BILIPSCHITZ APPROXIMATIONS OF QUASICONFORMAL MAPS

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ABSTRACT. We show that for any K-quasiconformal map of the upper half plane to itself and any $\epsilon > 0$, there is a $(K+\epsilon)$-quasiconformal map of the half plane with the same boundary values which is also biLipschitz with respect to the hyperbolic metric.

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1. Introduction

If f is a K-quasiconformal map of the upper half space, $\mathbb{H}$, to itself, then it is well known that one can find a quasiconformal $g : \mathbb{H} \to \mathbb{H}$ which agrees with f on the real line and which is also biLipschitz with respect to the hyperbolic metric. Moreover, the quasiconformal constant of g can be bounded in terms of K (e.g., [1], [5]). The purpose of this note is to show that the quasiconformal constant of g may be taken as close to K as we wish.

Theorem 1.1. *Given $K < \infty$ and $\epsilon > 0$, there is a $C < \infty$ so that if f is a K-quasiconformal map of the upper half-plane $\mathbb{H}$ to itself, then there is a $(K + \epsilon)$-quasiconformal map $g : \mathbb{H} \to \mathbb{H}$ which is C-biLipschitz with respect to the hyperbolic metric on $\mathbb{H}$ and which agrees with f on $\mathbb{R} = \partial\mathbb{H}$.*

One cannot take $\epsilon = 0$ in this result. A K-quasiconformal self-map of the upper half-plane is called uniquely extremal if it the only K-quasiconformal extension of its boundary values. If the result above held with $\epsilon = 0$ then every uniquely extremal map would be hyperbolically biLipschitz with constant depending only on K. But Theorem 11 of [4] implies that any K-quasiconformal map can be uniformly approximated by K-quasiconformal uniquely extremal maps. Since not every quasiconformal map is hyperbolically biLipschitz, neither can every uniquely extremal map, and hence the result above can't hold for $\epsilon = 0$. I thank V. Marković for showing this argument to me.

Theorem 1.1 arose from [3]. That paper considered finding the best constant in a theorem of Dennis Sullivan involving quasiconformal maps from a plane domain Ω to the disk and the question arises of whether the best constant is the same if we also require these maps to be biLipschitz with respect to the hyperbolic metrics. Theorem 1.1 shows that this is the case. I thank Al Marden for his comments on [3] which led me to formulate and prove the result.

Now, we sketch the proof of Theorem 1.1, leaving certain estimates to be proven later. We start by factoring $f = f_n \circ \cdots \circ f_1$ as a composition of n maps, each with quasiconformal constant $K^{1/n} \approx 1 + \frac{1}{n} \log K$ (Lemma 2.1). If $\mu_j = (f_j)_{\bar{z}}/(f_j)_z$ is the Beltrami coefficient associated to f_j, then we smooth μ_j out to get a new Beltrami coefficient $\tilde{\mu}_j$, by convolving with a smooth, radial bump function with respect to the hyperbolic metric in $\mathbb{H}$ which is supported in a disk of hyperbolic radius δ (and similarly for the lower half-plane). We will show that $\|\tilde{\mu}_j\|_\infty \leq \|\mu_j\|_\infty(1+C\delta^2)$ (Corollary 2.8) and so the corresponding map, denoted $\tilde{f}_j$, is quasiconformal with constant $K^{1/n}(1+C\log K\delta^2/n)$. We will also show $\tilde{f}_j$ is biLipschitz for the hyperbolic metric with constant $1+C\frac{1}{\delta}\frac{1}{n}\log K$ (Lemma 2.12). Although $\tilde{f}_j$ might not equal f_j on the boundary, we shall show that $g_j = f_j \circ \tilde{f}_j^{-1}$ is quasisymmetric with constant $1+C(\log K/n)^2$, independent of δ, if n is large enough (Corollary 2.11). Thus g_j can be extended to a quasiconformal mapping (also denoted g_j) of the plane which is quasiconformal and hyperbolically biLipschitz both with constant $1+C(\log K/n)^2$ (Lemma 2.3). Thus the map $G_n = (g_n \circ \tilde{f}_n) \circ \cdots \circ (g_1 \circ \tilde{f}_1)$ agrees with f on the boundary, has quasiconformal constant

$$K_n = [K^{1/n}(1+C\log K\delta^2/n)(1+C(\log K/n)^2)]^n = K^{1+C\delta^2}(1+O(1/n)),$$

and biLipschitz constant

$$B_n = [(1+C\log K/(n\delta))(1+C(\log K/n)^2)]^n = K^{C/\delta}(1+O(1/n)).$$

Taking n large enough and δ small enough, we can make K_n as close to K as we wish. This proves the theorem, except for proving the lemmas mentioned above. Throughout the paper C will denote a generic constant which may be different at different places. We will use subscripts when it is important to recall a particular value.

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2. Proof of the lemmas

Most of the facts we need are proven in Ahlfors' book [1]. In several cases we need a result with a sharper estimate than is stated there, so we give the necessary argument. We begin with some notation.

We let $B_r = B(0,r) \subset \mathbb{C}$ and define

$$\|f\|_{p,r} = (\int_{B_r} |f|^p dxdx)^{1/p}.$$

P denotes the Cauchy transform

$$Ph(w) = \frac{-1}{\pi}\int h(z)(\frac{1}{z-w} - \frac{1}{z})dxdy, \tag{1}$$

and T denotes the Beurling transform

$$Th(w) = -\frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|z-w|>\epsilon}\frac{h(w)}{(z-w)^2}dxdy,$$

which is known to be bounded on L^p with norm K_p for each $1 < p < \infty$. Moreover, $K_p \to 1$ as $p \to 2$ and for any $k < 1$, we define $k(p) > 2$ so that $kK_p < 1$ for $2 \leq p \leq k(p)$.

Given a measurable function μ on the plane with $\|\mu\|_\infty \leq k < 1$, there is a K-quasiconformal map (with $K = (k+1)/(k-1)$) of the plane to itself which satisfies the Beltrami equation $f_{\bar{z}} = \mu f_z$. We may normalize so that $0, 1, \infty$ are fixed points of the map and call this solution f^μ. If μ has compact support then we map also normalize so that $f(0) = 0$ and $f_z - 1 \in L^p$ for some $p > 2$ (depending on k). This solution will be denoted F^μ.

Lemma 2.1. *If f is a K-quasiconformal mapping of the plane then we can write $f = f_n \circ \cdots \circ f_1$ where each map is $K^{1/n}$-quasiconformal.*

For a proof, see page 99 of Ahlfors' book [1]. A map $h : \mathbb{R} \to \mathbb{R}$ is called k-quasisymmetric if

$$\frac{1}{k} \leq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leq k.$$

The following are standard results.

Lemma 2.2. *Suppose f is a K-quasiconformal map of the plane to itself which maps $\mathbb{H}$ to itself. Then the restriction to $\mathbb{R}$ is a k-quasisymmetric map for some k depending only on K. There is a $K_0 > 1$ and a $C_0 < \infty$, so that if $K = 1+\epsilon \leq K_0$ then we may take $k \leq 1 + C_0\epsilon$.*

The first part is Theorem 1 of Chapter IV of [1]. The small constant case is proven in [2].

Lemma 2.3. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a k-quasisymmetric map. Then f has an extension to a K-quasiconformal mapping of $\mathbb{H}$ to itself which is K-biLipschitz with respect to the hyperbolic metric, where K depends only on k. There is a $k_1 > 1$ and $C_1 < \infty$ so that if $k = 1+\epsilon \leq k_1$, then we may take $K \leq 1 + C_1\epsilon$.*

Proof. The first part is in Chapter IV of [1]. To prove the small constant case, associate to each dyadic interval $I = [j2^{-n}, (j+1)2^{-n}] = [a, b]$ on the line the "Whitney box" $Q_I = I \times [2^{-n-1}, 2^{-n}]$ in the upper half plane. Think of this a pentagon with the five vertices

$$(a, |I|), (b, |I|), (b, |I|/2), (\frac{a+b}{2}, |I|/2), (a, |I|/2).$$

To each dyadic interval $I = [a, b]$ we define the point $z_I = (a, |I|)$ (the "upper left corner" of the Whitney box Q_I) and define the extension f at the point z_I by

$$f(z_I) = (f(j2^{-n}), |f(I)|).$$

This defines f at every vertex of every Whitney box and we may extend it into the boxes in a piecewise linear way if k is close enough to 1. It is easy to check from the definition of quasisymmetry that the image of Q_I is a five sided polygon with angles and side lengths differing only $O(\epsilon)$ from those of Q, and this proves the result. $\square$

Recall from page 99 of [1] that if μ satisfies the symmetry condition $\mu(\bar{z}) = \overline{\mu(z)}$ then the corresponding map f^μ maps the real line into itself. From now on, we will assume we are dealing with such maps.

Lemma 2.4. *Given $\alpha < 1$, $R < \infty$, there is a $k < 1$ and a constant $C_2 = C_2(\alpha, R) < \infty$ so that the following holds. Suppose $\|\mu\|_\infty \leq k < 1$ and μ is supported in the ball $B_R = \{z : |z| < R\}$ and that $\mu(\bar{z}) = \overline{\mu(z)}$. Then for any point $z \in B_R$,*

$$|F^\mu(z) - z| \leq C_2 \|\mu\|_\infty |z|^\alpha.$$

and for any $z \in B_1$,

$$\frac{1}{C_2}|z|^{1/\alpha} \leq |F^\mu(z)| \leq C_2|z|^\alpha.$$

Proof. Equation (10) of Section V.B of [1] implies that

$$|F^\mu(z_1) - F^\mu(z_2)| \leq C(k, R)\|\mu\|_\infty |z_1 - z_2|^\alpha + |z_1 - z_2|, \tag{2}$$

for some $\alpha < 1$ which depends only on k and such that $\alpha \to 1$ as $k \to 0$ (one may take $\alpha = 1 - 2/p$ where p is such that $kK_p < 1$ with K_p the norm of the Beurling transform T on L^p, $p > 2$). Applying this to $z_2 = 0$ for both F^μ and its inverse clearly implies the second estimate, so we need only prove the first.

If we apply (2) to $z_1 = 0$ and $z_2 = x \in [-R, R]$ and set $\epsilon = \|\mu\|_\infty |z|^\alpha$, we see that

$$|F^\mu(x)| \leq C(k, R)\epsilon + |x| \leq |x| + C\epsilon.$$

Since F^μ is an orientation preserving homeomorphism on $\mathbb{R}$, and applying the same argument to the inverse map, we get $|F^\mu(x) - x| \leq C\epsilon$, as desired.

Now consider a point $z = x + iy \in B_R \cap \mathbb{H}$. The estimate above implies $|F^\mu(z)| \leq |z| + C\epsilon$. Considering the inverse map shows

$$-C\epsilon \leq |F^\mu(z)| - |z| \leq C\epsilon.$$

Let $t = |z|(x/|x|)$. As above, we can deduce

$$-C\epsilon \leq |F^\mu(z) - F^\mu(t)| - |z - t| \leq C\epsilon,$$

and hence by the previous paragraph,

$$-2C\epsilon \leq |F^\mu(z) - t| - |z - t| \leq 2C\epsilon.$$

Since the circles $\{w : |w| = |z|\}$ and $\{w : |w - t| = |z - t|\}$ intersect at an angle which is bounded away from zero, we see that the $2C\epsilon$ neighborhoods of these circles in $\mathbb{H}$ intersect in a set of diameter at most $20C\epsilon$ and this intersection contains both z and $F^\mu(z)$. This proves the lemma. $\square$

Lemma 2.5. *If $k < 1$ is small enough then there is a constant $C_3 = C_3(k)$ so that the following holds. Suppose $\|\mu\|_\infty \leq k < 1$. Then*

$$\|f^\mu_z - 1\|_{p,1} \equiv (\int_{B_1} |f^\mu_z - 1|^p dxdy)^{1/p} \leq C_3 \|\mu\|_\infty.$$

for all $2 \leq p \leq p(k)$.

Proof. This is the Lemma in Section V.C of [1], although the statement there only claims that $\|f^\mu_z - 1\|_p \to 0$ as $\|\mu\|_\infty \to 0$. We sketch the proof making the necessary changes.

First assume μ is supported in B_R and let $\epsilon = \|\mu\|_\infty$. It is proven on page 100 of [1] that $\|F^\mu_z - 1\|_p \leq C\|\mu\|_p \leq C\epsilon R^{2/p}$ if $p < p(k)$. Since $f^\mu = F^\mu/F^\mu(1)$, Lemma 2.4 implies

$$\begin{aligned}
\|f^\mu_z - 1\|_{p,1} &= \|\frac{F^\mu_z}{F^\mu(1)} - 1\|_{p,1} \\
&= |1 - \frac{1}{F^\mu(1)}| + \frac{1}{F^\mu(1)} \|F^\mu_z - 1\|_{p,1} \\
&\leq 2C_2\epsilon + \frac{C(R)}{1 - C\epsilon}\epsilon \\
&\leq C\epsilon.
\end{aligned}$$

Now write $\check{f}(z) = 1/f(1/z)$. We want to show

$$\|\check{f}^\mu_z - 1\|_{p,R} \leq C(R)\epsilon, \tag{3}$$

when μ has support in B_R. Just as above, it suffices to show $\|\check{F}^\mu_z - 1\|_{p,R} \leq C\epsilon$. Note that $\check{F}^\mu$ is analytic on $\{z : |z| < 3r\}$ where $r = 1/(3R)$. For an analytic function f on a ball $B(x,r)$ it is easy to see by the mean value property and Hölder's inequality that

$$|f(x)| \leq \frac{1}{\pi r^2}\int_{B(x,r)} |f| \leq \frac{1}{(\pi r^2)^{1/p}} \|f\|_{L^p(B(x,r))}.$$

Thus by the maximum principle,

$$\begin{aligned}
\int_{|z|<r} |\check{F}^\mu_z(z) - 1|^p dxdy &\leq C(r) \sup_{|z|=2r} |\check{F}^\mu_z(z) - 1|^p \\
&\leq C(r) \int_{r<|z|<3r} |\check{F}^\mu_z(z) - 1|^p dxdy.
\end{aligned}$$

On the other hand, changing variables from z to $1/z$ gives

$$\begin{aligned}
\int_{r<|z|<R} |\check{F}^\mu_z(z) - 1|^p dxdy &= \int_{1/R<|z|<1/r} |\frac{z^2 F^\mu_z(z)}{F^\mu(z)^2} - 1|^p \frac{dxdy}{|z|^4} \\
&= \int_{1/R<|z|<1/r} |\frac{z^2(F^\mu_z(z) - 1)}{F^\mu(z)^2} + \frac{z^2 - F^\mu(z)^2}{F^\mu(z)^2}|^p \frac{dxdy}{|z|^4} \\
&\leq C \int_{1/R<|z|<1/r} |\frac{z^2(F^\mu_z(z) - 1)}{F^\mu(z)^2}|^p + |\frac{z^2 - F^\mu(z)^2}{F^\mu(z)^2}|^p \frac{dxdy}{|z|^4} \\
&\leq C(R) \int_{1/R<|z|<1/r} |F^\mu_z(z) - 1|^p dxdy \\
&\qquad + C(R) \int_{1/R<|z|<1/r} |z - F^\mu(z)|^p dxdy \\
&\leq C(R)\epsilon^p.
\end{aligned}$$

Since the integral over $\{|z| < 3r\}$ was dominated by a constant (depending only on R) times this estimate, we have proven (3).

The general case now follows just as in [1]. Write $f = \check{g} \circ h$ where $\mu_h = \mu_f$ inside the unit disk and $\mu_h = 0$ outside the unit disk. Then

$$\|f_z - 1\|_{p,1} \leq \|[(\check{g}_z - 1) \circ h] h_z\|_{p,1} + \|h_z - 1\|_{p,1}.$$

The second term is bounded by $C\epsilon$ by the first paragraph and the first term is bounded using

$$\begin{aligned}
\|[(\check{g}_z - 1)\circ h]h_z\|_{p,1}^p &= \int_{B_1} |(\check{g}_z - 1)\circ h|^p |h_z|^p dxdy \\
&\le \frac{1}{1-k^2}\int_{h(B_1)} |\check{g}_z - 1|^p |h_z \circ h^{-1}|^{p-2} dxdy \\
&\le \frac{1}{1-k^2}(\int_{h(B_1)} |\check{g}_z - 1|^{2p} dxdy \int_{B_1} |h_z|^{2p-4} dxdy)^{1/2}
\end{aligned}$$

Clearly $h(B_1) \subset \{|z| < R\}$ for some R depending only on k. Thus using (3), the first integral is bounded by

$$\int_{B_R} |\check{g}_z - 1|^{2p} dxdy \le C\epsilon^{2p},$$

(assuming $2p < p(k)$; but since $p(k) \to \infty$ as $k \to 0$ this holds for some $p > 2$ if k is small enough). On the other hand

$$\int_{B_1} |h_z|^{2p-4} dxdy \le C(\int_{B_1} |h_z|^{2p} dxdy)^{1-2/p} \le \|\mu_h\|_{p,1} + \|1\|_{p,1} \le C,$$

since $\|h_z - 1\|_p \le C\|\mu_h\|_p$. □

Lemma 2.6. *There is a $0 < k < 1$ and a $C_4 < \infty$ so that the following holds. Suppose that f is a quasiconformal mapping of the plane to itself which preserves $\mathbb{H}$, fixing $0, 1$ and ∞ and the Beltrami coefficient of f is μ with $\|\mu\|_\infty \le k$. Then*

$$|f(w) - [w - \frac{1}{\pi}\int_{\mathbb{C}} \mu(z)R(z,w)dxdy]| \le C_4\|\mu\|_\infty^2,$$

for all $|w| \le 1$, where

$$R(z,w) = \frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z} = \frac{w(w-1)}{z(z-1)(z-w)}.$$

Proof. Again, we follow the proof in [1], Section V.C, simply inserting more explicit estimates at a few points. It is shown there that

$$f(w) = w - \frac{1}{\pi}\int_{B_1} f_{\bar{z}}(z)R(z,w)dxdy - \frac{1}{\pi}\int_{B_1} \frac{\check{f}_{\bar{z}}(z)}{\check{f}(z)^2} zS(z,w)dxdy,$$

where $S(z,w) = \frac{w^2}{1-wz} - \frac{w}{1-z}$ and as before $\check{f}(z) = 1/f(1/z)$. Using $f_{\bar{z}} = \mu f_z = \mu + \mu(f_z - 1)$, the first integral equals

$$\begin{aligned}\int_{B_1} \mu(z) R(z,w)dxdy \quad &+ \int_{B_1} \mu(z)(f_z(z)-1)R(z,w)dxdy \\ &= \int \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty \|f_z - 1\|_{p,1}\|R\|_{q,1}) \\ &= \int \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty^2),\end{aligned}$$

by Lemma 2.5 and the fact that $R \in L^q$, for every $q < 2$ (with a bound depending on q, but not on w for $|w| \leq 1$).

The second integral is estimated by writing $\check{f}_{\bar{z}} = \check{\mu} + \check{\mu}(\check{f}_z - 1)$ where $\check{\mu}(z) = (z/\bar{z})^2\mu(1/z)$. Repeating the argument above shows the second integral is equal to

$$\begin{aligned}\int_{B_1} \frac{\check{\mu}(z)}{\check{f}(z)^2} &+ \frac{\check{\mu}(z)(\check{f}_z(z)-1)}{\check{f}(z)^2} zS(z,w)dxdy \\ &= \int_{B_1} \mu(\frac{1}{z})[\frac{1}{\bar{z}^2} + \frac{z^2 - \check{f}(z)^2}{\bar{z}^2 \check{f}(z)^2}]zS(z,w)dxdy \\ &\qquad + \int_{B_1} \frac{\check{\mu}(\check{f}_z - 1)}{\check{f}(z)^2} zS(z,w)dxdy \\ &= \int_{B_1} \mu(\frac{1}{z})\frac{1}{\bar{z}^2} zS(z,w)dxdy + I + II\end{aligned}$$

Using Lemma 2.4, we see

$$\frac{1}{C}|z|^{1/\alpha} \leq |\check{f}(z)| \leq C|z|^\alpha,$$

$$|z - \check{f}(z)| \leq C\|\mu\|_\infty |z|^\alpha,$$

so we can estimate I by

$$\begin{aligned} I &\leq |\int_{B_1} \mu(1/z)\frac{z^2 - \check{f}(z)^2}{\bar{z}^2 \check{f}(z)^2} zS(z,w)dxdy| \\ &\leq C\|\mu\|_\infty^2 \int_{B_1} |z|^{2\alpha - 1 - \frac{2}{\alpha}} S(z,w)dxdy \\ &\leq C\|\mu\|_\infty^2 C(\alpha),\end{aligned}$$

if $2\alpha - 1 - \frac{2}{\alpha} > -2$ (recall that we may take α as close to 1 as we wish, if k is small enough). To estimate II, note that for $\frac{1}{p} + \frac{1}{q} = 1$, Lemma 2.5 implies

$$\begin{aligned} II &= \int_{B_1} \frac{\check{\mu}(z)(\check{f}_z(z) - 1)}{\check{f}(z)^2} zS(z,w)dxdy \\ &\leq C\|\mu\|_\infty \|\check{f}_z - 1\|_p \|\frac{zS(z,w)}{\check{f}(z)^2}\|_q \\ &\leq \|\mu\|_\infty^2 \|z^{1-\frac{2}{\alpha}} S(z,w)\|_q. \end{aligned}$$

Fix some $q < 2$, and take k so small that $\alpha > 2q/(2+q)$, which implies the L^q norm is finite (with bound depending only on α, hence only of k). Thus,

$$f(w) = w - \frac{1}{\pi}\int_{B_1} \mu(z)R(z,w)dxdy - \frac{1}{\pi}\int_{B_1} \mu(\frac{1}{z})\frac{1}{\bar{z}^2} zS(z,w)dxdy + O(\|\mu\|_\infty^2).$$

Changing variables from z to $1/z$ in the second integral converts the integrand to the same form as the first (but now over $\{|z| > 1\}$). Hence,

$$f(w) = w - \frac{1}{\pi}\int_{\mathbb{R}^2} \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty^2),$$

as desired. $\square$

Lemma 2.7. *There is a $\delta_0 > 0$ and a $C_5 < \infty$ such that the following holds. Suppose $0 < \delta \leq \delta_0$ and that φ is a decreasing continuous function of compact support on $[0, \delta)$. Then*

$$\int_{\mathbb{H}} \varphi(\rho(z,i))\frac{dxdy}{y^2} \leq \int_{\mathbb{H}} \varphi(\rho(z,i))dxdy \leq (1 + C_5\delta^2)\int_{\mathbb{H}} \varphi(\rho(z,i))\frac{dxdy}{y^2}.$$

Proof. If we can prove this when $\varphi(t) = \chi_{[0,s]}(t)$, $0 < s \leq \delta$, is the characteristic function of an interval, then by linearity it holds for decreasing step functions. By passing to uniform limits it holds for all decreasing continuous functions of compact support. Thus it suffices to prove that if $H(s)$ denotes the hyperbolic area of a hyperbolic ball $B \subset \mathbb{H}$ is a hyperbolic ball of radius s centered at i, and $E(s)$ is its Euclidean area, then $H(s) \leq E(s) \leq (1 + C\delta^2)H(s)$.

Considering where B hits the imaginary axis, and using $d\rho = |dz|/y$, it is easy to see the Euclidean diameter of B is $\frac{1}{y} - y$ where $y = e^{-s}$ and hence its Euclidean area is

$$E(s) = \frac{\pi}{4}(\frac{1}{y} - y)^2 = \frac{\pi}{4}(e^s - e^{-s})^2 = \pi(s^2 + \frac{1}{3}s^4 + O(s^6)).$$

To compute the hyperbolic area we move to the disk and assume B is a hyperbolic ball of radius s centered at the origin. Since $d\rho = 2|dz|/(1-|z|^2)$, we see that the Euclidean radius of B is $r = \frac{e^s-1}{e^s+1}$. Thus the hyperbolic area of B is

$$\begin{aligned} H(s) = 4\int_0^{2\pi}\int_0^r \frac{t}{(1-t^2)^2} dt d\theta &= 8\pi \int_0^{r^2} \frac{1}{(1-u)^2}\frac{du}{2} \\ &= 4\pi \frac{r^2}{1-r^2} \\ &= \pi(e^s - 2 + e^{-s}) \\ &= \pi(s^2 + \frac{s^4}{12} + O(s^6)), \end{aligned}$$

and hence

$$\frac{E(s)}{H(s)} = \frac{s^2 + \frac{1}{3}s^4 + O(s^6)}{s^2 + \frac{1}{12}s^4 + O(s^6)} = 1 + \frac{1}{4}s^2 + O(s^4),$$

which proves the lemma. □

Now suppose f is a K-quasiconformal selfmap of $\mathbb{H}$ which fixes 0, 1 and ∞. Let $\mu = \mu_f$ be the associated symmetric Beltrami coefficient. Given a $0 < \delta < 1$, choose a φ on $[0, \delta]$ so that $\varphi(z, i) = \varphi(\rho(z, i))$ is a smooth, positive function and $\int_{\mathbb{H}} \varphi(z, i)\frac{dxdy}{y^2} = 1$. This can clearly be done to that φ also satisfies

$$|\varphi| \leq \frac{C}{\delta^2}, \quad |\varphi'| \leq \frac{C}{\delta^3},$$

for some $C < \infty$ which is independent of δ. Suppose $z = x + iy$ and define

$$\psi(z, w) = \frac{1}{y^2}\varphi(\rho(z, w)).$$

If $w = u + iv$ then a simple linear change of variables shows

$$\int_{\mathbb{H}} \psi(z, w) dxdy = \int_{\mathbb{H}} \varphi(z, i)\frac{dxdy}{y^2} = 1,$$

$$\int_{\mathbb{H}} \psi(z, w) dudv = \int_{\mathbb{H}} \varphi(i, w) dudv \leq 1 + C\delta^2.$$

Now for $y > 0$, define

$$\tilde{\mu}(z) = \int_{\mathbb{H}} \mu(w)\psi(z, w) dw,$$

and symmetrically for $y < 0$.

Corollary 2.8. *With μ and $\tilde{\mu}$ as above, $\|\tilde{\mu}\|_\infty \leq \|\mu\|_\infty(1 + C_6\delta^2)$.*

This is immediate from the preceding remarks. Also note for later

Lemma 2.9. *$\tilde{\mu}$ is a smooth function off $\mathbb{R}$ with $|\nabla\tilde{\mu}(x+iy)| \leq \|\mu\|_\infty \frac{C}{\delta}\frac{1}{|y|}$.*

Proof. To prove this, let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $\epsilon = |z_1 - z_2|$ and assume, without loss of generality, that $y_1 \leq y_2$ and $\epsilon < \delta y_1$. Then (as a function of w) $|\psi(z_1, w) - \psi(z_2, w)|$ is supported on a hyperbolic 3δ ball B around z_1 (and hence has Euclidean area $\leq C\delta^2 y_1^2$). Hence

$$
\begin{aligned}
|\tilde{\mu}(z_1) - \tilde{\mu}(z_2)| &\leq \|\mu\|_\infty \int_{\mathbb{H}} |\psi(z_1, w) - \psi(z_2, w)| dw \\
&= \|\mu\|_\infty \int_{\mathbb{H}} |y_1^{-2}\varphi(z_1, w) - y_2^{-2}\varphi(z_2, w)| dw \\
&\leq \|\mu\|_\infty [\int_B y_1^{-2} |\varphi(z_1, w) - \varphi(z_2, w)| dw + \int_B |y_1^{-2} - y_2^{-2}| \varphi(z_2, w) dw] \\
&\leq \|\mu\|_\infty [C\epsilon\delta^{-3} y_1^{-1} \int_B y_1^{-2} dw + C\delta^{-2} \int_B \epsilon y_1^{-3} dw] \\
&\leq \|\mu\|_\infty [C\epsilon\delta^{-1} y_1^{-1} + C\epsilon y_1^{-1}].
\end{aligned}
$$

Dividing by $\epsilon = |z_1 - z_2|$ gives the desired esitmate.

□

Let $\tilde{f} = f^{\tilde{\mu}}$ be the quasiconformal map with Beltrami coefficient $\tilde{\mu}$ which fixes $0, 1, \infty$.

Lemma 2.10. *There is a $0 < k < 1$ and $C_7 < \infty$ so that the following holds. Suppose f is a K-quasiconformal selfmap of $\mathbb{H}$ with $\|\mu\|_\infty \leq k < 1$ and let $\tilde{f}$ be the map obtained by smoothing $\mu = \mu_f$ as above. Then*

$$
\max_{x \in [0,1]} |f(x) - \tilde{f}(x)| \leq C_7 \|\mu\|_\infty^2. \tag{4}
$$

Proof. To prove (4) we use Lemma 2.6. The main point is that for a fixed $w \in \mathbb{R}$, $R(z, w)$ is analytic in z (off the real line) and hence harmonic. In 2 dimensions, Euclidean harmonic functions are the same as hyperbolically harmonic functions (e.g. [6]). Thus R is hyperbolically harmonic and thus satisfies the mean value property with respect to averaging over hyperbolic balls. Thus if $\zeta \in [0, 1]$,

$$
\int_{\mathbb{H}} R(z, \zeta)\psi(z, w) dxdy = \int_{\mathbb{H}} R(z, \zeta)\varphi(\rho(z, w)) \frac{dxdy}{y^2} = R(w, \zeta).
$$

Hence

$$\begin{aligned}\int_{\mathbb{H}} \tilde{\mu}(z)R(z,\zeta)dxdy &= \int_{\mathbb{H}}[\int_{\mathbb{H}} \mu(w)\psi(z,w)dudv]R(z,\zeta)dxdy \\ &= \int_{\mathbb{H}} \mu(w)[\int_{\mathbb{H}} R(z,\zeta)\psi(z,w)dxdy]dudv \\ &= \int_{\mathbb{H}} \mu(w)R(w,\zeta)dudv.\end{aligned}$$

Thus if we apply Lemma 2.6 to both f and $\tilde{f}$, the integral terms are identical and hence cancel, giving

$$|f(\zeta) - \tilde{f}(\zeta)| \leq 2C_4\|\mu\|_\infty^2,$$

for all $\zeta \in [0,1]$. □

Corollary 2.11. *With f and $\tilde{f}$ as above, $g = f \circ \tilde{f}^{-1}$ is $(1+C_8\|\mu\|_\infty^2)$-quasisymmetric on the real line.*

Proof. For any two real numbers $a < b$, we want to estimate

$$\frac{g(b) - g(\frac{1}{2}(a+b))}{g(\frac{1}{2}(a+b)) - g(a)}.$$

After re-normalizing by linear maps, we may as well assume $a = g(a) = 0$ and $b = g(b) = 1$, so we want to show

$$\frac{1}{1+C\|\mu\|_\infty^2} \leq \frac{1-g(\frac{1}{2})}{g(\frac{1}{2})-0} \leq 1 + C\|\mu\|_\infty^2.$$

Thus it is enough to show that

$$|g(\frac{1}{2}) - \frac{1}{2}| = |f(\tilde{f}^{-1}(\frac{1}{2})) - \frac{1}{2}| = O(\|\mu\|_\infty^2).$$

This follows by taking $x = \tilde{f}^{-1}(\frac{1}{2})$ in (4). □

Lemma 2.12. *There is a $0 < k < 1$ and a $C_9 < \infty$ so that the following holds. Suppose f is a quasiconformal map of $\mathbb{H}$ to itself and its Beltrami coefficient μ satisfies $\|\mu\|_\infty \leq \epsilon \leq k$ and*

$$|\nabla\mu(x+iy)| \leq M\frac{\epsilon}{y}.$$

Then f is $(1 + C_9 M\epsilon)$-biLipschitz with respect to the hyperbolic metric

Proof. Since the inverse of f satisfies the same hypotheses, it suffices to show f is Lipschitz. Also, after rescaling by linear maps, we may just give a Lipschitz estimate at i.

Write $f = g \circ h$ where on $\mathbb{H}$ we have $\mu_h = \mu_f$ on $B_1 = B(i, \frac{1}{4})$, $\mu_h = 0$ off $B_2 = B(i, \frac{1}{2})$ and $|\nabla \mu_h(x+iy)| \leq 2M\frac{\epsilon}{y}$ (and symmetrically on the lower half-plane). Thus g is analytic on $h(B_1)$, which contains a ball B_3 of fixed radius (depending only on k) around $h(i)$. Thus by the mean value theorem for analytic functions

$$|g_z(h(i))| \leq \frac{1}{|B_3|}\int_{B_3} |g_z| dxdy \leq 1 + C\int_{B_3} |g_z - 1| dxdy \leq 1 + C\|\mu\|_\infty,$$

by Lemma 2.5.

To estimate the Lipschitz constant for h, we follow the proof from [1] (Lemma 3, Section V.B) that continuity of μ implies differentiability of h. It is shown there that $h_z = \lambda = e^\sigma$ and $h_{\bar{z}} = \mu\lambda = \mu e^\sigma$, with

$$\sigma = P(\mu_h v + (\mu_h)_z) + K, \tag{5}$$

where P denotes the Cauchy transform, K is a constant chosen so that $\sigma(w) \to 0$ as $w \to \infty$ and v satisfies

$$v = T(\mu_h v) + T((\mu_h)_z),$$

where T is the Beurling transform. Since μ_h and $(\mu_h)_z$ are in every L^p, $p < \infty$, this equation can be solved via the geometric series

$$v = T((\mu_h)_z) + T(\mu T((\mu_h)_z) + T\mu T(\mu T((\mu_h)_z) + \ldots,$$

and v satisfies

$$\|v\|_p \leq \frac{C\|(\mu_h)_z\|_p}{1 - C\|\mu_h\|_\infty}.$$

Thus if $\frac{1}{p} + \frac{1}{q} = 1$, $p > 2$,

$$|\sigma| \leq (\|\frac{1}{z-w}\|_q + \|\frac{1}{z}\|_q)(\|\mu v\|_p + \|(\mu_h)_z\|_p) \leq C_q(\epsilon\|v\|_p + CM\epsilon) \leq CM\epsilon.$$

Thus

$$|h_z| \leq e^\sigma \leq 1 + CM\epsilon,$$

and similarly for $|h_{\bar{z}}| = |\mu_h h_z| \leq CM\epsilon$.

Finally we have to estimate K in (5). Taking $w \to \infty$ in (1), we see that

$$K = \frac{1}{\pi}\int (\mu_h v + (\mu_h)_z)\frac{dxdy}{z}.$$

Hence, using the same estimates as above, we get

$$|K| \le \|\frac{1}{z}\|_q \|\mu_h\|_\infty \|v\|_p + \|\frac{1}{z}\|_q \|(\mu_h)_z\|_p \le C\epsilon.$$

This proves the desired estimate. □

We have now proven all the estimates used in the proof of Theorem 1.1, so the result is established.

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