QUASICONFORMAL MAPPINGS OF Y-PIECES

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ABSTRACT. In this paper we contruct quasiconformal mappings between Y-pieces so that the corresponding Beltrami coefficient has exponential decay away from the boundary. These maps are used in a companion paper to construct quasiFuchsian groups whose limit sets are non-rectifiable curves of dimension 1.

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1. Introduction

Given a simple closed geodesic γ on a Riemann surface R we can usually quasiconformally deform R in such a way that γ maps to a closed geodesic of shorter length. The purpose of this note is to record an explicit way of doing this, with a careful estimate of the complex dilatation μ of the corresponding map. In particular, we will do this so that μ has the optimal L^∞ norm (up to a bounded factor) and $|\mu|$ decays exponentially fast away from γ. This estimate is used in the companion paper [3] to construct quasiFuchisan groups whose limit sets are nonrectifiable curves of dimension 1.

A generalized Y-piece is a Riemann surface (with boundary) which is bounded by three closed geodesics (or punctures) and which is homeomorphic to a 2-sphere minus three disks (or points). It is called L bounded if each of the boundary components has length $\leq L$ (punctures count as length zero). Every finite area Riemann surface can be written as a finite union of such pieces and even general Riemann surfaces can be written as a union of Y-pieces, funnels and half-planes, [1].

Theorem 1.1. *Suppose Y_1 is an L bounded Y-piece with boundary components γ_a, γ_b and γ_c with lengths (a_1, b_1, c_1) respectively. Suppose Y_2 is another L-bounded generalized Y-piece with boundary lengths (a_2, b_1, c_1) respectively. Assume that $\epsilon = |\log a_1/a_2| \leq 2$. Then there is a $C = C(L)$ and a quasiconformal map $f : Y_1 \to Y_2$ with constant $K \leq 1 + C\epsilon$ which is affine on each of the boundary components (i.e. is isometric on γ_b and γ_c and multiplies length by a_2/a_1 on γ_a). Moreover, the Beltrami coefficient μ of f satisfies*

$$|\mu(z)| \leq C\epsilon \exp(-2\text{dist}(z, \gamma_a)).$$

Thus the Beltrami coefficient μ has L^∞ norm which is comparable to the optimal quasiconformal map from Y_1 to Y_2, but has rapid decay away from boundary curve which is being shrunk. This is important for our applications in [3] where we wish to estimate integrals involving μ.

An X-piece is the union of two (not necessarily distinct) Y-pieces which share a common boundary geodesic γ, which is called the central curve of X. As above an X-piece is called L-bounded if its boundary components and central curve all have length bounded by L. The following is an obvious consequence of the theorem and is the precise result used in [3]

Corollary 1.2. *Given an L-bounded X-piece X_1, its central curve γ and any $0 < \epsilon < 1/2$, there is an X-piece X_2 and a quasiconformal map $f : X_1 \to X_2$ so that $f(\gamma)$ is the central geodesic for X_2 and has length $1-\epsilon$ times that of γ. Moreover, f is an isometry on the boundary of X_1 and the corresponding Beltrami coefficient $\mu_{\gamma,\epsilon}$ satisfies $\|\mu_{\gamma,\epsilon}\|_\infty < C\epsilon$ and*

$$|\mu_{\gamma,\epsilon}(z)| \leq C\epsilon \exp(-2\text{dist}(z,\gamma)). \tag{1}$$

Using this we can construct a deformation of Riemann surface R by choosing an X-piece X_1 in R and replacing it by another X-piece X_2, to get a surface S. We then define a map $R \to S$ by taking the identity off X_1 and the map from the corollary on X_1. The deformations constructed in [3] are obtained by applying this idea simultaneously to certain infinite (disjoint) collections of X-pieces in R.

Other applications of our estimate are given in [7] and [8]. Let $\mathcal{T}$ denote the set of Riemann surfaces which can be written as a countable union of Y-pieces with the property that for any $\epsilon > 0$, all but finitely many of the pieces are ϵ bounded. In [8] our estimate used to show that Fuchsian groups corresponding to surfaces in $\mathcal{T}$ are δ-stable, i.e., for any quasiconformal deformation of the group $\delta = \dim(\Lambda)$, i.e. the critical exponent equals the Hausdorff dimension of the limit set. In [7], another application is given related to Ruelle's property. See [4] and [7].

Next we describe the structure of the rest of the paper.

Section 2: We review some hyperbolic trigonometry.

Section 3: We show that two hyperbolic triangles are quasiconformally equivalent with constant close to 1 if the ratio of their side lengths is close to 1.

Section 4: We construct quasiconformal maps between certain hyperbolic quadrilaterals.

Section 5: We show that certain hyperbolic hexagons can be written as a union of a bounded piece and at most three quadrilaterals.

Section 6: We prove Theorem 1.1.

I thank Michel Zinsmeister for his comments and questions on [6] which led me to consider the questions dealt with here. I also thank the two referees whose careful reading and numerous suggestions were greatly appreciated.

If A, B are quantities that depend on some parameter we write $A \lesssim B$ if the ratio B/A is bounded uniformly independent of the parameter. We will also sometimes write this as $A = O(B)$. Similarly for $\gtrsim$. We write $A \simeq B$ if both $A \lesssim B$ and $A \gtrsim B$ hold and say A and B are comparable.

2. Some hyperbolic trigonometry

In this section we record a few formulas of hyperbolic trigonometry which we will need later.

Recall the sine and cosine rules for hyperbolic geometry (e.g., see page 148 of Beardon's book [2]). Let T denote a hyperbolic triangle with angles α, β, γ and opposite side lengths denoted by a, b, c. See Figure 1. Then we have the Sin Rule,

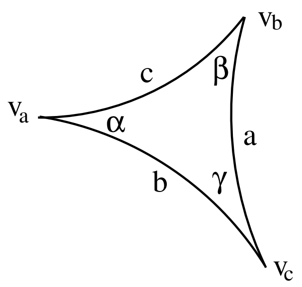

Figure 1. Definitions of a, b, c and α, β, γ

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma} \tag{2}$$

the First Cosine Rule,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma \tag{3}$$

and the Second Cosine Rule

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta} \tag{4}$$

Lemma 2.1. *If $\gamma = \pi/2$ then*

$$\sinh b = \sinh c \sin \beta, \tag{5}$$

$$\tanh b = \sinh a \tan \beta, \tag{6}$$

$$\tanh a = \tanh c \cos \beta. \tag{7}$$

Proof. The first of these is immediate by taking $\gamma = \pi/2$ in (2). To prove (6), note that

$$\begin{aligned}
\tanh b = \frac{\sinh b}{\cosh b} &= \frac{\sin \beta}{\sin \alpha} \frac{\sinh a}{\cosh b} \\
&= \frac{\sin \beta}{\sin \alpha} \frac{\sinh a \cosh a}{\cosh c} \\
&= \frac{\sin \beta}{\sin \alpha} \sinh a \cosh a \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta} \\
&= \tan \beta \sinh a [\cosh a \frac{\sin \beta}{\cos \alpha}] \\
&= \tan \beta \sinh a.
\end{aligned}$$

where the second, third and fourth equalities hold by (2), (3) and (4) respectively and the last equality holds by (4) (with the roles of a and c reversed). To prove (7), note that

$$\begin{aligned}
\tanh a = \frac{\sinh a}{\cosh a} &= \frac{\sinh a \cosh b}{\cosh c} \\
&= \frac{\tanh b \cos \beta \cosh b}{\sin \beta \cosh c} \\
&= \tanh c \cos \beta [\frac{\sinh b}{\sinh c \sin \beta}] \\
&= \tanh c \cos \beta,
\end{aligned}$$

where the second, third and last equalities hold by (3), (6) and (2) respectively. □

Formula (4) makes it clear that the side lengths of a hyperbolic triangle are determined by its angles. It is also easy to see that if all three angles are bounded below by some $\theta > 0$, then there is a finite upper bound on the side lengths which only depends on θ. By (4), if all three angles of a hyperbolic triangle are less than δ the the diameter of the triangle is $\geq M(\delta)$ where $M \to \infty$ as $\delta \to 0$.

By (2), if the three sides of a hyperbolic triangle have comparable lengths then the sin of the angles are also comparable. Since at most one of the angles can be close to π we see that if all the angles are bounded away from π then they must all be comparable in size.

3. Quasiconformal mappings of hyperbolic triangles

In this section we record a calculation which we will need later. It simply says that two hyperbolic triangles whose side lengths are pairwise close are quasiconformally equivalent with constant close to 1. This seems fairly obvious, but I have been unable to locate the result in the liturature, so it seems worth recording here. Our strategy will be to divide the triangles into small triangles which are approximately Euclidean and then apply the analogous result for Eucidean trangles (which is easy).

Lemma 3.1. *Suppose $T_1, T_2 \subset \mathbb{D}$ are two triangles in the hyperbolic disk with angles $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ respectively and opposite sides (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively. Suppose that all these angles are strictly positive, say all $\geq \theta > 0$ and let*

$$\epsilon = \max(|\log \frac{a_1}{a_2}|, |\log \frac{b_1}{b_2}|, |\log \frac{c_1}{c_2}|)$$

Suppose $\epsilon \leq A$. Then there is a constant $C_1 = C_1(\theta, A)$ and a $1 + C_1\epsilon$ quasiconformal map from T_1 to T_2 which maps each vertex to the corresponding vertex and which is affine on each edge of T_1 (i.e., it multiplies hyperbolic arclength on each edge by the appropriate factor a_2/a_1, b_2/b_1 or c_2/c_1).

Proof. The angle bisectors of each of the three angles of T_1 meet at a single point ζ_1 in the interior of T_1 by Theorem 7.14.1 of [2]. See Figure 2. This point is equidistant

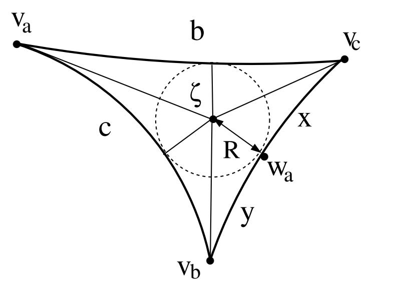

Figure 2. Angle bisectors meet at a point

from each of there three sides of T_1. Let R_1 denote this distance and let w_a^1, w_b^1, w_c^1 denote the closest point to ζ_1 on each of the sides a_1, b_1, c_1 respectively. Divide each edge of T_1 into 2^n equal length segments and connect the endpoints of these segments by geodesics segments to ζ_1. This divides T_1 into $3 \cdot 2^n$ sub-triangles $\{T_{1,k}\}$. See

Figure 3. Assume $\zeta_1 = 0$ and for each $k = 1, \dots, 3 \cdot 2^n$ let $\tilde{T}_{1,k}$ be the Euclidean triangle which has the same vertices as $T_{1,k}$ (since $\zeta_1 = 0$, two sides of $\tilde{T}_{1,k}$ will agree exactly with two sides of $T_{1,n}$ but the third side opposite ζ_1 will not). Let ζ_2, R_2, $\{T_{2,k}\}$ and $\{\tilde{T}_{2,k}\}$ be the corresponding objects for T_2 and assume $\zeta_2 = 0$ as well.

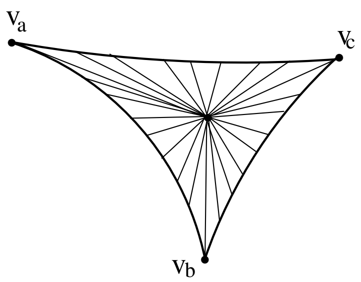

FIGURE 3. The triangles T_k's

We claim that there is a $C = C(\theta) < \infty$ so that each $\tilde{T}_{1,k}$ can be mapped to $\tilde{T}_{2,k}$ by a $1+C\epsilon$ quasiconformal map which is affine in the Euclidean sense. If so, then adjacent triangles have matching boundary values so we get a $1 + C\epsilon$ quasiconformal from $\cup_k \tilde{T}_{1,k}$ to $\cup_k \tilde{T}_{2,k}$. Then taking $n \to \infty$ we obtain in the limit a $1+C\epsilon$ quasiconformal map from T_1 to T_2. By definition this map is affine restricted to the vertices of $E \cap \cup_k \tilde{T}_{1,k}$ for each edge E of T_1, and so by continuity is affine on each edge, as desired. Thus it suffices to verify the claim.

Let L_1 denote an edge of $\tilde{T}_{1,k}$ which connects ζ_1 to an edge E_1 of T_1. Let θ_1 be the angle formed by L_1 and E_1 and let F_1 be the edge of $\tilde{T}_{1,k}$ which is opposite ζ_1. It is clear that θ_1 is minimized when L_1 is one of the angle bisectors of T_1 and hence $\frac{1}{2}\theta \leq \theta_1 \leq \frac{1}{2}\pi$ (where θ is assumed the lower bound on the angles of T_1). Let L_2, E_2, F_2 and θ_2 be the corresponding quantities in T_2. It is enough to check that the Euclidean lengths of these edges satisfy

$$|L_1|/|L_2| = 1 + O(\epsilon), \tag{8}$$

$$|F_1|/|F_2| = 1 + O(\epsilon) \tag{9}$$

$$\theta_1/\theta_2 = 1 + O(\epsilon). \tag{10}$$

(It is easy to check that Euclidean triangles satisfying these conditions are $1 + O(\epsilon)$ quasiconformally equivalent via an affine map.)

We start by showing $R_1/R_2 = 1 + O(\epsilon)$. Let $x_i = \rho(w_a^i, v_c^i)$ and $y_i = \rho(w_a^i, v_b^i)$ so that $x_i + y_i = a_i$ (see Figure 4). Using (6) we see that

$$\tanh R_i = \sinh x_i \tan(\alpha_i/2) = \sinh y_i \tan(\beta_i/2).$$

Since $(x_1 + y_1)/(x_2 + y_2) = 1 + O(\epsilon)$, we must have either $x_1/x_2 \geq 1 - O(\epsilon)$ or $y_1/y_2 \geq 1 - O(\epsilon)$. Without loss of generality assume the first holds. Then

$$\tanh R_1 / \tanh R_2 = \sinh x_1 \tan(\alpha_1/2)/\sinh x_2 \tan(\alpha_2/2) \geq 1 - O(\epsilon).$$

Since R_i, x_i are uniformly bounded (in terms of θ) and α_i is bounded away from π (depending only on θ), this implies that $R_1/R_2 \geq 1 - O(\epsilon)$ where the constant depends only on θ. Reversing the roles of R_1 and R_2 shows $R_1/R_2 = 1 + O(\epsilon)$, as desired.

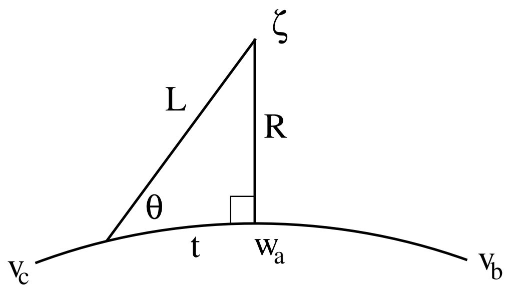

FIGURE 4. Definition of L_i and R_i

Next we prove (8). Suppose L_i connects ζ_i to the edge a_i of T_i and consider the hyperbolic triangle with edge L_i and third vertex w_a^i. See Figure 4. Let t_i be the signed hyperbolic distance from w_a^i to the endpoint of L_i on the edge a_i. Note that $|t_1 - t_2| = O(\epsilon a_1)$. Then by (3), with $\gamma = \pi/2$,

$$\cosh L_i = \cosh t_i \cosh R_i.$$

Now for $0 \leq s \leq 1$ let $t(s) = t_1(1-s) + t_2 s$, $R(s) = R_1(1-s) + R_2 s$ and $\cosh L(s) = \cosh t(s) \cosh R(s)$. Then clearly

$$|t'(s)| = O(\epsilon a_1), \quad |R'(s)| = O(\epsilon a_1),$$

and

$$|L'(s)| \leq \frac{1}{\sinh L(s)}[\sinh t(s) t'(s) \cosh R(s) + \cosh t(s) \sinh R(s) R'(s)] = O(\epsilon a_1).$$

Thus $|L_1 - L_2| = O(\epsilon a_1) = O(\epsilon L_1)$. Thus $|L_1|/|L_2| = 1 + O(\epsilon)$, as desired.

Now we prove (9). The segments F_1, F_2 have hyperbolic diameters which are comparable by a factor of $1+O(\epsilon)$, since by definition they have hyperbolic length $2^{-n}a_1$ and $2^{-n}a_2$ and $a_1/a_2 = 1+O(\epsilon)$ by assumption. Moreover, their hyperbolic distances from the origin are L_1 and L_2 which are comparable by the previous paragraph. Since $|dz| = (1-|z|^2)|d\rho|$ this implies their Euclidean lengths are also comparable by a factor of $1+O(\epsilon)$.

Finally we prove (10). By (7),

$$\cos\theta_i \tanh L_i = \tanh t_i,$$

and hence making a parameterized family as above we see

$$\sin\theta(s)\theta'(s)\tanh L(s) + \cos\theta(s)\mathrm{sech}^2 L(s)L'(s) = \mathrm{sech}^2 t(s)t'(s),$$

and since $\theta(s)$ is bounded away from zero in terms of θ, and $|t'(s)|, |L'(s)| = O(\epsilon a)$, we get

$$|\theta'(s)| = O(1)\frac{1}{\tanh L(s)}|\mathrm{sech}^2 t(s) - \cos\theta \,\mathrm{sech}^2 L(s)|O(\epsilon a) = O(\epsilon).$$

Thus $|\theta_1 - \theta_2| = O(\epsilon)$ and since these numbers are bounded by θ, we get $\theta_1/\theta_2 = 1+O(\epsilon)$, with constant depending only on θ. This ends the proof of Lemma 3.1. $\square$

The preceding lemma can be used to construct quasiconformal maps between polygons by triangulating the polygon and applying the lemma to each piece. Since the maps are affine on the edges, they must agree on the common boundary of any two triangles which share an edge. The following corollaries describe two specific situations where we will use this.

Corollary 3.2. *Suppose Q is a hyperbolic quadrilateral with sides a_1, a_2, a_3, a_4 (in clockwise direction) all bounded by $L < \infty$. Suppose the two angles adjacent to edge a_1 are right angles and the other two angles are bounded below by $\theta > 0$ and above by $\pi - \theta$. Suppose Q' is another quadrilateral with side lengths b_1, b_2, b_3, b_4 whose angles satisfy the same condition. Let $\epsilon = \max_i |\log b_i/a_i|$ and assume $\epsilon < A < \infty$. Then there is a $1 + C(L, \theta, A)\epsilon$ quasiconformal map from Q to Q' which is affine on each of the boundary edges.*

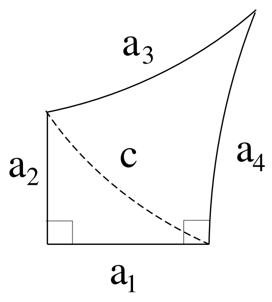

FIGURE 5.

Proof. Connect two opposite corners of Q forming two triangles with edge lengths a_1, a_2, c and a_3, a_4, c. See Figure 5. By (3),

$$\cosh c = \cosh a_1 \cosh a_2. \tag{11}$$

Do the same for Q', obtaining triangles of side length b_1, b_2, d and b_3, b_4, d. Now consider a parameterized family of right triangles on $[0, 1]$ with adjacent sides $sa_1 + (1-s)b_1$ and $sa_2 + (1-s)b_2$ and opposite side $c(s)$. Differentiating (11) we get

$$\sinh c \frac{dc}{ds} = \sinh a_1 \frac{da_1}{ds} \cosh a_2 + \cosh a_1 \sinh a_2 \frac{da_2}{ds}.$$

Since c is comparable to a_1 and a_2 we get that $\frac{dc}{da} = O(\epsilon a_1)$. The argument shows $c/d = 1 + O(\epsilon)$. Thus we can apply Lemma 3.1 once we know the angles of the triangles are bounded below uniformly.

However, one angle of the a_1, a_2, c triangle is $\pi/2$ and the side lengths are comparable. Thus by (2) all three angles are comparable (and hence are uniformly bounded below). A similar argument holds for the a_3, a_4, c triangle since one angle (the one between a_3 and a_4) is assumed to be bigger than θ and less than $\pi - \theta$. □

Corollary 3.3. *Suppose H is a hyperbolic hexagon such that all the edge lengths $a_1, \ldots, a_6$ (in clockwise direction) are $\leq B$ and are comparable with constant B. Also assume three alternating angles are $\pi/2$ and the remaining angles are bounded below by $\theta > 0$ and above by $\pi - \theta$. Then there is a $C = C(\theta, B)$ so that the following holds. If H' is another such hexagon with edge lengths $b_1, \ldots, b_6$ and $\epsilon = \max_i |\log a_i/b_i| \leq 2$, then there is a $1 + C\epsilon$ quasiconformal map from H to H' which is affine on each of the edges of H.*

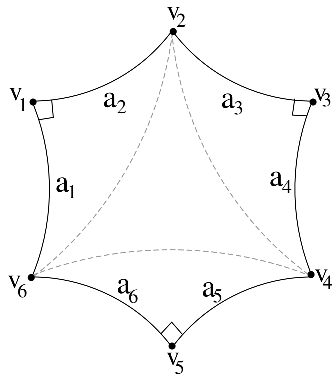

FIGURE 6.

Proof. Triangulate H by adding edges $[v_2v_4]$, $[v_4v_6]$ and $[v_6v_2]$, as in Figure 6. As in the previous corollary the new edges have length comparable to the edges of H and if the edges of H change by a factor of $1+\epsilon$ then the lengths of the new edges change by at most $1+O(\epsilon)$. Each of the three triangles which share edges with H has a right angle, and hence all their angles are bounded away from zero (since their edges are all comparable).

Thus we need only show the central triangle has angles bounded away from zero. However, all of its angles are smaller than $\pi-\theta$ (since this is true for the hexagon) and since the sides are comparable, so are the angles. Finally, not all three angles can be close to 0 since this would imply the triangle has large diameter (which is impossible since it is contained in H which has diameter bounded by $6B$). $\square$

4. Quasiconformal mappings of Lambert quadrilaterals

In this section we use the result about triangles from the previous section to prove a result about mappings of certain special quadrilaterals. In Section 6 on mappings of Y-pieces these quadrilaterals will correspond to neighborhoods of very short geodesic boundaries and the estimate we prove here will give the exponential decay we need to eventually show our deformations correspond to Carleson measures.

A hyperbolic quadrilateral is called a Lambert quadrilateral if it has three right angles (after J.H. Lambert, 1728-1777, see page 156 of [2]). We denote the fourth angle ϕ and the four sides (going clockwise, starting at ϕ) will be labeled b, x, a, y as in Figure 7.

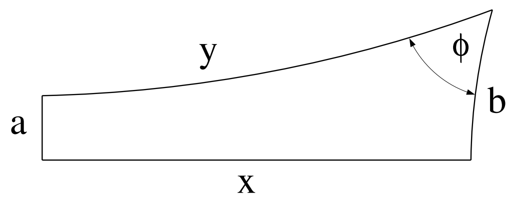

FIGURE 7. A Lambert quadrilateral

Given such a quadrilateral Q, we will sometimes let the letter a denote the edge of Q and sometimes let it denote the length of the edge. The usage will generally be clear from context (e.g. it is a length in $\sinh a$ but the side itself in $\rho(z, a)$), but we will write "a-side" when necessary. Similarly for the other sides of Q.

The quadrilateral is determined by any two side lengths or one side length and the angle ϕ, as is evident from the following two relations (Theorem 7.17.1 of [2])

$$\sinh a \sinh x = \cos \phi \tag{12}$$

$$\cosh a = \cosh b \sin \phi \tag{13}$$

We claim these imply

$$\cosh b \sinh x = \sinh y. \tag{14}$$

To prove this use (12) and (13) to get

$$\begin{aligned}
\sin \phi \cosh b &= \cosh a \\
&= (1 + \sinh^2 a)^{1/2} \\
&= (1 + \frac{\cos^2 \phi}{\sinh^2 x})^{1/2} \\
&= (\frac{\cosh^2 x - 1 + \cos^2 \phi}{\sinh^2 x})^{1/2} \\
&= (\frac{\cosh^2 y \sin^2 \phi - \sin^2 \phi}{\sinh^2 x})^{1/2} \\
&= \frac{\sin \phi \sinh y}{\sinh x},
\end{aligned}$$

which proves (14). We will think of b as fixed, but allow a to vary and consider ϕ, x and y as functions of a.

Lemma 4.1. *Suppose Q is a Lambert quadrilateral, labeled as above. If b is fixed and $a \le b/2$ is allowed to vary then*

$$\frac{b}{a} \simeq \sinh x \tag{15}$$

$$\frac{d\phi}{da} = O(a) \tag{16}$$

$$\frac{dx}{da} = \frac{-1}{\sinh a} + O(a) \tag{17}$$

$$\frac{dy}{da} = \frac{-1}{\sinh a} + O(a) \tag{18}$$

The constants depend only on an upper bound for b.

Proof. We recall an estimate which we will use frequently. Note that if $x > 0$,

$$1 + e^{-2x} \le \frac{\cosh x}{\sinh x} \le \frac{1 + e^{-2x}}{1 - e^{-2x}}, \tag{19}$$

and hence $\frac{\cosh x}{\sinh x} = 1 + O(e^{-2x})$ with a constant that depends only on a lower bound for x.

First we prove (15). By (12) and (13),

$$\begin{aligned} \sinh^2 x &= \frac{\cos^2 \phi}{\sinh^2 a} \\ &= \frac{1 - \sin^2 \phi}{\sinh^2 a} \\ &= \frac{\cosh^2 b - \cosh^2 a}{\sinh^2 a \cosh^2 b} \\ &\simeq \frac{b^2}{a^2}, \end{aligned}$$

where we have used $a \le b/2$ to deduce $\frac{1}{C} b^2 \le \cosh^2 b - \cosh^2 a \le C b^2$.

Next we prove (16). By (13)

$$\sin \phi = \frac{\cosh a}{\cosh b} \le \frac{\cosh b/2}{\cosh b} < 1, \tag{20}$$

with a bound depending only on b. Differentiating the equality gives

$$\frac{d\phi}{da} = \frac{\sinh a}{\cos \phi \cosh b} = O(a). \tag{21}$$

Also note for later use that if $t < s \leq b/2$ then

$$\phi(s) - \phi(t) = \int_s^t O(a)da = O(|s^2 - t^2|). \tag{22}$$

To prove (17), note that by (12) we have

$$\sinh x = \frac{\cos\phi}{\sinh a},$$

and so differentiating and using (12) and (21),

$$\begin{aligned}
\frac{dx}{da} &= -\frac{1}{\cosh x}\frac{-\sin\phi\frac{d\phi}{da}\sinh a - \cos\phi\cosh a}{\sinh^2 a} \\
&= -\frac{1}{\cosh x}\frac{-\sin\phi\sinh^2 a/(\cos\phi\cosh b) - \sinh a\sinh x\cosh a}{\sinh^2 a} \\
&= -\frac{1}{\sinh a}[\frac{-\sin\phi\sinh a}{\cos\phi\cosh b\cosh x} + \cosh a\frac{\sinh x}{\cosh x}] \\
&= \frac{-1}{\sinh a}[O(a^2) + (1+O(a^2))(1+O(e^{-2x}))] \\
&= \frac{-1}{\sinh a}(1+O(a^2)),
\end{aligned}$$

where the fourth equality uses the fact (20) that $\cos\phi$ is bounded away from zero (depending only on b), that $\cosh x \simeq 1/a$ and the fact that (15) implies

$$\frac{1}{C}a \leq e^{-x} \leq Ca \tag{23}$$

if $a \leq b/2$ (for some C depending on b). Also note that

$$|\frac{1}{a} - \frac{1}{\sinh a}| = O(a).$$

Integrating we get

$$|x(s) - x(t)| = \int_s^t \frac{1}{a} + O(a)da = |\log\frac{t}{s}| + O(|s^2 - t^2|). \tag{24}$$

Finally, we prove (18). If we differentiate (14) with respect to x we get

$$\cosh b\cosh x = \cosh y\frac{dy}{dx}.$$

Now divide by $\cosh y$ and use (13) to get

$$\frac{dy}{dx} = \cosh a = 1 + O(a^2).$$

The chain rule then gives

$$\frac{dy}{da} = \frac{dy}{dx}\frac{dx}{da} = \frac{-1}{\sinh a}(1+O(a^2))(1+O(a^2)) = \frac{-1}{\sinh a} + O(a),$$

as desired. □

Next we wish to prove the main estimate of this section:

Lemma 4.2. *Assume $b < 1/2$. Suppose Q_s, Q_t are Lambert quadrilaterals with a-sides of length s and t respectively and have equal length opposite sides b in both cases. Assume $s/10 \le t \le s \le b/2$. Let $\epsilon = (s-t)/s$. There is a $C < \infty$ depending only on b so that the following holds. There is a quasiconformal mapping $f : Q_s \to Q_t$ which is affine on the edges a and b and whose dilatation satisfies*

$$K(x) \le \begin{cases} 1 + C\epsilon & \rho(x,a) \le 1 \\ 1 + C\epsilon \exp(-2\rho(x,a)), & \rho(x,a) \ge 1. \end{cases}.$$

Proof. We know from (15) that $x = x(s)$ is bounded away from zero uniformly (in terms of b). If $x \le 1$, then the lemma follows from Corollary 3.2. Thus we may assume $x > 1$.

Choose points $x_0, \dots x_n$ along the x-side of Q_s inductively as follows. The point x_0 is the corner of Q_s where the x and b sides meet. The point y_0 is the other endpoint of the b-side. The point x_1 is the point on the x-side which is distance b from x_0. Consider the geodesic segment which is perpendicular to the x-side at x_1 and let y_1 be the point where it hits the y-side of Q_s and let d_1 be its length. Let x_2 be the point on the x-side of Q_s which is distance d_1 from x_1 (and so that x_1 separates x_2 and x_0). In general, given x_{n-1}, consider the geodesic segment perpendicular to x through x_{n-1}, let y_{n-1} be the point of intersection of this geodesic with the y-side of Q_s and let d_{n-1} be the distance from x_{n-1} and y_{n-1}. Then let x_n be the point on the x-side of Q_s which is distance d_{n-1} from x_{n-1} and which is separated from x_0 by x_{n-1}. See Figure 8.

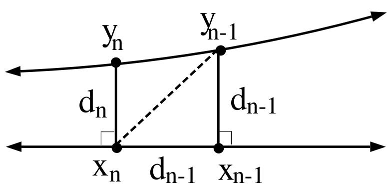

FIGURE 8. Definition of S_{n-1}

Continue until the first time that the distance from x_n to the a-side of Q_s is < 1. Let N denote the index when this occurs. For $n = 1, \dots N$, let $S_n \subset Q_s$ denote the quadrilateral with corners $x_{n-1}, x_n, y_n, y_{n-1}$. Let $T = Q_s \setminus \cup_1^N Q_n$. See Figure 9. Now divide the quadrilateral Q_t in a similar way. Choose points $\tilde{x}_n$ on its x-side so that $\tilde{x}_0$ is the corner of its b and x sides and $\rho(\tilde{x}_n, \tilde{x}_0) = \rho(x_n, x_0)$ for $n = 1, \ldots N$. Take the perpendiculars to the x-side of Q_t though these points and let $\tilde{y}_n$ be the points of intersection with the y-side of Q_t and $\tilde{d}$ be their lengths. Let $\tilde{S}_n$ be the corresponding quadrilaterals and let $\tilde{T} = Q_t \setminus \cup_1^N \tilde{S}_n$. See Figure 9.

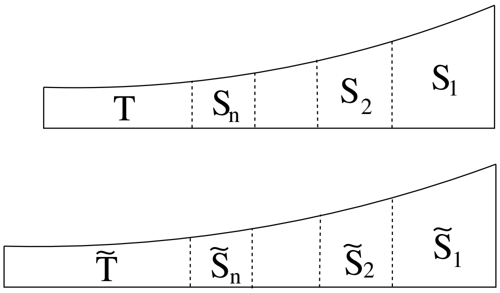

FIGURE 9. Decomposing Q_s and Q_t

By Lemma 4.1 we know $|\phi(s) - \phi(t)| \leq C(s^2 - t^2) = O(s(s-t))$. The segment d_n meets the y edges of Q_s and Q_t at angles between ϕ and $\pi/2$. Thus (12), (23) and (15) imply

$$d_n - \tilde{d}_n \leq Cs(s-t)e^{\rho(x_n,x_0)} = C(1 - \frac{t}{s})e^{-2x}e^{\rho(x_n,x_0)} = C\epsilon e^{-x-\rho(x_n,a)}.$$

See Figure 10 (in this figure, the vertex of ϕ is at the origin and the b-sides for the two quadrilaterals coincide). Using (15) and the fact that d_n is the b-side of a Lambert

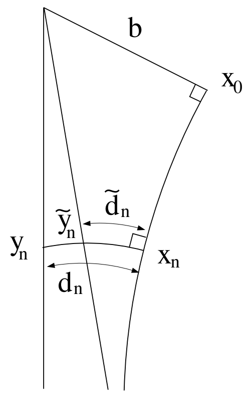

FIGURE 10. Estimating $d_n - \tilde{d}_n$

quadrilateral itself, we see

$$d_n \geq Ca \sinh \rho(x_n, a) \geq Ce^{-x} e^{\rho(x_n,a)} = Ce^{-\rho(x_n,x_0)}.$$

Thus

$$d_n/\tilde{d}_n = 1 + O(\epsilon e^{-2\rho(x_n,a)}).$$

Thus by Corollary 3.2 there is a $1 + O(\epsilon e^{-2\rho(x_n,a)})$-quasiconformal map from S_n to $\tilde{S}_n$. This gives the lemma on the S_n's.

Subdivide T (which by our choice of N has x-side of length between 1/2 and 1) into subquadrilaterials with equal length x-sides $\simeq a$. The corresponding side of $\tilde{T}$ is only longer by a factor of $\log \frac{s}{t} + O(a(s-t)) = O(s-t)$ by (24) and so we can apply Corollary 3.2 and get maps on each sub-quadrilateral with quasiconformal constant $1 + O(\epsilon)$. □

The estimate in the previous lemma for the case $\rho(x,a) \geq 1$ did not use the hypothesis that $t \geq s/10$, so that estimate is true in general. For $t \leq s/10$ the estimate for $\rho(x,a) < 1$ becomes $K(x) \leq Cs/t$. To prove this, note that because T and $\tilde{T}$ are both Lambert quadrilateral themselves, we may place them in the hyperbolic upper half-plane so that the a-sides lie on the imaginary axis, the x-side lies on the circle $|z| = 1$ and the y-sides line on circles of the form $|z| = r$ for $r = e^s$ and $r = e^t$. The remaining b sides lie on circles centered on the real axis. See Figure 11. Apply logarithims to conformally map to horizontal strips.

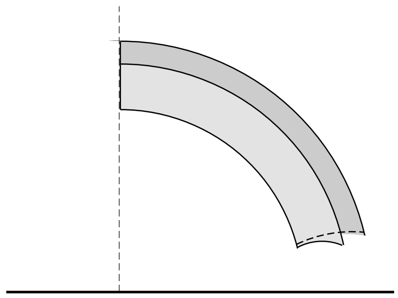

FIGURE 11. T and $\tilde{T}$

We now map T to $\tilde{T}$ in a series of steps as illustrated in Figure 12. First we map T to $W_1 \subset P_1$ with the b side going to a radial line segment and hyperbolic arclength on this edge being mapped affinely onto arclength of the image. This can be done with quasiconformal constant C_1 bounded only depending on b. Next

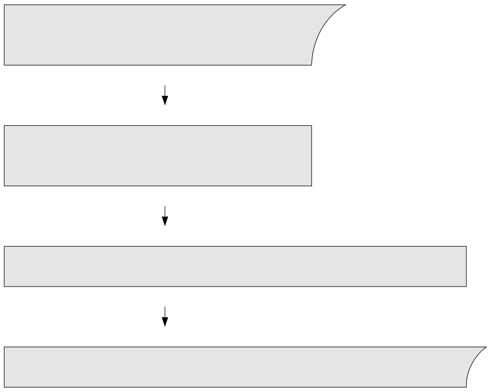

FIGURE 12. The map from T to $\tilde{T}$

we map $W_1 \to W_2 \subset \tilde{T}$ by the map $re^{i\theta} \to r^{s/t}e^{i(\frac{p}{i}2-a(\frac{\pi}{2}-\theta))}$ where a is chosen so that $\partial W_1 \cap \{|z| = 1\}$ is mapped onto the a_2 edge of P_λ. Since this edge of W_1 has (Euclidean) length uniformly bounded below, and the (Euclidean) length of the image is at most $\pi/2$, we see that a is uniformly bounded. Thus this map has quasiconformal constant $\leq C_2\frac{s}{t}$, where C_2 is absolute. Finally we map W_2 to P_λ by mapping the radial edge onto the b edge of $\tilde{T}$ with arclength mapping affinely. As above, this can be done with constant C_1 which depends only on b. Composing these steps gives the desired map with a quasiconformal constant of $C_1^2C_2\frac{s}{t}$. This finishes the proof of the estimate.

If $t > s$, then it is easy to formulate and prove the corresponding estimates by considering the inverse of the map constructed above.

5. Decomposing a L-bounded hexagon

Let H be a hyperbolic hexagon with all right angles and suppose the sides have lengths $a_1, b_3, a_2, b_1, a_3, b_2$, ordered counter-clockwise around the boundary (so a_j is opposite b_j). See Figure 13.

The following two relations are Theorems 7.19.1 and 7.19.2 of [2],

$$\frac{\sinh a_1}{\sinh b_1} = \frac{\sinh a_2}{\sinh b_2} = \frac{\sinh a_3}{\sinh b_3}, \tag{25}$$

$$\cosh b_1 \sinh a_2 \sinh a_3 = \cosh a_1 + \cosh a_2 \cosh a_3. \tag{26}$$

Note that this easily implies that right hexagon is determined by the lengths of three alternating sides.

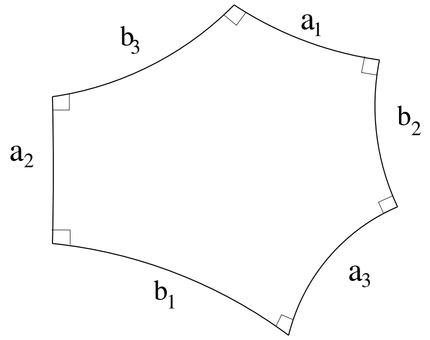

FIGURE 13. Labeling sides of a hexagon

Throughout this section we will assume that the three sides a_1, a_2, a_3 are all bounded by some $L < \infty$. Such a hexagon will be called L-bounded. However, if any of these sides is very short then some of the remaining b-edges will be long. In this section we want to show that any L-bounded hexagon H can be divided into a central hexagon H_0, all of whose edges are bounded and bounded away from zero in terms of L, and at most three Lambert quadrilaterals.

To accomplish this, suppose we have two numbers a, b (to be chosen later) with $a \leq \frac{1}{2}b < 1/4$. If $a_j \leq a$, $j = 1, 2, 3$, then form a Lambert quadrilateral, Q_j, with a_j as one side, two other sides lying along the edges b_{j-1} and b_{j+1} (here and below indices are taken mod 3 in $\{1, 2, 3\}$) and the fourth edge has length b and makes a right angle with b_{j-1}. See Figure 14. The unique non-right angle in Q_j will be denotes ϕ_j. Let x_{j-1} be the length of the edge of Q_j which lies in b_{j-1} and let y_{j+1} be the length of the edge of Q_j which lies on b_{j+1}. We shall see below that Q_j is a subset of H if a and b are chosen correctly. If $a_j > a$ then we set $Q_j = \emptyset$ and $x_j = y_j = 0$. Let $H_0 = H \setminus \cup_{j=1}^{3} Q_j$. The six sides of H_0 are in an obvious correspondence with those of H and will say "the side of H_0 in position a_j (or b_j)".

Lemma 5.1. *There is a choice of a, b so that $H_0 \subset H$ and the side lengths of H_0 are bounded and bounded away from zero depending only on L, a, b.*

Proof. Since the sides of H_0 in positions a_1, a_2, a_3 are all $\leq L$ and by construction have length $\geq a$, we only have to check the sides in positions b_1, b_2, b_3. Without loss of generality it suffices to consider just b_1. There are four cases depending on whether $a_2 > a$ and $a_3 > a$.

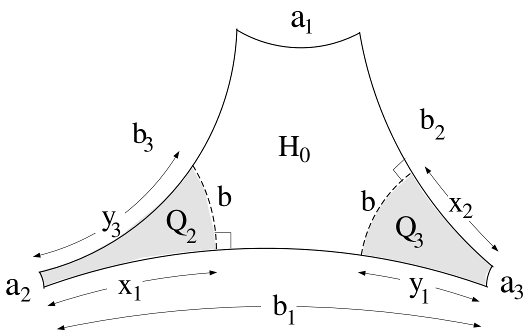

FIGURE 14. A hexagon with two short sides

Case 1: First suppose both $a_2, a_3 > a$, so that the b_1-edge of H_0 is the same as the corresponding edge of H, i.e.,has length b_1. By (26),

$$\cosh b_1 = \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_2 \sinh a_3} \leq \frac{2e^L + 2e^{2L}}{a^2},$$

and so b_1 is bounded above depending only on a and L. On the other hand, by (26) and (19)

$$\begin{aligned}\cosh b_1 = \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_2 \sinh a_3} \quad &\geq \quad e^{-2L} + (1+e^{-2a_2})(1+e^{-2a_3}) \\ &\geq \quad 1 + 2e^{-2a},\end{aligned}$$

which implies b_1 is bounded below depending only on a.

Case 2: Next suppose $a_2 \leq a$ and $a_3 > a$. Then the side of H_0 in position b_1 has length $b_1 - x_1$. By (12)

$$\sinh x_1 = \frac{\cos \phi_2}{\sinh a_2},$$

and by (26),

$$\begin{aligned}\cosh b_1 \quad &= \quad \frac{1}{\sinh a_2}\frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_3} \\ &\geq \quad \frac{1}{\sinh a_2}\frac{\cosh a_3}{\sinh a_3} \\ &\geq \quad \frac{1}{\sinh a_2}.\end{aligned} \tag{27}$$

This implies

$$2e^{b_1} \geq e^{b_1} + e^{-b_1} = 2\cosh b_1 \geq \frac{2}{\sinh a_2} \geq \frac{2C}{a_2}$$

for some $C = C(b)$ since $a_2 \leq a < b$. Hence

$$b_1 \geq \log \frac{1}{a_2} + \log C(b) \geq \log \frac{1}{b} + \log C(b), \tag{28}$$

independent of a or L. By (13)

$$\sin \phi_2 = \frac{\cosh a_2}{\cosh b} \geq \frac{1}{\cosh b},$$

so if b is small enough then ϕ_2 is a close as we wish to $\pi/2$ (independent of a). Assume we have fixed b so small that $\cos \phi_2 \leq \frac{1}{2}$ and $b_1 \geq 1$ (using (28). By (12) and (27)

$$\sinh x_1 = \frac{\cos \phi_2}{\sinh a_2} \leq \frac{1/2}{\sinh a_2} \leq \frac{1}{2} \cosh b_1, \tag{29}$$

and since $b_1 \geq 1$ this implies $b_1 - x_1 \geq C_1$ for some absolute C_1 which does not depend on a, b or L (as long as b satisfies the choices made above).

To prove the upper bound, note that if $a_1, a_2 \leq L$ and $a_3 \geq a$, then

$$\begin{aligned} \cosh b_1 &= \frac{1}{\sinh a_2} \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_3} \\ &\leq \frac{1}{\sinh a_2} \left(\frac{\cosh a_1}{\sinh a_3} + \cosh a_2 \frac{\cosh a_3}{\sinh a_3} \right) \\ &\leq \frac{1}{\sinh a_2} \left(\frac{e^L}{a} + e^L (1 + e^{-2a}) \right) \\ &\leq \frac{A(a, L)}{\sinh a_2}. \end{aligned} \tag{30}$$

Thus

$$\sinh x_1 = \frac{\cos \phi_2}{\sinh a_2} \geq \frac{\cos \phi_2}{A(a, L)} \cosh b_1,$$

which implies $b_1 - x_1 \leq C_2$ for some absolute C_2 which depends on a and L. This proves that $b_1 - x_1$ is bounded above and below in terms of a, b and L.

Case 3: Next assume $a_2 > a$, but $a_3 \leq a$. Since a_2 and a_3 play symmetric roles in (26), the argument above shows that $b_1 - x_2$ is bounded above and below uniformly. We shall now show that $x_2 - y_1$ is small, and hence that $b_1 - y_1$ is also bounded above and below. By (13)

$$\cosh y_1 \sin \phi_3 = \cosh x_2.$$

Since $\sin \phi_3 < 1$, this implies

$$y_1 > x_2. \tag{31}$$

Thus $b_1 - y_1$ is bounded above since $b_1 - x_2$ is. On the other hand,

$$\begin{aligned} \sin\phi_3 &= \frac{\cosh x_2}{\cosh y_1} \\ &= \frac{e^{x_2}+e^{-x_2}}{e^{y_1}+e^{-y_1}} \\ &= e^{x_2-y_1}\frac{1+e^{-2x_2}}{1+e^{-2y_1}}, \end{aligned}$$

and hence

$$\begin{aligned} y_1 &= x_2 + \log\frac{1}{\sin\phi_3} + \log(1+e^{-2x_2}) - \log(1+e^{-2y_1}) \\ &\le x_2 + \log\frac{1}{\sin\phi_3} + \log(1+e^{-2x_2}) \\ &\le x_2 + C, \end{aligned}$$

where C may be as small as we wish if b is small enough. Taking it to be less than $\frac{1}{2}C_1$ (which did not depend on b) we get

$$b_1 - y_1 \ge \frac{1}{2}C_1 \tag{32}$$

as our lower bound if b is small enough (independent of L).

Note that the proof of the previous two cases shows that $x_j \le b_j$ and $y_j \le b_j$ and hence the quadrilaterals Q_j are really subsets of H, as claimed in the lemma.

Case 4: Finally assume both $a_2, a_3 \le a$. By (26),

$$\cosh b_1 = \frac{\cosh a_1 + \cosh a_1 \cosh a_3}{\sinh a_2 \sinh a_3} \ge \frac{2}{\sinh a_2 \sinh a_3},$$

whereas by the middle of (29),

$$\sinh x_1 \le \frac{1}{2}\frac{1}{\sinh a_2},$$
$$\sinh x_2 \le \frac{1}{2}\frac{1}{\sinh a_3}.$$

Thus

$$\sinh x_1 \sinh x_2 \le \frac{1}{8}\cosh b_1,$$

and because $b_1 \ge 1$ this implies $b_1 - (x_1 + x_2) \ge C_3$, with C_3 independent of b. Choose b so small that $|x_2 - y_1| \le C_3/2$ and this gives the desired lower bound. On the other hand, by (26),

$$\cosh b_1 = \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_2 \sinh a_3} \le \frac{e^L + e^{2a}}{\sinh a_2 \sinh a_3},$$

which implies by (29),

$$\sinh x_1 \sinh x_2 \geq \frac{1}{A(L,a,b)} \cosh b_1,$$

which implies $b_1 - (x_1 + x_2)$ is bounded above, depending only on L, a, b. Since $y_1 \geq x_2$, this gives the desired upper bound.

Note that this shows the quadrilaterals Q_1, Q_2 and Q_3 are disjoint. This completes all the cases and hence finishes the proof of the lemma. □

Next we record a calculation which we will need later. We will think of the a_1 and a_3 sides of our right hexagon as having fixed length, but as a_2 as varying and we wish to compute the rate of change of the b-sides. Since b_1 and b_3 play symmetric roles we need only compute the derivatives for b_1 and b_2.

Lemma 5.2. *With notation as above*

$$\frac{db_1}{da_2} = \frac{-1}{\sinh a_2}(1 + O(e^{-2b_3})), \tag{33}$$

$$\frac{db_2}{da_2} = O(a_2), \tag{34}$$

where the constants depend only on L, the upper bound for a_1, a_2, a_3.

Proof. If we think of (26) as giving b_1 as a function of a_2 with a_1 and a_3 fixed, and differentiate, we get

$$\begin{aligned}
\frac{db_1}{da_2} &= \frac{\sinh^2 a_2 \cosh a_3 - (\cosh a_1 + \cosh a_2 \cosh a_3) \cosh a_2}{\sinh b_1 \sinh^2 a_2 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{\sinh a_2^2 \cosh a_3 - \cosh a_1 \cosh a_2 - \cosh^2 a_2 \cosh a_3}{\sinh b_1 \sinh a_2 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{(\sinh a_2^2 - \cosh a_2^2) \cosh a_3 - \cosh a_1 \cosh a_2}{\sinh b_1 \sinh a_2 \sinh a_3} \\
&= \frac{-\cosh a_3 - \cosh a_1 \cosh a_2}{\sinh a_2 \sinh b_1 \sinh a_2 \sinh a_3} \\
&= \frac{-\cosh b_3 \sinh a_1 \sinh a_2}{\sinh a_2 \sinh b_1 \sinh a_2 \sinh a_3} \\
&= \frac{-\cosh b_3 \sinh a_1}{\sinh a_2 \sinh b_1 \sinh a_3} \\
&= \frac{-\cosh b_3 \sinh a_3}{\sinh a_2 \sinh b_3 \sinh a_3} \\
&= \frac{-\cosh b_3}{\sinh a_2 \sinh b_3} \\
&= \frac{-1}{\sinh a_2}(1 + O(e^{-2b_3})).
\end{aligned}$$

where the fifth equality holds by (26) and the seventh holds by (25) and the constant in the last line only depends on a lower bound for b_3. We know by (26) that

$$\begin{aligned}
\cosh b_3 &= \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_2 \sinh a_3} \\
&\geq \frac{1}{\sinh a_2 \sinh a_3} + (\frac{\cosh a_2}{\sinh a_2} \frac{\cosh a_3}{\sinh a_3}) \\
&\geq e^{-2L} + 1,
\end{aligned}$$

since $a_2, a_3 \leq L$. Thus b_3 is bounded away from zero depending only on L and so our estimate of the derivative above depends only on L.

To see what happens to the side corresponding to b_2 think of (26) as giving b_2 as a function of a_2 with a_1, a_3 fixed. Differentiating, we get

$$\frac{db_2}{da_2} = \frac{\sinh a_2}{\sinh b_2 \sinh a_1 \sinh a_3}.$$

By Theorem 17.18.1 of [2],

$$\cosh t = \sinh b_2 \sinh a_3,$$

where t is the length of the common orthogonal between sides a_1 and b_1. See Figure 15.

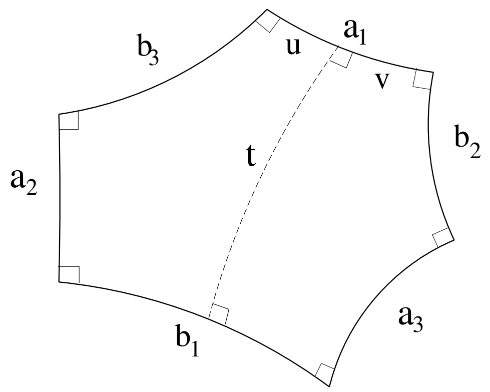

FIGURE 15. Definition of t

Let $u + v = a_1$ be the lengths of the subintervals of a_1 on either side of t as in Figure 15. On page 161 of [2] it is shown that

$$\sinh a_1 \sinh t \geq \sinh u \sinh t = \cosh a_2 \geq 1,$$

and hence

$$\frac{db_2}{da_2} = \frac{\sinh a_2}{\cosh t \sinh a_1} \leq \sinh a_2 = O(a_2). \tag{35}$$

□

Lemma 5.3. *Given $L < \infty$, there is a $C = C(L) < \infty$ so that the following holds. Suppose H_1 is right, hyperbolic hexagon and alternate side lengths $a_1, a_2 = s, a_3$ and suppose $s \leq L$. Suppose H_2 is another right hexagon with alternate side lengths $a_1, a_2 = t, a_3$. Assume s and t are both less than a. Then the central hexagons $H_{0,1}$ $H_{0,2}$ of H_1 and H_2 as described above have corresponding sides and angles which differ by less than $C|s^2 - t^2|$.*

Proof. Five of the angles in $H_{0,1}$ agree exactly with the corresponding angles in $H_{0,2}$. The only one that may disagree is the angle between the sides in the a_2 and b_3 positions. This angle is complementary to ϕ_2. So by (22) this angle in the two hexagons differs by less than $O(|s^2 - t^2|)$ as desired.

The sides of $H_{0,1}$ and $H_{0,2}$ which correspond to positions a_1, a_2, a_3 are the same length. Thus we need only consider the other three sides b_1, b_2, b_3 (and since b_1

and b_3 play symmetric roles, it is enough to consider only b_1 and b_2). By Lemma 5.2, $\frac{db_2}{da_2} = O(a_2)$. Moreover, the side lengths of the quadrilaterals Q_1 and Q_3 (if they occur) don't depend on a_2 and hence, integrating as in (22), we see that the side lengths of the central hexagons for H_1 and H_2 in position b_2 differ by at most $O(|s^2 - t^2|)$.

Next we consider the sides in position b_1. By the proof of Lemma 5.1 we know $b_3 \geq y_3 \geq x_1$. Thus by (23) we get

$$e^{-2b_3} \leq e^{-2x_1} \leq a_2^2.$$

Hence, from (17) and (33) we deduce

$$\frac{dx_1}{da_2} - \frac{db_1}{da_2} = \frac{-1}{\sinh a_2}(1 + O(a_2^2)) - \frac{-1}{\sinh a_2}(1 + O(a_2^2)) = O(a_2).$$

Since y_1 does not depend on a_2, this implies the sides of the central hexagons of H_1 and H_2 corresponding to b_1 differ by $O(|s^2 - t^2|)$.

Finally we have to consider the sides in position b_3. By (18),

$$\frac{dy_3}{da_2} = \frac{dy_3}{dx_1}\frac{dx_1}{da_2} = \frac{1}{\sinh a_2}(1 + O(a_2^2)),$$

with constant depending only on b. Since the roles of b_1 and b_3 are symmetric, we also have $\frac{db_3}{da_2} = (1 - O(a_2^2))/\sinh a_2$ and so we can complete the proof just as in the previous case. ☐

Lemma 5.4. *Suppose H_1 and H_2 are two L-bounded right hexagons with a-sides (a_1^1, a_2^1, a_3^1) and (a_1^2, a_2^2, a_3^2) respectively. Assume $a_1^1 = a_1^2$, $a_3^1 = a_3^2$ and $a_2^2 = (1-\epsilon)a_2^1 \leq a_2^1$. Then there is a quasiconformal map $f : H_1 \to H_2$ which is affine on the a-sides, is isometric on the boundary quadrilaterals associated to a_1^1 and a_3^1 and has dilatation satisfying*

$$K_f(z) \leq 1 + C\epsilon \exp(-2\rho(z, a_1^1)).$$

Proof. Since the quadrilaterals corresponding to sides a_1 and a_3 are identical for both hexagons, we may clearly take the map to be isometric in them. If $a_1^1 \geq a$ (where a is as in Lemma 5.1) then apply Lemma 3.3 to $H_1 \setminus (Q_1 \cup Q_3)$ using the estimates of Lemma 5.3. This gives the desired estimate since this region has bounded diameter. If $a_1^1 < a$ then apply Lemma 3.3 to $H_1 \setminus (Q_1 \cup Q_2 \cup Q_3)$ to get the estimate in the central hexagon and apply Lemma 4.2 to the quadrilateral Q_1. ☐

6. Quasiconformal mappings of Y-pieces

A Y-piece is a bordered Riemann surface which is topologically a sphere with three disks removed and in which each of the three boundary components is a hyperbolic geodesic. A generalized Y-piece is similarly defined, except that we also allow boundaries of length zero, i.e., instead of removing a disk we may remove a point.

A Y-piece can always be realized as a hyperbolic octagon with three side pairings. Moreover, there is a line of symmetry which divides the octagon into two isometric right hexagons. The alternate sides of these hexagons are given by $a_1 = a/2$, $a_2 = b/2$ and $a_3 = c/3$. See Figure 16. As noted following (26), a right hexagon is determined

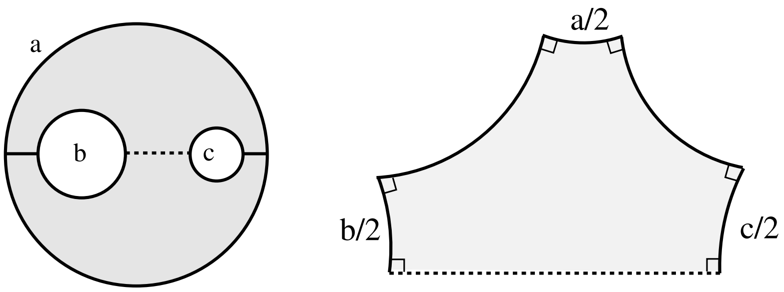

Figure 16. Parameterizing a Y-piece

by three alternating sides, and so we see that a Y piece is uniquely determined by its three side lengths.

The part of Y corresponding to the central hexagons will be called the central piece of Y. Note that the diameter of the central part is bounded depending only on L. The parts of Y corresponding to the quadrilaterals Q_j will be called the boundary pieces of Y. Given $\epsilon > 0$, the thin part of R consists of the points where the injectivity radius of is $\leq \epsilon$ and the thick part is where the injectivity radius is $> \epsilon$. The following is easy and its proof is left to the reader.

Lemma 6.1. *Given $L < \infty$, there is an $\epsilon > 0$ in the definition of the thick and thin parts of R and a $C < \infty$, so that for any L-bounded Y-piece, the ϵ-thin part of $R \cap Y$ is contained in the boundary pieces of Y and the ϵ-thick part of $R \cap Y$ is contained in a C-neighborhood of the central piece.*

We will think of two Y-pieces as being close to each other if their boundary lengths are close, i.e., we define

$$d(Y_1, Y_2) = \max(|\log \frac{a_1}{a_2}|, (|\log \frac{b_1}{b_2}|, (|\log \frac{c_1}{c_2}|).$$

For generalized Y-pieces we interpret $|\log \frac{a_1}{a_2}|$ as zero if $a_1 = a_2 = 0$ and as $+\infty$ if one is zero and the other is not. Similarly for the b and c terms.

Lemma 6.2. *Suppose Y_1 and Y_2 are two L-bounded generalized Y-pieces with boundary lengths (a_1, b_1, c_1) and (a_2, b_1, c_1) respectively. Assume that $d(Y_1, Y_2) = \epsilon = |\log a_1/a_2| \leq 2$. Then there is a $C = C(L)$ and a quasiconformal map $f : Y_1 \to Y_2$ with constant $K = 1 + C\epsilon$ which is affine on each of the boundary components. Moreover, the Beltrami coefficient μ of f is supported on the boundary piece associated to the a_1 boundary component γ and the central part of Y_1 and it satisfies*

$$|\mu(z)| \leq C\epsilon \exp(-2\mathrm{dist}(z, \gamma))$$

In other words, if a is small, the corresponding dilatation decays exponentially fast away from the boundary and is zero in the thin parts (if they exist) corresponding to the two other boundary pieces.

Proof. For Y-pieces with three non-degenerate boundary components, the proof is immediate from the results of the previous sections (write it as a union of two symmetric hexagons, apply the previous results and use symmetry to show the map is continuous on the Y-piece). If there are one or more punctures on the boundary, then we obtain the result by taking limits over a sequence of non-degenerate Y-pieces that approach the generalized Y-piece. □

The following is an easy consequence using compositions of the maps above which is used in the paper [5]

Lemma 6.3. *Suppose Y_1 and Y_2 are two L-bounded generalized Y-pieces and let $D = d(Y_1, Y_2)$. Then there is a quasiconformal map $f : Y_1 \to Y_2$ with constant $K = K(L, D)$ which is affine on each of the boundary components. Moreover, the dilatation K_f of f satisfies*

$$|K_f(z)| \leq 1 + C(L, D) \exp(-2\mathrm{dist}(z, \partial Y))$$

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