A CRITERION FOR THE FAILURE OF RUELLE'S PROPERTY

CHRISTOPHER J. BISHOP

Abstract. Ruelle proved that for quasiconformal deformations of cocompact Fuchsian groups the Hausdorff dimension of the limit set is an analytic function of the deformation. In this paper, we give a criterion for the failure of analyticity for certain infinitely generated groups. In particular, we show that it fails for any infinite abelian cover of a compact surface, answering a question of Astala and Zinsmeister.

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1. Introduction

We say a Fuchsian group G has the Ruelle property if whenever $\{G_t\}$ is an analytic family of quasiconformal deformations of G then $\dim(\Lambda(G_t))$ is an analytic function of t ($\dim(\Lambda)$ is the Hausdorff dimension of its limit set). Ruelle [12] proved all cocompact groups have this property and Astala and Zinsmeister showed it fails for certain infinitely generated convergence type groups. In this paper we give another condition on G which implies the failure of Ruelle's property and which applies to many divergence type examples. In particular, it answers a question of Astala and Zinsmeister by proving that Ruelle's property fails for all infinite Abelian covers of a compact surface.

Let $\delta(G)$ denote the critical exponent of the Poincaré series of G (see Section 2). In the finitely generated case, $\delta = \dim(\Lambda(G))$, but we only have $\delta \leq \dim$ in general, and so we speak separately of the Ruelle property for dimension and for the critical exponent.

If G is a Fuchsian group and μ is a bounded measurable function on the unit disk, $\mathbb{D}$, which satisfies $\|\mu\|_\infty < 1$ and $\mu(g(z)) = \mu(z)g'(z)/\overline{g'(z)}$, for every $g \in G$, then we say μ is a G-invariant Beltrami coefficient (or complex dilatation). It is well known that there is a corresponding quasiconformal mapping f_μ which is analytic outside the disk and which conjugates G to a quasi-Fuchsian group G_μ. For convenience, we will write $\delta(\mu) = \delta(G_\mu)$ and $\dim(\mu) = \dim(\Lambda(G_\mu))$ when G is clear from context. The function μ is defined on the unit disk, but is not a well defined function on the quotient $R = \mathbb{D}/G$. However, $|\mu|$ is well defined on R, so by an abuse of notation we will sometime refer to μ as being defined on R, e.g., "consider μ compactly supported on R" which really means that μ is G-invariant Betrami coefficient on the disk so that $|\mu|$ projects to a compactly supported function on R.

We say a Fuchsian group G has *big deformations near infinity* if there are $\epsilon, \delta > 0$ so that for any compact set $K \subset R = \mathbb{D}/G$, there is a G-invariant Beltrami coefficient μ supported in $R \setminus K$ with $\|\mu\|_\infty \leq 1 - \epsilon$ and $\delta(G_\mu) \geq 1 + \delta$.

Theorem 1.1. *If G is a torsion free, infinite area Fuchsian group with big deformations near infinity, then the Ruelle property fails for both the critical exponent and the Hausdorff dimension of the limit set.*

More precisely, we will show that if G has big deformations near infinity then there exist G-invariant dilatations μ_1 and μ_2 and an integer n so that if $\mu(t) = t\mu_1 + t^n\mu_2$ then at least one of $\delta(t\mu_1)$, $\delta(t^n\mu_2)$ or $\delta(\mu(t))$ is not real analytic on $(0,1)$.

The argument of Astala and Zinsmeister [2] shows that if G is a convergence group such that $\dim(\Lambda) = 1$ and μ is a G-invariant dilatation with compact support (modulo G) such that $\dim(\mu) > 1$, then the Ruelle property for dimension must fail. This is because they show $\dim(t\mu) \equiv 1$ on a non-degenerate interval containing 0 (but $\dim(t\mu) \geq \delta(t\mu) > 1$ for $t = 1$ by definition). If $\delta(G) = 1$, then their argument also applies to the critical exponent because if μ has compact support then $\dim(\mu) = \delta(\mu)$ by a result in [8]. See also [3] and [4].

One simple case of Theorem 1.1 is when $R = \mathbb{D}/G$ is an infinite cover of a surface $R_0 = \mathbb{D}/G_0$. If μ is a G_0-invariant dilatation then μ is also G-invariant (since $G \subset G_0$). Thus if G_0 has a big deformation so does G. Moreover we may assume such a μ is compactly supported on R (see Lemma 2.1 and our early remark about abuse of notation). Now take translates of μ by the covering group so that the supports leave every compact set. Clearly the critical exponent never changes so G has big deformations near infinity. Thus Ruelle's property fails. This answers the question of Astala and Zinsmeister from [4].

In [5] we give geometric conditions on the quotient surface $R = \mathbb{D}/G$ which implies there are big deformations near infinity. A special case (sufficient to give another proof of the covering surface case just discussed) is:

Proposition 1.2. *Suppose G is a torsion free, infinitely generated Fuchsian group and either of the following holds:*

(1) *the injectivity radius is bounded above and below*
(2) *G is divergence type and the injectivity radius is bounded below.*

Then G has big deformations near ∞ (and so Ruelle's property fails for G).

On the other hand, we also show in [5] that if R can be written as a countable union of Y-pieces and for any ϵ only finitely many of these pieces have a boundary component of length $> \epsilon$ then G has no big deformations near infinity. For such groups Ruelle's property remains open.

The remaining sections are organized as follows. In Section 2 we review some basic definition and results. In Section 3 we prove $\delta(\mu_1 + \mu_2) \approx \max(\delta(\mu_1), \delta(\mu_2))$, if the supports of μ_1 and μ_2 are far apart. In Section 4 we prove Theorem 1.1 for the critical exponent. In Section 5 we prove Theorem 1.1 for Hausdorff dimension.

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2. Background

Now we review some basic definitions. Given a set E, we define

$$\mathcal{H}^\alpha_\delta(E) = \inf\{\sum \operatorname{diam}(U_j)^\alpha : E \subset \cup_j U_j, \operatorname{diam}(U_j) \le \delta\},$$

where the infimum is over all coverings of E by sets of diameter $\le \delta$. Taking $\delta = \infty$ gives the Hausdorff content $\mathcal{H}^\alpha_\infty(E)$, and the Hausdorff measure of E is given by

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E).$$

The Hausdorff dimension of E is

$$\dim(E) = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\} = \inf\{\alpha : \mathcal{H}^\alpha_\infty(E) = 0\}.$$

We will consider two models for hyperbolic $d+1$ space. The first is the upper half-space $\mathbb{H}^{d+1} = \{(x, y) : x \in \mathbb{R}^d, y > 0\}$ with the metric $d\rho = ds/y$. The second is the ball model $\mathbb{B}^{d+1} = \{z \in \mathbb{R}^{d+1} : |z| < 1\}$ with the metric $2ds/(1 - |z|^2)$. A Kleinian group is a discrete group of isometries of the hyperbolic upper half-space $\mathbb{H}^3$. A Fuchsian group is a Kleinian group which fixes the unit disk $\mathbb{D} \subset \mathbb{R}^2$. Throughout this paper we shall assume that our groups are torsion free, i.e., there are are no elliptic elements.

If G is non-elementary then the accumulation set on the boundary of any orbit is independent of the orbit and is called the limit set $\Lambda(G)$. The conical limit set $\Lambda_c \subset \Lambda$ are the points x which are limits of an orbit intersected with a non-tangential cone with vertex at x. These are also the points such that the geodesic ray ending at x projects under the quotient to a curve which returns to some compact set infinitely often. The escaping limit set $\Lambda_e = \Lambda \setminus \Lambda_c$ corresponds to geodesics which project to curves which escape to infinity in the quotient.

The hyperbolic convex hull of Λ is denoted $C(\Lambda)$ and is a G-invariant set. Thus it projects to a well defined subset of the quotient manifold $M = \mathbb{H}^3/G$ call the convex core of M and denoted $C(M)$.

A Fuchsian group is cocompact if $\mathbb{D}/G$ is compact and is cofinite if the quotient has finite hyperbolic area. A Fuchsian group is called divergence type if

$$\sum_{g \in G} \exp(-\rho(0, g(0))) = \infty,$$

and otherwise it is called convergence type. There are several other conditions which are equivalent to divergence type. For example, a Fuchsian group G is divergence type iff the conical limit set Λ_c has full Lebesgue measure on the circle. Similarly G is divergence type iff the geodesic flow for $R = \mathbb{D}/G$ is ergodic (see [11]).

The group is called first kind if the limit set is the entire circle and is second kind otherwise. It is well known that cocompact $\subset$ cofinite $\subset$ divergence type $\subset$ first kind.

Given a Fuchsian group G and a point $z \in \mathbb{D}$, the injectivity radius of G at z is half the distance from z to the nearest distinct G-image of z, i.e.,

$$\text{inj}(z) = \frac{1}{2} \inf\{\rho(z, g(z)) : g(z) \neq z\}.$$

Given $\epsilon > 0$ we define the ϵ-thick and ϵ-thin parts of $R = \mathbb{D}/G$ to be the points where the injectivity radius is $\geq \epsilon$ or $< \epsilon$ respectively. There is a $\epsilon_M > 0$ (called the Margulis constant) independent of G so that for $\epsilon < \epsilon_M$ the ϵ-thin parts are disjoint and of exactly two possible types: the parabolic thin parts which lift to horoballs in $\mathbb{D}$ tangent to the unit circle at a parabolic fixed point and the hyperbolic thin parts which lift to fixed distance neighborhoods of hyperbolic geodesics.

The critical exponent (or Poincaré exponent) of the group is defined as

$$\delta(G) = \inf\{s : \sum_{g \in G} \exp(-s\rho(0, g(0)))\} < \infty.$$

It is shown in Lemma 2.2 [7] that if Ω is a bounded simply connected invariant component then one can also take

$$\delta(G) = \inf\{s : \sum_{g \in G} \text{dist}(g(z_0), \partial\Omega)^s < \infty\}, \tag{1}$$

for any $z_0 \in \Omega$ and where dist denotes Euclidean distance.

By Theorem 1.1 in [6], δ is equal to the Hausdorff dimension of the conical limit set. It is also proven there that if $G_n \to G$ (in the sense that each point of $G(0)$ is a limit of points from $\{G_n(0)\}$, then

$$\liminf_n \delta(G_n) \geq \delta(G).$$

A K-quasiconformal map from $\mathbb{R}^2$ to itself is one which satisfies

$$\limsup_{r\to 0} \frac{\max_{|x-y|=r} |f(x) - f(y)|}{\min_{|x-y|=r} |f(x) - f(y)|} \le K.$$

Such a map is well known to be differentiable almost everywhere and we let $\mu = f_z/f_{\bar{z}}$ denote its complex dilatation. It is also known that $\|\mu\|_\infty \le (K-1)/(K+1)$ and that the map f is determined (up to composition with Möbius transformations) by its dilatation. If we assume three points are fixed (i.e., 0, 1, ∞) then the map is uniquely determined by μ.

We shall say that a dilatation μ supported on the unit disk is G-invariant if

$$\mu(g(z)) = \mu(z)g'(z)/\overline{g'(z)},$$

for every $g \in G$. If μ is invariant under the action of a Fuchsian group G, then $f_\mu \circ g \circ f_\mu^{-1}$ is Möbius for every $g \in G$ and f_μ is called a quasi-conformal deformation of G. Given such a μ we let $\delta(\mu)$ denote the critical exponent of the group $G_\mu = f_\mu \circ G \circ f_\mu^{-1}$ and let $\dim(\mu) = \dim(\Lambda(G_\mu))$. Since any two groups associated to the same μ are conjugate by a Möbius transformation, these functions are well defined.

If G is a Fuchsian group then an analytic deformation of G is a family of quasi-Fuchsian groups G_t, together with group isomorphisms $\Phi_t : G \to G_t$ such that Φ_0 is the identity and $t \to \Phi_t(g)$ is analytic for each element $g \in G$. Using the analytic dependence of quasiconformal mappings on their dilatations (as discussed for example in Ahlfors book [1]) it is easy to see that if μ_1, μ_2 are fixed dilatations of disjoint support corresponding to deformations of G, f, g are analytic functions of t bounded by 1 and $\mu_t = f(t)\mu_1 + g(t)\mu_2$, then $\Phi_t = \Phi_{\mu_t}$ is an analytic deformation.

We will also use the fact that if $\{\mu_n\}$ is a sequence of dilatations with L^∞ norms uniformly bounded by $k < 1$ and which converge pointwise to μ, then the corresponding normalized quasiconformal maps converge uniformly to f_μ.

Lemma 2.1. *Suppose G is a Fuchsian group and μ is a G-invariant complex dilatation. Fix $z_0 \in R$ and let μ_r be the restriction of μ to the ball $B_r = \{z \in R : \rho(z, z_0) < r\}$. Then* $\liminf_{r\to\infty} \delta(G_{\mu_r}) \ge \delta(G_\mu)$.

Proof. This is immediate from our remarks above since convergence of the $\{\mu_n\}$ implies the corresponding quasiconformal maps converge uniformly. ☐

If f_μ is a quasiconformal deformation of a Fuchsian group G to a group G_μ then it is easy to check that $f_\mu(\Lambda_c) = \Lambda_c(G_\mu)$ (since quasiconformal maps on $\mathbb{R}^3$ mapping $\mathbb{R}^2$ to itself also preserve non-tangential convergence in the upper half-space). We shall use the result of Gehring and Väisälä [10] that for a K-quasiconformal mapping f,

$$K^{-1}\dim(E) \leq \dim(f(E)) \leq K\dim(E).$$

Since $\delta(G_\mu) = \dim(\Lambda_c(G))$, we see that we also have

$$K^{-1}\delta(G) \leq \delta(G_\mu) \leq K\delta(G).$$

In particular, if we consider a family $t\mu$ of G-invariant dilatations,

$$|\delta((t+\epsilon)\mu) - \delta(t\mu)| \leq C(\mu)\epsilon,$$

for $t \in [0,1]$, so that $\delta(t\mu)$ is Lipschitz with a constant depending only on $\|\mu\|_\infty$.

Given an interval $I \subset \mathbb{T}$, the corresponding Carleson "square" is

$$Q = Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1-|z| \leq |I|\}.$$

The "top half" of Q is defined as

$$T(Q) = \{z \in \mathbb{D} : z/|z| \in I, \frac{1}{2}|I| \leq 1-|z| \leq |I|\}.$$

A stopping time region is a domain of the form $W = Q_I \backslash \cup_j Q_{I_j}$ where I_j is a collection of pairwise disjoint intervals in I.

We will also need the following result from [8]. It basically says that at points far from the support of μ the corresponding map f_μ behaves like a Hölder function with exponent close to 1.

Lemma 2.2. *Given $K < \infty$ and $\eta > 0$ there are $C < \infty$ and $r < \infty$ so that the following holds. Suppose f is a conformal map of the disk with a K-quasiconformal extension to the plane with Beltrami coefficient μ. Also suppose that* $\operatorname{diam}(f(\mathbb{D})) \leq K|f'(0)|$. *Suppose $W = Q \setminus \cup_j Q_j$ is a stopping time region, $W^* = \{z^* : z \in W\}$, and $\rho(W^*, \operatorname{supp}(\mu)) \geq r$. Then for any collection $\mathcal{C} \subset \{Q_j\}$*

$$\sum_{\mathcal{C}} \operatorname{diam}(f(Q_j))^{1+\eta} \leq C \operatorname{diam}(f(Q))^{1+\eta} \sum_{\mathcal{C}} \frac{\operatorname{diam}(Q_j)}{\operatorname{diam}(Q)}.$$

3. Dilatations with distant supports

The proof of Theorem 1.1 is based on the simple observation that the maximum of two analytic functions need not be analytic. In fact, if f and g are functions on $[0,1]$ which cross (i.e., there are t_1, t_2 such that $f(t_1) < g(t_1)$ but $f(t_2) > g(t_2)$) then clearly not all three of $f, g, \max(f,g)$ can be analytic. In the proof of Theorem 1.1 we will build dilatations μ_1 and μ_2 so that $\delta(t\mu_1)$ and $\delta(t^n\mu_2)$ cross and such that

$$\delta(t\mu_1 + t^n\mu_2) = \max(\delta(t\mu_1), \delta(t^n\mu_2)) \equiv \max(\delta_1(t), \delta_2(t)), \tag{2}$$

(at least for an infinite set of t's which includes some points where $\delta_1 > \delta_2$ and others where $\delta_1 < \delta_2$). Thus at least one of $\delta, \delta_1, \delta_2$ must not be analytic in t, as desired. Equation 2 is based on the following result, which may be of interest on its own.

Lemma 3.1. *There is a function $\eta(r) \to 0$ as $r \to \infty$ so that the following is true. Suppose $R = \mathbb{D}/G$ is Riemann surface and $\{\mu_n\}$, $n = 1, 2\dots$ are G-invariant Beltrami coefficients supported on disjoint closed sets $\{K_n\}$ respectively such that $\inf_{n\neq m} \rho(K_n, K_m) \geq r$. Then if $\mu = \sum_n \mu_n$,*

$$|\delta(\mu) - \sup_n \delta(\mu_n)| \leq \eta(r).$$

Thus if we could have two Beltrami coefficients μ_1, μ_2 whose supports were infinitely far apart we would have

$$\delta(t(\mu_1 + \mu_2)) = \max(\delta(t\mu_1), \delta(t\mu_2)).$$

Of course, we cannot have two complex dilatations whose supports are infinitely far apart, but with only a little work we can achieve the same result with two sequences of dilatations whose supports tend towards infinity and which grow further and further apart. This is just what the "big deformations near infinity" hypothesis gives us. The remainder of this section is devoted to the proof of Lemma 3.1.

Given a hyperbolic 3-manifold $M = \mathbb{H}^3/G$ and a Lipschitz function f of compact support on M define

$$I(f) = \int_M |\nabla f|^2 / \int_M |f|^2,$$

and

$$\lambda_0 = \lambda_0(M) = \inf_f I(f).$$

where the infimum is over all Lipschitz functions of compact support (actually, the negative of this is more usually called the base eigenvalue, but for this paper we will take $\lambda_0 \geq 0$). It is also more usual to take the infimum over smooth functions, but it is easy to see this gives the same value in our situation. If $M = M_\mu$ is the quotient manifold associated to a quasi-Fuchsian group G_μ, we will sometimes simply write $\lambda_0(\mu)$ for $\lambda_0(M_\mu)$. By the Elstrodt-Patterson-Sullivan formula (e.g., Theorem 2.17 of [13]), $\lambda_0 = \delta(2-\delta)$, if $\delta \geq 1$ and $\lambda_0 = 1$ otherwise.

Lemma 3.2. *If $\{f_j\}$ are smooth functions with disjoint supports and $f = \sum f_j$, then $\inf_j I(f_j) \leq I(f) \leq \sup_j I(f_j)$.*

Proof. This is immediate from the corresponding inequality for real numbers
$$\inf_j \frac{x_j}{y_j} \leq \frac{\sum_j x_j}{\sum_j y_j} \leq \sup_j \frac{x_j}{y_j},$$
which is obvious. $\square$

Lemma 3.3. *There is a $C < \infty$ so that if $\epsilon < 1$ and $\varphi : M_1 \to M_2$ is a smooth, locally $(1+\epsilon)$-biLipschitz mapping then $|\lambda_0(M_1) - \lambda_0(M_2)| \leq C\epsilon$*

Proof. By the definition of λ_0 in terms of minimizing $I(f)$, it is easy to see that a $1+\epsilon$ biLipschitz map can change λ_0 by at most a factor bounded by a power of $(1+\epsilon)$. $\square$

Lemma 3.4. *There are constants d_0 and C such that the following holds. Suppose $\{\Omega_k\}$, $k = 1, 2\dots,$ are disjoint open sets in a hyperbolic 3-manifold M and $M = \Omega_0 \cup \cup_{k\geq 1}\Omega_k$. Also suppose that $\partial\Omega_0 \cap \Omega_k \neq \emptyset$ for every $k \geq 1$ and that $\rho(\partial\Omega_0 \cap \Omega_k, \partial\Omega_k) \geq d > d_0$. Then*
$$0 \leq \inf_{k\geq 0} \lambda_0(\Omega_k) - \lambda_0(M) \leqq \frac{C}{d}.$$

Proof. Since $\Omega_k \subset M$, the left hand inequality is obvious (since any f used in the definition of $\lambda_0(\Omega_k)$ also occurs in the definition of $\lambda_0(M)$). To prove the other direction, suppose f is a Lipschitz function of compact support on M, normalized so $\int_M |f|^2 = 1$ which almost minimizes in the definition of λ_0. Since M is a hyperbolic 3-manifold, $\lambda_0 \leq 1$. Thus we may assume that $\int_M |\nabla f|^2 + \int_M |f|^2 \leq 3$. For $k = 1, \dots, d$, define
$$A_k = \{z \in \Omega_0 : k \leq \rho(z, \partial\Omega_0) \leq k+1\}.$$

Choose k so that

$$\int_{A_k} |f|^2 \leq \frac{3}{d}, \quad \text{and} \quad \int_{A_k} |\nabla f|^2 \leq \frac{3}{d}.$$

Define

$$\varphi(z) = \max(0, 1 - 4\rho(z, A_k^c)).$$

Then φ is Lipschitz, is equal to 1 off A_k and equals 0 on a set which separates $\partial\Omega_0 \cap \Omega_k$ from the complement of Ω_k for every $k \geq 1$.

Thus $F = f\varphi$ is a Lipschitz function of compact support and

$$\int_M F^2 \geq \int_{M \setminus A_k} |f|^2 \geq 1 - \frac{3}{d},$$

$$\int_M |\nabla F|^2 \leq \int_M |\nabla f|^2 + \sup |\varphi'| \int_{A_k} |f|^2 \leq \int_M |\nabla f|^2 + \frac{6}{d} = I(f) + \frac{6}{d}.$$

Thus

$$I(F) \leq \frac{I(f) + \frac{6}{d}}{1 - \frac{2}{d}} \leq I(f) + \frac{C}{d},$$

if d is large enough (and this defines d_0). Moreover, $F = \sum_k F_k$ is a sum of functions of disjoint support with F_k supported in Ω_k. Thus by Lemma 3.2

$$I(f) \geq I(F) - \frac{C}{d} \geq \inf_k I(F_k) - \frac{C}{d} \geq \inf_k \lambda_0(\Omega_k) - \frac{C}{d}.$$

Now taking the infimum over all such f's gives the desired result. □

Next we have to show that the hypothesis of Lemma 3.1 implies that there is a decomposition as above. To do this we will make use of the Douady-Earle extension of a quasiconformal mapping f of $\mathbb{R}^2$ to a (hyperbolic) biLipschitz map of $\mathbb{H}^3$. We need that if f is close to the identity on $\mathbb{R}^2$ then so is the extension in the following sense.

Lemma 3.5. *Let $z_0 = (0, 0, 1) \in \mathbb{H}^3$. Given any $\delta > 0$ and $\lambda < 1$ there is an $\epsilon > 0$ so that the following holds. Suppose μ is a G-invariant dilatation supported in $\mathbb{D}$ on a set E of area $\leq \epsilon$ and $||\mu||_\infty \leq \lambda$, and that f is the corresponding quasiconformal map fixing $0, 1, \infty$. Then f has a G-invariant extension to F to $\mathbb{H}^3$ such that*

$$\sup_{||u||=1} ||F'(z_0)u|| \leq 1 + \delta,$$

$$\inf_{||u||=1} ||F'(z_0)u|| \geq 1 - \delta,$$

Proof. This is essentially the same as Lemma 7 in the paper of Douady and Earle [9]. Actually, one must repeat the proof with a small change. In Proposition $2'$ of that paper, Douady and Earle prove that there is a conformally natural extension operator (at least for dilatations with small norm) which is continuous from the supremum norm on homeomorphisms of the boundary (in the spherical metric) to C^∞ functions on $\mathbb{H}^3$. Moreover, the identity map on the boundary extends to the identity map on the interior. If the lemma above was false, then we could take a sequence of maps $\{f_n\}$ with $\epsilon \to 0$. By standard estimates this implies that $\{f_n\}$ converges uniformly to the identity mapping and thus by continuity $F_n'(z_0)$ converges to the identity tangent map. In particular, the quantities in the lemma tend to 1 as $\epsilon \to 0$, as desired.

If $\|\mu\|_\infty$ is not small enough for the Douady-Earle extension to be a homeomorphism then we factor μ into a finite number N (which depends only on $\|\mu\|_\infty$) of dilatations of small enough norm and take the composition of the extensions of the corresponding maps. The resulting map is an G-invariant extension of f_μ which clearly satisfies the correct estimates at z_0 (possibly with different constants depending on N). □

Given a set $K \subset \mathbb{R}^2$ we let $\omega_z(K)$ be the measure of K viewed from $z \in \mathbb{H}^3$, i.e., the normalized area of K after mapping to the ball model with z going to the origin. This is also the same as the harmonic measure of K with respect to z and is a (hyperbolic) harmonic function of z for a fixed K. The following is then immediate from the previous lemma.

Corollary 3.6. *Given $\delta > 0$ and $\lambda < 1$ there is an $\epsilon > 0$ so that the following holds. If μ is dilatation supported in a set K and $\|\mu\|_\infty \le \lambda$, then f_μ has an extension to $\mathbb{H}^3$ which is locally $1+\delta$ biLipschitz on the set*

$$\hat{K} = \{z \in \mathbb{H}^3 : \omega_z(K) < \epsilon\}.$$

We are now ready to show how to decompose M as required by Lemma 3.4.

Lemma 3.7. *Given any $d > 0$, $\eta > 0$ and $\lambda < 1$ there is a $r > 0$ so that the following holds. Suppose $\{\mu_n\}_{n\ge1}$ are G-invariant dilatations in $\mathbb{D}$ with $\|\mu_n\|_\infty \le \lambda$ for all n and suppose μ_n is supported on a set K_n with $\rho(K_n, K_m) \ge r$ for $n \ne m$. Let $M = \mathbb{H}^3/G$ be the 3-manifold associated to the Fuchsian group G. Let $\mu = \sum_n \mu_n$ and let G_μ, $M_\mu = \mathbb{H}^3/G_\mu$ be the corresponding quasi-Fuchsian group and hyperbolic* *manifold. Similarly, let $M_n = M_{\mu_n}$. Then we can choose $\epsilon_1 > \epsilon_2 > 0$ so that*

$$\Omega_0 = \{z \in \mathbb{H}^3 : \omega_z(f(K_n)) < \epsilon_1 \forall n\},$$

$$\Omega_n = \{z \in \mathbb{H}^3 : \omega_z(f(K_n)) > \epsilon_2\}, n \geq 1,$$

satisfy

(i) $M_\mu = \Omega_0 \cup \cup_{n\geq 1} \Omega_n$,
(ii) $\partial\Omega_0 \cap \Omega_n \neq \emptyset$ *for all* $n \geq 1$,
(iii) $\rho(\partial\Omega_0 \cap \Omega_n, \Omega_n^c) \geq d$,
(iv) $\lambda_0(\Omega_0) \geq 1 - \eta$,
(v) $\{\Omega_n\}$ *are pairwise disjoint for* $n \geq 1$,
(vi) *either* $\lambda_0(M_n) \geq 1 - \eta$ *or* $|\lambda_0(\Omega_n) - \lambda_0(M_n)| \leq \eta$.

Proof. Conditions (i) and (ii) are obvious since $\epsilon_1 > \epsilon_2$. Condition (iii) is also clear by Harnack's inequality (considering $u(z) = \omega_z(f(K_n))$) if the ratio ϵ_1/ϵ_2 is large enough. Condition (iv) follows immediately from Lemma 3.3 and Corollary 3.6 if ϵ_1 is small enough since this implies f^{-1} is a locally biLipschitz map (with constant close to 1 if ϵ_1 is small enough) of Ω_0 to a subset Ω_0' of M. Because of the infimum definition of λ_0 we obviously have $\lambda_0(\Omega_0') \geq \lambda_0(M) = 1$. Thus $\lambda_0(\Omega) \geq 1 - \eta$ if ϵ_1 is small enough.

Next we prove Condition (v), i.e., that the domains $\{\Omega_n\}_{n\geq 1}$ are pairwise disjoint. Let $X_n = \{z \in \mathbb{H}^3 : \omega_z(K_n) > \epsilon_3\}$. By the absolute continuity of quasiconformal mappings of the 2-sphere with respect to area, we see that for any $\epsilon_2 > 0$ there is an $\epsilon_3 > 0$ so that $\Omega_n \subset F(X_n)$, where F is the extension of f to $\mathbb{H}^3$. Thus given ϵ_3 we need only show that the sets $\{X_n\}$ are pairwise disjoint if r is large enough.

To prove this, choose r so that $2\pi e^{-r/4} < \epsilon_3$. Then if $z_0 = (0, 0, 1) \in X_n$, the set K_n must hit the disk $D(0, 1-e^{-r/4}) \subset \mathbb{D}$. Thus if $m \neq n$, k_m must lie in $\mathbb{D} \backslash D(0, 1-e^{-r/2})$. and hence has Euclidean area $\leq 2\pi e^{-r/2}$. Thus any point $w \in X_m$ must be within at least Euclidean distance $e^{-r/4}$ of the boundary. Thus the hyperbolic distance from X_n to X_m, $m \neq n$ (up to an additive constant) is $r/4$, and so they are disjoint.

To prove Condition (vi) holds, we write $f = g_n \circ f_n$ where f_n fixes $0, 1, \infty$ and has dilatation μ_n. Let $W_1 = g_n^{-1}(\Omega_n)$ and $W_2 = g_n^{-1}(\Omega_0 \cup \cup_{m\neq n} \Omega_m)$. Then Condition (v) and Corollary 3.6 implies g_n^{-1} is a locally biLipschitz map (with constant close to 1) of Ω_n to W_1. Furthermore, $W_1 \cup W_2 = M_n$ and the boundaries of these two sets are at least distance d/A apart (since g_n can be taken to be biLipschitz in the hyperbolic metric with constant $A = A(\lambda)$). Thus we can apply Lemma 3.4 with just these two sets to deduce

$$\min(\lambda_0(W_1), \lambda_0(W_2)) - \frac{CA}{d} \leq \lambda_0(M_n) \leq \min(\lambda_0(W_1), \lambda_0(W_2)).$$

Choose d so large that $CA/d \leq \eta/2$ and hence

$$\min(\lambda_0(W_1), \lambda_0(W_2)) - \eta/2 \leq \lambda_0(M_n) \leq \min(\lambda_0(W_1), \lambda_0(W_2)). \tag{3}$$

Using the absolute continuity of f_n with respect to area (as in the proof of Condition (v)) we see that

$$f_n^{-1}(W_2) \subset \{z \in \mathbb{H}^3 : \omega_z(K_n) < \epsilon_4\},$$

for some ϵ_4 depending only on ϵ_1 and tending to zero as $\epsilon_1 \to 0$). Thus if ϵ_1 is small enough, ϵ_4 will be small and f_n^{-1} will be locally biLipschitz on W_2 with constant close to 1. Thus we have $\lambda_0(W_2) \geq 1 - \eta/2$ if ϵ_1 is small enough. This and (3) imply

$$\min(\lambda_0(W_1), 1 - \eta) \leq \lambda_0(M_n) \leq \lambda_0(W_1), \tag{4}$$

and clearly this implies Condition (vi) holds. □

Proof of Lemma 3.1. Notation is as in the proof of Lemma 3.7. Given $\eta > 0$ we want to show that there is an r which makes the lemma true. There are two cases depending on whether $\inf_n \lambda_0(M_n) \leq 1 - \eta/3$ or not. If this is true, then Condition (5) of Lemma 3.7 and Lemma 3.4 easily imply that for r large enough,

$$|\lambda_0(M) - \inf_{n \geq 0} \lambda_0(M_n)| \leq |\lambda_0(M) - \inf_{n \geq 0} \lambda_0(\Omega_n)| + \eta/3 \leq 2\eta/3.$$

Otherwise, $\lambda_0(M_n) > 1 - \eta/3$ for all n and this implies

$$\lambda_0(\Omega_n) \geq \lambda_0(W_1) - \eta/3 \geq \lambda_0(M_n) - \eta/3 \geq 1 - 2\eta/3,$$

for all $n \geq 1$. By Lemma 3.4 this implies $\lambda_0(M) \geq 1 - \eta$. Since all our manifolds have $\lambda_0 \leq 1$,

$$|\lambda_0(M) - \inf_n \lambda_0(M_n)| \leq \eta,$$

holds trivially. This finishes the proof of Lemma 3.1. □

Now for some corollaries which we will use later.

Corollary 3.8. *Suppose $\{\mu_n\}$ is a sequence of G-invariant dilatations with disjoint supports and let $r_n = \inf_{m\neq n} \rho(\mathrm{supp}(\mu_n), \mathrm{supp}(\mu_m))$. Let $\mu = \sum_n \mu_n$. If $r_n \to \infty$ then*

$$\delta(\mu) \geq \limsup_n \delta(\mu_n).$$

Proof. Write $\mu = \mu_n + \sum_{m\neq n} \mu_m$. By Lemma 3.1,

$$\delta(\sum_n^{\infty} \mu_n) \geq \delta(\mu_n) - \eta(r_n).$$

Since $r_n \to \infty$, we have $\eta(r_n) \to 0$, which gives the result. □

Corollary 3.9. *Suppose $\{\mu_n\}$ and μ are as in Corollary 3.8 and in addition $\{r_n\}$ is increasing and*

$$\eta(r_n) + \delta(\mu_n) \leq \delta(\mu_{n+1}),$$

for all n. Then $\delta(\mu) = \lim_n \delta(\mu_n)$.

Proof. The condition implies $\delta(\mu_n)$ is an increasing sequence so the limit exists. Moreover, by Corollary 3.8, $\delta(\mu) \geq \lim_n \delta(\mu_n)$, so we only have to show the opposite direction. First we will show by induction that

$$\delta(\sum_{k=1}^{n} \mu_k) \leq \delta(\mu_{n+1}).$$

This is clearly true for $n = 1$ by hypothesis. In general, Lemma 3.1 implies

$$\begin{aligned} \delta(\mu_n + \sum_{k=1}^{n-1} \mu_k) &\leq \max(\delta(\mu_n), \delta(\sum_{k=1}^{n-1} \mu_k)) + \eta(r_n) \\ &\leq \delta(\mu_n) + \eta(r_n) \\ &\leq \delta(\mu_{n+1}), \end{aligned}$$

as desired. Finally, write $\mu = \sum_{k=1}^{n} \mu_k + \mu_{n+1} + \mu_{n+2} + \dots$ and apply Lemma 3.1 to get

$$\delta(\mu) \leq \sup\{\delta(\sum_{k=1}^{n} \mu_k), \delta(\mu_{n+1}), \delta(\mu_{n+2}), \dots\} + \eta(r_n) \leq \sup_n \delta(\mu_n) + \eta(r_n).$$

Since the left hand side is independent of n we get the desired result as $r_n \to \infty$. □

4. Proof of Theorem 1.1 for the critical exponent

In this section we will show how to construct an analytic family of deformations so that the critical exponent is not analytic. In the next section we will show that the same holds for the Hausdorff dimension of the limit set. We start with a few definitions and simple lemmas.

We say that two real-valued functions f, g on $[0,1]$ cross if there are $t_1, t_2 \in [0,1]$ such that $f(t_1) > g(t_1)$ and $f(t_2) < g(t_2)$. For our purposes, we will need the following more technical looking condition. We say that f crosses g strongly if there is an interval $I = [t_1, t_2]$ such that $f > g$ on I and $f(t_1) < f(t_2)$ and there is a $t_3 \in [0,1]$ such that $f(t_3) < g(t_3)$. (If would also be o.k. to take $f(t_1) > f(t_2)$ in the definition, but this would require some modifications in what follows.)

Lemma 4.1. *Suppose f is a continuous, real-valued function on $[0,1]$ such that $f(1) > f(0)$. Then there is an $A < 1$ and a positive integer m so that $f(t^m)$ crosses $f(At)$ strongly.*

Proof. Choose numbers a, b, c so that $f(0) < a < b < c < f(1)$. Let $t_a = \inf\{t : f(t) = a\}$ and $t_b = \sup\{t : f(t) = b\}$ and $t_c = \inf\{t > t_b : f(t) = c\}$. Note that $0 < t_a < t_b < t_c < 1$. Let $A = t_a$ and choose $t_3 < 1$ so close to 1 that $f(At_3) > a/2$. Then choose m so large that $f(t_3^m) < a/2$. Thus if we set $t_1 = t_b^{1/m}$ and $t_2 = t_c^{1/m}$ we see that $f(t^m)$ crosses $f(At)$ strongly. $\square$

Lemma 4.2. *Suppose f is a continuous, real-valued function on $[t_1, t_2]$ and $f(t_2) > f(t_1)$. Then there is an infinite, compact set $E \subset (t_1, t_2)$ and $t_0 > 0$ such that $f(x - t) < f(x)$ for every $x \in E$ and $0 < t < t_0$.*

Proof. Let $F(x) = \sup_{0 \le t \le x} f(t)$. Then F is continuous and increasing. Let $\{I_j\}$ be the closed intervals on which F is constant (possibly the empty collection). Expand each to an open interval $\{J_j\}$ so that $F(J_j)$ has length $\le 4^{-j}|f(t_2) - f(t_1)|$. Then the set $E = [t_2, t_1] \setminus \cup J_j$ is compact and must be uncountable since it maps under F to a set of positive length. Moreover, if $x \in E$ then $f(x - t) < f(x)$ for every $0 < t \le x - t_1$, since x is the first point where f attains the value $f(x)$. By removing a neighborhood of t_1 and t_2 if necessary we may assume $E \subset (t_1, t_2)$. Now let t_0 be the distance from t_1 to E and we are done. $\square$

Lemma 4.3. *Suppose we are given a sequence of real-valued, uniformly-continuous functions $\{f_n\}$ which converge to a function f on $[t_1, t_2]$ with $f(t_2) > f(t_1)$. Then there is a subsequence of $\{f_n\}$, a compact, infinite set $E \subset [t_1, t_2]$, a sequence $\{\epsilon_k\} \searrow 0$ and a sequence $\{a_n\} \nearrow 1$ such that $f_n(a_n x) \nearrow f(x)$ for every $x \in E$ and $\epsilon_n < f(x) - f_n(a_n x) < \epsilon_{n-1}$.*

Proof. Apply Lemma 4.2 to obtain an infinite compact set E and an increasing sequence $a_k \to 1$ such that $f(a_n x) \nearrow f(x)$ for all $x \in E$. Using the compactness of E and Lemma 4.2, we can find a subsequence $\{a_k\}$ of $\{a_n\}$ and $\epsilon_k \to 0$ such that $\epsilon_{k-1} \geq f(x) - f(a_k x) \geq 3\epsilon_k$ for all $x \in E$. Now choose a subsequence $\{f_k\} \subset \{f_n\}$ such that $\sup_{[0,1]} |f - f_k| \leq \epsilon_k$. Thus for $x \in E$,

$$f(x) - \epsilon_k > f(a_k x) + 2\epsilon_k \geq f_k(a_k x) + \epsilon_k > f(a_k x) \geq f(x) - \epsilon_{k-1},$$

for all $x \in E$. $\square$

Next we want to apply these results to particular functions of the form $\delta(f(t)\mu)$.

Lemma 4.4. *Suppose $\{\mu_n\}$ and $\{\nu_n\}$ are sequences of G-invariant dilatations with pairwise disjoint compact supports, such that the supports tend to infinity. Suppose that $f(t) = t^{m_1}$ and $g(t) = t^{m_2}$ are powers of t and that*

$$F_1(t) = \lim_{n\to\infty} \delta(f(t)\mu_n),$$

$$F_2(t) = \lim_{n\to\infty} \delta(g(t)\nu_n),$$

exist for all $0 \leq t \leq 1$ and that F_1 crosses F_2 strongly. Then there exist subsequences $\{\mu_k\}$ and $\{\nu_k\}$, a sequence of real numbers $\{b_k\}$ which increases to 1, and an infinite set $E \subset [0, 1]$ such that if

$$F(t) \equiv \delta(\sum_k b_k f(t)\mu_k),$$

then $F_1 = F$ for all t in E and $F(t_0) < F_2(t_0)$ for some $t_0 \notin E$. Furthermore, if

$$G(t) \equiv \delta(\sum_k b_k f(t)\mu_k + \sum_k g(t)\nu_k),$$

then $G = F$ on E and $G(t) \geq F_2(t)$ for every $t \in [0, 1]$. In particular, not both of G and F can be analytic functions since they agree on an infinite set, but have different values somewhere off that set.

Proof. By Lemmas 4.2 and 4.3 applied to the sequence $\{f_n\}$ where $f_n(t) = \delta(f(t)\mu_n)$, we get an infinite set E and a sequence $\{a_n\} \nearrow 1$ such that $\delta(a_n f(t)\mu_n) \nearrow F_1(t)$ for all t in an infinite set E. Set $b_n = (a_n)^{1/m_2}$, so that $f(b_n t) = a_n f(t)$.

In fact, we have a sequence $\epsilon_n \to 0$ such that

$$\epsilon_{n+1} < F_1(t) - \delta(a_n f(t)\mu_n) < \epsilon_n.$$

By Corollary 3.9 we can choose a subsequence of $\{\mu_n\}$ so that r_n grows so quickly that

$$\eta(r_n) < \epsilon_{n+1},$$

which implies

$$\delta(a_n f(t)\mu_n) + \epsilon_{n+1} \leq \delta(a_{n+1} f(t)\mu_{n+1}).$$

Thus by Corollary 3.9,

$$\delta(\sum_k a_n f(t)\mu_n) = \lim_k \delta(a_n f(t)\mu_n) = F_1(t),$$

for all t in the set E. This proves the first part of the lemma.

Next we want to see that $F(t) < F_2(t)$ for some t. Choose a t and an $\epsilon > 0$ so that $F_1(t) < F_2(t) - \epsilon$ (such exists since these functions cross) and truncate a finite number of terms from $\{\mu_n\}$ if necessary so we may assume

$$\sup_n \delta(f(t)a_n\mu_n) \leq F_1(t) + \epsilon/4,$$

$$\sup_n \eta(r_n) \leq \epsilon/4.$$

Then Lemma 3.1 implies

$$F(t) = \delta(\sum_k f(t)a_n\mu_n) \leq F_1(t) + \epsilon/2 < F_2(t),$$

as desired.

Note that Corollary 3.8 implies $G(t) \geq F_2(t)$ for all t. Since $F_2 > F_1 = F$ for some point (by the the argument above) we get $G > F$ somewhere, as desired.

Finally, we have to show $G = F_1 = F$ on E. By taking subsequences of both $\{\mu_n\}$ and $\{\nu_n\}$ we may assume

$$\inf_{n,m} \rho(\mathrm{supp}(\mu_n), \mathrm{supp}(\nu_m)) > r_0$$

$$\inf_{m \neq n} \rho(\mathrm{supp}(\mu_n), \mathrm{supp}(\nu_m)) > r_n$$

$$\inf_{m \neq n} \rho(\mathrm{supp}(\mu_n), \mathrm{supp}(\mu_m)) > r_n$$

and that $r_n \nearrow \infty$ satisfies

$$\eta(r_0) + \delta(\sum_n g(t)\nu_n) \leq \delta(f(t)a_1\mu_1),$$

$$\eta(r_n) + \delta(f(t)a_n\mu_n) \leq \delta(f(t)a_{n+1}\mu_{n+1}), \quad n = 1, 2, \ldots$$

for all $t \in E$. Now apply Corollary 3.9 to the sequence,

$$(\sum_n g(t)\nu_n), f(t)a_1\mu_1, f(t)a_2\mu_2, \ldots,$$

deduce $G = F_1$ on E. This proves the lemma. □

Proof of Theorem 1.1 for the critical exponent. Suppose G is a Fuchsian group and $\{\mu_n\}$ is a sequence of dilatations on $R = \mathbb{D}/G$ such that $\sup_n \|\mu_n\|_\infty < 1$ and $\inf_n \delta(\mu_n) > 1 + \delta_0 > 1$. By Lemma 2.1 we may also assume each dilatation has compact support. Thus by passing to a subsequence, if necessary, we may assume that the supports leave every compact set and are pairwise disjoint. For each n define $f_n(t) = \delta(t\mu_n)$. Since the critical exponent equals the Hausdorff dimension of the conical limit set and since the Hausdorff dimension can only be changed by a factor of $1 + C\epsilon$ by a $1 + \epsilon$ quasiconformal map, we see that these functions are uniformly continuous, independent of n. Thus they form an equicontinuous family and so by passing to yet another subsequence if necessary, we may assume that $\{f_n\}$ converges uniformly to a limit f with $f(0) = 1$ and $f(1) \geq 1 + \delta_0 > 1$.

Apply Lemma 4.1 to the function f to find values A, m such that $f(t^m)$ crosses $f(At)$ strongly. Now apply Lemma 4.4 with $f(t) = t^m$, $g(t) = t$, and letting $\{\mu_n\}$ and $\{\nu_n\}$ be the even and odd terms of our sequence, respectively. The lemma implies that either

$$\delta(t^m \sum a_k\mu_n),$$

or

$$\delta(t^m \sum a_k\mu_n + t\sum_k b_k A\nu_k),$$

is not an analytic function of t, as desired. □

5. Theorem 1.1 for $\dim(\Lambda)$

Next we prove Theorem 1.1 for the Hausdorff dimension of the limit set. If G is convergence type with $\dim(\Lambda) = 1$, and μ is a G-invariant complex dilatation of compact support with $\delta > 1$ (which certainly exists if G has big deformations near infinity) then the arguments of [8] show that $\dim(t\mu) = \max(\dim(\Lambda_0), \delta(t\mu))$ and the argument of Astala and Zinsmeister shows $\delta(t) \leq 1$ for t close enough to 0. Thus if $\delta(0) \leq 1$, then dimension cannot be analytic in t.

Next, we must consider the other cases: (1) divergence groups and (2) convergence groups with $\dim(\Lambda) < 1$. To handle these cases we first need a lemma. Recall that if Λ denotes the limit set of G and $\Lambda_c \subset \Lambda$ is the conical limit set then $\Lambda_e = \Lambda \setminus \Lambda_e$ denotes the "non-conical" or "escaping" part of the limit set.

Lemma 5.1. *Given a Fuchsian group G such that Λ_e has zero Lebesgue measure and a $\lambda < 1$ there is an increasing function $s(r)$ mapping $[0, \infty)$ to itself with the following property. Let $R = \mathbb{D}/G$, $z_0 \in R$ and $B_r = \{z \in R : \rho_R(z, z_0) < r\}$. Suppose $\{\mu_n\}$ are complex dilatations with compact supports $K_n \subset R$, that $\|\mu_n\|_\infty \leq \lambda$ for all n and that for each n there is an $r_n > 0$ so that $K_n \subset B(r_n) \setminus B(s(r_{n-1}))$ (so the support of μ_n is very far from the supports of $\mu_1, \dots, \mu_{n-1}$). If $\mu = \sum_n \mu_n$, then $\delta(\mu) = \dim(\mu)$.*

Suppose $\{\mu_n\}$ and $\{\nu_n\}$ are the sequences of Beltrami coefficients used in the proof of Theorem 1.1 for the critical exponents. By passing to a subsequence, we may assume that their supports satisfy the conditions of Lemma 5.1, and hence we have that $\dim(\Lambda) = \delta$ for our analytic families. Since δ is not analytic, the same is true for dimension. This completes the proof of Theorem 1.1 except for the proof of Lemma 5.1.

The rest of this section is devoted to the proof of Lemma 5.1. Recall that every geodesic ray either returns to some compact set infinitely often (the recurrent case) or leaves every compact set eventually (the escaping case). This dichotomy divides the limit set into two disjoint subsets; the conical limit set Λ_c which corresponds to the recurrent rays and $\Lambda_e = \Lambda \setminus \Lambda_c$ which corresponds to the escaping rays. It is a result from [6] that $\delta(G) = \dim(\Lambda_c)$ for every non-elementary Kleinian group and hence $\dim(\Lambda) = \max(\delta, \dim(\Lambda_e))$. Therefore, in order to prove Lemma 5.1 we must prove that $\dim(\Lambda_e) \leq \delta$. Consider a sequence $\{\epsilon_n\} \to 0$. We will show that the Hausdorff content satisfies $\mathcal{H}_\infty^{\delta+\epsilon_n}(\Lambda_e) \leq \epsilon_n$ for all n. This clearly implies $\dim(\Lambda_e) \leq \delta$.

For two points $z, w \in \mathbb{D}$ we will let $\rho_R(z, w)$ denote the distance of their projections in $R = \mathbb{D}/G$. Fix a point $z_0 \in R$ and a number $r > 0$ and assume $Q = Q_I \subset \mathbb{D}$ is a Carleson square such that $r < \rho_R(z, z_0) \leq r + 1$ for every point in $T(Q)$. Form a stopping time region $W(r, s) = Q \setminus \cup_j Q_j$ by removing the maximal dyadic subintervals of I where either $\rho_R(z, z_0) > s$ for some $z \in T(Q)$ (type 1 subsquares), or $T(Q)$ is outside the convex core (type 2 subsquares) or $\rho_R(z, z_0) < r$ for some $z \in T(Q)$ (type 3). Every escaping geodesic is eventually at least distance s from z_0, so by taking a countable disjoint union of such regions $\{W_j\}$ and the corresponding type 1 stopping squares, we get a cover of all of $\Lambda_e(G)$.

Suppose $\mu = \mu_n$ is the dilatation described in the statement of Lemma 5.1 (supported in $\mathbb{D}^* = \{|z| > 1\} \cup \{\infty\}$) and let $f = f_\mu$ be the corresponding quasiconformal map fixing $0, 1, \infty$. Then inside $\mathbb{D}$, f is conformal. According to the hypotheses of Lemma 5.1, none of of the μ_n's have supports intersecting $\{z : r_n < \rho_R(z, z_0) < s(r_n)\}$. Choose a large number $d = d_n$ and assume

$$r_n < r_n + d < r < s < s(r_n) - d < s(r_n).$$

Thus for any point z in our stopping time region W is at least distance d from the support of any dilatation. Suppose $\{Q_j\}$ denotes the type 1 stopping squares in W. Then $\sum_j \mathrm{diam}(Q_j)/\mathrm{diam}(W) \to 0$ as $s \to \infty$ (otherwise there would be a set of escaping geodesics of positive measure, contrary to hypothesis), i.e.,

$$\sum_j |Q_j| \leq C(s, r)\mathrm{diam}(W),$$

where $C(s, r) \to 0$ as $s \to 0$ (with r fixed). Thus by Lemma 2.2,

$$\sum_j [\frac{\mathrm{diam}(f(Q))}{\mathrm{diam}(f(W))}]^{1+\eta} \leq C \sum_j \frac{\mathrm{diam}(Q)}{\mathrm{diam}(W)} \leq C(s, r).$$

Choose η so that $1 + \eta \leq \delta + \epsilon$ and take d so large that the estimates above hold for this η. If $\{x_j\}$ are numbers in $[0, 1]$ and $1 \leq \alpha \leq \beta$ then clearly $\sum x_j^\beta \leq \sum x_j^\alpha$. Thus

$$\sum_j [\frac{\mathrm{diam}(f(Q))}{\mathrm{diam}(f(W))}]^{\delta+\epsilon} \leq \sum_j [\frac{\mathrm{diam}(f(Q))}{\mathrm{diam}(f(W))}]^{1+\eta} \leq C(s, r),$$

or equivalently,

$$\sum_j \mathrm{diam}(f(Q_j))^{\delta+\epsilon} \leq C(r, s)\mathrm{diam}(f(W))^{\delta+\epsilon}.$$

Thus summing over all regions W_j that occur in this way gives

$$\sum_{j,k} \mathrm{diam}(f(Q_{j,k}))^{\delta+\epsilon} \leq C(r, s) \sum_j \mathrm{dist}(z_j, \partial\Omega)^{\delta+\epsilon},$$

where $\{z_j\}$ are points chosen in the top half of each W_j. These points can be chosen from the orbit of a finite set $E \subset A_r = \{z : r \leq \rho_R(z, z_0) \leq r+1\}$ and maximal with respect to the property that $z, w \in E$ implies $\rho(z, w) \geq 1$. This set has at most $C_1(r)$ points (in fact, $\leq Ce^r$ since A_r has hyperbolic area $\leq Ce^r$) and by (1) in Section 2,

$$\sum_{g \in G} \operatorname{dist}(f(g(z)), \partial\Omega)^{\delta+\epsilon} \leq C_2(\epsilon, r),$$

for all $z \in E$, since this sum is finite for each fixed z. Thus

$$\sum_{j,k} \operatorname{diam}(f(Q_{j,k}))^{\delta+\epsilon} \leq C_1(r) C_2(\epsilon, r) C(r, s),$$

can be made as small as we wish by taking s large enough (depending only on r and ϵ). The sets $\{f(I_{j,k})\}$ form a cover of $\Lambda_e(G')$, so we see that Λ_e has small $\delta+\epsilon$ content if $s(r_n)$ is large enough. Taking a sequence $\epsilon_n \to 0$ and repeating the argument for all n shows that the escaping geodesics of G_μ have small $\delta + \epsilon$ content for every $\epsilon > 0$ and hence this set has dimension $\leq \delta$, as desired. This completes the proof of Lemma 5.1.

References

[1] L.V. Ahlfors. *Lectures on quasiconformal mappings.* Math. Studies 10. Van Nostrand, 1966.

[2] K. Astala and M. Zinsmeister. Mostow rigidity and Fuchsian groups. *Comptes Rendu Acad. Sci., Paris*, 311:301–306, 1990.

[3] K. Astala and M. Zinsmeister. Holomorphic families of quasi-Fuchsian groups. *Ergod. Th. & Dynam. Sys.*, 14:207–212, 1994.

[4] K. Astala and M. Zinsmeister. Abelian coverings, Poincaré exponent of convergence and holomorphic deformations. *Ann. Acad. Sci. Fenn.*, 20:81–86, 1995.

[5] C.J. Bishop. Big deformations near infinity. *Ill. J. of Math.*, 47:977–996, 2003.

[6] C.J. Bishop and P.W. Jones. Hausdorff dimension and Kleinian groups. *Acta. Math*, 179:1–39, 1997.

[7] C.J. Bishop and P.W. Jones. Wiggly sets and limit sets. *Arkiv Mat.*, 35:201–224, 1997.

[8] C.J. Bishop and P.W. Jones. Compact deformations of Fuchsian groups. *J. Anal. Math.*, 87:5–36, 2002. Dedicated to the memory of Thomas H. Wolff.

[9] A. Douady and C. J. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.*, 157(1-2):23–48, 1986.

[10] F. W. Gehring and J. Väisälä. Hausdorff dimension and quasiconformal mappings. *J. London Math. Soc. (2)*, 6:504–512, 1973.

[11] P.J. Nicholls. *The ergodic theory of discrete groups.* LMS Lecture Notes 143. Cambridge University Press, 1989.

[12] D. Ruelle. Repellers for real analytic maps. *Ergod. Th. & Dym. Sys.*, 2:99–107, 1982.

[13] D. Sullivan. Related aspects of positivity in Riemannian geometry. *J. Differential Geometry*, 25:327–351, 1987.

C.J. Bishop, Mathematics Department, SUNY at Stony Brook, Stony Brook, NY 11794-3651
E-mail address: bishop@math.sunysb.edu