10}r)} w(t)^p |dt|, \tag{6.35}$$

$$\int_{(F\setminus\gamma)(x,r)} \tilde{w}(t)^{-q} |dt| \le M_2 \int_{\gamma(x,M_{10}r)} w(t)^{-q} |dt|, \tag{6.36}$$

where M_2 is the constant from (6.15). These two estimates imply that

$$\begin{aligned}
&\left(\frac{1}{r}\int_{F\cap D(x,r)} W(t)^p |dt|\right)^{1/p} \left(\frac{1}{r}\int_{F\cap D(x,r)} W(t)^{-q} |dt|\right)^{1/q} \\
&= \left(\frac{1}{r}\int\limits_{(F\setminus\gamma)(x,r)} \tilde{w}(t)^p |dt| + \frac{1}{r}\int\limits_{\gamma(x,r)} w(t)^p |dt|\right)^{1/p} \\
&\quad \times \left(\frac{1}{r}\int\limits_{(F\setminus\gamma)(x,r)} \tilde{w}(t)^{-q} |dt| + \frac{1}{r}\int\limits_{\gamma(x,r)} w(t)^{-q} |dt|\right)^{1/q} \\
&\le (1+M_2)\left(\frac{1}{r}\int_{\gamma(x,M_{10}r)} w(t)^p |dt|\right)^{1/p} \left(\frac{1}{r}\int_{\gamma(x,M_{10}r)} w(t)^{-q} |dt|\right)^{1/q},
\end{aligned}$$

and this is uniformly bounded since $w \in A_p(\gamma)$.

We have

$$\begin{aligned}
\int_{(F\setminus\gamma)(x,r)} \tilde{w}(t)^p |dt| &\le \sum_n w_n^p |\Gamma_n| = \sum_n \left(\frac{1}{|A_n|}\int_{A_n} w(t)^p |dt|\right) |\Gamma_n| \\
&\le M_2 \sum_n \int_{A_n} w(t)^p |dt| \le M_2 \int_{\gamma(x,M_{10}r)} w(t)^p |dt|,
\end{aligned}$$

which is (6.35). Further, Hölder's inequality with the conjugate exponents $(\alpha+1)/\alpha$ and $\alpha+1$ shows that if m is a probability measure, then

$$1 = \left(\int f^{\frac{\alpha}{\alpha+1}} f^{\frac{-\alpha}{\alpha+1}} \, dm\right)^{\alpha+1} \le \left(\int f \, dm\right)^{\alpha} \left(\int f^{-\alpha} \, dm\right),$$

whence

$$\left(\int f \, dm\right)^{-\alpha} \le \int f^{-\alpha} \, dm$$

for every $\alpha > 0$. Taking into account the latter inequality we get

$$\int_{(F\setminus\gamma)(x,r)} \tilde{w}(t)^{-q} |dt| \le \sum_n w_n^{-q} |\Gamma_n| = \sum_n \left(\frac{1}{|A_n|}\int_{A_n} w(t)^p |dt|\right)^{-q/p} |\Gamma_n|$$

$$\leq \sum_n \left(\frac{1}{|A_n|} \int_{A_n} w(t)^{-q} |dt| \right) |\Gamma_n| \leq M_2 \sum_n \int_{A_n} w(t)^{-q} |dt|$$
$$\leq M_2 \int_{\gamma(x, M_{10} r)} w(t)^{-q} |dt|$$

and this is (6.36). □

7. Local algebras

Let Γ be a composed Carleson curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. Fix $t \in \Gamma$ and define the local algebra $\mathcal{B}_t(\Gamma)$ as in Section 3. Proposition 3.3 implies that $\mathcal{B}_t(\Gamma)$ contains $\pi_t(S_\Gamma)$ and $\pi_t(aI)$ for every $a \in PC(\Gamma)$. Let $\mathcal{A}_t(\Gamma)$ be the smallest closed subalgebra of $\mathcal{B}_t(\Gamma)$ containing the set $\{\pi_t(S_\Gamma)\} \cup \{\pi_t(aI) : a \in PC(\Gamma)\}$. Clearly, if $A \in \text{alg}\,(S_\Gamma, PC(\Gamma))$ then $\pi_t(A) \in \mathcal{A}_t(\Gamma)$.

Suppose $\Gamma_0(t, \varepsilon) = \eta_1 \cup \ldots \cup \eta_{N(t)}$ is as in (4.3). Put $N := N(t)$ and abbreviate the limits (4.4) to a_j $(j = 1, \ldots, N)$. Let $\chi_1, \ldots, \chi_N$ denote the characteristic functions of the arcs $\eta_1, \ldots, \eta_N$. It is easily seen that if φ is a function in $PC(\Gamma)$, if φ is continuous at the point t and $\varphi(t) = 0$, then $\pi(\varphi I) \in \mathcal{J}_t(\Gamma)$. We have

$$a = a_1 \chi_1 + \ldots + a_N \chi_N + \varphi$$

where $\varphi \in PC(\Gamma)$, φ is continuous at t and $\varphi(t) = 0$. It follows that

$$\pi(aI) - \sum_{j=1}^{N} a_j \pi(\chi_j I) = \pi(\varphi I) \in \mathcal{J}_t(\Gamma)$$

and hence,

$$\pi_t(aI) = a_1 \pi_t(\chi_1 I) + \ldots + a_N \pi_t(\chi_N I). \tag{7.1}$$

Because $\pi_t(S_\Gamma) = 2\pi_t(P_\Gamma) - \pi_t(I)$, we see that $\mathcal{A}_t(\Gamma)$ is generated by $P := \pi_t(P_\Gamma)$ and the N elements $p_1 := \pi_t(\chi_1 I), \ldots, p_N := \pi_t(\chi_N I)$. Since obviously $\chi_j^2 = \chi_j$, we have $p_j^2 = p_j$ for all j. Thus, $p_1, \ldots, p_N$ are idempotents. In general, P is not an idempotent. However, it is well known that this happens if $\Gamma = F$ is a Carleson flower. For the reader's convenience, we will give a proof.

Theorem 7.1. *If F is a Carleson flower then $S_F^2 = I$ and hence $P_F^2 = P_F$.*

Proof. Let F be comprised of the Carleson Jordan curves $\Gamma_1, \ldots, \Gamma_N$. Denote by D_j the bounded connected component of $\mathbf{C} \setminus \Gamma_j$, and let D_∞ stand for the unbounded connected component of $\mathbf{C} \setminus F$. For $j = 1, \ldots, N$, let $R_0^-(D_j)$ be the set of all functions in $R(F)$ which have no poles outside D_j and vanish at infinity. Further, let $R^-(D_\infty)$ stand for the functions in $R(F)$ which have no poles outside D_∞.

Every function $r \in R(F)$ is of the form $r = r_1 + \ldots + r_N + r_\infty$ with $r_j \in R_0^-(D_j)$ for $j = 1, \ldots, N$ and $r_\infty \in R^-(D_\infty)$. Denote by χ_j^0 the characteristic function of Γ_j.

As in [33, Chapter 1, Theorem 1.1] one can show that if $j = 1, \ldots, N$, then

$$\chi_k^0 S_F \chi_i^0 r_j = \begin{cases} -\chi_k^0 r_j & \text{if } k = i = j, \\ \chi_k^0 r_j & \text{if } k = i \neq j, \\ 0 & \text{if } k \neq i \neq j, \\ -2\chi_k^0 r_j & \text{if } k \neq i = j, \end{cases} \tag{7.2}$$

while

$$\chi_k^0 S_F \chi_i^0 r_\infty = \begin{cases} \chi_k^0 r_\infty & \text{if } k = i, \\ 0 & \text{if } k \neq i; \end{cases} \tag{7.3}$$

we remark that in these equalities the notation $\chi_k^0 f = \chi_k^0 g$ means that $f(t) = g(t)$ for all $t \in \Gamma_k$ at which Γ_k has a tangent – for rectifiable Γ_k this is the case almost everywhere. For $j = 1, \ldots, N$, it follows from (7.2) that

$$\begin{aligned} S_F r_j &= \sum_{k,i} \chi_k^0 S_F \chi_i^0 r_j = \chi_j^0 S_F \chi_j^0 r_j + \sum_{k=i\neq j} \chi_k^0 S_F \chi_i^0 r_j + \sum_{k\neq i=j} \chi_k^0 S_F \chi_i^0 r_j \\ &= -\chi_j^0 r_j - 2\sum_{k\neq j} \chi_k^0 r_j + \sum_{k\neq j} \chi_k^0 r_j = -r_j. \end{aligned} \tag{7.4}$$

Thus, $S_F^2 r_j = r_j$ for $j = 1, \ldots, N$. Similarly, using (7.3) one can show that

$$S_F r_\infty = r_\infty, \tag{7.5}$$

whence $S_F^2 r_\infty = r_\infty$. Consequently, $S_F r = r$ for all $r \in R(F)$. Theorems 2.1 and 2.2 finally give $S_F^2 f = f$ for all $f \in L^p(F, w)$. $\square$

The following theorem is our main tool for studying invertibility in the local algebras $\mathcal{A}_t(F)$ and $\mathcal{B}_t(F)$ at the center of a Carleson flower.

Theorem 7.2. *Let $\mathcal{B}$ be a (complex) Banach algebra with identity e. Let $p_1, p_2, \ldots, p_{2N}$ and P be nonzero elements of $\mathcal{B}$ satisfying*

$$p_i p_j = \delta_{ij} p_i \ \textit{for all } i, j \ \textit{ and } p_1 + p_2 + \ldots + p_{2N} = e, \tag{7.6}$$

where δ_{ij} is the Kronecker delta, and

$$Q(p_{2j-1} + p_{2j})P = P(p_{2j} + p_{2j+1})Q = 0 \ \textit{ for all } j = 1, \ldots, N, \tag{7.7}$$

where $Q := e - P$ and $p_{2N+1} := p_1$. Denote by $\mathcal{A}$ the smallest closed subalgebra of $\mathcal{B}$ containing P and $p_1, p_2, \ldots, p_{2N}$. Put

$$X := \sum_{j=1}^{N} (p_{2j-1} P p_{2j-1} + p_{2j} Q p_{2j}) \tag{7.8}$$

and denote by $\mathrm{sp}_{\mathcal{B}}(X)$ the spectrum of X in $\mathcal{B}$. Suppose finally that

$$\textit{0 and 1 are cluster points of } \mathrm{sp}_{\mathcal{B}}(X). \tag{7.9}$$

Put $E := \operatorname{diag}(1,-1,1,-1,\ldots,1,-1)$*. For* $x \in \mathbf{C}$*, define the map*

$$\sigma_x : \{P, p_1, p_2, \ldots, p_{2N}\} \to \mathbf{C}^{2N\times 2N} \tag{7.10}$$

as follows. For $x \notin \{0,1\}$*, let*

$$\sigma_x(P) = E \begin{pmatrix} x & x-1 & x-1 & x-1 & \ldots & x-1 & x-1 \\ x & x-1 & x-1 & x-1 & \ldots & x-1 & x-1 \\ x & x & x & x-1 & \ldots & x-1 & x-1 \\ x & x & x & x-1 & \ldots & x-1 & x-1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x & x & x & \ldots & x & x-1 \\ x & x & x & x & \ldots & x & x-1 \end{pmatrix}, \tag{7.11}$$

for $x \in \{0,1\}$*, let*

$$\sigma_x(P) = \operatorname{diag}(x,\ 1-x,\ x,\ 1-x,\ \ldots,\ x,\ 1-x), \tag{7.12}$$

and for all x*, let*

$$\sigma_x(p_j) = \operatorname{diag}(0,\ldots,0,1,0,\ldots,0) \ \text{ with } 1 \text{ at the } j\text{th position} \tag{7.13}$$

(a) *For each* $x \in \operatorname{sp}_{\mathcal{B}}(X)$*, the map* σ_x *extends to a Banach algebra homomorphism* σ_x *of* $\mathcal{A}$ *into* $\mathbf{C}^{2N\times 2N}$*.*

(b) *An element* $a \in \mathcal{A}$ *is invertible in* $\mathcal{B}$ *if and only if* $\sigma_x(a)$ *is invertible in* $\mathbf{C}^{2N\times 2N}$ *for every* $x \in \operatorname{sp}_{\mathcal{B}}(X)$*.*

This theorem is a special case of the union of Theorem 9 and Proposition 9(d) of the paper [50]. □

Adding the second equalities in (7.7) and taking into account the second equality in (7.6) we get

$$0 = \sum_{j=1}^{N} P(p_{2j} + p_{2j+1})Q = P\biggl(\sum_{i=1}^{2N} p_i\biggr)Q = PQ,$$

whence $0 = P(e-P) = P - P^2$. Thus, if the hypotheses of Theorem 7.2 are satisfied, then P must necessarily be an idempotent.

We will apply Theorem 7.2 to the case where $\mathcal{B} := \mathcal{B}_t(F)$ and $\mathcal{A} := \mathcal{A}_t(F)$ are the local algebras at the center t of a Carleson flower F. Let $F = \Gamma_1 \cup \ldots \cup \Gamma_N$ with Carleson Jordan curves $\Gamma_1, \ldots, \Gamma_N$. For sufficiently small $\varepsilon > 0$, the connected component $F_0(t,\varepsilon)$ of $F(t,\varepsilon)$ containing t may be written as

$$F_0(t,\varepsilon) = \bigcup_{j=1}^{N} (\eta^*_{2j-1} \cup \eta^*_{2j})$$

where $\eta^*_{2j-1} \subset \Gamma_j$ and $\eta^*_{2j} \subset \Gamma_j$ are outgoing and incoming arcs, respectively. Let χ^*_i be the characteristic function of η^*_i $(i = 1, \ldots, 2N)$ and put

$$p_i := \pi_t(\chi^*_i I), \quad P := \pi_t(P_F), \quad Q := \pi_t(I - P_F). \tag{7.14}$$

Conditions (7.6) are obviously satisfied.

Lemma 7.3. *If F is a Carleson flower and p_i, P, Q are defined by* (7.14) *then* (7.7) *holds.*

We remark that for Lyapunov flowers (carrying a power weight) such a result is already in [30] and that just this result motivated the "axiom" (7.7) (see [50]).

Proof. We have $\chi^*_{2j-1} + \chi^*_{2j} = \chi^0_j + c_j$ where χ^0_j is the characteristic function of Γ_j and $c_j \in PC(F)$ vanishes on $F_0(t,\varepsilon)$. Hence, $\pi(c_j I) \in \mathcal{J}_t(F)$, $\pi_t(c_j I) = 0$, and

$$Q(p_{2j-1} + p_{2j})P = \pi_t((I - P_F)(\chi^0_j + c_j)P_F) = \pi_t((I - P_F)\chi^0_j P_F). \tag{7.15}$$

We claim that

$$T := (I - P_F)\chi^0_j P_F = 0. \tag{7.16}$$

Let $r_i \in R^-_0(D_i)$ be as in the proof of Theorem 7.1. Then $P_F r_i = (1/2)(r_i + S_F r_i) = 0$ due to (7.4) and hence, $Tr_i = 0$ for $i = 1, 2, \ldots, N$. On the other hand, if $r_\infty \in R^-(D_\infty)$, then (7.5) and (7.3) imply that

$$Tr_\infty = (I - P_F)\chi^0_j P_F r_\infty = (I - P_F)\chi^0_j r_\infty = \chi^0_j r_\infty - \chi^0_j r_\infty = 0.$$

This completes the proof of (7.16) and thus shows that (7.15) vanishes, as desired.

We now prove that $P(p_{2j} + p_{2j+1})Q = 0$ for every $j = 1, \ldots, N$. Denote by $\tilde{\eta}^*_i$ ($i = 1, \ldots, 2N$) the arc which results from η^*_i by changing the orientation and consider the Carleson star

$$\tilde{F}_0(t,\varepsilon) := \bigcup_{j=1}^{N} (\tilde{\eta}^*_{2j} \cup \tilde{\eta}^*_{2j+1}).$$

By virtue of Theorem 5.2, we may extend $\tilde{F}_0(t,\varepsilon)$ to a Carleson flower $\tilde{F}$ and $w|\tilde{F}_0(t,\varepsilon)$ to a weight $W \in A_p(\tilde{F})$ so that $\tilde{F} = \tilde{\Gamma}_1 \cup \ldots \cup \tilde{\Gamma}_N$ with Carleson Jordan curves $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_N$ and so that $\tilde{\eta}^*_{2j}$ and $\tilde{\eta}^*_{2j+1}$ are the outgoing and incoming arcs of $\tilde{\Gamma}_j$, respectively. From what was already proved we know that

$$\pi_t((I - P_{\tilde{F}}))\pi_t(\chi^*_{2j} I + \chi^*_{2j+1} I)\pi_t(P_{\tilde{F}}) = 0. \tag{7.17}$$

Due to the change of orientation we have

$$\pi_t(P_{\tilde{F}}) = \pi_t((I + S_{\tilde{F}})/2) = \pi_t((I - S_F)/2) = \pi_t(I - P_F).$$

Consequently, (7.17) implies that

$$0 = \pi_t(P_F)\pi_t(\chi^*_{2j} I + \chi^*_{2j+1} I)\pi_t(I - P_F) = P(p_{2j} + p_{2j+1})Q.$$

□

In summary, conditions (7.6) and (7.7) are satisfied if $\mathcal{B} = \mathcal{B}_t(F)$ is the local algebra at the center of a flower F and $p_1, p_2, \ldots, p_{2N}, P$ are given by (7.14). Our next concern is to say something about the set

$$\mathcal{X}_t(F) := \mathrm{sp}_{\mathcal{B}}(X) \tag{7.18}$$

where X is the element (7.8).

8. Local spectra

In Section 4, we constructed the indicator leaf $\mathcal{L}^0(p;\alpha_t,\beta_t)$ for each point t of a Carleson Jordan curve.

Theorem 8.1. *If Γ is a Carleson Jordan curve, $1<p<\infty$, and $w\in A_p(\Gamma)$, then for every $t\in\Gamma$,*

$$\mathcal{X}_t(\Gamma)=\mathcal{L}^0(p;\alpha_t,\beta_t)$$

where α_t,β_t are the indicator functions of Γ,w at t.

Proof. If $\varepsilon>0$ is sufficiently small, then $\Gamma_0(t,\varepsilon)=\eta_1^*\cup\eta_2^*$ with an outgoing arc η_1^* and an incoming arc η_2^*. The element (7.8) now is

$$X=p_1Pp_1+p_2Qp_2$$

In [8] it is shown that for each $x\in\mathcal{L}^0(p;\alpha_t,\beta_t)$ the map

$$\mathrm{Sym}\,^0_{t,x}:=\{P,p_1,p_2\}\to\mathbf{C}^{2\times 2}$$

sending P,p_1,p_2 to

$$\begin{pmatrix} x & \sqrt{x(1-x)}\\ \sqrt{x(1-x)} & 1-x\end{pmatrix},\ \begin{pmatrix}1&0\\0&0\end{pmatrix},\ \begin{pmatrix}0&0\\0&1\end{pmatrix},$$

respectively, extends to a Banach algebra homomorphism of $\mathcal{A}_t(\Gamma)$ to $\mathbf{C}^{2\times2}$ and that an element $a\in\mathcal{A}_t(\Gamma)$ is invertible in $\mathcal{B}_t(\Gamma)$ if and only if $\mathrm{Sym}\,^0_{t,x}(a)$ is invertible for all $x\in\mathcal{L}^0(p;\alpha_t,\beta_t)$. An easy computation yields

$$\mathrm{Sym}\,^0_{t,x}(X-\lambda e)=\begin{pmatrix} x-\lambda & 0\\ 0 & x-\lambda\end{pmatrix}. \tag{8.1}$$

Since $\mathcal{X}_t(\Gamma)$ is the set of all $\lambda\in\mathbf{C}$ for which the determinant (8.1) vanishes for some $x\in\mathcal{L}^0(p;\alpha_t;\beta_t)$, it follows that $\mathcal{X}_t(\Gamma)=\mathcal{L}^0(p;\alpha_t,\beta_t)$. □

Lemma 8.2. *Let Γ be a composed Carleson curve, $1<p<\infty$, and $w\in A_p(\Gamma)$. Let further $t\in\Gamma$ and $A\in\Lambda(\Gamma)$. Then $\pi_t(A)=0$ if and only if there exist operators $K_n\in\mathcal{K}(\Gamma)$, $E_n\in\Lambda(\Gamma)$, and functions $c_n\in C(\Gamma)$ such that $c_n(t)=0$ and*

$$\|A-K_n-c_nE_n\|_{\mathcal{B}(L^p(\Gamma,w))}\to 0 \ \textit{ as } \ n\to\infty.$$

Proof. The ideal $\mathcal{J}_t(\Gamma)$ is the closure in $\Lambda(\Gamma)/\mathcal{K}(\Gamma)$ of all finite sums

$$\sum_{j=1}^{n}\pi(c_jE_j)\ \text{ with }\ c_j\in C(\Gamma),\ c_j(t)=0,\ E_j\in\Lambda(\Gamma). \tag{8.2}$$

We claim that every sum (8.2) is actually of the form $\pi(cE)$ with $c\in C(\Gamma)$, $c(t)=0$, $E\in\Lambda(\Gamma)$. Obviously, it suffices to prove this for $n=2$.

So let $c_1, c_2 \in C(\Gamma)$, $c_1(t) = c_2(t) = 0$. Put $c := \sqrt{|c_1| + |c_2|}$. Then $c \in C(\Gamma)$ and $c(t) = 0$. Let $N_c := \{\tau \in \Gamma : c(\tau) = 0\}$ and for $j \in \{1, 2\}$, define

$$g_j(\tau) = \begin{cases} c_j(\tau)/c(\tau) & \text{if } \tau \notin N_c, \\ 0 & \text{if } \tau \in N_c. \end{cases}$$

If $\tau \notin N_c$, then

$$|g_j(\tau)| = \frac{|c_j(\tau)|}{c(\tau)} = \frac{|c_j(\tau)|}{|c_1(\tau)| + |c_2(\tau)|} c(\tau) \le c(\tau)$$

and hence, $g_j \in C(\Gamma)$. Since $c_1 = cg_1$ and $c_2 = cg_2$, it follows that $c_1 E_1 + c_2 E_2 = c(g_1 E_1 + g_2 E_2)$, which proves our claim.

We have $\pi_t(A) = 0$ if and only if $\pi(A) \in \mathcal{J}_t(\Gamma)$. From what was proved above, we infer that $\pi(A) \in \mathcal{J}_t(\Gamma)$ if and only if there are $c_n \in C(\Gamma)$ and $E_n \in \Lambda(\Gamma)$ such that $c_n(t) = 0$ and $\|\pi(A - c_n E_n)\| \to 0$ as $n \to 0$. Because $\|\pi(B)\| = \inf\{\|B + K\| : K \in \mathcal{K}(\Gamma)\}$, we arrive at the assertion. □

In what follows we have frequently to change curves and thus local algebras. This requires some caution, and we will therefore occasionally denote the homomorphism $\pi_t : \Lambda(\Gamma) \to \mathcal{B}_t(\Gamma)$ by π_t^Γ.

Lemma 8.3. *Let $\Gamma_1, \dots, \Gamma_N$ be Carleson Jordan curves and let $F = \Gamma_1 \cup \dots \cup \Gamma_N$ be a Carleson flower. Denote the characteristic function of Γ_j by χ_j^0 and let t be the center of F. Let further $A_j \in \mathcal{B}(\Gamma_j) := \mathcal{B}(L^p(\Gamma_j, W|\Gamma_j))$ for $j = 1, \dots, N$ and define $A \in \mathcal{B}(F) := \mathcal{B}(L^p(F, W))$ by*

$$A = \sum_{j=1}^{n} \chi_j^0 A_j \chi_j^0 I.$$

Then the following hold.

(a) $A \in \mathcal{K}(F) \Longleftrightarrow A_j \in \mathcal{K}(\Gamma_j)$ *for all* j.

(b) $A \in \Lambda(F) \Longleftrightarrow A_j \in \Lambda(\Gamma_j)$ *for all* j.

(c) $\operatorname{sp} \pi_t^F(A) = \bigcup_{j=1}^{N} \operatorname{sp} \pi_t^{\Gamma_j}(A_j)$.

Remark. In this lemma and in several similar situations that will follow, we identify the operator $A_j \in \mathcal{B}(\Gamma_j)$ with $A_j \oplus 0 \in \mathcal{B}(F)$, i.e. with its extension by zero to all of the flower F.

P r o o f. (a) Obvious.

(b) Let $A \in \Lambda(F)$. If $\psi \in C(\Gamma_j)$, then $\psi = \chi_j^0 \varphi$ with some $\varphi \in C(F)$ and hence

$$\begin{aligned} A_j \psi I - \psi A_j &= \chi_j^0 (A \chi_j^0 \varphi I - \chi_j^0 \varphi A) \chi_j^0 I \\ &= \chi_j^0 (\varphi A \chi_j^0 I + K - \chi_j^0 \varphi A) \chi_j^0 I = \chi_j^0 K \chi_j^0 \end{aligned}$$

with some $K \in \mathcal{K}(F)$. So $A_j \in \Lambda(\Gamma_j)$ by virtue of (a). Conversely, suppose $A_j \in \Lambda(\Gamma_j)$ for all j. If $\varphi \in C(F)$, then $\varphi|\Gamma_j \in C(\Gamma_j)$ and thus

$$A\varphi I - \varphi A = \sum_{j=1}^{N} \chi_j^0 (A_j \varphi I - \varphi A_j) \chi_j^0 I \in \Lambda(F)$$

again due to (a).

(c) Suppose $\pi_t^F(A - \lambda I)$ is invertible. Then there is a $B \in \Lambda(F)$ such that $\pi_t^F(B(A - \lambda I) - I) = 0$. From Lemma 8.2 we obtain $K_n \in \mathcal{K}(F)$, $E_n \in \Lambda(F)$, $c_n \in C(F)$ such that $c_n(t) = 0$ and

$$\|B(A - \lambda I) - I - K_n - c_n E_n\|_{\mathcal{B}(F)} \to 0.$$

It follows that

$$\|\chi_j^0 B(A - \lambda I)\chi_j^0 I - \chi_j^0 I - \chi_j^0 K_n \chi_j^0 I - c_n \chi_j^0 E_n \chi_j^0 I\|_{\mathcal{B}(\Gamma_j)} \to 0.$$

We have $\chi_j^0 K_n \chi_j^0 I \in \mathcal{K}(\Gamma_j)$ and $\chi_j^0 E_n \chi_j^0 I \in \Lambda(\Gamma_j)$ by (a) and (b). Further,

$$\chi_j^0 B(A - \lambda I)\chi_j^0 I = \chi_j^0 B \sum_{k=1}^{N} \chi_k^0 (A_k - \lambda I)\chi_k^0 \chi_j^0 I = \chi_j^0 B \chi_j^0 (A_j - \lambda I)\chi_j^0 I.$$

Thus, we deduce from Lemma 8.2 that

$$\pi_t^{\Gamma_j}(\chi_j^0 B \chi_j^0 I)\pi_t^{\Gamma_j}(A_j - \lambda I) = \pi_t^{\Gamma_j}(I).$$

Analogously one can show that $\pi_t^{\Gamma_j}(A_j - \lambda I)$ is invertible from the right.

Conversely, assume $\pi_t^{\Gamma_j}(A_j - \lambda I)$ is invertible for all j. Then there are $B_j \in \Lambda(\Gamma_j)$ such that $\pi_t^{\Gamma_j}(B_j(A_j - \lambda I) - I) = 0$ and from Lemma 8.2 we get $K_n^{(j)} \in \mathcal{K}(\Gamma_j)$, $E_n^{(j)} \in \Lambda(\Gamma_j)$, $c_n^{(j)} \in C(\Gamma_j)$ such that $c_n^{(j)}(t) = 0$ and

$$\|B_j(A_j - \lambda I) - I - K_n^{(j)} - c_n^{(j)} E_n^{(j)}\|_{\mathcal{B}(\Gamma_j)} \to 0.$$

Put $c_n := \sum\limits_{j=1}^{N} c_n^{(j)} \chi_j^0$ and

$$B := \sum_{j=1}^{N} \chi_j^0 B_j \chi_j^0 I, \quad K_n := \sum_{j=1}^{N} \chi_j^0 K_n^{(j)} \chi_j^0 I, \quad E_n := \sum_{j=1}^{N} \chi_j^0 E_n^{(j)} \chi_j^0 I.$$

Clearly, $c_n \in C(F)$ and $c_n(t) = 0$. From (a) and (b) we deduce that $K_n \in \mathcal{K}(F)$ and $E_n \in \Lambda(F)$. Since

$$\|\sum_{j=1}^{N} \chi_j^0 B_j \chi_j^0 (A_j - \lambda I)\chi_j^0 - I - K_n - c_n E_n\|_{\mathcal{B}(F)} \to 0$$

and, as above,

$$\sum_{j=1}^{N}\chi_j^0 B_j\chi_j^0(A_j-\lambda I)\chi_j^0 I = B(A-\lambda I),$$

Lemma 8.2 implies that $\pi_t^F(B)\pi_t^F(A-\lambda I)=\pi_t^F(I)$. $\square$

Theorem 8.4. *Let $\Gamma_1,\ldots,\Gamma_N$ be Carleson Jordan curves and let $F=\Gamma_1\cup\ldots\cup\Gamma_N$ be a Carleson flower with the center t. Suppose $1<p<\infty$ and $W\in A_p(F)$. Then*

$$\mathcal{X}_t(F)=\mathcal{L}^0(p;\alpha_t^1,\beta_t^1)\cup\ldots\cup\mathcal{L}^0(p;\alpha_t^N,\beta_t^N) \tag{8.3}$$

where α_t^j,β_t^j are the indicator functions of $\Gamma_j, W|\Gamma_j$ at t.

Proof. By (7.18), (7.8), (7.14),

$$\begin{aligned}\mathcal{X}_t(F) &= \operatorname{sp}\pi_t^F\left(\sum_{j=1}^{N}\left(\chi_{2j-1}^*P_F\chi_{2j-1}^*I+\chi_{2j}^*Q_F\chi_{2j}^*I\right)\right)\\ &= \operatorname{sp}\pi_t^F\left(\sum_{j=1}^{N}\chi_j^0\left(\chi_{2j-1}^*P_{\Gamma_j}\chi_{2j-1}^*I+\chi_{2j}^*Q_{\Gamma_j}\chi_{2j}^*I\right)\chi_j^0 I\right).\end{aligned}$$

Lemma 8.3(c) therefore gives

$$\mathcal{X}_t(F)=\bigcup_{j=1}^{N}\operatorname{sp}\pi_t^{\Gamma_j}(\chi_{2j-1}^*P_{\Gamma_j}\chi_{2j-1}^*I+\chi_{2j}^*Q_{\Gamma_j}\chi_{2j}^*I)=\bigcup_{j=1}^{N}\mathcal{X}_t(\Gamma_j),$$

and Theorem 8.1 then implies (8.3). $\square$

We will show that actually $\mathcal{L}^0(p;\alpha_t^1,\beta_t^1)=\ldots=\mathcal{L}^0(p;\alpha_t^N,\beta_t^N)$. At the present moment we only remark that, by (4.13) and (4.15), each leaf $\mathcal{L}^0(p;\alpha_t^j,\beta_t^j)$ is a connected set containing the points 0 and 1, so that (7.9) is satisfied in view of (8.3).

9. Symbol calculus

Let Γ be a composed Carleson curve, $1<p<\infty$, and $w\in A_p(\Gamma)$. We fix a point $t\in\Gamma$ and denote by $\gamma:=\Gamma_0(t,\varepsilon)$ the connected component of the portion $\Gamma(t,\varepsilon)$ containing t. If $\varepsilon>0$ is small enough, then γ is a Carleson star comprised of $N(t)$ simple Carleson arcs $\eta_1,\ldots,\eta_{N(t)}$. By Theorem 6.9, we may find a Carleson flower F consisting of $N(t)$ Carleson Jordan curves $\Gamma_1,\ldots,\Gamma_{N(t)}$ and a weight $W\in A_p(F)$ such that $\gamma\subset F$ and $W|\gamma=w|\gamma$. We may assume that $\eta_j\subset\Gamma_j$. Write $(\Gamma_j)_0(t,\varepsilon)=\eta_{2j-1}^*\cup\eta_{2j}^*$ where η_{2j-1}^* is the outgoing arc and η_{2j}^* is the incoming arc. Thus, if η_j is outgoing ($\Longleftrightarrow\varepsilon_j(t)=1$) then $\eta_j=\eta_{2j-1}^*$ and η_{2j}^* is not contained in γ, while if η_j is incoming ($\Longleftrightarrow \varepsilon_j(t) = -1$) then $\eta_j = \eta^*_{2j}$ and η^*_{2j-1} is not a subset of γ. Clearly,

$$\gamma = \Gamma_0(t,\varepsilon) = \eta_1 \cup \ldots \cup \eta_{N(t)},$$
$$F_0(t,\varepsilon) = \bigcup_{j=1}^{N(t)} (\Gamma_j)_0(t,\varepsilon) = \bigcup_{j=1}^{N(t)} (\eta^*_{2j-1} \cup \eta^*_{2j}).$$

We have the three local algebras

$$\mathcal{B}_t(\Gamma) := \mathcal{B}_t(\Gamma, p, w), \quad \mathcal{B}_t(\gamma) := \mathcal{B}_t(\gamma, p, w|\gamma), \quad \mathcal{B}_t(F) := \mathcal{B}_t(F, p, W)$$

and their subalgebras $\mathcal{A}_t(\Gamma), \mathcal{A}_t(\gamma), \mathcal{A}_t(F)$. The results of Sections 7 and 8 tell us a lot about $\mathcal{A}_t(F)$. We now connect $\mathcal{A}_t(\Gamma)$ with $\mathcal{A}_t(F)$ via $\mathcal{A}_t(\gamma)$.

Lemma 9.1. *The map*

$$\Phi^\pi_t : \mathcal{B}_t(\Gamma) \to \mathcal{B}_t(\gamma),\ \pi^\Gamma_t(A) \mapsto \pi^\gamma_t(\chi_\gamma A \chi_\gamma I)$$

is a well-defined Banach algebra isomorphism.

Proof. As in the proof of Lemma 8.3(b) one can verify that $\chi_\gamma A \chi_\gamma I \in \Lambda(\gamma)$ whenever $A \in \Lambda(\Gamma)$. Clearly, the map

$$\Phi : \Lambda(\Gamma) \to \mathcal{B}_t(\gamma),\ A \mapsto \pi^\gamma_t(\chi_\gamma A \chi_\gamma I)$$

is a bounded linear operator. We claim that $\Phi(AB) = \Phi(A)\Phi(B)$, i.e. that

$$\pi^\gamma_t(\chi_\gamma AB \chi_\gamma I) = \pi^\gamma_t(\chi_\gamma A \chi_\gamma B \chi_\gamma I)$$

for all $A, B \in \Lambda(\Gamma)$. Let $\varphi \in C(\Gamma)$ be any function such that $\varphi(t) = 1$ and $\varphi|\Gamma \setminus \gamma = 0$. Then $\pi^\gamma_t(\varphi I) = \pi^\gamma_t(I)$, and since $\chi_\gamma \varphi = \varphi \in C(\Gamma)$, we get

$$\begin{aligned} \pi^\gamma_t(\chi_\gamma A \chi_\gamma B \chi_\gamma I) &= \pi^\gamma_t(\chi_\gamma A \chi_\gamma B \chi_\gamma \varphi I) = \pi^\gamma_t(\chi_\gamma A \chi_\gamma \varphi B \chi_\gamma I) \\ &= \pi^\gamma_t(\chi_\gamma AB \chi^2_\gamma \varphi I) = \pi^\gamma_t(\chi_\gamma AB \chi_\gamma I), \end{aligned}$$

which proves our claim and shows that Φ is a Banach algebra homomorphism.

Obviously, $\mathcal{K}(\Gamma) \subset \operatorname{Ker} \Phi$. Therefore

$$\Phi^\pi : \Lambda(\Gamma)/\mathcal{K}(\Gamma) \to \mathcal{B}_t(\gamma),\ \pi^\Gamma(A) \mapsto \pi^\gamma_t(\chi_\gamma A \chi_\gamma I)$$

is a well-defined Banach algebra homomorphism. It is readily seen that $\mathcal{J}_t(\Gamma) \subset \operatorname{Ker} \Phi^\pi$, which implies that Φ^π_t is a well-defined Banach algebra homomorphism.

Every element in $\mathcal{B}_t(\gamma)$ is of the form $\pi^\gamma_t(B)$ with $B \in \Lambda(\gamma)$. Define $A \in \mathcal{B}(\Gamma)$ by $A := \varphi B \varphi I$ where φ is as above. If $\psi \in C(\Gamma)$, then

$$A\psi I - \psi A = (\varphi B)(\psi \varphi I) - (\psi \varphi)(B \varphi I) \in \mathcal{K}(\Gamma)$$

and hence, $A \in \Lambda(\Gamma)$. Because $\pi^\gamma_t(\chi_\gamma A \chi_\gamma I) = \pi^\gamma_t(\varphi B \varphi I) = \pi^\gamma_t(B)$, we conclude that Φ^π_t is surjective.

Now assume $\pi_t^\gamma(\chi_\gamma A\chi_\gamma I)=0$ for some $A\in\Lambda(\Gamma)$. Lemma 8.2 gives $K_n\in\mathcal{K}(\gamma)$, $E_n\in\Lambda(\gamma)$, $c_n\in C(\gamma)$ such that $c_n(t)=0$ and

$$\|\chi_\gamma A\chi_\gamma I-K_n-c_nE_n\|_{\mathcal{B}(\gamma)}\to 0.$$

Again let $\varphi\in C(\Gamma)$ be as above. Then

$$\|\varphi^2\chi_\gamma A\chi_\gamma\varphi I-\varphi^2K_n\varphi I-\varphi c_n\varphi E_n\varphi\|_{\mathcal{B}(\Gamma)}\to 0,$$

and because $\varphi^2K_n\varphi I\in\mathcal{K}(\Gamma)$, $\varphi E_n\varphi I\in\Lambda(\Gamma)$, $\varphi c_n\in C(\Gamma)$, $(\varphi c_n)(t)=0$, we deduce from Lemma 8.2 that

$$0=\pi_t^\Gamma(\varphi^2\chi_\gamma A\chi_\gamma\varphi I)=\pi_t^\Gamma(A).$$

This proves that Φ_t^π is injective. □

Lemma 9.2. *The map*

$$\Psi_t^\pi:\mathcal{B}_t(\gamma)\to\mathcal{B}_t(F),\ \pi_t^\gamma(A)\mapsto\pi_t^F(\chi_\gamma A\chi_\gamma I)$$

is a well-defined injective Banach algebra homomorphism enjoying the property that $\pi_t^\gamma(A)$ *is invertible in* $\mathcal{B}_t(\gamma)$ *if and only if* $\pi_t^F(\chi_\gamma A\chi_\gamma I)+\pi_t^F(\chi_{F\setminus\gamma}I)$ *is invertible in* $\mathcal{B}_t(F)$.

Proof. The map

$$\Psi:\Lambda(\gamma)\to\mathcal{B}_t(F),\ A\mapsto\pi_t^F(\chi_\gamma A\chi_\gamma I)$$

is a bounded linear operator. If $A,B\in\Lambda(\gamma)$, then $\chi_\gamma AB\chi_\gamma I=\chi_\gamma A\chi_\gamma B\chi_\gamma I$ and thus, Ψ is a Banach algebra homomorphism. Taking into account that $\mathcal{K}(\gamma)\subset\operatorname{Ker}\Psi$, we see that

$$\Psi^\pi:\Lambda(\gamma)/\mathcal{K}(\gamma)\to\mathcal{B}_t(F),\ \pi^\gamma(A)\mapsto\pi_t^F(\chi_\gamma A\chi_\gamma I)$$

is a well-defined Banach algebra homomorphism. Since every function $c\in C(\gamma)$ vanishing at t may be extended to a function $c\in C(F)$ taking on the value zero at t, it follows that $\mathcal{J}_t(\gamma)\subset\operatorname{Ker}\Psi^\pi$ and hence that Ψ_t^π is a well-defined Banach algebra homomorphism. Notice, however, that Ψ_t^π maps the identity of $\mathcal{B}_t(\gamma)$ to $\pi_t^F(\chi_\gamma I)$, which is not the identity of $\mathcal{B}(F)$.

To show that Ψ_t^π is injective, assume $A\in\Lambda(\gamma)$ and $\pi_t^F(\chi_\gamma A\chi_\gamma I)=0$. By Lemma 8.2., there are $K_n\in\mathcal{K}(F)$, $E_n\in\Lambda(F)$, $c_n\in C(F)$ such that $c_n(t)=0$ and

$$\|\chi_\gamma A\chi_\gamma I-K_n-c_nE_n\|_{\mathcal{B}(F)}\to 0.$$

It follows that

$$\|\chi_\gamma A\chi_\gamma I-\chi_\gamma K_n\chi_\gamma I-c_n\chi_\gamma E\chi_\gamma I\|_{\mathcal{B}(\gamma)}\to 0,$$

and because $\chi_\gamma K_n\chi_\gamma I\in\mathcal{K}(\gamma)$ and $\chi_\gamma E\chi_\gamma I\in\Lambda(\gamma)$, we infer from Lemma 8.2 that

$$0=\pi_t^\gamma(\chi_\gamma A\chi_\gamma I)=\pi_t^\gamma(A).$$

This proves the injectivity of Ψ_t^π.

Now suppose $A \in \Lambda(\gamma)$ and $\pi_t^\gamma(A)$ is invertible. Then there is an operator $B \in \Lambda(\gamma)$ such that $\pi_t^\gamma(BA) = \pi_t^\gamma(I)$. We claim that

$$\pi_t^F(\chi_\gamma B\chi_\gamma I + \chi_{F\setminus\gamma}I)\pi_t^F(\chi_\gamma A\chi_\gamma I + \chi_{F\setminus\gamma}I) = \pi_t^F(I).$$

Indeed,

$$\begin{aligned}
&\pi_t^F(\chi_\gamma B\chi_\gamma I + \chi_{F\setminus\gamma}I)\pi_t^F(\chi_\gamma A\chi_\gamma I + \chi_{F\setminus\gamma}I) - \pi_t^F(I)\\
&= \pi_t^F(\chi_\gamma B\chi_\gamma A\chi_\gamma I + \chi_{F\setminus\gamma}I - \chi_F I)\\
&= \pi_t^F(\chi_\gamma B\chi_\gamma A\chi_\gamma I - \chi_\gamma I)\\
&= \pi_t^F(\chi_\gamma B\chi_\gamma I)\pi_t^F(\chi_\gamma A\chi_\gamma I) - \pi_t^F(\chi_\gamma I),
\end{aligned}$$

and this equals zero because the map $\pi_t^\gamma(C) \mapsto \pi_t^F(\chi_\gamma C\chi_\gamma I)$ was shown to be a homomorphism and $\pi_t^\gamma(B)\pi_t^\gamma(A) - \pi_t^\gamma(I) = 0$ by assumption. Right invertibility can be disposed of analogously.

Conversely, suppose now that $A \in \Lambda(\gamma)$ and that $\pi_t^F(\chi_\gamma A\chi_\gamma I + \chi_{F\setminus\gamma}I)$ is invertible. Then there exists a $B \in \Lambda(F)$ such that

$$\pi_t^F(B)\pi_t^F(\chi_\gamma A\chi_\gamma I + \chi_{F\setminus\gamma}I) = \pi_t^F(I).$$

It results that

$$\begin{aligned}
0 &= \pi_t^F(\chi_\gamma I)\pi_t^F\Big(B(\chi_\gamma A\chi_\gamma I + \chi_{F\setminus\gamma}I) - \chi_F I\Big)\\
&= \pi_t^F(\chi_\gamma B\chi_\gamma A\chi_\gamma I - \chi_\gamma I).
\end{aligned}$$

Since the map $\pi_t^\gamma(C) \mapsto \pi_t^F(\chi_\gamma C\chi_\gamma I)$ is injective, we obtain that

$$0 = \pi_t^\gamma(\chi_\gamma B\chi_\gamma A\chi_\gamma I - \chi_\gamma I) = \pi_t^\gamma(\chi_\gamma B\chi_\gamma I)\pi_t^\gamma(A) - \pi_t^\gamma(I),$$

which shows that $\pi_t^\gamma(\chi_\gamma B\chi_\gamma I)$ is a left inverse of $\pi_t^\gamma(A)$. In the same way one can show that $\pi_t^\gamma(A)$ is invertible from the right. $\square$

Put

$$D_t := \operatorname{diag}\,(\delta_1(t), \delta_2(t), \ldots, \delta_{2N(t)}(t)) \tag{9.1}$$

where $\delta_i(t) = 1$ if $\eta_i^* \subset \gamma$ and $\delta_i(t) = 0$ if η_i^* is not contained in γ. In other words, if η_j is outgoing ($\Longleftrightarrow \varepsilon_j(t) = 1$) then $\delta_{2j-1}(t) = 1$ and $\delta_{2j}(t) = 0$, while if η_j is incoming ($\Longleftrightarrow \varepsilon_j(t) = -1$) then $\delta_{2j-1}(t) = 0$ and $\delta_{2j}(t) = 1$. Recall that $\mathcal{X}_t(F)$ is given by formula (8.3).

Lemma 9.3. (a) *For each $x \in \mathcal{X}_t(F)$, the map*

$$H_{t,x} : \{P_\Gamma\} \cup \{aI : a \in PC(\Gamma)\} \to \mathbf{C}^{2N(t)\times 2N(t)}$$

given by

$$H_{t,x}(aI) = D_t \operatorname{diag}\Big(a_1(t), a_1(t), \ldots, a_{2N(t)}(t), a_{2N(t)}(t)\Big)D_t, \tag{9.2}$$

$$H_{t,x}(P_\Gamma) = D_t M_x D_t, \tag{9.3}$$

M_x being the matrix (7.11), (7.12), *extends to a Banach algebra homomorphism of the algebra* $\operatorname{alg}(S_\Gamma, PC(\Gamma))$ *to* $\mathbf{C}^{2N(t)\times 2N(t)}$.

(b) *For* $A \in \operatorname{alg}(S_\Gamma, PC(\Gamma))$ *and* $x \in \mathcal{X}_t(F)$, *put*

$$\operatorname{Sym}{}^0_{t,x}(A) = H_{t,x}(A) + I - D_t. \tag{9.4}$$

Then $\pi_t^\Gamma(A)$ *is invertible in* $\mathcal{B}_t(\Gamma)$ *if and only if* $\operatorname{Sym}{}^0_{t,x}(A)$ *is invertible in* $\mathbf{C}^{2N(t)\times 2N(t)}$ *for all* $x \in \mathcal{X}_t(F)$.

Proof. Combining Lemmas 9.1 and 9.2 we see that the map

$$H_t := \Psi_t^\pi \circ \Phi_t^\pi \circ \pi_t^\Gamma : \Lambda(\Gamma) \to \mathcal{B}_t(F),\ A \mapsto \pi_t^F(\chi_\gamma A \chi_\gamma I)$$

is a Banach algebra homomorphism with the property that $\pi_t^\Gamma(A)$ is invertible in $\mathcal{B}_t(\Gamma)$ if and only if $H_t(A) + \pi_t^F(\chi_{F\setminus\gamma} I)$ is invertible in $\mathcal{B}_t(F)$. Clearly, H_t maps $\operatorname{alg}(S_\Gamma, PC(\Gamma))$ into $\mathcal{A}_t(F)$.

The algebra $\mathcal{A}_t(F)$ is generated by $P := \pi_t^F(P_F)$ and $p_i := \pi_t^F(\chi_i^* I)$ where χ_i^* is the characteristic function of η_i^* ($i = 1, 2, \ldots, 2N(t)$). Lemma 7.3 and the remark at the end of Section 8 show that the hypotheses of Theorem 7.2 are satisfied for $\mathcal{B} := \mathcal{B}_t(F)$. We have $\operatorname{sp}_{\mathcal{B}}(X) = \mathcal{X}_t(F)$ by the definition (7.18) and we have some preliminary information about $\mathcal{X}_t(F)$ from Theorem 8.4. Thus, Theorem 7.2 implies that for each $x \in \mathcal{X}_t(F)$ the map

$$\sigma_{t,x} : \{P, p_1, p_2, \ldots, p_{2N(t)}\} \to \mathbf{C}^{2N(t)\times 2N(t)}$$

which sends p_i and P to the matrices (7.13) and (7.11), respectively, extends to a Banach algebra homomorphism

$$\sigma_{t,x} : \mathcal{A}_t(F) \to \mathbf{C}^{2N(t)\times 2N(t)}$$

and that $\pi_t^F(A)$ is invertible in $\mathcal{B}_t(F)$ if and only if $\sigma_{t,x}(\pi_t^F(A))$ is invertible in $\mathbf{C}^{2N(t)\times 2N(t)}$ for all $x \in \mathcal{X}_t(F)$.

Define

$$\begin{aligned} H_{t,x} := \sigma_{t,x} \circ H_t :\ & \operatorname{alg}(S_\Gamma, PC(\Gamma)) \to \mathbf{C}^{2N(t)\times 2N(t)}, \\ & A \mapsto \sigma_{t,x}(\pi_t^F(\chi_\gamma A \chi_\gamma I)) \end{aligned} \tag{9.5}$$

and

$$\begin{aligned} \operatorname{Sym}{}^0_{t,x} :\ & \operatorname{alg}(S_\Gamma, PC(\Gamma)) \to \mathbf{C}^{2N(t)\times 2N(t)}, \\ & A \mapsto H_{t,x}(A) + \sigma_{t,x}(\pi_t^F(\chi_{F\setminus\gamma} I)). \end{aligned} \tag{9.6}$$

We know that $H_{t,x}$ is a Banach algebra homomorphism and we know that $\pi_t^\Gamma(A)$ is invertible in $\mathcal{B}_t(\Gamma)$ if and only if $\operatorname{Sym}{}^0_{t,x}(A)$ is invertible in $\mathbf{C}^{2N(t)\times 2N(t)}$ for all $x \in \mathcal{X}_t(F)$. The lemma will follow once we have shown that the map $H_{t,x}$ defined by (9.5) sends aI and P_Γ to the matrices (9.2) and (9.3), respectively, and that (9.6) is nothing but the map (9.4).

Obviously,

$$\sigma_{t,x}(\pi_t^F(\chi_\gamma I)) = D_t, \;\; \sigma_{t,x}(\pi_t^F(\chi_{F\setminus\gamma} I)) = I - D_t$$

where D_t is the matrix (9.1). Moreover, $\sigma_{t,x}(\pi_t^F(\chi_\gamma a \chi_\gamma I))$ equals

$$\begin{aligned}&\sum_{j=1}^{N(t)} \sigma_{t,x}(\pi_t^F(\delta_{2j-1}(t)a_j(t)\chi_{2j-1}^* I + \delta_{2j}(t)a_j(t)\chi_{2j}^* I))\\ &= D_t \operatorname{diag}\Big(a_1(t), a_1(t), a_2(t), a_2(t), \dots, a_{N(t)}(t), a_{N(t)}(t)\Big),\end{aligned}$$

which is (9.2), and $\sigma_{t,x}(\pi_t^F(\chi_\gamma P_\Gamma \chi_\gamma I))$ is equal to

$$\sigma_{t,x}(\pi_t^F(\chi_\gamma P_F \chi_\gamma I)) = \sigma_{t,x}(\pi_t^F(\chi_\gamma I))\sigma_{t,x}(\pi_t^F(P_F))\sigma_{t,x}(\pi_t^F(\chi_\gamma I)),$$

which coincides with (9.3). □

The previous lemma is already almost the thing we are looking for. Its flaws are the following: although the maps $H_{t,x}$ are homomorphisms, they do not answer for invertibility; the maps $\operatorname{Sym}_{t,x}^0$ answer for invertibility but are no homomorphisms; the "symbol matrices" are of order $2N(t)$ and not of order $N(t)$; the set $\mathcal{X}_t(F)$ is not yet identified with the leaf $\mathcal{L}_t(\Gamma)$. The first three of these spots will be remedied by the following corollary. The equality $\mathcal{X}_t(F) = \mathcal{L}_t(\Gamma)$ will be shown in Section 10.

Corollary 9.4. (a) *For each $x \in \mathcal{X}_t(F)$, the map*

$$\operatorname{Sym}_{t,x} : \{S_\Gamma\} \cup \{aI : a \in PC(\Gamma)\} \to \mathbf{C}^{N(t)\times N(t)}$$

sending S_Γ and aI to the matrices (4.5), (4.6) *and* (4.7), *respectively, extends to a Banach algebra homomorphism of* $\operatorname{alg}(S_\Gamma, PC(\Gamma))$ *to* $\mathbf{C}^{N(t)\times N(t)}$.

(b) *If $A \in \operatorname{alg}(S_\Gamma, PC(\Gamma))$ then*

$$\operatorname{sp} \pi_t^\Gamma(A) = \bigcup_{x\in\mathcal{X}_t(F)} \operatorname{sp} \operatorname{Sym}_{t,x}(A).$$

In short, Theorem 4.2 holds with $\mathcal{L}_t(\Gamma)$ replaced by $\mathcal{X}_t(F)$.

Proof. From (9.2) and (9.3) we infer that

$$H_{t,x}(A) = D_t H_{t,x}(A) D_t \tag{9.7}$$

for every $A \in \operatorname{alg}(S_\Gamma, PC(\Gamma))$. Appropriately changing rows and columns of $H_{t,x}(A)$, one can produce a matrix of the form $\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$ where $*$ is an $N(t)\times N(t)$ matrix. To be more precise, let Π_t be the permutation matrix for which

$$\Pi_t D_t \Pi_t^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

From (9.7) we then get

$$\Pi_t H_{t,x}(A)\Pi_t^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \Pi_t H_{t,x}(A)\Pi_t^{-1} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Denote the nonzero $N(t) \times N(t)$ block of the latter matrix by $\mathrm{Sym}_{\,t,x}(A)$. Thus,

$$\Pi_t H_{t,x}(A)\Pi_t^{-1} = \begin{pmatrix} \mathrm{Sym}_{\,t,x}(A) & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $H_{t,x}$ is a Banach algebra homomorphism, the map

$$\mathrm{Sym}_{\,t,x} : \mathrm{alg}\,(S_\Gamma, PC(\Gamma)) \to \mathbf{C}^{N(t)\times N(t)}$$

must also be a Banach algebra homomorphism. Because

$$\begin{aligned} &\Pi_t \mathrm{Sym}^{\,0}_{\,t,x}(A)\Pi_t^{-1} = \Pi_t(H_{t,x}(A) + I - D_t)\Pi_t^{-1} \\ &= \begin{pmatrix} \mathrm{Sym}_{\,t,x}(A) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \mathrm{Sym}_{\,t,x}(A) & 0 \\ 0 & I \end{pmatrix}, \end{aligned}$$

it follows from Lemma 9.3(b) that $\pi_t^\Gamma(A - \lambda I)$ is invertible in $\mathcal{B}_t(\Gamma)$ if and only if the matrix $\mathrm{Sym}_{\,t,x}(A - \lambda I)$ is invertible in $\mathbf{C}^{N(t)\times N(t)}$ for all $x \in \mathcal{X}_t(F)$.

What we are left with is to show that if $x \notin \{0, 1\}$, then $\mathrm{Sym}_{\,t,x}(S_\Gamma)$ and $\mathrm{Sym}_{\,t,x}(aI)$ are just the matrices (4.5) and (4.7) respectively. This is easy for aI:

$$\Pi_t H_{t,x}(aI)\Pi_t^{-1} = \mathrm{diag}\,(a_1(t), \ldots, a_{N(t)}(t), 0, \ldots, 0)$$

and thus, $\mathrm{Sym}_{\,t,x}(aI) = \mathrm{diag}\,(a_1(t), \ldots, a_{N(t)}(t))$. A little computation shows that

$$\Pi_t H_{t,x}(P_\Gamma)\Pi_t^{-1} = \Pi_t D_t \Pi_t^{-1} \Pi_t M_x \Pi_t^{-1} \Pi_t D_t \Pi_t^{-1}$$

has the north-west $N(t) \times N(t)$ block

$$\begin{aligned} \mathrm{Sym}_{\,t,x}(P_\Gamma) &= \mathrm{diag}\,(\varepsilon_1(t), \varepsilon_2(t), \ldots, \varepsilon_{N(t)}(t)) \times \\ &\times \begin{pmatrix} x + \frac{\varepsilon_1(t)-1}{2} & x-1 & \ldots & x-1 \\ x & x + \frac{\varepsilon_2(t)-1}{2} & \ldots & x-1 \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \ldots & x + \frac{\varepsilon_{N(t)}(t)-1}{2} \end{pmatrix}. \end{aligned} \tag{9.8}$$

This implies that $\mathrm{Sym}_{\,t,x}(S_\Gamma) = 2\mathrm{Sym}_{\,t,x}(P_\Gamma) - I$ is the matrix (4.5). □

10. Identification of leaves

We are now in a position to complete the proofs of the four theorems stated in Section 4.

Proof of Theorem 4.3. Part (a) is a special case of Theorem 6.9. So let the situation be as in part (b). By definition,

$$\mathcal{L}_t(\eta) = \text{sp}\ \pi_t^\eta(P_\eta^+) = \text{sp}\ \pi_t^\eta((I + S_\eta^+)/2).$$

Lemma 9.2 (with $\gamma := \eta$ and $F := \Gamma$) tells us that

$$\{1\} \cup \text{sp}\ \pi_t^\eta \left(\frac{I + S_\eta^+}{2} \right) = \text{sp}\ \pi_t^\Gamma \left(\chi_\eta \frac{I + S_\eta^+}{2} \chi_\eta I + \chi_{\Gamma \setminus \eta} I \right) =: \text{sp}\ \pi_t^\Gamma (A_\eta).$$

The (counter-clockwise) orientation of Γ induces an orientation on η. If t is the starting point of η with respect to this orientation, then

$$A_\eta = \chi_\eta \frac{I + S_\Gamma}{2} \chi_\eta I + \chi_{\Gamma \setminus \eta} I, \tag{10.1}$$

while if t is the terminating point of η in this orientation, we have

$$A_\eta = \chi_\eta \frac{I - S_\Gamma}{2} \chi_\eta I + \chi_{\Gamma \setminus \eta} I. \tag{10.2}$$

For $x \in \mathcal{X}_t(\Gamma)$, we construct $\text{Sym}_{\,t,x}(A_\eta)$ as in Corollary 9.4. Let $x \notin \{0, 1\}$. In the case (10.1), $\text{Sym}_{\,t,x}(A_\eta)$ equals

$$\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2x-1 & 2x-2 \\ 2x & 2x-1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and in the case (10.2), $\text{Sym}_{\,t,x}(A_\eta)$ is equal to

$$\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2x-1 & 2x-2 \\ 2x & 2x-1 \end{pmatrix} \right] \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The latter two matrices are

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix},$$

respectively. In either case, the eigenvalues of the resulting matrices are x and 1. The same is also true if $x \in \{0, 1\}$. Thus, by Corollary 9.4(b),

$$\text{sp}\ \pi_t^\Gamma (A_\eta) = \bigcup_{x \in \mathcal{X}_t(F)} \{x, 1\} = \mathcal{X}_t(\Gamma) \cup \{1\}.$$

Because $\mathcal{X}_t(\Gamma) = \mathcal{L}^0(p; \alpha_t, \beta_t)$ due to Theorem 8.1, we see that

$$\mathcal{L}_t(\eta) \cup \{1\} = \mathcal{L}^0(p; \alpha_t, \beta_t) \cup \{1\} = \mathcal{L}^0(p; \alpha_t, \beta_t).$$

The connectedness of $\mathcal{L}^0(p, \alpha_t, \beta_t)$ and the closedness of $\mathcal{L}_t(\eta)$ finally show that $\mathcal{L}_t(\eta) = \mathcal{L}^0(p, \alpha_t, \beta_t)$. $\square$

Proof of Theorem 4.1. By virtue of Theorem 6.9, there exists a Carleson Jordan curve Γ and a weight $W \in A_p(\Gamma)$ such that $\eta_1 \cup \eta_2 \subset \Gamma$ and $W|\eta_1 \cup \eta_2 = w$. Theorem 4.3(b) implies that $\mathcal{L}_t(\eta_1) = \mathcal{L}^0(p; \alpha_t, \beta_t)$ and $\mathcal{L}_t(\eta_2) = \mathcal{L}^0(p; \alpha_t, \beta_t)$. $\square$

Proof of Theorem 4.2. As already said, part (c) is a consequence of part (b) and Propositions 3.2 and 3.3. Let the notation be as in the first paragraph of Section 9. Corollary 9.4 tells us that parts (a) and (b) of Theorem 4.2 are true with $\mathcal{L}_t(\eta)$ replaced by $\mathcal{X}_t(F)$. From Theorem 8.4 we deduce that

$$\mathcal{X}_t(F) = \mathcal{L}^0(p; \alpha_t^1, \beta_t^1) \cup \ldots \cup \mathcal{L}^0(p; \alpha_t^{N(t)}, \beta_t^{N(t)}).$$

Theorem 4.3(b) then implies that $\mathcal{X}_t(F) = \mathcal{L}_t(\eta_1) \cup \ldots \cup \mathcal{L}_t(\eta_{N(t)})$, and since $w \in A_p(\eta_i \cup \eta_j)$ for each pair (i, j) by the boundedness of $S_{\eta_i \cup \eta_j}$, we infer from Theorem 4.1 that $\mathcal{L}_t(\eta_1) = \ldots = \mathcal{L}_t(\eta_{N(t)}) =: \mathcal{L}_t(\Gamma)$. Thus, $\mathcal{X}(F) = \mathcal{L}_t(\Gamma)$. $\square$.

Proof of Theorem 4.4. Put $\varepsilon_j := \varepsilon_j(t)$ and denote the matrix (4.5) by A_x. From Theorem 4.3 we deduce that

$$\text{sp}\, \pi_t(S_\Gamma) = \bigcup_{x \in \mathcal{L}^0(p; \alpha_t, \beta_t)} \text{sp}\, A_x. \tag{10.3}$$

Let $\Delta_N(\varepsilon_1, \ldots, \varepsilon_N; \lambda) := \det(A_x - \lambda I)$. If $\varepsilon_j + \varepsilon_{j+1} = 0$ for some j, we first add the jth row of $A_x - \lambda I$ to the $j+1$st row and then subtract the $j+1$st column from the jth column to obtain

$$\Delta_N(\varepsilon_1, \ldots, \varepsilon_N, \lambda) = (\lambda^2 - 1)\Delta_{N-2}(\varepsilon_1, \ldots, \varepsilon_{j-1}, \varepsilon_{j+2}, \ldots, \varepsilon_N; \lambda).$$

Repeating this procedure we get the formula

$$\Delta_N(\varepsilon_1, \ldots, \varepsilon_N, \lambda) = \begin{cases} (\lambda^2 - 1)^{(N-\varepsilon)/2}\Delta_\varepsilon(1, \ldots, 1; \lambda) & \text{if } \varepsilon > 0, \\ (\lambda^2 - 1)^{(N-|\varepsilon|)/2}\Delta_{|\varepsilon|}(-1, \ldots, -1; \lambda) & \text{if } \varepsilon < 0. \end{cases} \tag{10.4}$$

If $\varepsilon = 0$, then $\Delta_N(\varepsilon_1, \ldots, \varepsilon_N; \lambda) = (\lambda^2 - 1)^{N/2}$, which gives (4.19). Assume $\varepsilon > 0$. Using the identity

$$\left| \begin{matrix} c & a & a & \ldots & a \\ b & c & a & \ldots & a \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b & b & b & \ldots & c \end{matrix} \right|_{(\varepsilon \times \varepsilon)} = \frac{a}{a-b}(c-b)^\varepsilon + \frac{b}{b-a}(c-a)^\varepsilon$$

we arrive at the equality

$$\Delta_\varepsilon(1, \ldots, 1; \lambda) = x(1-\lambda)^\varepsilon - (x-1)(-1-\lambda)^\varepsilon. \tag{10.5}$$

If $x = 0$ then (10.5) vanishes for $\lambda = -1$, and if $x = 1$ then (10.5) is zero for $\lambda = 1$. Hence,

$$\text{sp}\, A_0 \cup \text{sp}\, A_1 = \{-1, 1\}. \tag{10.6}$$

So let $x \notin \{0,1\}$. Then (10.4) and (10.5) give

$$\{-1,1\} \cup \operatorname{sp} A_x = \{-1,1\} \cup \left\{ \lambda \in \mathbf{C} : \Big(\frac{\lambda+1}{\lambda-1}\Big)^{\varepsilon} = \frac{x}{x-1} \right\}. \tag{10.7}$$

We have

$$\left\{ \frac{x}{x-1} : x \in \mathcal{L}^0(p;\alpha_t,\beta_t) \setminus \{0,1\} \right\} = \left\{ e^{2\pi z} : z \in Y(p;\alpha_t,\beta_t) \right\}$$

and therefore, by (10.3), (10.6), (10.7),

$$\begin{aligned} \operatorname{sp} \pi_t(S_\Gamma) &= \{-1,1\} \cup \left\{ \operatorname{sp} A_x : x \in \mathcal{L}^0(p;\alpha_t,\beta_t) \setminus \{0,1\} \right\} \\ &= \{-1,1\} \cup \bigcup_{k=0}^{\varepsilon-1} \left\{ \lambda \in \mathbf{C} : \frac{\lambda+1}{\lambda-1} = e^{2\pi(z+ik)/\varepsilon} \text{ for some } z \in Y(p;\alpha_t,\beta_t) \right\}. \end{aligned}$$

Since $1/p + \alpha_t(x) \le y \le 1/p + \beta_t(x)$ if and only if

$$\frac{1}{p} + \left(\frac{1}{p\varepsilon} - \frac{1}{p} + \frac{k}{\varepsilon} + \frac{\alpha_t(\varepsilon x/\varepsilon)}{\varepsilon} \right) \le \frac{y+k}{\varepsilon} \le \frac{1}{p} + \left(\frac{1}{p\varepsilon} - \frac{1}{p} + \frac{k}{\varepsilon} + \frac{\beta_t(\varepsilon x/\varepsilon)}{\varepsilon} \right),$$

we see that $z \in Y(p;\alpha_t,\beta_t)$ if and only if

$$\frac{z+ik}{\varepsilon} \in Y\left(p; \frac{1}{\varepsilon}\Big(\frac{1}{p} - \frac{\varepsilon}{p} + k + T_\varepsilon \alpha_t\Big), \frac{1}{\varepsilon}\Big(\frac{1}{p} - \frac{\varepsilon}{p} + k + T_\varepsilon \beta_t\Big) \right) =: Y_k.$$

Because

$$\begin{aligned} &\left\{ \lambda \in \mathbf{C} : \frac{\lambda+1}{\lambda-1} = e^{2\pi\zeta} \text{ for some } \zeta \in Y_k \right\} \\ &= \left\{ \lambda \in \mathbf{C} : \lambda = 2\frac{e^{2\pi\zeta}}{e^{2\pi\zeta}-1} - 1 \text{ for some } \zeta \in Y_k \right\}, \end{aligned}$$

we finally get (4.20). The case $\varepsilon < 0$ may be tackled analogously or may be reduced to the case $\varepsilon > 0$ by changing the orientation of Γ and taking into account that $S_{-\Gamma} = -S_\Gamma$. $\square$

11. Matrix case and semi-Fredholmness

Let $L^{p,n}(\Gamma,w)$ be the direct sum of n copies of $L^p(\Gamma,w)$ and provide $L^{p,n}(\Gamma,w)$ with the norm

$$\|f\| = \|(f_1,\ldots,f_n)\| := (\|f_1\|^p + \ldots + \|f_n\|^p)^{1/p}$$

or any norm equivalent to this one. The operator S_Γ is defined on $L^{p,n}(\Gamma, w)$ elementwise, that is,

$$S_\Gamma(f_1, \ldots, f_n) := (S_\Gamma f_1, \ldots, S_\Gamma f_n).$$

Denote by $PC^n(\Gamma)$ the algebra of all $n \times n$ matrix functions with entries in $PC(\Gamma)$. We think of $L^{p,n}(\Gamma, w)$ as a space of column vectors and may therefore identify elements $a \in PC^n(\Gamma)$ with the multiplication operators they induce on $L^{p,n}(\Gamma, w)$:

$$(af)(t) = a(t)f(t) \text{ a.e. on } \Gamma.$$

Let $\operatorname{alg}(S_\Gamma, PC^n(\Gamma))$ stand for the closed algebra generated by S_Γ and $PC^n(\Gamma)$. Let also $\mathcal{B}_t^n(X)$, $\mathcal{A}_t^n(X)$ be similar modifications of $\mathcal{B}_t(X)$, $\mathcal{A}_t(X)$ (with $X \in \{\Gamma, F, \gamma\}$).

Theorem 7.2 can be extended to a theorem applicable to the case $\mathcal{B} = \mathcal{B}_t^n(F)$, $\mathcal{A} = \mathcal{A}_t^n(F)$. Namely, Theorem 13 of [50] implies the following.

Theorem 11.1. *Let $\mathcal{B}$ be a (complex) Banach algebra with identity e containing a subalgebra $\mathcal{G}$ isomorphic to $\mathbf{C}^{n\times n}$ and nonzero elements $p_1, \ldots, p_{2N}$, P such that*

$$p_i g = g p_i \text{ and } gP = Pg \text{ for all } i = 1, \ldots, 2N \text{ and } g \in \mathcal{G},$$

$$p_i p_j = \delta_{ij} p_i \text{ for all } i, j \text{ and } p_1 + p_2 + \ldots + p_{2N} = e,$$

where δ_{ij} is the Kronecker delta, and

$$Q(p_{2j-1} + p_{2j})P = P(p_{2j} + p_{2j+1})Q = 0 \text{ for all } j = 1, \ldots, N,$$

where $Q := e - P$ and $p_{2N+1} := p_1$. Denote by $\mathcal{A}$ the smallest closed subalgebra of $\mathcal{B}$ containing $\mathcal{G} \cup \{P, p_1, p_2, \ldots, p_{2N}\}$. Put

$$X := \sum_{j=1}^{N} (p_{2j-1} P p_{2j-1} + p_{2j} Q p_{2j})$$

and let $\operatorname{sp}_{\mathcal{B}}(X)$ denote the spectrum of X in $\mathcal{B}$. Also suppose that

$$0 \text{ and } 1 \text{ are cluster points of } \operatorname{sp}_{\mathcal{B}}(X).$$

Let $\varphi : \mathcal{G} \to \mathbf{C}^{n\times n}$ be any isomorphism and let 0 and I stand for the $n \times n$ zero and identity matrices, respectively. Put $E := \operatorname{diag}(I, -I, I, -I, \ldots, I, -I)$, and given $x \in \mathbf{C}$, define the map

$$\sigma_x : \mathcal{G} \cup \{P, p_1, p_2, \ldots, p_{2N}\} \to \mathbf{C}^{2Nn\times 2Nn}$$

by

$$\begin{aligned} \sigma_x(g) &= \operatorname{diag}(\varphi(g), \ldots, \varphi(g)) \ (g \in \mathcal{G}), \\ \sigma_x(p_j) &= \operatorname{diag}(0, \ldots, 0, I, 0, \ldots, 0), \text{ the } I \text{ located at the } j\text{th position}, \end{aligned}$$

for all x and by

$$\sigma_x(P) = E \begin{pmatrix} xI & (x-1)I & (x-1)I & (x-1)I & \dots & (x-1)I & (x-1)I \\ xI & (x-1)I & (x-1)I & (x-1)I & \dots & (x-1)I & (x-1)I \\ xI & xI & xI & (x-1)I & \dots & (x-1)I & (x-1)I \\ xI & xI & xI & (x-1)I & \dots & (x-1)I & (x-1)I \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ xI & xI & xI & xI & \dots & xI & (x-1)I \\ xI & xI & xI & xI & \dots & xI & (x-1)I \end{pmatrix}$$

if $x \notin \{0,1\}$ and

$$\sigma_x(P) = \operatorname{diag}\,(xI, (1-x)I, xI, (1-x)I, \dots, xI, (1-x)I)$$

if $x \in \{0,1\}$.

(a) *For each $x \in \operatorname{sp}_{\mathcal{B}}(X)$, the map σ_x extends to a Banach algebra homomorphism σ_x of $\mathcal{A}$ into $\mathbf{C}^{2Nn\times 2Nn}$.*

(b) *An element $a \in \mathcal{A}$ is invertible in $\mathcal{B}$ if and only if $\sigma_x(a)$ is invertible in $\mathbf{C}^{2Nn\times 2Nn}$ for every $x \in \operatorname{sp}_{\mathcal{B}}(X)$.*

Lemmas 9.1 - 9.3 also extend to the matrix case with obvious adjustments (for example, the Banach algebra homomorphism in Lemma 9.3 acts now from $\operatorname{alg}(S_\Gamma, PC^n(\Gamma))$ to $\mathbf{C}^{2N(t)n\times 2N(t)n}$). Hence, our main result (Theorem 4.2) carries over to the matrix setting.

Theorem 11.2. *Suppose Γ is a composed Carleson curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. Given $t \in \Gamma$, let $E(t) := \operatorname{diag}\left(\varepsilon_1(t)I, \dots, \varepsilon_{N(t)}(t)I\right)$. For $x \in \mathbf{C}$, define the map*

$$\operatorname{Sym}_{t,x} : \{S_\Gamma\} \cup \{aI : a \in PC^n(\Gamma)\} \to \mathbf{C}^{N(t)n\times N(t)n}$$

by

$$\operatorname{Sym}_{t,x}(aI) := \operatorname{diag}\left(a_1(t), \dots, a_{N(t)}(t)\right)$$

for all x and by

$$\operatorname{Sym}_{t,x}(S_\Gamma) := E(t) \begin{pmatrix} (2x-1)I & (2x-2)I & (2x-2)I & \dots & (2x-2)I \\ 2xI & (2x-1)I & (2x-2)I & \dots & (2x-2)I \\ 2xI & 2xI & (2x-1)I & \dots & (2x-2)I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2xI & 2xI & 2xI & \dots & (2x-1)I \end{pmatrix}$$

if $x \notin \{0,1\}$ and

$$\operatorname{Sym}_{t,x}(S_\Gamma) := (2x-1)\operatorname{diag}\left(\varepsilon_1(t)I, \dots, \varepsilon_{N(t)}(t)I\right)$$

if $x \in \{0,1\}$.

(a) *For each* $x \in \mathcal{L}_t(\Gamma)$*, the map* $\mathrm{Sym}_{\,t,x}$ *extends to a Banach algebra homomorphism*

$$\mathrm{Sym}_{\,t,x} : \mathrm{alg}\,(S_\Gamma, PC^n(\Gamma)) \to \mathbf{C}^{N(t)n \times N(t)n}.$$

(b) *If* $A \in \mathrm{alg}\,(S_\Gamma, PC^n(\Gamma))$*, then*

$$\mathrm{sp}\,\pi_t(A) = \bigcup_{x \in \mathcal{L}_t(\Gamma)} \mathrm{sp}\,\mathrm{Sym}_{\,t,x}(A).$$

(c) *An operator* $A \in \mathrm{alg}\,(S_\Gamma, PC^n(\Gamma))$ *is Fredholm on* $L^{p,n}(\Gamma, w)$ *if and only if* $\mathrm{Sym}_{\,t,x}(A)$ *is an invertible matrix for all* (t, x) *in the leaf bundle*

$$\mathcal{M} := \mathcal{M}(\Gamma, p, w) := \bigcup_{t \in \Gamma} \Big(\{t\} \times \mathcal{L}_t(\Gamma) \Big).$$

Equivalently,

$$\mathrm{sp}_{\mathrm{ess}}\, A = \bigcup_{(t,x) \in \mathcal{M}} \mathrm{sp}\,\mathrm{Sym}_{\,t,x}(A) = \bigcup_{t \in \Gamma} \bigcup_{x \in \mathcal{L}_t(\Gamma)} \mathrm{sp}\,\mathrm{Sym}_{\,t,x}(A).$$

Theorems 4.2 and 11.2 give necessary and sufficient conditions for operators from the algebras $\mathrm{alg}\,(S_\Gamma, PC(\Gamma))$ and $\mathrm{alg}\,(S_\Gamma, PC^n(\Gamma))$ to be Fredholm. The next natural question to ask is when an operator A from these algebras is *semi-Fredholm*, which means that $\mathrm{Im}\, A := A(X)$ is closed and at least one of the dimensions $\dim \mathrm{Ker}\, A := \dim\{x \in X : Ax = 0\}$, $\dim \mathrm{Coker} A := \dim X/\mathrm{Im}\, A$ is finite. Of course, any Fredholm operator is always semi-Fredholm, and the converse is obviously not true. Moreover, there exist singular integral operators on $L^p(\mathbf{T})$ which are semi-Fredholm but not Fredholm: for example, if a is an infinite Blaschke product and

$$P_\Gamma := (I + S_\Gamma)/2, \;\; Q_\Gamma := (I - S_\Gamma)/2,$$

then $P_\Gamma + aQ_\Gamma$ is surjective with an infinite dimensional kernel while $aP_\Gamma + Q_\Gamma$ is injective and has a closed image of infinite codimension. However, as the following theorem shows, this phenomenon does not occur for operators with piecewise continuous symbols.

Theorem 11.3. *All semi-Fredholm operators in* $\mathrm{alg}\,(S_\Gamma, PC^n(\Gamma))$ *are Fredholm.*

For piecewise Lyapunov curves, this result was established in [35]. There the general case was first reduced to the case of operators of the form $aP_\Gamma + Q_\Gamma$ on Jordan curves Γ, and in the latter case it had already been shown in [58] that semi-Fredholmness implies Fredholmness. Due to the extension results for Carleson curves and Muckenhoupt weights established in Section 6, the first part of the proof remains literally the same. The second part, however, was based on the fact that for each a invertible in $PC(\Gamma)$ the set of all $p \in (1, \infty)$ for which $aP_\Gamma + Q_\Gamma$ is Fredholm on $L^p(\Gamma, w)$ has cluster points at 1 and ∞. Interestingly enough, this property fails for general Carleson Jordan curves.

Theorem 11.4. *Let* Γ *be a Carleson Jordan curve, let* $t \in \Gamma$*, and let* a *be an invertible function in* $PC(\Gamma)$ *continuous on* $\Gamma \setminus \{t\}$*. Then*

(a) *$aP_\Gamma + Q_\Gamma$ is Fredholm on $L^p(\Gamma)$ for all but maybe one value of $p \in (1,\infty)$ if $\delta_t^- = \delta_t^+$;*

(b) *$aP_\Gamma + Q_\Gamma$ is not Fredholm on any $L^p(\Gamma, w)$ $(1 < p < \infty,\ w \in A_p(\Gamma))$ if $\delta_t^+ \neq \delta_t^-$ and $|a(t-0)/a(t+0)| \geq e^{4\pi/(\delta_t^+ - \delta_t^-)}$.*

Proof. Taking into account (4.8) and (4.10), we see that for every $(\tau,\xi) \in \Gamma \times \mathcal{L}_\tau(\Gamma)$ such that $\xi \notin \{0,1\}$ we have

$$\begin{aligned}&\mathrm{Sym}_{\,\tau,\xi}(aP_\Gamma + Q_\Gamma) = \mathrm{Sym}_{\,\tau,\xi}\left(\frac{a+1}{2}I + \frac{a-1}{2}S_\Gamma\right)\\ &= \begin{pmatrix} \Big(a(\tau+0)-1\Big)\xi + 1 & \Big(a(\tau+0)-1\Big)\Big(\xi-1\Big) \\ \Big(1-a(\tau-0)\Big)\xi & \Big(1-a(\tau-0)\Big)\xi + a(\tau-0)\end{pmatrix}.\end{aligned}$$

The determinant of this matrix is $a(\tau+0)\xi + a(\tau-0)(1-\xi)$. For $\xi \in \{0,1\}$ we get the same determinant. If $\tau \in \Gamma \setminus \{t\}$, then

$$a(\tau+0)\xi + a(\tau-0)(1-\xi) = a(\tau)\xi + a(\tau)(1-\xi) \neq 0$$

because a is invertible. Thus, Theorem 4.2(c) tells us that $aP_\Gamma + Q_\Gamma$ is Fredholm on $L^p(\Gamma, w)$ if and only if

$$a(t+0)\xi + a(t-0)(1-\xi) \neq 0 \ \text{ for all } \ \xi \in \mathcal{L}_t(\Gamma). \tag{11.1}$$

Combining (4.18), (4.17), and (4.16), we arrive at the conclusion that (11.1) holds if and only if

$$\frac{a(t+0)e^{2\pi z} - a(t-0)}{e^{2\pi z}-1} \neq 0 \ \text{ for all } \ z \in Y(p;\alpha_t,\beta_t),$$

which is equivalent to saying that

$$\frac{1}{2\pi}\log\frac{a(t-0)}{a(t+0)} \notin Y(p;\alpha_t,\beta_t) \ \text{ for all values of the log.} \tag{11.2}$$

We now consider the two cases separately.

(a) Suppose $\delta_t^- = \delta_t^+ =: \delta_t$. Because $L^p(\Gamma)$ involves no weight, we have $\mu_t = \nu_t = 0$. Therefore $\alpha_t(x) = \beta_t(x) = \delta_t x$ by virtue of (5.4), and consequently, (11.2) assumes the form

$$\frac{1}{p} \neq F\left(\frac{1}{2\pi}\arg\frac{a(t-0)}{a(t+0)} - \delta_t\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right)$$

where $F(x)$ stands for the fractional part of x. This gives the assertion.

(b) Now let $\delta_t^- \neq \delta_t^+$. Consider the sector

$$\Delta := \{z = x+iy \in \mathbf{C} : 1 + \delta_t^- x \leq y \leq \delta_t^+ x\}.$$

Theorem 3.2 of [8], which contains more precise information about the indicator functions than (4.13) to (4.15), says that there exist numbers μ_t^- and ν_t^+ in $(-1/p, 1/q)$ such that if $x+iy \in \Delta$, then

$$\frac{1}{p} + \alpha_t(x) \leq \frac{1}{p} + \mu_t^- + \delta_t^- x < 1 + \delta_t^- x \leq y \leq \delta_t^+ x < \frac{1}{p} + \nu_t^+ + \delta_t^+ x \leq \frac{1}{p} + \beta_t(x).$$

This shows that $\Delta \subset Y$. Put

$$\Delta_0 := \{z = x + iy \in \Delta : x \geq 2/(\delta_t^+ - \delta_t^-)\}.$$

Since $\Delta_0 \subset \Delta$, it follows from (11.2) that the operator $aP_\Gamma + Q_\Gamma$ cannot be Fredholm for any $p \in (1, \infty)$ and $w \in A_p(\Gamma)$ provided that

$$\frac{1}{2\pi} \log \frac{a(t-0)}{a(t+0)} \in \Delta_0.$$

Separating the real and imaginary part in the latter condition, we may rewrite it as

$$\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| = x \geq 2/(\delta_t^+ - \delta_t^-), \tag{11.3}$$

$$\frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)} = y \in [1 + \delta_t^- x, \delta_t^+ x] + \mathbf{Z}. \tag{11.4}$$

Condition (11.3) is satisfied by assumption. Since the length of the interval $[1 + \delta_t^- x, \delta_t^+ x]$ is at least 1 if (11.3) holds, condition (11.4) is also satisfied in the case at hand. □

Because of Theorem 11.4, the proof of Theorem 11.3 proposed in [35] goes through if and only if $\delta_t^+ = \delta_t^-$ for all $t \in \Gamma$, that is, for so-called spiralic curves. The result itself, however, remains valid in the most general setting. As already observed, we need prove it only for operators $aP_\Gamma + Q_\Gamma$ acting on Jordan curves. This is done by the following lemma.

Lemma 11.5. *Let Γ be a Carleson Jordan curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. Let also a be an invertible function in $PC(\Gamma)$ such that the operator $aP_\Gamma + Q_\Gamma$ is semi-Fredholm on $L^p(\Gamma, w)$. Then $aP_\Gamma + Q_\Gamma$ is Fredholm on $L^p(\Gamma, w)$.*

Proof. Suppose that $aP_\Gamma + Q_\Gamma$ is semi-Fredholm but not Fredholm. Then one of the numbers $\dim \operatorname{Ker}(aP_\Gamma + Q_\Gamma)$, $\dim \operatorname{Coker}(aP_\Gamma + Q_\Gamma)$ is infinite. Without loss of generality we may consider the case when $\dim \operatorname{Ker}(aP_\Gamma + Q_\Gamma) = \infty$; otherwise, we may pass to the adjoint operator

$$(aP_\Gamma + Q_\Gamma)^* = Q_\Gamma \overline{a} + P_\Gamma = (Q_\Gamma + P_\Gamma \overline{a}^{\,-1})\overline{a}$$

acting on $L^q(\Gamma, w^{-1})$ and may have recourse to the identity

$$bP_\Gamma + Q_\Gamma = (I - P_\Gamma b Q_\Gamma)(Q_\Gamma + P_\Gamma b)(I + Q_\Gamma b P_\Gamma),$$

in which $(I - P_\Gamma b Q_\Gamma)^{-1} = I + P_\Gamma b Q_\Gamma$ and $(I + Q_\Gamma b P_\Gamma)^{-1} = I - Q_\Gamma b P_\Gamma$. Thus, assume $aP_\Gamma + Q_\Gamma \in \Phi^-$, where Φ^- is the set of all semi-Fredholm operators with infinite dimensional kernel. The set Φ^- is open, and hence we may suppose that a has only finitely many jumps, say at $t_1, \ldots, t_M$. Write $a = a_1 \cdots a_M$ where a_j is continuous on $\Gamma \setminus \{t_j\}$. Since

$$aP_\Gamma + Q_\Gamma = (P_\Gamma a P_\Gamma + Q_\Gamma)(I + Q_\Gamma a P_\Gamma) \tag{11.5}$$

and $I+Q_\Gamma aP_\Gamma$ has the inverse $I-Q_\Gamma aP_\Gamma$, the operator $aP_\Gamma+Q_\Gamma$ is Φ^- if and only if so is $P_\Gamma aP_\Gamma+Q_\Gamma$. With an appropriate continuous partition of the unity, we deduce from Proposition 3.3 that the difference

$$P_\Gamma aP_\Gamma+Q_\Gamma-(P_\Gamma a_1P_\Gamma+Q_\Gamma)\cdots(P_\Gamma a_MP_\Gamma+Q_\Gamma)$$

is compact and that the operators on the right commute modulo compact operators. This implies that all operators $P_\Gamma a_jP_\Gamma+Q_\Gamma$ are semi-Fredholm and that at least one of them belongs to Φ^-. Again using (11.5), we may assume that at least one of the operators $a_jP_\Gamma+Q_\Gamma$ is in Φ^-. Considering the latter operator instead of the original $aP_\Gamma+Q_\Gamma$, we may suppose that a itself has exactly one discontinuity, say at t_0. Since $a(t_0\pm 0)\neq 0$, we may find a power function t^α continuous on $\Gamma\setminus\{t_0\}$ such that $f(t):=a(t)/t^\alpha$ is continuous and nonzero on Γ (we here suppose for simplicity that 0 lies in the interior D_+ of Γ; otherwise we would have to choose a point $z_0\in D_+$ and consider $(t-z_0)^\alpha$ instead of t^α). The difference

$$aP_\Gamma+Q_\Gamma-(t^\alpha P_\Gamma+Q_\Gamma)(fP_\Gamma+Q_\Gamma)=(t^\alpha-1)Q_\Gamma fP_\Gamma$$

is compact due to Proposition 3.3, and since $fP_\Gamma+Q_\Gamma$ is Fredholm, it follows that the operator $t^\alpha P_\Gamma+Q_\Gamma$ belongs to Φ^- simultaneously with $aP_\Gamma+Q_\Gamma$. In particular, $\dim\operatorname{Ker}(t^\alpha P_\Gamma+Q_\Gamma)=\infty$, that is, there exist infinitely many linearly independent functions $\phi^+\in PL^p(\Gamma,w)$ such that $t^\alpha\phi^+=\phi^-\in QL^p(\Gamma,w)$. To show that this is not possible, we use the factorization

$$t^\alpha=(t-t_0)^\alpha(1-t_0/t)^{-\alpha}$$

in which $(t-t_0)^\alpha$ is any branch analytic in D_+, and $(1-t_0/t)^{-\alpha}$ is the corresponding branch analytic in the exterior D_- of Γ. The equality $\phi^-=t^\alpha\phi^+$ can then be rewritten as

$$(t-t_0)^\alpha\phi^+(t)=(1-t_0/t)^\alpha\phi^-(t),\quad t\in\Gamma,$$

or

$$(t-t_0)^{\alpha+m}\phi^+(t)=(1-t_0/t)^\alpha(t-t_0)^m\phi^-(t),\quad t\in\Gamma \tag{11.6}$$

for any $m\in\mathbf{Z}$. Since Γ is a Carleson curve, we have

$$\arg(t-t_0)/(-\log|t-t_0|)=O(1)\ \text{ as }\ t\to t_0,\ \ t\in\Gamma$$

(see [53]). As in [6, Proof of Lemma 6.1] we conclude from here that

$$\arg(z-t_0)/(-\log|z-t_0|)=O(1)\ \text{ as }\ z\to t_0,\ \ z\in D_+,$$

so that $|\arg(z-t_0)|\le -M\log|z-t_0|$ with a certain constant $M<\infty$ whenever $z\in D^+$ is sufficiently close to t_0. Thus, for these z we have

$$|(z-t_0)^{\alpha+m}|=|z-t_0|^{m+\operatorname{Re}\alpha}e^{-\operatorname{Im}\alpha\arg(z-t_0)}\le|z-t_0|^{m+\operatorname{Re}\alpha-M|\operatorname{Im}\alpha|}.$$

It follows that for every $m>M|\operatorname{Im}\alpha|-\operatorname{Re}\alpha$ the function $(z-t_0)^{\alpha+m}$ is analytic and bounded in D_+. Similarly, the function $(1-t_0/z)^\alpha(z-t_0)^m$ is analytic and bounded on bounded subsets of D_- if only m is large enough. At infinity the latter function has a pole of order m. What results is that the function on the left of (11.6) belongs to $P_\Gamma L^p(\Gamma, w)$ together with ϕ^+ and that, up to a polynomial summand q of degree at most $m-1$, the function on the right of (11.6) lies in $Q_\Gamma L^p(\Gamma, w)$ together with ϕ^-. Consequently, after subtracting $q(t)$ from both sides of (11.6) we get a function in $P_\Gamma L^p(\Gamma, w) \cap Q_\Gamma L^p(\Gamma, w) = \{0\}$. Therefore both sides of (11.6) are a polynomial of degree at most $m-1$, and since

$$\phi^+(t) = (t - t_0)^{-\alpha - m} q(t),$$

we obtain that $\dim \operatorname{Ker} (t^\alpha P_\Gamma + Q_\Gamma) \le m < \infty$. This contradiction completes the proof. $\square$

Theorem 11.3 in its full generality follows from Lemma 11.5 and the reasoning of the paper [35].

12. Index formulas

Once Theorems 4.2 and 11.2 are available, one can establish index formulas for Fredholm operators in $\operatorname{alg}(S_\Gamma, PC(\Gamma))$ and $\operatorname{alg}(S_\Gamma, PC^n(\Gamma))$ using the method developed by Gohberg and Krupnik for operators on composed Lyapunov curves. This method is now standard, and therefore we restrict ourselves to the main steps and omit the details. For composed Lyapunov curves, index formulas are already in [30, 32, 33] (power weights) and in [35] (general weights). These formulas differ from ours, since they are based on representations like (5.6), (5.7) and Theorems 7.2 and 11.1 were not available at that time.

Let X be a Banach space. An operator $A \in \mathcal{B}(X)$ is Fredholm if and only if its image $\operatorname{Im} A := A(X)$ is closed and both its kernel $\operatorname{Ker} A := \{x \in X : Ax = 0\}$ and its cokernel $\operatorname{Coker} A := X / \operatorname{Im} A$ are finite-dimensional. In that case the *index* of A is defined as

$$\operatorname{Ind} A := \dim \operatorname{Ker} A - \dim \operatorname{Coker} A.$$

Several properties of the index can be found in [33]. We only remark that the index is stable under small or compact perturbations and that $\operatorname{Ind}(AB) = \operatorname{Ind} A + \operatorname{Ind} B$ whenever A and B are Fredholm.

12.1. Structure of the index formula

Let Γ be a composed Carleson curve, $p \in (1, \infty)$, and $w \in A_p(\Gamma)$. We may write $\Gamma = \eta_1 \cup \ldots \cup \eta_m$ where $\eta_1, \ldots, \eta_m$ are simple oriented arcs each pair of which have at most endpoints in common. A point $t \in \Gamma$ is called a *node* if t is endpoint of at least one of the arcs $\eta_1, \ldots, \eta_m$. The set of all nodes is denoted by $T = T(\Gamma)$. Note that the partition of Γ into simple oriented arcs is not unique. Hence, neither is the set T. For example, if Γ is a Jordan curve, we may represent it as the union of m simple oriented arcs for every number $m \ge 2$. In what follows we always assume that we are given a fixed representation $\Gamma = \eta_1 \cup \ldots \cup \eta_m$. In that case T is uniquely determined. Of course, in practice one strives to keep m minimal.

Denote by $\Phi(S_\Gamma, PC^n(\Gamma))$ the set of all operators in alg $(S_\Gamma, PC^n(\Gamma))$ which are Fredholm on $L^{p,n}(\Gamma, w)$. We will construct functions φ_{η_j} $(j = 1, \ldots, m)$ and φ_t $(t \in T)$ of $\Phi(S_\Gamma, PC^n(\Gamma))$ into $\mathbf{R}$ such that

$$\text{Ind}\, A = \sum_{j=1}^{m} \varphi_{\eta_j}(A) + \sum_{t \in T} \varphi_t(A) \tag{12.1}$$

for every $A \in \Phi(S_\Gamma, PC^n(\Gamma))$. It should be emphasized that in fact φ_{η_j} and φ_t (and thus also Ind A) depend on Γ, p, w:

$$\varphi_{\eta_j} = \varphi_{\eta_j, \Gamma, p, w}, \quad \varphi_t = \varphi_{t, \Gamma, p, w}.$$

We also remark that $\varphi_{\eta_j}(A)$ and $\varphi_t(A)$ need not be integers. The sum on the right of (12.1) will however always be an integer.

Let first $\eta \in \{\eta_1, \ldots, \eta_m\}$, denote the endpoints of η by t_1 and t_2, and put $\eta^0 := \eta \setminus \{t_1, t_2\}$. For $A \in \Phi(S_\Gamma, PC^n(\Gamma))$, define $\text{Sym}_{\,t,x}(A)$ as in Theorem 11.2. If

$$(t, x) \in \mathcal{M}(\eta^0, p, w) := \bigcup_{t \in \eta^0} \Big(\{t\} \times \mathcal{L}_t(\Gamma) \Big),$$

then $\text{Sym}_{\,t,x}(A)$ is a $2n \times 2n$ matrix and hence

$$\text{Sym}_{\,t,x}(A) = \begin{pmatrix} A_{11}(t,x) & A_{12}(t,x) \\ A_{21}(t,x) & A_{22}(t,x) \end{pmatrix}$$

with $n \times n$ matrices $A_{ij}(t,x)$. The matrices $\text{Sym}_{\,t,0}(A)$ and $\text{Sym}_{\,t,1}(A)$ are block-diagonal and hence $A_{11}(t,0)$ and $A_{22}(t,1)$ are invertible whenever $A \in \Phi(S_\Gamma, PC^n(\Gamma))$. For $(t,x) \in \mathcal{M}(\eta^0, p, w)$, put

$$A(t,x) := \frac{\det \text{Sym}_{\,t,x}(A)}{\det \big(A_{11}(t,0) A_{22}(t,1)\big)} \tag{12.2}$$

and $A_0(t) := A(t,0)$. The function A_0 is an invertible function in $PC(\eta)$. Since $\text{Sym}_{\,t,0}(A)$ and $\text{Sym}_{\,t,1}(A)$ are always block-diagonal, we conclude that

$$A(t,0) = \frac{\det A_{22}(t,0)}{\det A_{22}(t,1)}, \quad A(t,1) = \frac{\det A_{11}(t,1)}{\det A_{11}(t,0)} \tag{12.3}$$

and that the maps

$$\begin{aligned} A &\mapsto A_{22}(t,0), \quad A \mapsto A_{11}(t,1), \\ A &\mapsto A_{22}(t,1), \quad A \mapsto A_{11}(t,0) \end{aligned}$$

are Banach algebra homomorphisms. If $A \in \{S_\Gamma\} \cup \{aI \,:\, a \in PC^n(\Gamma)\}$, it can be straightforwardly verified that

$$\begin{aligned} A_{22}(t-0,0) &= A_{22}(t,0), \quad A_{11}(t+0,1) = A_{11}(t,1), \\ A_{22}(t-0,1) &= A_{22}(t,1), \quad A_{11}(t+0,0) = A_{11}(t,0). \end{aligned}$$

It follows that the latter equalities hold for all $A \in \operatorname{alg}(S_\Gamma, PC^n(\Gamma))$ and therefore (12.3) gives

$$A_0(t-0) = A(t-0,0) = A(t,0), \quad A_0(t+0) = A(t+0,0) = A(t,1) \tag{12.4}$$

for all $t \in \eta^0$ and all $A \in \Phi(S_\Gamma, PC^n(\Gamma))$. Denote by $\Lambda[A_0]$ the points of η^0 at which A_0 has a jump. Theorem 11.2 implies that if $t \in \Lambda[A_0]$, then

$$A(t,x) \neq 0 \text{ for all } x \in \mathcal{L}_t(\Gamma) = \mathcal{L}^0(p;\alpha_t,\beta_t). \tag{12.5}$$

Let $\mathcal{L}_t^{\#}$ be any continuous curve joining 0 to 1 and staying within $\mathcal{L}^0(p;\alpha_t,\beta_t)$. For example, we can take $\mathcal{L}_t^{\#} = \mathcal{L}^0(p;\psi_t,\psi_t)$ where $\psi_t : \mathbf{R} \to \mathbf{R}$ is any continuous function such that $\alpha_t(x) \le \psi_t(x) \le \beta_t(x)$ for all $x \in \mathbf{R}$. Note that $\psi_t = (\alpha_t + \beta_t)/2$ is such a function. For each $t \in \Lambda[A_0]$ join the endpoints (12.3) of the jump of A_0 at t by the curve $\{A(t,x) : x \in \mathcal{L}_t^{\#}\}$ and orient this curve from $A(t,0)$ to $A(t,1)$. This construction gives us a continuous and oriented curve

$$A_\eta^{\#} := A_0(\eta \setminus \Lambda[A_0]) \cup \bigcup_{t\in\Lambda[A_0]} \{A(t,x) \ : \ x \in \mathcal{L}_t^{\#}\}$$

joining

$$A_0(t_1+0) := \lim_{t\to t_1} A_0(t) \quad \text{and} \quad A_0(t_2-0) := \lim_{t\to t_2} A_0(t)$$

(see [39] for the continuity of the curve). The origin does not belong to $A_\eta^{\#}$. Let $\{\arg z\}_{z\in A_\eta^{\#}}$ denote the increment of any continuous branch of the argument as $A_\eta^{\#}$ is traced out from $A_0(t_1+0)$ to $A_0(t_2-0)$. We finally put

$$\varphi_\eta(A) := -\frac{1}{2\pi}\{\arg z\}_{z\in A_\eta^{\#}} \tag{12.6}$$

and have thus defined the functions φ_{η_j} in (12.1).

Now let $t \in T$. If $A \in \Phi(S_\Gamma, PC^n(\Gamma))$, then the $N(t)n \times N(t)n$ matrix $\operatorname{Sym}_{t,x}(A)$ is invertible for all $x \in \mathcal{L}_t(\Gamma) = \mathcal{L}^0(p;\alpha_t,\beta_t)$ by virtue of Theorem 11.2. Define a curve $\mathcal{L}_t^{\#} \subset \mathcal{L}^0(p;\alpha_t,\beta_t)$ joining 0 to 1 as in the preceding paragraph. Then

$$A_t^{\#} := \{\det \operatorname{Sym}_{t,x}(A) \ : \ x \in \mathcal{L}_t^{\#}\}$$

is a continuous curve joining $\det \operatorname{Sym}_{t,0}(A)$ to $\det \operatorname{Sym}_{t,1}(A)$ which does not pass through the origin (for the continuity of the curve we refer again to [39]). Denote by $\{\arg z\}_{z\in A_t^{\#}}$ the increment of any continuous argument as $A_t^{\#}$ is traced out from $\det \operatorname{Sym}_{t,0}(A)$ to $\det \operatorname{Sym}_{t,1}(A)$ and define

$$\varphi_t(A) := -\frac{1}{2\pi}\{\arg z\}_{z\in A_t^{\#}}. \tag{12.7}$$

We so also know the functions φ_t in (12.1).

Theorem 12.1. *Let Γ be a composed Carleson curve, $p \in (1,\infty)$, and $w \in A_p(\Gamma)$. If $A \in \operatorname{alg}(S_\Gamma, PC^n(\Gamma))$ is Fredholm on $L^{p,n}(\Gamma,w)$, then the index of A is given by* (12.1), (12.6), (12.7).

The (outline of the) proof will be divided into several steps: we first show Theorem 12.1 for Jordan curves, then for flowers and stars, and finally for general composed curves.

12.2. Jordan curves

In the case where Γ is a Jordan curve, the right-hand side of (12.1) may be written in another form. Given $A \in \Phi(S_\Gamma, PC^n(\Gamma))$, define $A(t,x)$ for $(t,x) \in \mathcal{M}(\Gamma, p, w)$ by (12.2) and put $A_0(t) := A(t,0)$ for $t \in \Gamma$. Denote by $A_\Gamma^\#$ the closed, continuous, and naturally oriented curve resulting from the essential range of A_0 by filling in the curve

$$\{A(t,x)\ :\ x \in \mathcal{L}_t^\#\}$$

between $A(t,0) = A_0(t-0)$ and $A(t,1) = A_0(t+0)$ for each $t \in \Lambda[A_0]$. The curve $A_\Gamma^\#$ does not pass through the origin, and hence its winding number about the origin, wind $(A_\Gamma^\#, 0)$, is well-defined. Note that both $A_\Gamma^\#$ and wind $(A_\Gamma^\#, 0)$ actually depend also on p and w.

Proposition 12.2. *Let Γ be a Carleson Jordan curve, $p \in (1,\infty), w \in A_p(\Gamma)$, and $A \in \Phi(S_\Gamma, PC^n(\Gamma))$. If $\Gamma = \eta_1 \cup \ldots \cup \eta_m$ is any partition of Γ into oriented simple arcs, then the right-hand side of* (12.1) *equals* $-\mathrm{wind}\,(A_\Gamma^\#, 0)$.

Proof. For simplicity, choose two different points t_1 and t_2 on Γ, let η_1 be the arc from t_1 to t_2, and denote by η_2 the arc from t_2 to t_1. Then $T = \{t_1, t_2\}$. We have to show that

$$\varphi_{\eta_1}(A) + \varphi_{\eta_2}(A) + \varphi_{t_1}(A) + \varphi_{t_2}(A) = -\mathrm{wind}\,(A_\Gamma^\#, 0). \tag{12.8}$$

For $t \in \Gamma$, put $c_t := \det\big(A_{11}(t,0)A_{22}(t,1)\big)$. The curve $A_{\eta_1}^\#$ joins $A_0(t_1+0) = A(t_1,1)$ to $A_0(t_2-0) = A(t_2,0)$ and coincides with $A_\Gamma^\#$ between these points. The curve $A_{t_2}^\#$ joins $\det \mathrm{Sym}_{\,t_2,0}(A) = c_{t_2}A(t_2,0)$ to $\det \mathrm{Sym}_{\,t_2,1}(A) = c_{t_2}A(t_2,1)$ along

$$\{\det \mathrm{Sym}_{\,t_2,x}(A) : x \in \mathcal{L}_{t_2}^\#\} = \{c_{t_2}A(t_2,x) : x \in \mathcal{L}_{t_2}^\#\},$$

while $A_\Gamma^\#$ joins $A_0(t_2-0) = A(t_2,0)$ to $A_0(t_2+0) = A(t_2,1)$ along $\{A(t_2,x) : x \in \mathcal{L}_{t_2}^\#\}$. Clearly, the increment of $\arg z$ as z traces out $\{c_{t_2}A(t_2,x) : x \in \mathcal{L}_{t_2}^\#\}$ is equal to the increment of $\arg z$ as moves on $\{A(t_2,x) : x \in \mathcal{L}_{t_2}^\#\}$. The arc η_2 and the point t_1 may be considered analogously. Consequently, the left-hand side of (12.8) is

$$-\frac{1}{2\pi}\{\arg z\}_{z\in A_\Gamma^\#} = -\mathrm{wind}\,(A_\Gamma^\#, 0).$$

□

We now consider a special case. Let $A = aP_\Gamma + Q_\Gamma$ with a (scalar-valued) function $a \in PC(\Gamma)$. For $x \notin \{0,1\}$ we have

$$\mathrm{Sym}_{\,t,x}(A) = \begin{pmatrix} a(t+0)x + (1-x) & (1-a(t+0))(1-x) \\ (1-a(t-0))x & x + a(t-0)(1-x) \end{pmatrix},$$

hence $A(t,x) = a(t+0)x + a(t-0)(1-x)$, and thus $A_\Gamma^\#$ is the closed, continuous, and naturally oriented curve resulting from the essential range of a by filling in the curve

$$\{a(t+0)x + a(t-0)(1-x) \;:\; x \in \mathcal{L}_t^\#\}$$

between $a(t-0)$ and $a(t+0)$ for each t at which a has a jump.

Proposition 12.3. *Let Γ be a Carleson Jordan curve, $p \in (1,\infty)$, $w \in A_p(\Gamma)$, and $a \in PC(\Gamma)$. The operator $A := aP_\Gamma + Q_\Gamma$ is Fredholm on $L^p(\Gamma, w)$ if and only if $0 \notin A_\Gamma^\#$, in which case*

$$\text{Ind}\, A = -\text{wind}\,(A_\Gamma^\#, 0). \tag{12.9}$$

Proof. This is Theorem 3.1 of [8] (note that the Fredholm criterion is also an immediate consequence of Theorem 4.2). □

Now let $A := aP_\Gamma + Q_\Gamma$ with $a \in PC^n(\Gamma)$. Then

$$\text{Sym}_{\,t,x}(A) = \begin{pmatrix} a(t+0)x + (1-x)I & (I - a(t+0))(1-x) \\ (I - a(t-0))x & Ix + a(t-0)(1-x) \end{pmatrix},$$

for $x \notin \{0,1\}$, and it is easily seen that

$$A(t,x) = \det \text{Sym}_{\,t,x}(A) = \det(a(t+0)x + a(t-0)(1-x)).$$

Consequently, $A_\Gamma^\#$ is now obtained from the essential range of $\det a$ by filling in the curve

$$\{\det(a(t+0)x + a(t-0)(1-x)) \;:\; x \in \mathcal{L}_t^\#\}$$

whenever $a(t-0) \neq a(t+0)$.

Proposition 12.4. *Let Γ be a Carleson Jordan curve, $p \in (1,\infty)$, $w \in A_p(\Gamma)$, $a \in PC^n(\Gamma)$. The operator $A := aP_\Gamma + Q_\Gamma$ is Fredholm on $L^{p,n}(\Gamma, w)$ if and only if $0 \notin A_\Gamma^\#$. In that case* (12.9) *holds.*

Proof. Assume first that the set $\Lambda(a)$ of the points $t \in \Gamma$ at which a has a jump is finite. Let A be Fredholm. Then $0 \notin A_\Gamma^\#$ by Theorem 11.2. In particular, $a \in PC^n(\Gamma)$ is invertible. By a result of [29] (also see [15, p. 171] or [42, p. 199]), we may represented a in the form

$$a = fbg \tag{12.10}$$

where f and g are invertible matrix functions in $C^n(\Gamma)$ and b is an invertible upper-triangular matrix function in $PC^n(\Gamma)$ whose diagonal entries $b_1, \dots, b_n$ are continuous on $\Gamma \setminus \Lambda(a)$. As in the proof of Lemma 11.5 we get

$$P_\Gamma a P_\Gamma + Q_\Gamma = (P_\Gamma f P_\Gamma + Q_\Gamma)(P_\Gamma b P_\Gamma + Q_\Gamma)(P_\Gamma g P_\Gamma + Q_\Gamma) + K \tag{12.11}$$

with $K \in \mathcal{K}(\Gamma)$. An invertible matrix function $h \in C^n(\Gamma)$ is homotopic to the diagonal matrix $\text{diag}\,(\det h, 1, \dots, 1)$ within the invertible matrix functions of $C^n(\Gamma)$ (see, e.g., [43, Appendix 6 by B. Bojarski]). This implies that the operators $P_\Gamma f P_\Gamma + Q_\Gamma$ and

$P_\Gamma g P_\Gamma + Q_\Gamma$ are Fredholm of index $-\text{wind}\,(\det f, 0)$ and $-\text{wind}\,(\det g, 0)$, respectively. Since $P_\Gamma a P_\Gamma + Q_\Gamma$ is Fredholm together with $aP_\Gamma + Q_\Gamma$ and both operators have the same index (again recall the proof of Lemma 11.5), we see from (12.11) that $B := bP_\Gamma + Q_\Gamma$ is Fredholm and that

$$\text{Ind}\, A = -\text{wind}\,(\det f, 0) + \text{Ind}\, B - \text{wind}\,(\det g, 0). \tag{12.12}$$

Put $B_j := b_j P_\Gamma + Q_\Gamma$ $(j = 1, \dots, n)$. Because

$$\det \text{Sym}_{\,t,x}(B) \neq 0 \quad \text{for all} \quad (t, x) \in \mathcal{M},$$

it follows that

$$\det \text{Sym}_{\,t,x}(B_j) \neq 0 \quad \text{for all} \quad (t, x) \in \mathcal{M} \quad \text{and all} \quad j.$$

Thus, the operators B_j are Fredholm and

$$\text{Ind}\, B = \sum_{j=1}^{n} \text{Ind}\, B_j .$$

By Proposition 12.3, $\text{Ind}\, B_j = -\text{wind}\,((B_j)_\Gamma^\#, 0)$. Consequently,

$$\text{Ind}\, B = -\sum_{j=1}^{n} \text{wind}\,((B_j)_\Gamma^\#, 0) = -\text{wind}\,(B_\Gamma^\#, 0). \tag{12.13}$$

(the last equality of (12.13) needs a justification we renounce to give here). Combining (12.12) and (12.13) we arrive at (12.9).

For general $\Lambda[a]$ the assertion follows from the stability of $\text{Ind}\, A$ and $-\text{wind}\,(A_\Gamma^\#, 0)$ under perturbations by operators with small norm. □

By virtue of Proposition 12.2, the following theorem is Theorem 12.1 for Jordan curves.

Theorem 12.5. *Let Γ be a Carleson Jordan curve, $p \in (1, \infty)$, and $w \in A_p(\Gamma)$. If $A \in \Phi(S_\Gamma, PC^n(\Gamma))$ then $0 \notin A_\Gamma^\#$ and the index of A on $L^{p,n}(\Gamma, w)$ is given by* (12.9).

Proof. Assume first that $A = aP_\Gamma + bQ_\Gamma$ with $a, b \in PC^n(\Gamma)$. Since then

$$\det \big(A_{11}(t, 0) A_{22}(t, 1)\big) = \det \big(b(t + 0) b(t - 0)\big),$$

we get $A(t, x) = C(t, x)$ where $C := cP_\Gamma + Q_\Gamma$ with $c := b^{-1}a$. Hence, by Proposition 12.4,

$$\text{Ind}\, A = \text{Ind}\, C = -\text{wind}\,(C_\Gamma^\#, 0) = -\text{wind}\,(A_\Gamma^\#, 0).$$

Let now

$$A = \sum_{i=1}^{k} A_{i1} \dots A_{ir} \tag{12.14}$$

where $A_{ij} = a_{ij}P_\Gamma + b_{ij}Q_\Gamma$ and $a_{ij}, b_{ij} \in PC^n(\Gamma)$. Following [31], we associate an operator $\tilde{A} = \tilde{a}P_\Gamma + \tilde{b}Q_\Gamma$ with A such that $\tilde{a}, \tilde{b} \in PC^{n(k(r+1)+1)}(\Gamma)$ and

$$\text{(12.15)} \qquad \operatorname{Ind} A = \operatorname{Ind} \tilde{A}, \quad \det \operatorname{Sym}_{t,x}(A) = \det \operatorname{Sym}_{t,x}(\tilde{A}),$$

$$\text{(12.16)} \qquad \det A_{11}(t,0) = \det \tilde{A}_{11}(t,0), \quad \det A_{22}(t,1) = \det \tilde{A}_{22}(t,1).$$

The operator $\tilde{A}$ is called the linear dilation of A, and it will explicitly be given below. The equalities (12.15), (12.16) imply that $A(t,x) = \tilde{A}(t,x)$ and $A_\Gamma^\# = \tilde{A}_\Gamma^\#$. As Proposition 12.4 proves the index formula for $\tilde{A}$, we therefore get it for A.

Here is the linear dilation of A. Denote by O_m and I_m the $m \times m$ zero and identity matrix, respecively. For $l = 1, \ldots, r$, let B_l be the $kn \times kn$ matrix

$$B_l = \operatorname{diag}(A_{1l}, A_{2l}, \ldots, A_{kl}),$$

then define the $kn(r+1) \times kn(r+1)$ matrix Z by

$$Z = \begin{pmatrix} I_{kn} & B_1 & O_{kn} & \ldots & O_{kn} \\ O_{kn} & I_{kn} & B_2 & \ldots & O_{kn} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{kn} & O_{kn} & O_{kn} & \ldots & B_r \\ O_{kn} & O_{kn} & O_{kn} & \ldots & I_{kn} \end{pmatrix},$$

put

$$\begin{aligned} X &= \operatorname{column}(\underbrace{O_n, \ldots, O_n}_{kr}, \underbrace{-I_n, \ldots, -I_n}_{k}), \\ Y &= (\underbrace{I_n, \ldots, I_n}_{k}, \underbrace{O_n, \ldots, O_n}_{kr}), \end{aligned}$$

and finally, let

$$\tilde{A} = \begin{pmatrix} Z & X \\ Y & O_n \end{pmatrix}.$$

To establish (12.15) and (12.16), put

$$M_0 = (\underbrace{I_n, \ldots, I_n}_{k}),$$

for $l \in \{1, \ldots, r\}$ let

$$M_l = (A_{11}A_{12}\ldots A_{1l}, A_{21}A_{22}\ldots A_{2l}, \ldots, A_{k1}A_{k2}\ldots A_{kl}),$$

and define $W = (M_0, M_1, \ldots, M_r)$. It can be verified straightforwardly that

$$\text{(12.17)} \quad \begin{pmatrix} I_{nk(r+1)} & O \\ W & I_n \end{pmatrix} \begin{pmatrix} I_{nk(r+1)} & O \\ O & A \end{pmatrix} \begin{pmatrix} Z & X \\ O & I_n \end{pmatrix} = \begin{pmatrix} Z & X \\ Y & O_n \end{pmatrix} =: \tilde{A}.$$

The outer matrices on the left of (12.17) are obviously invertible. This gives (12.15) and (12.16).

To complete the proof, we remark that an operator $A \in \operatorname{alg}(S_\Gamma, PC^n(\Gamma))$ can be approximated by operators of the form (12.14) and that $\operatorname{Ind} A$ as well as $\operatorname{wind}(A_\Gamma^\#, 0)$ are stable in the operator norm; the stability of $\operatorname{wind}(A_\Gamma^\#, 0)$ is shown in [39]. $\square$

12.3. Flowers

Let F be a Carleson flower comprised of $N \geq 2$ counter-clockwise oriented Jordan curves $\Gamma_1, \dots, \Gamma_N$ and let t be the center of F. On each Γ_j choose a point $t_j \in \Gamma_j \setminus \{t\}$, let $\eta^*_{2j-1} \subset \Gamma_j$ be the positively oriented arc from t to t_j, and denote by $\eta^*_{2j} \subset \Gamma_j$ the positively oriented arc from t_j to t. Let further $p \in (1, \infty)$ and $w \in A_p(\Gamma)$.

Let $\Pi^n(F)$ stand for the set of all matrix functions a on F which are continuous on $F \setminus \{t\}$ and have finite limits

$$a_i := \lim_{\tau \to t, \tau \in \eta^*_i} a(\tau) \quad (i = 1, \dots, 2N).$$

Lemma 12.6. *If $a, b \in \Pi^n(F)$ and $A := aP_F + bQ_F$ is Fredholm on $L^{p,n}(F, w)$ then*

$$\begin{aligned} \det \operatorname{Sym}_{t,x}(A) &= \det \left(b_1 a_2 b_3 a_4 \dots b_{2N-1} a_{2N}\right) \times \\ &\times \det \left(x b_1^{-1} a_1 a_2^{-1} b_2 \dots b_{2N-1}^{-1} a_{2N-1} a_{2N}^{-1} b_{2N} + (1-x) I\right) \end{aligned} \tag{12.18}$$

for all $x \in \mathcal{L}_t(F)$.

Proof. Since

$$\operatorname{Sym}_{t,0}(A) = \operatorname{diag}(b_1, a_2, b_3, \dots, a_{2N}), \quad \operatorname{Sym}_{t,1}(A) = \operatorname{diag}(a_1, b_2, a_3, \dots, b_{2N}),$$

and these matrices are invertible in view of Theorem 11.2, the equality (12.18) is trivial for $x \in \{0, 1\}$. For $x \in \mathcal{L}_t(F) \setminus \{0, 1\}$, the matrix $\operatorname{Sym}_{t,x}(A)$ is

$$\begin{pmatrix} a_1 x + b_1(1-x) & (a_1 - b_1)(x-1) & (a_1 - b_1)x & \dots & (a_1 - b_1)(x-1) \\ (b_2 - a_2)x & a_2(1-x) + b_2 x & (b_2 - a_2)x & \dots & (b_2 - a_2)(x-1) \\ (a_3 - b_3)x & (a_3 - b_3)x & a_3 x + b_3(1-x) & \dots & (a_3 - b_3)(x-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (b_{2N} - a_{2N})x & (b_{2N} - a_{2N})x & (b_{2N} - a_{2N})x & \dots & a_{2N}(1-x) + b_{2N} x \end{pmatrix}.$$

Put

$$N_x = \begin{pmatrix} I & O & O & \dots & O & (1-x)I \\ -I & I & O & \dots & O & O \\ O & -I & I & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & I & O \\ O & O & O & \dots & -I & xI \end{pmatrix}.$$

Obviously, $\det N_x = 1$. A straightforward calculation yields

$$\operatorname{Sym}_{t,x}(A) \cdot N_x = \begin{pmatrix} a_1 & O & O & \dots & O & b_1(1-x) \\ -a_2 & b_2 & O & \dots & O & O \\ O & -b_3 & a_3 & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & a_{2N-1} & O \\ O & O & O & \dots & -a_{2N} & b_{2N} x \end{pmatrix}, \tag{12.19}$$

and the determinant of the matrix on the right of (12.19) coincides with the right-hand side of (12.18). □

Proposition 12.7. *Let $a, b \in \Pi^n(F)$ and suppose $A := aP_F + bQ_F$ is Fredholm on $L^{p,n}(F, w)$. If $F = \eta_1 \cup \ldots \cup \eta_m$ is any partition of F into oriented simple arcs, then the right-hand side of* (12.1) *equals the sum of*

$$-\frac{1}{2\pi}\sum_{j=1}^{N}\{\arg\det(b^{-1}a)\}_{\Gamma_j\setminus\{t\}} \tag{12.20}$$

and

$$-\frac{1}{2\pi}\{\arg\det(xb_1^{-1}a_1a_2^{-1}b_2\ldots b_{2N-1}^{-1}a_{2N-1}a_{2N}^{-1}b_{2N} + (1-x)I)\}_{\mathcal{L}_t^{\#}}. \tag{12.21}$$

Proof. Since A is Fredholm, $\det \mathrm{Sym}_{\tau,x}(A) \neq 0$ for all $(\tau, x) \in \mathcal{M}(F, p, w)$. If $\tau \in \Gamma \setminus \{t\}$, then

$$A(\tau, 0) = \frac{\det \mathrm{Sym}_{\tau,0}(A)}{\det\big(A_{11}(\tau,0)A_{22}(\tau,1)\big)} = \frac{\det\big(b(\tau)a(\tau)\big)}{\det\big(b(\tau)b(\tau)\big)} = \det\big(b^{-1}(\tau)a(\tau)\big).$$

This shows that the first sum in (12.1) is independent of the partition $F = \eta_1 \cup \ldots \cup \eta_m$ and always equals (12.20). Taking $F = \eta_1^* \cup \ldots \cup \eta_{2N}^*$ as above, we see that the second sum in (12.1) reduces to

$$-\frac{1}{2\pi}\{\arg\det \mathrm{Sym}_{t,x}(A)\}_{\mathcal{L}_t^{\#}},$$

which, by Lemma 12.6, is just (12.21). □

Put

$$\begin{aligned}
\Pi_+^n(F) &:= \{a \in \Pi^n(F) \, : \, a_1 = a_2, \; a_3 = a_4, \; \ldots, \; a_{2N-1} = a_{2N}\}, \\
\Pi_-^n(F) &:= \{a \in \Pi^n(F) \, : \, a_2 = a_3, \; a_4 = a_5, \; \ldots, \; a_{2N} = a_1\}.
\end{aligned}$$

Of course, $\Pi_+^n(F) \cup \Pi_-^n(F) = C(F)$. The following result was established in [35] for nice curves. Note that this result also proves Lemma 7.3.

Lemma 12.8. *If $a \in \Pi_+^n(F)$ then $Q_F a P_F$ is compact, and if $a \in \Pi_-^n(F)$ then $P_F a Q_F$ is compact.*

Proof. Let $a \in \Pi_+^n(F)$. Then $a = \sum_{j=1}^N a_{2j}\chi_j^0 + c$ where χ_j^0 is the characteristic function of $\Gamma_j, c \in C(F)$, and $c(t) = 0$. Since $Q_F\chi_j^0 P_F = 0$ due to (7.16), we obtain that $Q_\Gamma a P_F = Q_\Gamma c P_F$. As $cS_F - S_F cI$ is compact for every $c \in C^n(F)$ and $Q_F P_F = 0$ (Theorem 7.1), it follows that $Q_F c P_F = (cS_F - S_F cI)P_F/2$ is compact.

In the case where $a \in \Pi_-^n(F)$ we can proceed as in the proof of Proposition 1.2 of [33, Chapter 10]; notice that in this proof the equality $g = g_1 \ldots g_N$ must be replaced by $g = g_1 \ldots g_{N-1}$. □

Lemma 12.9. *Formula* (12.1) *holds if $A := aP_F + bQ_\Gamma$ is Fredholm on $L^{p,n}(F, w)$ and $a, b \in \Pi_+^n(F)$ or $a, b \in \Pi_-^n(F)$.*

Proof. If $a, b \in \Pi_+^n(F)$ or $a, b \in \Pi_-^n(F)$, then the determinant in (12.21) is identically 1 and hence, by Proposition 12.7, we are left with verifying that

$$\text{Ind}\, A = -\frac{1}{2\pi}\sum_{j=1}^{N}\{\arg\det(b^{-1}a)\}_{\Gamma_j\setminus\{t\}}. \tag{12.22}$$

This can be done as in the proof of Theorem 1.1 of [33, Chapter 10] (also see [32]). □

Proposition 12.10. *If $a, b \in \Pi^n(F)$ and $A := aP_F + bQ_F$ is Fredholm then the index formula* (12.1) *is true.*

Proof. Due to Proposition 12.7, we have to show that Ind A is given by the sum of (12.20) and (12.21).

Suppose A is Fredholm. Then a and b are invertible matrix functions. Put $c := b^{-1}a$. In [32, Lemma 4.1] and [35, Lemma 2.4] it is shown that if $y \in \Pi^n(F)$ is any invertible matrix function such that $y_1 y_2^{-1} \dots y_{2N-1} y_{2N}^{-1}$ is similar to $c_1 c_2^{-1} \dots c_{2N-1} c_{2N}^{-1}$, then there is a representation $c = fyg$ with invertible matrix functions $f \in \Pi_-^n(F)$ and $g \in \Pi_+^n(F)$. Thus, if $y \in \Pi^n(F)$ is any invertible matrix function such that

$$y\big|\Gamma\setminus\Gamma_1 = I, \quad y_2 = I, \quad y_1 = c_1 c_2^{-1} \dots c_{2N-1} c_{2N}^{-1}, \tag{12.23}$$

then $c = fyg$ with invertible $f \in \Pi_-^n(F)$ and $g \in \Pi_+^n(F)$. We have

$$\begin{aligned} A &= b(cP_F + Q_F) = b(fygP_F + Q_F) \\ &= b\Big[f(yP_F + Q_F)(gP_F + f^{-1}Q_F) + (fy - f)(Q_F g P_F - P_F f^{-1} Q_F)\Big]. \end{aligned}$$

The operators $Q_F g P_F$ and $P_F f^{-1} Q_F$ are compact by Lemma 12.9. Thus, $(yP_F + Q_F)(gP_F + f^{-1}Q_F)$ is Fredholm together with A and has the same index as A. The operator $gP_F + f^{-1}Q_F$ is Fredholm, a regularizer being $g^{-1}P_F + fQ_F$. It follows that $yP_F + Q_F$ is Fredholm and that

$$\text{Ind}\, A = \text{Ind}\,(yP_F + Q_F) + \text{Ind}\,(gP_F + f^{-1}Q_F). \tag{12.24}$$

Since $y = I$ on $F \setminus \Gamma_1$ and the operator $yP_F + Q_F$ is Fredholm on $L^{p,n}(\Gamma, w)$, its restriction $yP_{\Gamma_1} + Q_{\Gamma_1}$ to $L^{p,n}(\Gamma_1, w|\Gamma_1)$ is Fredholm and

$$\text{Ind}\,(yP_F + Q_F) = \text{Ind}\,(yP_{\Gamma_1} + Q_{\Gamma_1}).$$

Proposition 12.4 in conjunction with (12.23) shows that

$$\begin{aligned} &\text{Ind}\,(yP_{\Gamma_1} + Q_{\Gamma_1}) = -\frac{1}{2\pi}\{\arg\det y\}_{\Gamma_1\setminus\{t\}} - \frac{1}{2\pi}\{\arg\det(xy_1 + (1-x)y_2)\}_{\mathcal{L}_t^{\#}} \\ &= -\frac{1}{2\pi}\sum_{j=1}^{N}\{\arg\det y\}_{\Gamma_j\setminus\{t\}} - \frac{1}{2\pi}\{\arg\det(xc_1c_2^{-1}\dots c_{2N-1}c_{2N}^{-1} + (1-x)I)\}_{\mathcal{L}_t^{\#}}. \end{aligned}$$

We now consider the second term on the right of (12.24). By Lemma 12.8,

$$\begin{aligned} gP_F + f^{-1}Q_F &= (gP_F + Q_F)(P_F + f^{-1}Q_F) + K_1 \\ &= (P_F + f^{-1}Q_F)(gP_F + Q_F) + K_2 \end{aligned}$$

with compact operators K_1, K_2. Thus, since $gP_F + f^{-1}Q_F$ is Fredholm, so also are $gP_F + Q_F$ and $P_F + f^{-1}Q_F$. Because $g \in \Pi^n_+(F)$ and $f \in \Pi^n_-(F)$, we can use Lemma 12.9 and (12.22) to get

$$\begin{aligned}
\operatorname{Ind}(gP_F + f^{-1}Q_F) &= \operatorname{Ind}(gP_F + Q_F) + \operatorname{Ind}(P_F + f^{-1}Q_F) \\
&= -\frac{1}{2\pi}\sum_{j=1}^{N}\left(\{\arg\det g\}_{\Gamma_j\setminus\{t\}} + \{\arg\det f\}_{\Gamma_j\setminus\{t\}}\right).
\end{aligned}$$

Inserting the expressions for $\operatorname{Ind}(yP_{\Gamma_1} + Q_{\Gamma_1})$ and $\operatorname{Ind}(gP_F + f^{-1}Q_F)$ we have obtained in (12.24), we obtain that $\operatorname{Ind} A$ equals

$$-\frac{1}{2\pi}\sum_{j=1}^{N}\{\arg\det(fyg)\}_{\Gamma_j\setminus\{t\}} - \frac{1}{2\pi}\sum_{j=1}^{N}\{\arg\det(xc_1c_2^{-1}\dots c_{2N-1}c_{2N}^{-1} + (1-x)I)\}_{\mathcal{L}_t^{\#}}.$$

Since $fyg = c = b^{-1}a$, this is the sum of (12.20) and (12.21). □

12.4. General composed curves

Lemma 12.11. *Let $\gamma = \bigcup_{j=1}^{N}\eta_j$ $(N \ge 1)$ be a Carleson star centered at t, let $p \in (1,\infty)$ and $w \in A_p(\gamma)$. Suppose $a, b \in PC^n(\gamma)$ are continuous on $\gamma\setminus\{t\}$ and $a(t_j) = b(t_j) = I$ at the endpoints t_j of η_j different from t. If the operator $A := aP_\gamma + bQ_\gamma$ is Fredholm on $L^{p,n}(\gamma, w)$ then*

$$\operatorname{Ind} A = -\frac{1}{2\pi}\sum_{j=1}^{N}\{\arg\det(b^{-1}a)\}_{\eta_j} - \frac{1}{2\pi}\{\arg\det \operatorname{Sym}_{t,x}(A)\}_{\mathcal{L}_t^{\#}}.$$

Proof. By Theorem 6.9, we can extend the Carleson star γ and the weight $w \in A_p(\gamma)$ to a Carleson flower $F = \bigcup_{j=1}^{N}\Gamma_j$ and a weight $W \in A_p(F)$. Assume $\eta_j \subset \Gamma_j$. Let

$$\tilde{a}(\tau) := \begin{cases} a(\tau) & \text{if } \tau \in \gamma \\ I & \text{if } \tau \in F\setminus\gamma \end{cases}, \quad \tilde{b}(\tau) := \begin{cases} b(\tau) & \text{if } \tau \in \gamma \\ I & \text{if } \tau \in F\setminus\gamma \end{cases}.$$

Then $\tilde{a}$ and $\tilde{b}$ belong to $\Pi^n(F)$. It is clear that $A = aP_\gamma + bQ_\gamma$ is Fredholm on $L^{p,n}(\gamma, w)$ if and only if $\tilde{A} := \tilde{a}P_F + \tilde{b}Q_F$ is Fredholm on $L^{p,n}(F, W)$. In that case $\operatorname{Ind} A = \operatorname{Ind}\tilde{A}$. One can easily verify that

$$\det \operatorname{Sym}_{t,x}(\tilde{A}) = \begin{cases} \det \operatorname{Sym}_{t,x}(A) & \text{if } (t,x) \in \mathcal{M}(\gamma, p, w) \\ 1 & \text{if } (t,x) \in \mathcal{M}(F, p, W)\setminus\mathcal{M}(\gamma, p, w) \end{cases}.$$

Thus, taking into account that $a(t_j) = b(t_j) = I$, we obtain from Proposition 12.10 that

$$\begin{aligned}
\operatorname{Ind} A = \operatorname{Ind}\tilde{A} &= -\frac{1}{2\pi}\sum_{j=1}^{N}\{\arg\det(\tilde{b}^{-1}\tilde{a})\}_{\Gamma_j\setminus\{t\}} - \frac{1}{2\pi}\{\arg\det \operatorname{Sym}_{t,x}(\tilde{A})\}_{\mathcal{L}_t^{\#}} \\
&= -\frac{1}{2\pi}\sum_{j=1}^{N}\{\arg\det(b^{-1}a)\}_{\eta_j} - \frac{1}{2\pi}\{\arg\det \operatorname{Sym}_{t,x}(A)\}_{\mathcal{L}_t^{\#}}. \quad \square
\end{aligned}$$

Proof of Theorem 12.1. We follow the proof of Theorem 4.2 of [35]. Suppose first that $A := cP_\Gamma + Q_\Gamma$ with an invertible matrix function $c \in PC^n(\Gamma)$.

Let $T_0 = \{z_k \,:\, k = 1, \ldots, r\}$ be a subset of Γ which contains the set T of all nodes and exactly one point $t_j \in \eta_j \setminus T$ for each $j = 1, \ldots, m$. For $z_k \in T_0$, choose open neighborhoods γ_k and γ_k' on Γ such that $\overline{\gamma_k'} \subset \gamma_k$, γ_k is a star with the center z_k, $\gamma_k \cap (T_0 \setminus \{z_k\}) = \emptyset$, and $\Gamma \subset \gamma_1' \cup \ldots \cup \gamma_r'$.

Assume c is continuous on $\Gamma \setminus T$. Let $c^{(1)}$ be any invertible matrix function in $PC^n(\Gamma)$ which equals c on γ_1', is I on $\Gamma \setminus \gamma_1$, and is continuous on the closure of $\gamma_1 \setminus \gamma_1'$. Of course, $c^{(1)}$ is continuous on $\Gamma \setminus \{z_1\}$, its limits $c_i^{(1)}(z_1)$ at z_1 coincide with the limits $c_i(z_1)$, and the limits of the matrix function $\left(c^{(1)}\right)^{-1} c$ at the points $z_2, \ldots, z_r$ are equal to the corresponding limits of c. For points in γ_1', the latter limits are I. Repeating the construction with the point z_2 and the matrix $\left(c^{(1)}\right)^{-1} c$ in place of c and then with the points $z_3, \ldots, z_r$, we arrive at a representation $c = c^{(1)} c^{(2)} \ldots c^{(r)}$ in which $c^{(k)}$ is continuous and invertible on $\Gamma \setminus \{z_k\}$,

$$c^{(k)} \big| \Gamma \setminus \gamma_k = I, \tag{12.25}$$

and $c^{(k)} \big| \gamma_k' = c \big| \gamma_k'$.

Put $A_k := c^{(k)} P_\Gamma + Q_\Gamma$. Since the matrices $c^{(1)}, c^{(2)}, \ldots, c^{(r)}$ do not have common points of discontinuity, the Fredholmness of A implies that the operators $A_1, A_2, \ldots, A_r$ are Fredholm and that

$$\operatorname{Ind} A = \operatorname{Ind} A_1 + \ldots + \operatorname{Ind} A_r.$$

From (12.25) we deduce that $A_k' := A_k \big| L^{p,n}(\gamma_k, w|\gamma_k)$ is Fredholm and that $\operatorname{Ind} A_k = \operatorname{Ind} A_k'$ for each k. Lemma 12.11 is applicable to the operators A_k' and gives

$$\operatorname{Ind} A_k' = -\frac{1}{2\pi} \sum_{j=1}^{N(z_k)} \{\arg \det c^{(k)}\}_{\eta_j} - \frac{1}{2\pi} \{\arg \det \operatorname{Sym}_{z_k, x}(A_k')\}_{\mathcal{L}^{\#}_{z_k}}, \tag{12.26}$$

where $N(z_k)$ is the number of simple arcs composing the star γ_k. In (12.26) we may delete the primes and replace $N(z_k)$ by m. Adding the resulting equalities we get

$$\operatorname{Ind} A = -\frac{1}{2\pi} \sum_{j=1}^{m} \{\arg \det c\}_{\eta_j} - \frac{1}{2\pi} \sum_{k=1}^{r} \{\arg \det \operatorname{Sym}_{z_k, x}(A_k)\}_{\mathcal{L}^{\#}_{z_k}}.$$

Since $\operatorname{Sym}_{z_k,x}(A_k) = \operatorname{Sym}_{z_k,x}(A)$ for all $x \in \mathcal{L}_{z_k}(\Gamma)$ and c is continuous at the points of $T_0 \setminus T$, we therefore have

$$\operatorname{Ind} A = -\frac{1}{2\pi} \sum_{j=1}^{m} \{\arg \det c\}_{\eta_j} - \frac{1}{2\pi} \sum_{t \in T} \{\arg \det \operatorname{Sym}_{t,x}(A)\}_{\mathcal{L}^{\#}_{t}}.$$

Equality (12.1) now follows from the observation that

$$\{\arg \det c\}_{\eta_j} = \{\arg z\}_{z \in A^{\#}_{\eta_j}}.$$

This completes the proof for operators $A = cP_\Gamma + Q_\Gamma$ whose coefficient is continuous on $\Gamma \setminus T$.

If c has a finite number Λ'_c of jumps at points of $\Gamma \setminus T$, we include Λ'_c into the set of nodes and may proceed as above to get the index formula. The case of infinitely many jumps can be disposed of by the usual approximation and stability argument.

The case where $A = aP_\Gamma + bQ_\Gamma$ with invertible matrix function $a, b \in PC^n(\Gamma)$ may be reduced to the above case by writing $A = b(b^{-1}aP_\Gamma + Q_\Gamma)$. If A is of the form (12.14), one can construct a linear dilation $\tilde{A} = \tilde{a}P_\Gamma + \tilde{b}Q_\Gamma$ which leaves both sides of (12.1) invariant (recall the proof of Theorem 12.5). Finally, taking into account that (12.1) is stable under small perturbations (see [39] for the case of Jordan curves), one gets the assertion for general $A \in \text{alg}\,(S_\Gamma, PC^n(\Gamma))$. □

13. Coburn's lemma

The following result is known as Coburn's lemma. It is in Coburn's paper [16] for $L^p(\Gamma, w) = L^2(\mathbf{T})$. For piecewise Lyapunov Jordan curves Γ, it was established by Simonenko [55, 57] (who did not know of the Coburn paper), and for composed Lyapunov curves Γ one can find the theorem in Gohberg and Krupnik's book [33, Chapter 7.5]. Gohberg and Krupnik also conjectured that the result is true in the general case.

Theorem 13.1. *Let Γ be a composed Carleson curve, $p \in (1, \infty)$, and $w \in A_p(\Gamma)$. Suppose a and b are functions in $L^\infty(\Gamma)$. Then the operator $aP_\Gamma + bQ_\Gamma$ is invertible on $L^p(\Gamma, w)$ if and only if it is Fredholm of index zero.*

This theorem divides the problem of deciding whether an operator of the form $aP_\Gamma + bQ_\Gamma$ is invertible into two "halves": into finding out whether it is Fredholm and into computing its index. Unfortunately, a theorem like Theorem 13.1 is not true for operators with matrix coefficients nor for generic operators in the algebra $\text{alg}\,(S_\Gamma, PC(\Gamma))$.

Theorem 13.1 can be proved by the arguments of [33, Chapter 7.5] provided Theorem 13.3 below is at our disposal. To prove this theorem, we need one more result on the extension of Muckenhoupt weights.

Theorem 13.2. *Let $\gamma = \eta_1 \cup \ldots \cup \eta_N$ be a Carleson star and let $F = \Gamma_1 \cup \ldots \cup \Gamma_N$ be any Carleson flower such that the arc η_k is contained in the Jordan curve Γ_k ($k = 1, \ldots, N$). If $w \in A_p(\gamma)$, then there exists a weight $W \in A_p(F)$ such that $W|\gamma = w$.*

Thus, while in Theorem 6.9 we extend γ and w to F and W, we now are given F and have to extend w from γ to F. Obviously, Theorem 6.2 and Theorem 13.2 imply Theorem 6.9. Since Theorem 6.9 is just the result we needed to prove the theorems of Section 4, we accompanied it with a detailed proof. As for the proof of Theorem 13.2, we confine ourselves to sketching the basic idea.

Proof. (outline) Using the arguments of Section 6, all we need to do is to show that $F \setminus \gamma$ can be divided into pieces Γ_k and that to each Γ_k we can associate a set $A_k \subset \gamma$ so that

the sets A_k have bounded overlaps,
$\operatorname{diam}(\Gamma_k) \simeq \operatorname{dist}(\Gamma_k, \gamma)$,
$|A_k| \geq C_1 \operatorname{diam}(\Gamma_k)$,
$\operatorname{diam}(A_k) \leq C_2 \operatorname{diam}(\Gamma_k)$,
$\operatorname{dist}(A_k, F \setminus \gamma) \leq C_3 \operatorname{diam}(\Gamma_k)$,

where C_1, C_2, C_3 are finite positive constants independent of k. Given this, the definition of $\tilde{w}$ and Hölder's inequality argument go through exactly as in Section 6.

Put $\Omega := \mathbf{C} \setminus \gamma$ and let $\{Q_j\}$ be a Whitney decomposition of Ω, i.e. suppose each Q_j is a square, Q_j and Q_k have at most pieces of edges in common if $j \neq k, \Omega = \bigcup_j Q_j$, and $\operatorname{diam}(Q_j) \simeq \operatorname{dist}(Q_j, \partial\Omega)$. Let λQ_j denote the concentric square with $\operatorname{diam}(\lambda Q_j) = \lambda \operatorname{diam}(Q_j)$.

Now consider just the Whitney squares Q_j which hit $F \setminus \gamma$. Let Γ_k be any measurable pairwise disjoint subsets of $F \setminus \gamma$ such that

$$F \setminus \gamma = \bigcup_k \Gamma_k \quad \text{and} \quad \Gamma_k \subset (F \setminus \gamma) \cap \frac{4}{3} Q_k.$$

Given Γ_k, denote by $\mathcal{C}_k$ the Whitney squares $Q_j \neq Q_k$ such that $\operatorname{diam}(Q_j) \leq \operatorname{diam}(Q_k)$ and $10Q_j \cap 10Q_k \neq \emptyset$. Define

$$n(x) := \sum_{Q_j \in \mathcal{C}_k} \chi_{\gamma \cap 10Q_j}(x) \quad \text{and} \quad A_k := \{x \in \gamma \cap 10Q_j \,:\, n(x) < \lambda\}.$$

One can show that if $\lambda > 0$ is large enough, then the sets Γ_k and A_k we have constructed possess all the desired properties. □

Given a Jordan curve Γ, denote by $D(\Gamma)$ the bounded connected component of $\mathbf{C} \setminus \Gamma$ and let $\overline{D(\Gamma)} := D(\Gamma) \cup \Gamma$. Recall that we always assume that a Jordan curve is oriented counter-clockwise. We call a composed curve G a *loop curve* if it may be represented in the form $G = \Gamma_1 \cup \ldots \cup \Gamma_m$ where $\Gamma_1, \ldots, \Gamma_m$ are Jordan curves such that $\overline{D(\Gamma_j)}$ and $\overline{D(\Gamma_k)}$ have at most finitely many points in common whenever $j \neq k$. Obviously, flowers are examples of loop curves. Here is a generalizations of Theorem 6.9.

Theorem 13.3. *Let Γ be a composed Carleson curve, $p \in (1, \infty)$, and $w \in A_p(\Gamma)$. There exists a Carleson loop curve G and a weight $W \in A_p(G)$ such that $\Gamma \subset G$ and $W|\Gamma = w$.*

Proof. Write $\Gamma = \eta_1 \cup \ldots \cup \eta_m$ with simple oriented arcs having at most endpoints in common, put $E := \{\eta_1, \ldots, \eta_m\}$, and denote by T the set of nodes of Γ (recall the first paragraph of Section 12.1). Let Ω be an arbitrary connected component of $\mathbf{C} \setminus \Gamma$. We say that an arc $\eta_j \in E$ as of type Ω_+ if $\eta_j \subset \partial\Omega$ and there exists an arc $\eta_j' \subset \Omega$ such that $\eta_j \cup \eta_j'$ is a Jordan curve. Otherwise we call η_j an arc of type Ω_-. Note that clearly every η_j may be extended to a (counter-clockwise oriented!) Jordan curve $\eta_j \cap \eta_j'$, but that it is precisely the arcs of type Ω_+ for which this is possible with $\eta_j' \subset \Omega$. Since Ω is a simply connected domain in $\mathbf{C} \cup \{\infty\}$, we can use Theorem 6.1 and proceed as in the proof of Theorem 6.2 to extend each arc of type Ω_+ to Carleson

Jordan curve $\eta_j \cup \eta_j'$ such that $\eta_j' \subset \Omega$ and the closures of $D(\eta_i \cup \eta_i')$ and $D(\eta_l \cup \eta_l')$ have at most finitely many points in common if $i \neq l$.

Each arc $\eta_j \in E$ is of type Ω_+ for exactly one connected component of $\mathbf{C} \setminus \Gamma$. Thus, doing the above construction for each connected component of $\mathbf{C} \setminus \Gamma$ we arrive at a Carleson loop curve $G = (\eta_1 \cup \eta_1') \cup \ldots \cup (\eta_m \cup \eta_m')$ containing Γ.

Let $\eta_j \in E$ and denote by t_1 and t_2 the endpoints of η_j. We divide the associated arc η_j' into three disjoint subarcs

$$\eta_j' = \eta_j'(t_1) \cup \mu_j' \cup \eta_j'(t_2)$$

so that $t_1 \in \eta_j'(t_1)$ and $t_2 \in \eta_j'(t_2)$.

We now extend $w \in A_p(\Gamma)$ to a weight $W \in A_p(G)$. Let $t \in T$. Denote by $\eta_{i_1}, \ldots, \eta_{i_n}$ the arcs in E having the endpoint t. Then $\gamma_t := \eta_{i_1} \cup \ldots \cup \eta_{i_n}$ is a Carleson star and

$$F_t := (\eta_{i_1} \cup \eta_{i_1}') \cup \ldots \cup (\eta_{i_n} \cup \eta_{i_n}')$$

is a Carleson flower containing γ_t. By Theorem 13.2, we can extend $w|\gamma_t$ to a weight $W_t \in A_p(F_t)$ such that $W_t|\gamma_t = w|\gamma_t$. Denote by W_t' the restriction of W_t to

$$\gamma_t \cup \eta_{i_1}'(t) \cup \ldots \cup \eta_{i_n}'(t).$$

Repeating this construction for each node $t \in T$, we get a weight W' on

$$G' := \Gamma \cup \bigcup_{j=1}^{m} (\eta_j' \setminus \mu_j')$$

such that $W'|\Gamma = w$ and $W' \in A_p(G')$.

It remains to define W on the middle pieces μ_j' of the arcs η_j' we have added. If η_j has the endpoints t_1 and t_2, then $\gamma_j := \eta_j'(t_1) \cup \eta_j \cup \eta_j'(t_2)$ is a simple arc (and thus a star) and $\eta_j \cup \eta_j' = \gamma_j \cup \mu_j'$ is a Jordan curve (and thus a flower) extending γ_j. By virtue of Theorem 13.2, the weight $W'|\gamma_j$ can be extended to a weight W_j on $\gamma_j \cup \mu_j'$ such that $W_j \in A_p(\gamma_j \cup \mu_j')$. Clearly, the weight W on G defined by $W|\eta_j \cup \eta_j' = W_j$ extends w and belongs to $A_p(G)$. □

As already said, once Theorem 13.3 is available, Theorem 13.1 can be proved by the methods developed in [33] for the case of composed Lyapunov curve. Given $A := aP_\Gamma + bQ_\Gamma$, one first extends Γ and w to a Carleson loop curve G and a weight $W \in A_p(G)$ and then one considers $\tilde{A} := \tilde{a}P_G + \tilde{b}Q_G \in \mathcal{B}(L^p(G, W))$ with

$$\tilde{a} := a\chi_\Gamma + \chi_{G\setminus\Gamma}, \quad \tilde{b} := b\chi_\Gamma + \chi_{G\setminus\Gamma}.$$

One can easily see that

$$\dim \operatorname{Ker} A = \dim \operatorname{Ker} \tilde{A} \quad \text{and} \quad \dim \operatorname{Ker} A^* = \dim \operatorname{Ker} \tilde{A}^*,$$

where the adjoint operators act on $L^q(\Gamma, w^{-1})$ and $L^q(G, W^{-1})$ $(1/p + 1/q = 1)$. This reduces the proof of Theorem 13.1 to the proof of this theorem for the case where $\Gamma = G$ is a loop curve. Proceeding as in the proof of Theorem 7.1 one obtains that if G is a Carleson loop curve, then $S_G^2 = I$ and thus $P_G^2 = P_G$. Taking into account that

$$\tilde{A}^* = H_G(Q_G\tilde{a} + P_G\tilde{b})H_G \quad \text{on} \quad L^q(G, W^{-1}),$$

where $(H_G f)(t) := e^{-i\Theta_G(t)}\overline{f(t)}$ and $\Theta_G(t)$ is the angle between the (almost everywhere existing) tangent to G at t and the positively oriented real axis, and using the Lusin-Privalov theorem, one finally gets the following theorem.

Theorem 13.4. *Let Γ be a composed Carleson curve, $p \in (1,\infty)$, $w \in A_p(\Gamma)$. Let further a and b be functions in $L^\infty(\Gamma)$, consider the operator $A := aP_\Gamma + bQ_\Gamma$ on $L^p(\Gamma,w)$ and its adjoint operator $A^* = H_\Gamma(Q_\Gamma a + P_\Gamma b)H_\Gamma$ on $L^q(\Gamma,w^{-1})$. If neither a nor b vanishes identically and*

$$|\{t \in \Gamma \ : \ a(t) = b(t) = 0\}| = 0,$$

then $\dim \operatorname{Ker} A = 0$ *or* $\dim \operatorname{Ker} A^* = 0$.

If $a, b \in L^\infty(\Gamma)$ and $aP_\Gamma + bQ_\Gamma$ is Fredholm, then a and b can be shown to be invertible (this is the Hartman-Winter-Simonenko theorem). Thus, Theorem 13.4 implies Theorem 13.1.

14. Spectrum of the Cauchy singular integral operator

Let Γ be a composed Carleson curve, $p \in (1,\infty)$, and $w \in A_p(\Gamma)$. The essential spectrum of S_Γ on $L^p(\Gamma,w)$ is given by Theorem 4.4. Since $S_\Gamma = P_\Gamma - Q_\Gamma$, we may invoke Theorems 12.1 and 13.1 to determine the spectrum of S_Γ on $L^p(\Gamma,w)$:

$$\operatorname{sp} S_\Gamma = \operatorname{sp}_{\mathrm{ess}} S_\Gamma \cup \{\lambda \in \mathbf{C} \setminus \operatorname{sp}_{\mathrm{ess}} S_\Gamma \ : \ \operatorname{Ind}(S_\Gamma - \lambda I) \neq 0\}.$$

So let us compute the index of $S_\Gamma - \lambda I$.

An indicator leaf $\mathcal{L}^0(p;\alpha,\beta)$ is always a connected compact set containing the points 0 and 1. Given points $z, w \in \mathbf{C}$, we put

$$\mathcal{L}^0(z,w;p;\alpha,\beta) := \{(1-x)z + xw \ : \ x \in \mathcal{L}^0(p;\alpha,\beta)\}. \tag{14.1}$$

For $t \in \Gamma$, let $\mathcal{L}_t^\# \subset \mathcal{L}_t(\Gamma) = \mathcal{L}^0(p;\alpha_t,\beta_t)$ be a curve as in Section 12.1. For z and w in $\mathbf{C}$, we define

$$\mathcal{L}_t^\#(z,w) := \{(1-x)z + xw \ : \ x \in \mathcal{L}_t^\#\},$$

and we think of $\mathcal{L}_t^\#(z,w)$ as being oriented from z to w.

Write $\Gamma = \eta_1 \cup \ldots \cup \eta_m$ as in Section 12.1 and denote by $T = \{t_1,\ldots,t_n\}$ the set of nodes. Let ε_j be the valency of t_j, put $\sigma_0 = 0$ and $\sigma_j = \varepsilon_1 + \ldots + \varepsilon_j$ for $j \in \{1,\ldots,n\}$. The points -1 and 1 always belong to $\operatorname{sp}_{\mathrm{ess}} S_\Gamma$. So let $\lambda \in \mathbf{C} \setminus \{-1,1\}$ and put $\mu := (\lambda-1)/(\lambda+1)$. Consider the set

$$\Phi_\Gamma^\#(\lambda) := \Phi_{\Gamma,w,\lambda}^\#(\lambda) := \bigcup_{j=1}^n \mathcal{L}^0(\mu^{\sigma_{j-1}},\mu^{\sigma_j};p;\alpha_{t_j},\beta_{t_j})$$

and the curve

$$C_\Gamma^\#(\lambda) := C_{\Gamma,p,w}^\#(\lambda) := \bigcup_{j=1}^n \mathcal{L}_{t_j}^\#(\mu^{\sigma_{j-1}},\mu^{\sigma_j}).$$

Obviously, the curve $C_\Gamma^\#(\lambda)$ is entirely contained in the "festoon" $\Phi_\Gamma^\#(\lambda)$. The curve $C_\Gamma^\#(\lambda)$ is a continuous oriented curve joining $1 = \mu^0$ via $\mu^{\sigma_1}, \mu^{\sigma_2}, \ldots, \mu^{\sigma_{n-1}}$ to μ^{σ_n}. Since $\sigma_n = \varepsilon_1 + \ldots + \varepsilon_n = 0$, it follows that $C_\Gamma^\#(\lambda)$ is a closed curve starting and terminating at the point 1.

Theorem 14.1. *Let Γ be a composed Carleson curve, $p \in (1,\infty)$, and $w \in A_p(\Gamma)$. A point $\lambda \in \mathbf{C} \setminus \{-1,1\}$ lies in the complement of the essential spectrum of S_Γ on $L^p(\Gamma, w)$ if and only if $\Phi_\Gamma^\#(\lambda)$ does not contain the origin. In that case*

$$\operatorname{Ind}(S_\Gamma - \lambda I) = -\operatorname{wind}(C_\Gamma^\#(\lambda), 0). \tag{14.2}$$

Proof. Let first $t \in \Gamma \setminus T$. Then, by Theorem 4.2,

$$\operatorname{Sym}_{t,x}(S_\Gamma - \lambda I) = \begin{pmatrix} 2x-1-\lambda & 2x-2 \\ -2x & 1-2x-\lambda \end{pmatrix}$$

and the determinant of this matrix is $\lambda^2 - 1$. Thus, $\operatorname{Sym}_{t,x}(S_\Gamma - \lambda I)$ is invertible for all $x \in \mathcal{L}_t(\Gamma)$ and from (12.2) we get

$$(S_\Gamma - \lambda I)(t,x) = \frac{\lambda^2-1}{(-1-\lambda)(-1-\lambda)} = \frac{\lambda-1}{\lambda+1} =: \mu,$$

whence $(S_\Gamma - \lambda I)_0(t) = \mu$ for all $t \in \Gamma \setminus T$. This shows that the first sum in (12.1) is zero.

Now pick $t_j \in T$. From the proof of Theorem 4.4 we know that $\det \operatorname{Sym}_{t_j,x}(S_\Gamma - \lambda I)$ equals

$$\begin{array}{ll} \left(\lambda^2-1\right)^{(N_j-\varepsilon_j)/2}\left(x(1-\lambda)^{\varepsilon_j} + (1-x)(-1-\lambda)^{\varepsilon_j}\right) & \text{if} \quad \varepsilon_j > 0, \\ \left(\lambda^2-1\right)^{(N_j+\varepsilon_j)/2}\left((1-x)(1-\lambda)^{-\varepsilon_j} + x(-1-\lambda)^{-\varepsilon_j}\right) & \text{if} \quad \varepsilon_j < 0, \\ \left(\lambda^2-1\right)^{N_j/2} & \text{if} \quad \varepsilon_j = 0. \end{array}$$

Thus, if $\varepsilon_j > 0$, letting $\Delta_j(\lambda) := (\lambda^2-1)^{(N_j-\varepsilon_j)/2}$ we see that $(S_\Gamma - \lambda I)_{t_j}^\#$ is equal to

$$\begin{aligned} &\left\{\det \operatorname{Sym}_{t_j,x}(S_\Gamma - \lambda I) \; : \; x \in \mathcal{L}_{t_j}^\#\right\} \\ &= \Delta_j(\lambda)\left\{x(1-\lambda)^{\varepsilon_j} + (1-x)(-1-\lambda)^{\varepsilon_j} \; : \; x \in \mathcal{L}_{t_j}^\#\right\} \\ &= \Delta_j(\lambda)(-1-\lambda)^{\varepsilon_j}\left\{x\mu^{\varepsilon_j} + (1-x)\cdot 1 \; : \; x \in \mathcal{L}_{t_j}^\#\right\} \\ &= \Delta_j(\lambda)(-1-\lambda)^{\varepsilon_j}\mu^{-\sigma_{j-1}}\left\{x\mu^{\sigma_j} + (1-x)\mu^{\sigma_{j-1}} \; : \; x \in \mathcal{L}_{t_j}^\#\right\} \\ &=: \Delta_j(\lambda)(-1-\lambda)^{\varepsilon_j}\mu^{-\sigma_{j-1}}\mathcal{L}_{t_j}^\#(\mu^{\sigma_{j-1}}, \mu^{\sigma_j}). \end{aligned} \tag{14.3}$$

Analogously, for $\varepsilon_j > 0$ we deduce from Theorem 4.2(b) that

$$\lambda \notin \operatorname{sp}\pi_{t_j}(S_\Gamma) \iff 0 \notin \Delta_j(\lambda)(-1-\lambda)^{\varepsilon_j}\mu^{-\sigma_{j-1}}\mathcal{L}^0(\mu^{\sigma_{j-1}}, \mu^{\sigma_j}; p; \alpha_{t_j}, \beta_{t_j}). \tag{14.4}$$

Hence, if $\lambda \notin \mathrm{sp}_{\mathrm{ess}}\, S_\Gamma$ then (14.3) gives

$$-\frac{1}{2\pi}\{\arg z\}_{z\in(S_\Gamma-\lambda I)^{\#}_{t_j}} = -\frac{1}{2\pi}\{\arg z\}_{z\in\mathcal{L}^{\#}_{t_j}(\mu^{\sigma_{j-1}},\mu^{\sigma_j})}. \tag{14.5}$$

Both (14.4) and (14.5) also hold for $\varepsilon_j \le 0$. From (14.4) we deduce that $S_\Gamma - \lambda I$ is Fredholm if and only if $0 \notin \Phi^{\#}_\Gamma(\lambda)$, and from (14.5) and (12.1), (12.7) we obtain

$$\mathrm{Ind}\,(S_\Gamma - \lambda I) = -\frac{1}{2\pi}\sum_{j=1}^{n}\{\arg z\}_{z\in\mathcal{L}^{\#}_{t_j}(\mu^{\sigma_{j-1}},\mu^{\sigma_j})} = -\mathrm{wind}\,(C^{\#}_\Gamma(\lambda),0).$$

□

The following theorem provides another index formula. For a node $t \in T$, let $\mathcal{L}_t(\Gamma) = \mathcal{L}^0(p;\alpha_t,\beta_t)$ be as above and for $k = 0,1,\dots,n-1$, define

$$\mathcal{L}^0_{t,k,n} := \mathcal{L}^0_{k,n} := \mathcal{L}^0\left(p;\frac{1}{n}\Big(\frac{1}{p}-\frac{n}{p}+k+T_n\alpha_t\Big),\frac{1}{n}\Big(\frac{1}{p}-\frac{n}{p}+k+T_n\beta_t\Big)\right) \tag{14.6}$$

as in Theorem 4.4. Given $z, w \in \mathbf{C}$, we put

$$\mathcal{L}^{\#}_{t,k,n}(z,w) := \left\{(1-x)z + xw \;:\; x \in \mathcal{L}^{\#}_{t,k,n}\right\},$$

where $\mathcal{L}^{\#}_{t,k,n} \subset \mathcal{L}^0_{t,k,n}$ is a curve as in Section 12.1, and we think of $\mathcal{L}^{\#}_{t,k,n}$ as being oriented from z to w.

Theorem 14.2. *Let Γ be a composed Carleson curve, $p \in (1,\infty)$, and $w \in A_p(\Gamma)$. Denote by T_* the set of all nodes $t \in T$ with valency $\varepsilon(t) \ne 0$. Then*

$$C^{\#}_\Gamma := C^{\#}_{\Gamma,p,w} := \bigcup_{t\in T_*}\bigcup_{k=0}^{|\varepsilon(t)|-1}\mathcal{L}^{\#}_{t,k,|\varepsilon(t)|}(-\mathrm{sign}\,\varepsilon(t),\mathrm{sign}\,\varepsilon(t))$$

is a closed, continuous, and oriented curve which is contained in $\mathrm{sp}_{\mathrm{ess}}\, S_\Gamma$. If $\lambda \in \mathbf{C}\setminus\mathrm{sp}_{\mathrm{ess}}\, S_\Gamma$, then

$$\mathrm{Ind}\,(S_\Gamma - \lambda I) = -\mathrm{wind}\,(C^{\#}_\Gamma,\lambda). \tag{14.7}$$

Remark. Since $\sum_{t\in T_*}\varepsilon(t) = 0$, the number of arcs of $C^{\#}_\Gamma$ joining -1 to 1 is equal to the number of arcs of $C^{\#}_\Gamma$ going from 1 to -1. Thus, the arcs of $C^{\#}_\Gamma$ may be traced out in a succession so that $C^{\#}_\Gamma$ becomes a closed, continuous, and oriented curve.

Proof. It is immediate from Theorem 4.4 that $C^{\#}_\Gamma \subset \mathrm{sp}_{\mathrm{ess}}\, S_\Gamma$. Let $\lambda \in \mathbf{C}\setminus\mathrm{sp}_{\mathrm{ess}}\, S_\Gamma$. Then the equality (14.2) of Theorem 14.1 holds. Fix a node $t_j \in T$. We have $\sigma_j = \sigma_{j-1} + \varepsilon_j$, where ε_j is the valency of t_j. If $\varepsilon_j = 0$, then $\mathcal{L}^{\#}_{t_j}(\mu^{\sigma_{j-1}},\mu^{\sigma_j})$ is a point and does not contribute to the winding number. We therefore need only consider nodes $t_j \in T_*$.

Suppose $\varepsilon_j > 0$. We have

$$\mathcal{L}^{\#}_{t_j}(\mu^{\sigma_{j-1}}, \mu^{\sigma_j}) = \mu^{\sigma_j} \mathcal{L}^{\#}_{t_j}(\mu^{-\varepsilon_j}, 1) = \mu^{\sigma_j} \mathcal{L}^{\#}_{t_j}\left(\left(\frac{\lambda+1}{\lambda-1}\right)^{\varepsilon_j}, 1\right),$$

and hence, the increment of $\arg z$ along $\mathcal{L}^{\#}_{t_j}(\mu^{\sigma_{j-1}}, \mu^{\sigma_j})$ is equal to the increment of $\arg z$ along

$$\mathcal{L}^{\#}_{t_j}\left(\left(\frac{\lambda+1}{\lambda-1}\right)^{\varepsilon_j}, 1\right) := \left\{\left(\frac{\lambda+1}{\lambda-1}\right)^{\varepsilon_j}(1-x) + x \;:\; x \in \mathcal{L}^{\#}_{t_j}\right\}, \tag{14.8}$$

where

$$\mathcal{L}^{\#}_{t_j} := \mathcal{L}^0(p; \psi_{t_j}, \psi_{t_j}) := \left\{\frac{e^{2\pi\gamma}}{e^{2\pi\gamma}-1} \;:\; \gamma \in Y(p; \psi_{t_j}, \psi_{t_j})\right\} \cup \{0, 1\}.$$

If ε is a positive integer, $\gamma \in \mathbf{C}$, and $x/(x-1) = e^{2\pi\gamma}$, then

$$\begin{aligned}
\left(\frac{\lambda+1}{\lambda-1}\right)^{\varepsilon}(1-x) + x &= (1-x)\prod_{k=0}^{\varepsilon-1}\left(\frac{\lambda+1}{\lambda-1} - e^{2\pi k i/\varepsilon}\left(\frac{x}{x-1}\right)^{1/\varepsilon}\right) \\
&= \frac{1}{1-e^{2\pi\gamma}} \cdot \prod_{k=0}^{\varepsilon-1}\left(\frac{\lambda+1}{\lambda-1} - e^{2\pi(\gamma+ki)/\varepsilon}\right) \\
&= \left(\prod_{k=0}^{\varepsilon-1}\frac{1}{1-e^{2\pi(\gamma+ki)/\varepsilon}}\right) \cdot \prod_{k=0}^{\varepsilon-1}\left(\frac{\lambda+1}{\lambda-1} - e^{2\pi(\gamma+ki)/\varepsilon}\right) \\
&= \prod_{k=0}^{\varepsilon-1}\left(\frac{\lambda+1}{\lambda-1} \cdot \frac{1}{1-e^{2\pi(\gamma+ki)/\varepsilon}} + \frac{e^{2\pi(\gamma+ki)/\varepsilon}}{e^{2\pi(\gamma+ki)/\varepsilon}-1}\right).
\end{aligned} \tag{14.9}$$

If γ traces out $Y(p; \psi_t, \psi_t)$, then $(\gamma + ki)/\varepsilon$ traverses

$$Y\left(p; \frac{1}{\varepsilon}\Big(\frac{1}{p} - \frac{\varepsilon}{p} + k + T_\varepsilon\psi_t\Big), \frac{1}{\varepsilon}\Big(\frac{1}{p} - \frac{\varepsilon}{p} + k + T_\varepsilon\psi_t\Big)\right)$$

(recall the proof of Theorem 4.4), and therefore

$$w := \frac{e^{2\pi(\gamma+ki)/\varepsilon}}{e^{2\pi(\gamma+ki)/\varepsilon}-1}$$

describes the curve $\mathcal{L}^{\#}_{t,k,n}$. Since the kth factor of the product (14.9) is $\frac{\lambda+1}{\lambda-1}(1-w)+w$, we obtain from (14.8) and (14.9) that the increment of $\arg z$ along $\mathcal{L}^{\#}_{t_j}(\mu^{\sigma_{j-1}}, \mu^{\sigma_j})$ is equal to the sum of the increments of $\arg z$ along the arcs

$$\mathcal{L}^{\#}_{t_j,k,\varepsilon_j}\left(\frac{\lambda+1}{\lambda-1}, 1\right) := \left\{\frac{\lambda+1}{\lambda-1}(1-w) + w \;:\; w \in \mathcal{L}^{\#}_{t_j,k,\varepsilon_j}\right\}, \tag{14.10}$$

k ranging over $\{0, 1, \dots, \varepsilon_j - 1\}$. Because (14.10) is

$$\frac{1}{1-\lambda}\left\{(-1-\lambda)(1-w) + (1-\lambda)w \;:\; w \in \mathcal{L}^{\#}_{t_j,k,\varepsilon_j}\right\},$$

the increment of $\arg z$ along the arc (14.10) coincides with the increment of $\arg z$ along $\mathcal{L}^{\#}_{t_j,k,\varepsilon_j}(-1-\lambda, 1-\lambda)$ and therefore with the increment of $\arg(z-\lambda)$ as z traces out

$$\mathcal{L}^{\#}_{t_j,k,\varepsilon_j}(-1,1) = \mathcal{L}^{\#}_{t_j,k,\varepsilon_j}(-\operatorname{sign}\varepsilon_j, \operatorname{sign}\varepsilon_j).$$

The case $\varepsilon_j < 0$ may be disposed of analogously. □

Thus, in order to decide whether a given connected component Ω of $\mathbf{C} \setminus \operatorname{sp}_{\text{ess}} S_\Gamma$ is contained in $\operatorname{sp} S_\Gamma$, we need only pick a particular $\lambda_0 \in \Omega$ and apply Theorems 14.1 or 14.2 to compute $\operatorname{Ind}(S_\Gamma - \lambda_0 I)$. The advantage of formula (14.2) is that it is in terms of the original leaves $\mathcal{L}_t$, while formula (14.7) involves the "root leaves" $\mathcal{L}^0_{t,k,|\varepsilon(t)|}$ ($k = 0, 1, \dots, |\varepsilon(t)| - 1$). On the other hand, formula (14.2) is very inconvenient in order to find the spectrum of S_Γ. In contrast to this, formula (14.7) enables us to *see* the spectrum of S_Γ at a glance once $\operatorname{sp}_{\text{ess}} S_\Gamma$ has been plotted.

To illustrate Theorems 14.1 and 14.2, suppose $\Gamma = \eta_1 \cup \eta_2$ where η_1 and η_2 are simple arcs with exactly one common endpoint and with no other common points. Denote the common endpoint by t_0, and let t_1 and t_2 be the other endpoints of η_1 and η_2. Suppose η_1 and η_2 are oriented from t_0 to t_1 and t_2, respectively. We have $\varepsilon(t_0) = 2$ and $\varepsilon(t_1) = \varepsilon(t_2) = -1$. Let α_t and β_t be any functions such that $\mathcal{L}_t(\Gamma) = \mathcal{L}^0(p; \alpha_t, \beta_t)$ and choose continuous functions ψ_t such that $\alpha_t \le \psi_t \le \beta_t$. By Theorem 4.4,

$$\operatorname{sp}_{\text{ess}} S_\Gamma = \mathcal{L}^0_{t_1}(1,-1) \cup \mathcal{L}^0_{t_2}(1,-1) \cup \mathcal{L}^0_{t_0,0,2}(-1,1) \cup \mathcal{L}^0_{t_0,1,2}(-1,1)$$

where

$$\mathcal{L}^0_t(1,-1) := 1 - 2\mathcal{L}^0(p;\alpha_t,\beta_t) = \mathcal{L}^0(1,-1;p;\alpha_t,\beta_t),$$
$$\mathcal{L}^0_{t,k,2}(-1,1) := 2\mathcal{L}^0_{k,2} - 1 = \mathcal{L}^0(-1,1;p;\alpha_t^{(k)},\beta_t^{(k)}),$$

and $\alpha_t^{(k)}, \beta_t^{(k)}$ are given by

$$\alpha_t^{(k)} := \frac{1}{n}\left(\frac{1}{p} - \frac{n}{p} + k + T_n\alpha_t\right), \quad \beta_t^{(k)} := \frac{1}{n}\left(\frac{1}{p} - \frac{n}{p} + k + T_n\beta_t\right)$$

with $n := |\varepsilon(t)|$. We have

$$\mathcal{L}^{\#}_t := \mathcal{L}^0(p;\psi_t,\psi_t), \quad \mathcal{L}^{\#}_{t,k,2} := \mathcal{L}^0(p;\psi_t^{(k)},\psi_t^{(k)}).$$

Now let $\lambda \in \mathbf{C} \setminus \operatorname{sp}_{\text{ess}} S_\Gamma$. The curve $C^{\#}_\Gamma$ of Theorem 14.1 with $T := \{t_1, t_2, t_0\}$ is

$$\mathcal{L}^{\#}_{t_1}\left(1, \frac{\lambda+1}{\lambda-1}\right) \cup \mathcal{L}^{\#}_{t_2}\left(\frac{\lambda+1}{\lambda-1}, \left(\frac{\lambda+1}{\lambda-1}\right)^2\right) \cup \mathcal{L}^{\#}_{t_0}\left(\left(\frac{\lambda+1}{\lambda-1}\right)^2, 1\right), \tag{14.11}$$

whereas the curve $C_\Gamma^\#$ of Theorem 14.2 is

$$\mathcal{L}_{t_1}^\#(1,-1) \cup \mathcal{L}_{t_2}^\#(1,-1) \cup \mathcal{L}_{t_0,0,2}^\#(-1,1) \cup \mathcal{L}_{t_0,1,2}^\#(-1,1) \tag{14.12}$$

$$= \mathcal{L}_{t_1}^\#(1,-1) \cup \mathcal{L}_{t_0,0,2}^\#(-1,1) \cup \mathcal{L}_{t_2}^\#(1,-1) \cup \mathcal{L}_{t_0,1,2}^\#(-1,1). \tag{14.13}$$

Note that (14.13) is nothing but a rearrangement of (14.12) which indicates a possibility of traversing (14.12) along a closed path. If, for example, all occuring leaves are circular arcs or horns, it follows almost at once from (14.13) that the possible values of Ind $(S_\Gamma - \lambda I)$ are $-2, -1, 0, 1, 2$. This is less obvious from (14.11).

Finally, let η be a simple Carleson arc. Denote the endpoints of η by t_1 and t_2 and suppose η is oriented from t_1 to t_2. By Theorem 4.4,

$$\begin{aligned} \mathrm{sp}_{\mathrm{ess}}\, S_\eta &= \big(2\mathcal{L}_{t_1}^0(p;\alpha_{t_1},\beta_{t_1}) - 1\big) \cup \big(1 - 2\mathcal{L}_{t_2}^0(p;\alpha_{t_2},\beta_{t_2})\big) \\ &= \mathcal{L}_{t_1}^0(-1,1;p;\alpha_{t_1},\beta_{t_1}) \cup \mathcal{L}_{t_2}^0(1,-1;p;\alpha_{t_2},\beta_{t_2}), \end{aligned}$$

where we used the notation (14.1). The curve of Theorem 14.2 is

$$C_\eta^\# = \mathcal{L}_{t_1}^\#(-1,1) \cup \mathcal{L}_{t_2}^\#(1,-1)$$

and thus, Ind $(S_\eta - \lambda I) = -\mathrm{wind}\,(C_\eta^\#, \lambda)$ for every $\lambda \in \mathbf{C} \setminus \mathrm{sp}_{\mathrm{ess}}\, S_\eta$. If $\mathcal{L}_{t_1}(\eta)$ and $\mathcal{L}_{t_2}(\eta)$ are circular arcs or horns, which happens necessarily if η is piecewise Lyapunov, then $\mathcal{L}_{t_1}^\#(-1,1) \cup \mathcal{L}_{t_2}^\#(1,-1)$ may be chosen to be the union of two circular arcs. In this case Theorems 14.2 implies that the only possible values of Ind $(S_\eta - \lambda I)$ are $-1, 0, +1$. However, if $\eta := \{re^{i\delta \log r} \,:\, 0 < r < 1\}$ $(\delta \in \mathbf{R})$, then η is (a piece of) a logarithmic spiral and thus the simplest nontrivial example of a Carleson arc. In that case $t_1 = 0,\ t_2 = 1,\ \mathcal{L}_{t_1}(\eta)$ is a logarithmic double spiral, and $\mathcal{L}_{t_2}(\eta)$ is a circular arc. From Theorem 14.2 we infer that for every integer k there exists a point $\lambda \in \mathbf{C} \setminus \mathrm{sp}_{\mathrm{ess}}\, S_\eta$ such that Ind $(S_\Gamma - \lambda I) = k$.

Notice how drastically things change when passing from Lyapunov curves to Carleson curves. Without considering spectra, such phenomena stay in the dark.

Note added in proof. In the meanwhile the book [10] was accomplished. This book is a self-contained exposition of the spectral theory of Toeplitz operators with piecewise continuous symbols and singular integral operators with piecewise continuous coefficients. It includes an introduction to Carleson curves, Muckenhoupt weights, weighted norm inequalities, local principles, Wiener-Hopf factorization, and Banach algebras generated by idempotents.

Acknowledgements

Bishop is partially supported by NSF Grant DMS 95-00557 and an Alfred P. Sloan research fellowship. Böttcher is partially supported by the Alfried Krupp Förderpreis für junge Hochschullehrer of the Krupp Foundation. Spitkovsky is partially supported by NSF Grant DMS 94-01848. Böttcher, Karlovich, and Spitkovsky are partially supported by NATO Collaborative Grant 950332.

References

[1] A.V. Aizenshtat, Yu.I. Karlovich, and G.S. Litvinchuk: On a theorem of conformal sewing and its application. *Trudy Inst. Prikl. Matem. im. I.N. Vekua* **3** (1988), 9–12 [Russian].

[2] A.V. Aizenshtat, Yu.I. Karlovich, and G.S. Litvinchuk: The method of conformal gluing for the Haseman boundary value problem on an open contour. *Complex Variables* **28** (1996), 313–346.

[3] G.R. Allan: Ideals of vector-valued functions. *Proc. London Math. Soc.*, 3rd ser., **18** (1968), 193–216.

[4] C.J. Bishop and P.W. Jones: Harmonic measure and arclength. *Ann. Math.* **132** (1990), 511–547.

[5] C.J. Bishop and P.W. Jones: Harmonic measure, L^2 estimates and the Schwarzian derivative. *J. d'Analyse Math.* **62** (1994), 77–113.

[6] A. Böttcher and Yu.I. Karlovich: Toeplitz and singular integral operators on Carleson curves with logarithmic whirl points. *Integral Equations and Operator Theory* **22** (1995), 127–161.

[7] A. Böttcher and Yu.I. Karlovich: Toeplitz and singular integral operators on general Carleson Jordan curves. *Operator Theory: Advances and Applications* **90** (1996), 119–152.

[8] A. Böttcher and Yu.I. Karlovich: Toeplitz operators with PC symbols on general Carleson Jordan curves with arbitrary Muckenhoupt weights. *Trans. Amer. Math. Soc.* (to appear).

[9] A. Böttcher and Yu.I. Karlovich: Submultiplicative functions and spectral theory of Toeplitz operators. *Integral Transforms and Special Functions* **4** (1996), 181–202.

[10] A. Böttcher and Yu.I. Karlovich: *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators.* Birkhäuser Verlag, Basel (book to appear).

[11] A. Böttcher, Yu.I. Karlovich, and V.S. Rabinovich: Emergence, persistence, and disappearance of logarithmic spirals in the spectra of singular integral operators. *Integral Equations and Operator Theory* **25** (1996), 404–444.

[12] A. Böttcher, Yu.I. Karlovich, and V.S. Rabinovich: Mellin pseudodifferential operators with slowly varying symbols and singular integrals on Carleson curves with Muckenhoupt weights. *Math. Annalen* (to appear).

[13] A. Böttcher, Yu.I. Karlovich, and V.S. Rabinovich: The method of limit operators for one-dimensional singular integrals with slowly oscillating data. Preprint, TU Chemnitz-Zwickau 1997.

[14] A. Böttcher and B. Silbermann: *Analysis of Toeplitz Operators.* Akademie–Verlag, Berlin 1889 and Springer–Verlag, Berlin, Heidelberg, New York 1990.

[15] K. Clancey and I. Gohberg: *Factorization of Matrix Functions and Singular Integral Operators.* Birkhäuser Verlag, Basel, Boston, Berlin 1981.

[16] L.A. Coburn: Weyl's theorem for non-normal operators. *Michigan Math. J.* **13** (1966), 285–286.

[17] R.R. Coifman and C. Fefferman: Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51** (1974), 241–249.

[18] E.A. Danilov: *The Riemann boundary value problem on contours with unbounded distortion.* Cand. Dissertation, Odessa 1984 [Russian].

[19] G. David: L'integrale de Cauchy sur le courbes rectifiables. Prepublication Univ. Paris-Sud, Dept. Math. 82T05, 1982.

[20] G. David: Opérateurs intégraux singuliers sur certaines courbes du plan complexe. *Ann. Sci. École Norm. Super.* **17** (1984), 157–189.

[21] R.G. Douglas: *Banach Algebra Techniques in Operator Theory.* Academic Press, New York 1972.

[22] R.G. Douglas: Local Toeplitz operators. *Proc. London Math. Soc.*, 3rd ser., **36** (1978), 243–272.

[23] E.M. Dynkin and B.P. Osilenker: Weighted norm estimates for singular integrals and their applications. *J. Sov. Math.* **30** (1985), 2094–2154.

[24] J.L. Fernández, J. Heinonen, and O. Martio: Quasilines and conformal mappings. *J. d'Analyse Math.* **52** (1989), 117–132.

[25] J.L. Fernández and M. Zinsmeister: Ensembles de niveau des représentations conformes. *Comptes Rendus de l'Acad. Sci.* **305** (1987), 449–452.

[26] T. Finck, S. Roch, and B. Silbermann: Two projections theorems and symbol calculus for operators with massive local spectra. *Math. Nachr.* **162** (1993), 167–185.

[27] J.B. Garnett, F. Gehring, and P.W. Jones: Conformally invariant length sums. *Indiana Univ. Math. J.* **32** (1983), 809–829.

[28] I. Gohberg: On an application of the theory of normed rings to singular integral equations. *Uspekhi Matem. Nauk* **7** (1952), 149–156 [Russian].

[29] I. Gohberg and N. Krupnik: Systems of singular integral equations in weighted L^p spaces. *Soviet Math. Dokl.* **10** (1969), 688–691.

[30] I. Gohberg and N. Krupnik: On singular integral operators on a composed curve. *Soobshch. Akad. Nauk Gruz. SSR* **64** (1971), 21–24 [Russian].

[31] I. Gohberg and N. Krupnik: Singular integral operators with piecewise continuous coefficients and their symbols. *Math. USSR Izv.* **5** (1971), 955–979.

[32] I. Gohberg and N. Krupnik: On the symbol of singular integral operators on a composed contour. *Trudy Simpoz. Mech. Sploshnych Sred* **1** (1973), 3–12 [Russian].

[33] I. Gohberg and N. Krupnik: *One-Dimensional Linear Singular Integral Equations.* Vols. I,II. Birkhäuser Verlag, Basel, Boston, Berlin 1992 [Russian original: Shtiintsa, Kishinev 1973].

[34] I. Gohberg and N. Krupnik: Extension theorems for Fredholm and invertibility symbols. *Integral Equations and Operator Theory* **16** (1993), 514–529.

[35] I. Gohberg, N. Krupnik, and I. Spitkovsky: Banach algebras of singular integral operators with piecewise continuous coefficients. *Integral Equations and Operator Theory* **17** (1993), 322–337.

[36] R. Hagen, S. Roch, and B. Silbermann: *Spectral Theory of Approximation Methods for Convolution Equations.* Birkhäuser Verlag, Basel, Boston, Berlin 1995.

[37] W.K. Hayman and J.-M. Wu: Level sets of univalent functions. *Comment. Math. Helv.* **56** (1981), 366–403.

[38] R. Hunt, B. Muckenhoupt, and R. Wheeden: Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176** (1973), 227–251.

[39] A.Yu. Karlovich: On the index of singular integral operators in reflexive Orlicz spaces. *Matem. Zametki* (submitted) [Russian].

[40] B.V. Khvedelidze: Linear discontinuous boundary value problems of function theory, singular integral equations and some of their applications. *Trudy Tbil. Inst. Akad. Nauk Gruz. SSR* **23** (1956), 3–158 [Russian].

[41] B.V. Khvedelidze: The method of Cauchy type integrals for discontinuous boundary value problems of the theory of holomorphic functions of one complex variable. *J. Sov. Math.* 7 (1977), 309–414 [Russian original: in *Itogi Nauki Tekh., Sovr. Probl. Matem.*, Vol. 7, pp. 5–162, Moscow 1975].

[42] G.S. Litvinchuk and I. Spitkovsky: *Factorization of Measurable Matrix Functions.* Akademie–Verlag, Berlin 1987 and Birkhäuser Verlag, Basel, Boston, Berlin 1987.

[43] N.I. Muskhelishvili: *Singular Integral Equations.* Nordhoff, Groningen 1953 [German translation: Akademie–Verlag, Berlin 1965; Bojarski's appendix is in the 3rd Russian edition, Nauka, Moscow 1968].

[44] K. Øyma: Harmonic measure and conformal length. *Proc. Amer. Math. Soc.* **115** (1992), 687–689.

[45] V.A. Paatashvili and G.A. Khuskivadze: On the boundedness of the Cauchy singular integral on Lebesgue spaces in the case of non-smooth contours. *Trudy Tbilisk. Mat. Inst. Akad.Nauk Gruz. SSR* **69** (1982), 93–107 [Russian].

[46] CH. POMMERENKE: *Univalent Functions.* Vandenhoeck and Ruprecht, Göttingen 1975.

[47] I.I. PRIWALOW: *Randeigenschaften analytischer Funktionen.* Deutscher Verlag der Wissenschaften, Berlin 1956 [Russian original: GITTL, Moscow 1950].

[48] V.S. RABINOVICH: Singular integral operators on a complex contour with oscillating tangent, and Mellin pseudodifferential operators *Soviet Math. Dokl.* **44** (1992), 791-796.

[49] V.S. RABINOVICH: Singular integral operators on composed contours and pseudodifferential operators. *Matem. Zametki*, **58** (1995), 67–85 [Russian].

[50] S. ROCH et al.: Banach algebras generated by N idempotents and applications. *Operator Theory: Advances and Applications* **90** (1996), 19–54.

[51] S. ROCH AND B. SILBERMANN: Algebras generated by idempotents and the symbol calculus for singular integral operators. *Integral Equations and Operator Theory* **11** (1988), 385–419.

[52] S. ROCH AND B. SILBERMANN: *Algebras of Convolution Operators and Their Image in the Calkin Algebra.* Report R-MATH-05/90 of the Karl Weierstrass Institute for Mathematics, 156 pages, Berlin 1990.

[53] R.K. SEIFULLAYEV: The Riemann boundary value problem on non-smooth open curves. *Matem. Sb.* **112** (1980), 147–161 [Russian] (English transl. in *Math USSR Sb.* **40** (1981)).

[54] E. SHAMIR: Solution of the Riemann-Hilbert system with piecewise continuous coefficients. *Dokl. Akad. Nauk SSSR* **167** (1960), 1000–1003 [Russian].

[55] I.B. SIMONENKO: The Riemann boundary value problem for n pairs of functions with measurable coefficients and its application to the investigation of singular integrals in the spaces L^p with weight. *Izv. Akad. Nauk SSSR, Ser. Mat.*, **28** (1964), 277–306 [Russian].

[56] I.B. SIMONENKO: A new general method for studying linear operator equations of the type of singular integral equations. Part I: *Izv. Akad. Nauk SSSR, Ser. Mat.*, **29** (1965), 567–586; Part II: *ibidem*, 757–782 [Russian].

[57] I.B. SIMONENKO: Some general questions of the theory of the Riemann boundary value problem. *Math. USSR Izv.* **2** (1968), 1091–1099.

[58] I. SPITKOVSKY: Singular integral operators with PC symbols on the spaces with general weights. *J. Funct. Analysis* **105** (1992), 129–143.

[59] J.-O. STRÖMBERG AND A. TORCHINSKY: *Weighted Hardy Spaces.* Lect. Notes in Math., Vol. 1381, Springer-Verlag, Berlin 1989.

[60] L.N. TREFETHEN: Pseudospectra of linear operators. Preprint, Cornell University 1995.

[61] H. WIDOM: Singular integral equations in L^p. *Trans. Amer. Math. Soc.* **97** (1960), 131–160.

[62] M. ZINSMEISTER: Les domains de Carleson. *Michigan Math. J.* **36** (1989), 213–220.

Mathematics Department
SUNY at Stony Brook
Stony Brook, NY 11794-3651
USA

bishop@math.sunysb.edu

TU Chemnitz
Fakultät für Mathematik
09107 Chemnitz
Germany

aboettch@mathematik.tu-chemnitz.de

TU Chemnitz
Fakultät für Mathematik
09107 Chemnitz
Germany

karlovic@mathematik.tu-chemnitz.de

Department of Mathematics
The College of William and Mary
Williamsburg, VA 23187-8795
USA

ilya@cs.wm.edu