GEOMETRIC EXPONENTS AND KLEINIAN GROUPS

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Abstract. We define a number of different geometric critical exponents associated to a compact set in the plane which generalize the index of Besicovitch and Taylor on the line. Although these exponents may differ for general sets, we show that they are all equal when Λ is the limit set of a non-elementary, analytically finite Kleinian group and they agree with the classical Poincaré exponent. We also prove that if $\text{area}(\Lambda) = 0$ then the exponents equal $\dim_H(\Lambda)$ and give an example to show this can fail when $\text{area}(\Lambda) > 0$. We show that for analytically finite groups the Hausdorff and Minkowski dimensions of the limit set must agree. Another consequence of our argument is that if $\{\Omega_j\}$ is an enumeration of the components of $\Omega = S^2 \setminus \Lambda$, then

$$\sum_j \text{diam}(\Omega_j)^2 < \infty.$$

This had been conjectured by Maskit in 1974.

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1. Statement of results

The purpose of this paper is to simplify and extend results of Patterson and Sullivan (and more recent results of Jones and myself) concerning the dimension of Kleinian limit sets. We will see that a variety of measures of the 'size' of a compact set all agree for limit sets of a wide class of groups (although they can differ for general sets). The quantities we will consider include Hausdorff dimension, Minkowski (or box counting) dimension, and several geometric exponents: δ_{Whit}, δ_{diam}, δ_{area}, δ_{inrad}, δ_{cir}. An easy consequence of our arguments is that the Hausdorff and Minkowski dimensions agree; thus limit sets of analytically finite groups are 'dimension regular' in the sense of Tricot [34] and 'fractal' in the sense of Taylor [32].

We start by reviewing the definitions of the various dimensions and exponents.

Suppose K is a compact set in $\mathbb{R}^2$ and let $N(K,\epsilon)$ be the minimal number of ϵ balls needed to cover K. We define the *upper Minkowski dimension* as

$$\overline{\dim_M}(K) = \limsup_{\epsilon\to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon},$$

and the *lower Minkowski dimension*

$$\underline{\dim_M}(K) = \liminf_{\epsilon\to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon}.$$

If the two values agree, the common value is simply called the Minkowski dimension of K and is denoted $\dim_M(K)$. We define the Hausdorff content of K as

$$H^\infty_\alpha(K) = \inf\{\sum r_j^\alpha : K \subset \cup_j D(x_j, r_j)\},$$

(the infimum is over all coverings of K by disks) and

$$\dim_H(K) = \inf\{\alpha : H^\infty_\alpha(K) = 0\}.$$

This is the Hausdorff dimension of K and it is easy to see that $\dim_H(K) \le \underline{\dim}_M(K)$. For general sets, the Minkowski dimension need not even exist and if it does exist, it can be strictly larger than the Hausdorff dimension. Recall that a dyadic interval $I \subset \mathbb{R}$ is an interval of the form $[j2^{-n}, (j+1)2^{-n}]$ and a dyadic square in $\mathbb{R}^2$ is simply any product of equal length dyadic intervals. It is easy to see that in the definition of either dimension we need only consider coverings of K by dyadic squares.

For $K \subset [0,1]$, Besicovitch and Taylor [6] considered the following critical exponent. Suppose $\{I_j\}$ are the complementary components of K in $[0,1]$ and $|I|$ denotes the length of I. They showed that if K has zero length then

$$\dim_H(K) \le \delta_{\mathrm{BT}}(K) \equiv \inf\{s : \sum_j |I_j|^s < \infty\}.$$

Later Tricot [33] sharpened this by observing that $\delta_{\mathrm{BT}} = \overline{\dim}_M(K)$ if $|K| = 0$.

We want to generalize the Besicovitch-Taylor index to the plane (similar definitions clearly make sense in higher dimensions as well). Suppose $K \subset \mathbb{R}^2$ is compact and $\{\Omega_j\}$ is an enumeration of the bounded complementary components. We define

$$\delta_{\mathrm{inrad}}(K) = \inf\{s : \sum_j \mathrm{inrad}(\Omega_j)^s < \infty\},$$

$$\delta_{\mathrm{area}}(K) = \inf\{s : \sum_j \mathrm{area}(\Omega_j)^{s/2} < \infty\},$$

$$\delta_{\mathrm{diam}}(K) = \inf\{s : \sum_j \mathrm{diam}(\Omega_j)^s < \infty\},$$

(Here "inrad" denotes the radius of the largest disk in Ω_j.) If K is a compact set on the sphere S^2, then these quantities are measured with respect to the spherical metric. Each exponent generalizes δ_{BT}, but they disagree with each other for general sets (see Section 13). Moreover they may all disagree with Minkowski dimension.

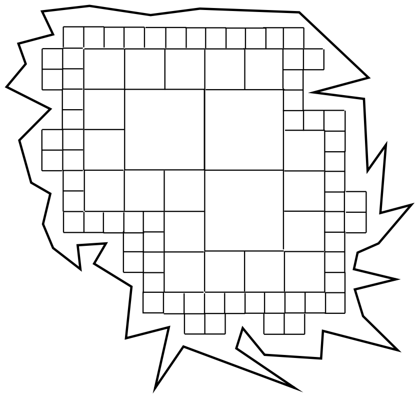

FIGURE 1.1. A Whitney decomposition

Next we define an exponent generalizing $\delta_{\rm BT}$ which maintains the relationship to Minkowski dimension. A Whitney decomposition of a open set $\Omega \subset \mathbb{R}^n$ is a collection of cubes $\{Q_j\}$ (disjoint except for boundaries) so that $\Omega = \cup_j Q_j$ and such that

$$\operatorname{diam}(Q_j) \simeq \operatorname{dist}(Q_j, \partial\Omega).$$

(In this paper, $a \simeq b$ means that the ratio a/b is bounded and bounded away from zero.) The existence of a Whitney decomposition for any open set is a standard fact in real analysis (e.g., Theorem VI.1 of [27]). One can simply take a maximal collection of dyadic cubes in Ω such that $\operatorname{diam}(Q) \leq \operatorname{dist}(Q, \partial\Omega)$. In $\mathbb{R}^2$, this gives a decomposition satisfying $\operatorname{diam}(Q) \leq \operatorname{dist}(Q_j, \partial\Omega) \leq 4\operatorname{diam}(Q_j)$.

Given a compact set K let $\Omega = \mathbb{R}^n \setminus K$ and suppose $\{Q_j\}$ is a Whitney decomposition of Ω. Define

$$\delta_{\rm Whit}(K) = \inf\{s : \sum_{Q_j, \operatorname{diam}(Q_j)\leq 1} \operatorname{diam}(Q_j)^s < \infty\}.$$

We need to omit the Whitney squares of large diameter, or the series would not converge. It is easy to check that the definition does not depend on the particular choice of Whitney decomposition. Moreover, for any compact set $K \subset \mathbb{R}^n$ we will see $\delta_{\rm Whit}(K) \leq \overline{\dim}_M(K)$ with equality if $\operatorname{area}(K) = 0$. See Section 2.

If $I \subset \mathbb{R}$ is an open interval and $\{J_j\}$ the corresponding Whitney decomposition, it is easy to see that

$$\sum |J_j|^s \simeq C_s |I|^s,$$

for any $s > 0$. Thus for $K \subset [0,1]$, the sum in the definition of δ_{Whit} is comparable to the sum in the definition the Besicovitch-Taylor index. Thus $\delta_{\mathrm{Whit}} = \delta_{\mathrm{BT}}$.

We now turn our attention to a special class of sets in $\mathbb{R}^2$ for which all these different exponents agree.

Consider a group G of Möbius transformations acting on the two sphere S^2. Such transformations are identified elements of $\mathrm{PSL}(2,\mathbb{C})$ in a natural way and G is called *Kleinian* if it is discrete in this topology (i.e., the identity is isolated in G). Such a group G also acts as isometries on the hyperbolic 3-ball $\mathbb{B}^3$. G is called *elementary* if it contains a finite index Abelian subgroup. In this paper we will consider only non-elementary groups. For a non-elementary group, the limit set, $\Lambda(G)$, is the accumulation set (on S^2) of the orbit of any point $z_0 \in S^2$ (and is independent of the point). The complement $\Omega(G) = S^2 \setminus \Lambda$ is called the ordinary set. In this paper we will assume Ω is non-empty and that the group is normalized so $\infty \in \Omega$.

For any Kleinian group, the quotient $R = \Omega/G$ is a union of (branched) Riemann surfaces. We say that G is *analytically finite* if $R = R_1 \cup \cdots \cup R_s$ is a finite union of finite type surfaces (i.e., each R_j is compact or compact with a finite number of punctures and branch points). The Ahlfors finiteness theorem ([2], [5]) says that if G is finitely generated then G is analytically finite.

If $z_0 \in \Omega(G)$ then the Poincaré exponent is defined as

$$\delta_{\mathrm{Poin}}(G) = \inf\{s : \sum_{g\in G} \mathrm{dist}(g(z_0), \Lambda)^s < \infty\}, \tag{1.1}$$

where distance is in the spherical metric. It is easy to show it does not depend on the choice of z_0. Usually this exponent is simply denoted $\delta(G)$, but in this paper we will always call it δ_{Poin} to avoid confusion with the many other exponents we consider.

The exponent $\delta_{\text{Poin}}(G)$ is can also be defined by extending G's action on S^2 to a group of isometries of the 3-ball $\mathbb{B}^3 \subset \mathbb{R}^3$ (with its hyperbolic metric ρ), and then considering any of the series

$$\sum_G (1-|g(0)|)^s, \quad \sum_G |g'(0)|^s, \quad \sum_G e^{-s\rho(0,g(0))}.$$

It is easy to see that these series all give the same critical exponent, δ_{Poin}, (e.g. Lemma 3.2 of [7]). Jones and I proved that $\delta_{\text{Poin}}(G) \leq \dim_H(\Lambda)$ for any non-elementary Kleinian group (Theorem 1.1 of [7]).

Theorem 1.1. *Suppose G is an analytically finite Kleinian group such that $\Lambda \neq S^2$. Then the Minkowski dimension of Λ exists and $\delta_{Poin} = \delta_{Whit} \leq dim_M(\Lambda) = dim_H(\Lambda)$. If area$(\Lambda) = 0$ then $\delta_{Poin} = dim_H(\Lambda)$. If $\Omega(G) = S^2 \setminus \Lambda(G)$ has infinitely many components, then*

$$\delta_{Poin} = \delta_{inrad} = \delta_{area} = \delta_{diam}. \tag{1.2}$$

If, in addition, area$(\Lambda) = 0$ then they are all equal to $dim_H(\Lambda)$.

For Fuchsian and geometrically finite groups the equality $\delta_{\text{Poin}} = \dim_H(\Lambda)$ is due to Patterson and Sullivan respectively (see [24] and its references, particularly [26] and [30]). The equality $\dim_M = \dim_H$ was proven for geometrically finite groups by Stratmann and Urbanski in [28]. The equalities $\delta_{\text{Poin}} = \dim_H(\Lambda) = \dim_M(\Lambda)$ were established for finite generated groups with zero area limit set of by Jones and myself in [7]. Besides giving new results, the proof of Theorem 1.1 will also give simpler, more direct, proofs of these previously known cases.

It is obvious that $\delta_{\mathrm{area}} \leq 2$ for any Kleinian group and hence by the theorem $\delta_{\mathrm{diam}} \leq 2$ for any analytically finite group. Thus if $\{\Omega_j\}$ is an enumeration of the components of $\Omega(G)$, we have

$$\sum_j \mathrm{diam}(\Omega_j)^s < \infty,$$

for any $s > 2$ (the diameter is in the spherical metric, or in the Euclidean metric if $\infty \in \Omega(G)$ and we sum over the bounded components). Maskit proved this with $s = 4$ in [22] and conjectured it should be true with $s = 2$ (he also proved it with $s = 3$; personal communication).

Theorem 1.2. *If G is an analytically finite Kleinian group then*

$$\sum_j \mathit{diam}(\Omega_j)^2 < \infty.$$

If every component of Ω was merely a Euclidean rescaling of one of a finite number of "large" components this would be trivial since the sum could be compared with the sum of areas of the components. This is what happens for geometrically finite groups and groups without parabolics. The idea of the proof of the general case is to associate to each component a measure of how much it is distorted from a simple rescaling and then estimate the number of components with large distortion.

An interesting special case occurs when G is a circle packing group, that is, every component of $\Omega(G)$ is a Euclidean disk. (Note: some authors use the term "circle packing" only if the components are disks and Λ has zero area, but here we only mean that the components are disks.) See Figure 1.2. In this case we automatically get $\delta_{\mathrm{inrad}} = \delta_{\mathrm{area}} = \delta_{\mathrm{diam}}$. If $K = S^2 \setminus \cup_j D(x_j, r_j)$ is the complement of a collection of

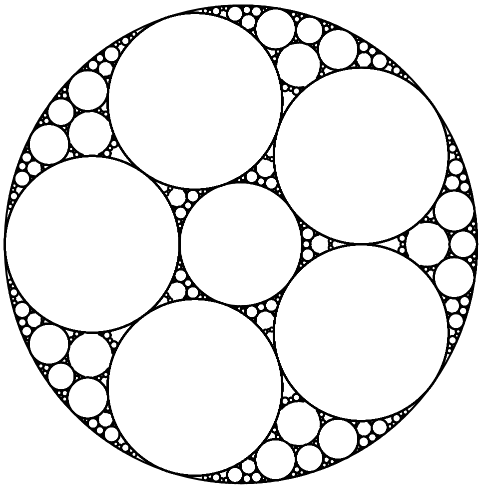

FIGURE 1.2. A circle packing limit set

open disks we define the circle packing exponent as

$$\delta_{\text{cir}} = \inf\{s : \sum_j r_j^s < \infty\}.$$

If we sum the Whitney decomposition of a single disk D_j we get

$$\sum_{Q_k \subset D_j} \text{diam}(Q_j)^s \simeq r_j^s,$$

if $s > 1$ and equals ∞ if $s \leq 1$. Thus $\delta_{\text{Whit}}(K) = \max(1, \delta_{\text{cir}})$. However, since circle packing groups have connected Λ it is known that $\delta_{\text{Poin}} > 1$ ([7]) and thus

$$\delta_{\text{Whit}} = \delta_{\text{inrad}} = \delta_{\text{area}} = \delta_{\text{diam}} = \delta_{\text{cir}}, \tag{1.3}$$

for any analytically finite circle packing group. Furthermore, they all equal $\dim_H(\Lambda)$ if $\text{area}(\Lambda) = 0$. For finitely generated circle packing groups this had been proven by Parker [25], using the results of [7] to reduce to the geometrically finite case. Circle packing groups are also considered in [19] and [8].

Notation: In this paper $A \simeq B$ means that A/B is bounded and bounded away from 0. Given a square S in the plane and $\lambda > 0$, λS denotes the concentric square with $\text{diam}(\lambda S) = \lambda \text{diam}(S)$. $D(x,r)$ denotes a disk in $\mathbb{R}^2$ with center x and radius r. $B(x,r)$ denotes a ball on S^2 or $\mathbb{B}^3$ in the spherical metric. $B_\rho(x,r)$ denotes a ball in $\mathbb{B}^3$ (or $M = \mathbb{B}^3/G$) with respect to the hyperbolic metric.

The remaining sections of this paper are organized as follows:

Section 2: $\delta_{\text{Whit}}(K) \leq \overline{\dim}_M(K)$ for any compact K, and equality holds if $\text{area}(K) = 0$.

Section 3: $\delta_{\text{Poin}} \leq \delta_{\text{Whit}}$ for analytically finite groups with equality if $\Omega(G)/G$ is compact. We also show $\delta_{\text{Poin}} \leq \delta_{\text{inrad}} \leq \delta_{\text{diam}}$.

Section 4: We define the isometric circle of a Möbius transformation and give a lemma of Maskit.

Section 5: Proof of Theorem 1.1 for geometrically finite groups and groups without parabolics.

Section 6: Proof of Maskit's conjecture.

Section 7: We state an estimate for the Poincaré sum which is used in the next two sections and proven in Section 12.

Section 8: $\delta_{\text{diam}} \leq \delta_{\text{Poin}}$.

Section 9: $\delta_{\text{Whit}} \leq \delta_{\text{Poin}}$.

Section 10: We show all the exponents equal $\dim_H(\Lambda)$ when $\text{area}(\Lambda) = 0$.

Section 11: We give an example to show $\delta_{\text{Poin}} < 2$ and $\text{area}(\Lambda) > 0$ is possible for analytically finite groups.

Section 12: Review some facts about the heat kernel on $M = \mathbb{B}^3/G$ and prove the estimate on the Poincaré sum used earlier.

Section 13: Describe the restrictions between δ_{inrad}, δ_{area}, δ_{diam} and δ_{Whit} which hold for general compact sets.

I thank Bernie Maskit for several very helpful conversations. This paper was partial motivated by his question from [22] as well as the results of [25]

2. Whitney squares and Minkowski dimension

Here we recall the proof of the following lemma. This is a known fact, but the proof is very short and important for what follows, so we include it for the convenience of the reader.

Lemma 2.1. *If K is compact then $\delta_{Whit}(K) \leq \overline{dim_M}(K)$. If, in addition, area$(K) = 0$ then $\delta_{Whit}(K) = \overline{dim_M}(K)$.*

Proof. Suppose $\{Q_j\}$ is a Whitney decomposition of $\Omega = \mathbb{R}^n \setminus K$. For each Q_j with $\mathrm{diam}(Q_j) \leq \mathrm{diam}(K)$, there is a dyadic cube Q_j' of the same size which hits K and satisfies $\mathrm{dist}(Q_j, Q_j') \leq C\mathrm{diam}(Q_j)$. Clearly each Q_j' is associated to only a bounded number of Whitney cubes. Therefore the number of dyadic cubes of size 2^{-n} which hit K is at least $C2^{n(\delta_{\mathrm{Whit}}-\epsilon)}$ (for n large enough, depending on ϵ). Thus $\delta_{\mathrm{Whit}}(K) \leq \overline{\mathrm{dim}_M}(K)$.

Conversely, if K has zero area, Q is a dyadic square hitting K and $\{Q_k\}$ is the collection of Whitney squares for Ω contained in Q, then

$$\sum_k \mathrm{diam}(Q_k)^2 = \mathrm{diam}(Q)^2.$$

Hence for any $s \leq 2$, (since $\mathrm{diam}(Q) \leq 1$),

$$\sum_k \mathrm{diam}(Q_k)^s \geq \mathrm{diam}(Q)^s.$$

Since there are more than $C2^{n(\overline{\mathrm{dim}_M}(K)-\epsilon)}$ such squares Q, the sum over the whole Whitney collection is greater than

$$C2^{-ns}2^{n(\overline{\mathrm{dim}_M}(K)-\epsilon)},$$

which diverges if $s < \overline{\dim_M}(K) - \epsilon$. Thus $\delta_{\text{Whit}}(K) \geq \overline{\dim_M}(K)$, as desired. $\square$

We can have strict inequality if K has positive area. For example, one can choose a set of disjoint disks $D(x_j, r_j) \subset D(0,1)$, so that $K = \overline{D(0,1)} \setminus \cup_j D_j$, is nowhere dense, has positive area and $r_j \to 0$ as fast as we wish. If we sum the Whitney decomposition of a single disk we get

$$\sum_{Q_k \subset D_j} \operatorname{diam}(Q_j)^s \simeq r_j^s,$$

if $s > 1$ and equals ∞ if $s \leq 1$. Thus $\delta_{\text{Whit}}(K) = \max(1, \delta_{\text{cir}})$. By taking $r_j \to 0$ very fast, we can get

$$\delta_{\text{Whit}}(K) = 1 < 2 = \overline{\dim_M}(K).$$

3. Whitney squares and the Poincaré series

In this section we explain the elementary relations between δ_{Poin} and δ_{Whit}.

Suppose Ω is a domain in S^2 with more than two boundary points. Then Ω has a hyperbolic metric ρ defined by the covering map from the disk to Ω. Let $d(z) = \operatorname{dist}(z, \partial\Omega)$. For a general domain (e.g., [11], Theorem 4.3),

$$\frac{(1+o(1))|dz|}{d(z)\log 1/d(z)} \leq |d\rho(z)| \leq 2\frac{|dz|}{d(z)}. \tag{3.1}$$

A set $K \subset \mathbb{R}^2$ is called *uniformly perfect* if there is a constant $C < \infty$ so that

$$\frac{1}{C}\frac{|dz|}{d(z)} \leq |d\rho(z)| \leq 2\frac{|dz|}{d(z)},$$

on each component Ω of $S^2 \setminus K$. (This is one of many equivalent definitions; see [15] and [17].)

Canary [9] has shown that the limit set of any finitely generated group is uniformly perfect. In fact, his proof shows this is true under the weaker assumption that there is an $\epsilon_0 > 0$ so that any closed geodesic on Ω/G has length $\geq \epsilon_0$. This is certainly true

Ω/G is a finite union of finite type surfaces, so his result is still true for analytically finite groups. (Also see Lemma 3.1 of [7].)

Lemma 3.1. *If G is any non-elementary Kleinian group with $\Lambda \neq S^2$ then $\delta_{Poin} \leq \delta_{Whit}$. If $\Omega(G)/G$ is compact then $\delta_{Poin} = \delta_{Whit}$.*

Proof. Fix a point $z_0 \in \Omega(G)$ (not an elliptic fixed point). There is a small hyperbolic disk around z_0 (with radius r_0 depending on z_0) which projects injectively to $R = \Omega/G$ under the quotient map. Thus points in $G(z_0)$, the orbit of z_0 under G, are separated by at least r_0 in the hyperbolic metric. By (3.1) each Whitney square has a uniformly bounded hyperbolic diameter and area. Thus each Whitney square for $\Omega = S^2 \setminus \Lambda$ contains at most a bounded number M (depending on z_0 and G) of points in $G(z_0)$. Therefore,

$$\sum_{g\in G} \operatorname{dist}(g(z_0), \Lambda)^s \quad \leq \quad M \sum_j \operatorname{diam}(Q_j)^s$$

and hence $\delta_{\mathrm{Poin}}(G) \leq \delta_{\mathrm{Whit}}(\Lambda(G))$.

Now suppose $R = \Omega(G)/G = R_1 \cup \cdots \cup R_s$ is a finite union of compact Riemann surfaces. We can choose points $E = \{z_1, \dots, z_s\} \subset \Omega$, so that z_j projects into R_j, $j = 1, \dots, s$ under the quotient map. By compactness, any point $z \in \Omega$ is a bounded hyperbolic distance from $G(E)$, the orbit of E under G. For each square Q with $\operatorname{dist}(Q, \Lambda) \leq 1$, choose a closest point $z_Q \in G(E)$. Then z_Q is only a bounded hyperbolic distance from Q so the uniform perfectness of Λ implies

$$\operatorname{diam}(Q) \leq C \operatorname{dist}(z_Q, \Lambda).$$

Furthermore, only a bounded number (say M) of the Q_j's can associated to any given point of $G(E)$. Thus

$$\sum_j \operatorname{diam}(Q_j)^s \quad \leq \quad MC^s \sum_{z_j \in E} \sum_{g \in G} \operatorname{dist}(g(z_j), \Lambda)^s,$$

and therefore $\delta_{\text{Whit}}(\Lambda(G)) \leq \delta_{\text{Poin}}(G)$. $\square$

Suppose that $\Omega(G) = S^2 \setminus \Lambda(G)$ has infinitely many components. For each component Ω_j choose the point $z_j \in G(E) \cap \Omega_j$ which maximizes $\operatorname{dist}(z, \partial\Omega)$. Define

$$\delta_{\text{orbit}}(G) = \inf\{s : \sum_j \operatorname{dist}(z_j, \Lambda)^s < \infty\}.$$

It's easy to see that this is independent of the choice of E. δ_{orbit} is the group analog of the index δ_{inrad}, since it is the radius of the largest disk in Ω which is centered at a point of $G(E)$. In particular, $\delta_{\text{orbit}} \leq \delta_{\text{inrad}}$.

Lemma 3.2. *If G is an analytically finite Kleinian group such that $\Omega(G)$ has infinitely many components, then $\delta_{Poin} = \delta_{orbit}$.*

Proof. The direction $\delta_{\text{orbit}} \leq \delta_{\text{Poin}}$ is obvious since δ_{orbit} is defined by summing over a strict subseries of the Poincaré series defining δ_{Poin}. To prove the other direction we simply note that if $\Omega(G)$ has infinitely many components then we can find two distinct components Ω_1, Ω_2 which are at positive (spherical) distance from each other. This is because if we fix a component Ω_0 of $\Omega(G)$ then there are infinitely many disjoint quasidisks in $S^2 \setminus \Omega_0$ each of which contain components of $\Omega(G)$ (see Maskit's book [23]). Moreover the quasidisks have diameters tending to zero and accumulate at every point of $\partial\Omega_0$. Thus is possible to choose two at positive distance from each other, and so the corresponding components of $\Omega(G)$ will also be at a positive distance.

Fix a point $z_0 \in \Omega(G)$ in the definition of the Poincaré series. Fix orbit points $z_1 \in G(z_0) \cap \Omega_1$ and $z_2 \in G(z_0) \cap \Omega_2$ and balls $B_i \subset \Omega_i$ for $i = 1, 2$. Consider the cone C_i of hyperbolic geodesics in $\mathbb{B}^3$ which connect the origin $0 \in \mathbb{B}^3$ to B_i. Since the components are at positive distance from each other, these cones are separated by a definite positive angle at 0. For any $g \in G$ consider the image cones $g(C_1)$ and $g(C_2)$ which connect $g(0)$ to $g(B_1)$ and $g(B_2)$. Since these cones are separated by a positive angle (independent of g), at least one of them makes a positive angle with the radial segment $[0, g(0)]$. Call this index $i_g \in \{1, 2\}$. Simple hyperbolic geometry now implies that

$$\mathrm{dist}(g(z_{i_g}), \partial g(\Omega_{i_g})) \simeq \mathrm{diam}(g(\Omega_{i_g})) \simeq 1 - |g(0)|.$$

This implies that each component of $\Omega(G)$ can contain only a bounded number of point of the form $g(z_{i_g})$. Thus

$$\sum_{g \in G} (1 - |g(0)|)^s \leq C \sum_{g \in G} \mathrm{dist}(g(z_{i_g}))^s,$$

and only a bounded number of distinct $g(z_{i_g})$'s lie in any one component of G. Thus the right hand side is bounded by a constant times the series defining δ_{orbit}. Thus $\delta_{\mathrm{Poin}} = \delta_{\mathrm{orbit}}$. $\square$

Since we have already noted that $\delta_{\mathrm{orbit}} \leq \mathrm{inrad}$, we get

Corollary 3.3. *If G is an analytically finite Kleinian group so that $\Omega(G)$ has infinitely many components, then $\delta_{Poin} \leq \delta_{inrad} \leq \delta_{diam}$.*

4. Isometric circles

Consider the Möbius transformation

$$g(z) = \frac{az+b}{cz+d},$$

and its derivative

$$g'(z) = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2},$$

(since we can identify Möbius transformations with elements of $\mathrm{PSL}(2,\mathbb{C})$ we may normalize them so $ad-bc=1$; assume that we have done this). The circle $\{z : |cz+d| = 1\}$, with center $c_\tau = -\frac{d}{c} = \tau^{-1}(\infty)$ and radius $r_\tau = |c|^{-1/2}$ is called the isometric circle of τ. The mapping τ can be written as a composition $\tau(z) = f(h(z))$, where h is the inversion through the isometric circle and f is a (Euclidean) isometry of the plane. (see [16], Sections 17 and 47).

Suppose G is a Kleinian group and Ω_0 is some component of the ordinary set $\Omega(G) = S^2 \setminus \Lambda(G)$. Let $G_0 = \{g \in G : g(\Omega_0) = \Omega_0\}$ be stabilzer ofΩ_0. Normalize the group G so that $\infty \in \Omega_0$ and $\mathrm{diam}(\Lambda(G)) = 1$. Suppose that Ω_0 is not invariant under all of G and let $\Omega_j = g(\Omega_0)$ be a distinct component of $\Omega(G)$ which is equivalent to Ω_0. Let $S_j = \{g \in G : g(\Omega_0) = \Omega_j\} = gG_0$ be the corresponding coset. Let C_j be the set of centers of isometric circles of elements in S_j. Note that each point of C_j is of the form $w = g^{-1}(\infty)$ for $g \in S_j$ and hence $w \notin \Omega_0$ since $\infty \in \Omega_0$ and $g^{-1}(\Omega_0) \neq \Omega_0$ by assumption.

Let $h_1 \in G_0$ and $w_1 = (gh_1)^{-1}(\infty) \in C_j$. If $h_2 \in G_0$, then

$$h_2(w) = h_2(h_1^{-1}(g^{-1}(\infty))) = (gh_1h_2^{-1})^{-1}(\infty),$$

is in C_j as well. In other words, the set C_j is invariant under G_0.

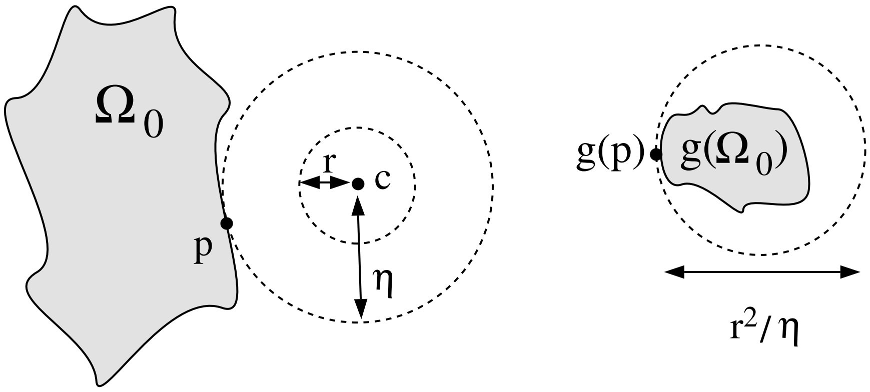

FIGURE 4.1. Bounding the diameter of Ω_j

Now let F be a fundamental region in $\Omega(G_0)$ for the action of G_0. Let $F_0 = F \setminus \Omega_0$. By our remarks above, given any component $\Omega_j \neq \Omega_0$ of $\Omega(G)$ which is equivalent to Ω_0 in G, there is an element $g_j \in G$ so that $g(\Omega_0) = \Omega_j$ and such that the center c_j of g_j's isometric circle lies in F_0. Define

$$\eta_j = \eta(\Omega_j) = \text{dist}(c_j, \Omega_0).$$

This will serve as a measure of how distorted Ω_j is compared to Ω_0. In particular, if G has no parabolics, then $\overline{F_0} \cap \overline{\Omega_0} = \emptyset$, so η is bounded away from zero independently of j and we shall see in the next section that this implies Ω_j is (up to a Euclidean similarity) a bounded distortion of Ω_0.

Let P be the finite set of cusp points on the boundary of F_0. The following is due to Maskit [22].

Lemma 4.1. *Normalize G so that $\infty \in \Omega_0$ and $diam(\Lambda) = 1$. Suppose $\Omega_j = g(\Omega_0)$, $c_j \in F_0$ is the center of the isometric circle of g, $\eta_j = dist(c_j, P)$ and r_j is the radius of the isometric circle of g. Then there is a $C = C(G)$ so that*

$$C^{-1} \frac{r_j^2}{\eta_j} \leq diam(\Omega_j) \leq C \frac{r_j^2}{\eta_j}$$

Proof. Let $p \in P$ be a parabolic cusp closest to c_j. See Figure 4.1. Let D be the largest disk in $\Omega(G_0)$ which is centered at c_j. By our remarks above, its radius is uniformly comparable to η_j. Thus when we invert it through the circle centered at c_j of radius r_j it is mapped to the exterior of the disk (also centered at c_j) of radius comparable to r_j^2/η_j. Thus Ω_0, which lies outside of D, is mapped by the inversion into a disk of this radius. Since g_j can be written as this inversion followed by a Euclidean isometry we deduce that $\Omega_j = g_j(\Omega_0)$ has diameter at most Cr_j^2/η_j. On the other hand, note that the inversion maps ∞ to c_j and maps the point p to a point at distance $\geq Cr_j^2/\eta_j$ from c_j. Since $\infty \in \Omega_0$ and $p \in \partial\Omega_0$, this gives the lower bound for $\mathrm{diam}(\Omega_j)$. $\square$

Later we will want to apply the lemma in a slightly different setting when Ω is replaced by a Euclidean ball $B \subset \Omega_0$ which has p on its boundary. The argument above implies that if $\mathrm{diam}(B) \simeq 1$ then

$$\mathrm{diam}(g(B)) \simeq \frac{r^2}{\eta}, \tag{4.1}$$

where r is the radius of the isometric circle of g and η is the distance of its center from p.

5. The geometrically finite case

Recall that if we fix a component Ω_0 of $\Omega(G)$ and consider a distinct component $\Omega_j = g(\Omega_0)$, then we can choose $g = g_j$ so that the isometric circle for g_j has radius r_j and center $c_j \in F_0$. The set F_0 may be taken to be a finite union of components each of which is bounded by a piecewise smooth curve which hits Λ at only a finite number of points, each a rank 1 parabolic fixed point of G_0. At each such point p there is a component of $\Omega(G_0)$ intersecting F_0 and which contains a disk which is

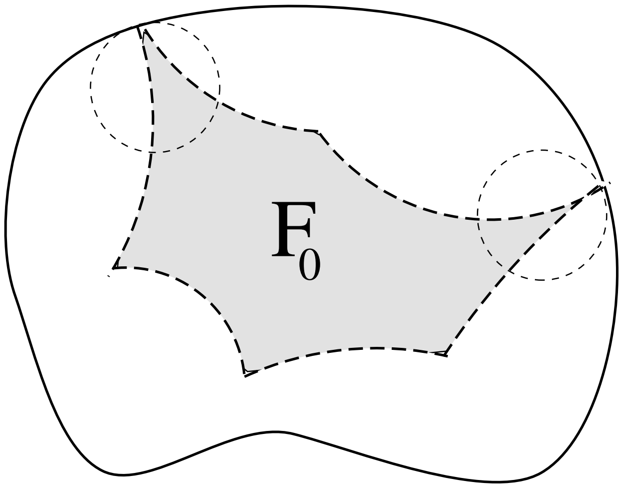

FIGURE 5.1. The geometry of F

invariant under h and so that there is a neighborhood of p in which F_0 looks like a cusp bounded by two circles tangent at p. In particular, there is a uniform bound C (depending only on F_0, hence only on the choice of Ω_0) so that for each $z \in F_0$, the component of $\Omega(G_0)$ contains a disk D_z centered at z whose radius is comparable to $\operatorname{dist}(z, P)$. (This is clear near the parabolic cusps of F_0 and true by compactness outside the cusp regions.) See Figure 5.1.

Lemma 5.1. *Suppose $p \in \partial F_0$ is a rank one parabolic fixed point of G_0 (the stabilizer of Ω_0) and h is a parabolic element fixing p. If $\Omega(G) \setminus \Omega_0$ contains a disk D invariant under h, then $dist(c_g, p) > \eta(G)$ is bounded away from zero for all elements $g \in G$ with $c_g \in F_0$.*

Proof. If $\Omega(G)$ has such a disk, it corresponds to a neighborhood of a puncture on the surface $\Omega(G)/G$. By taking the disk smaller we may assume that ∞ is not contained in any G-image of the disk D. If $D \cap F_0$ contains the center c_j of the isometric circle corresponding to some g_j then g_j maps the disk D to the exterior of a disk of radius $\simeq r_j^2/\eta_j$ and hence $g_j(D)$ contains ∞, a contradiction. Thus the centers of isometric

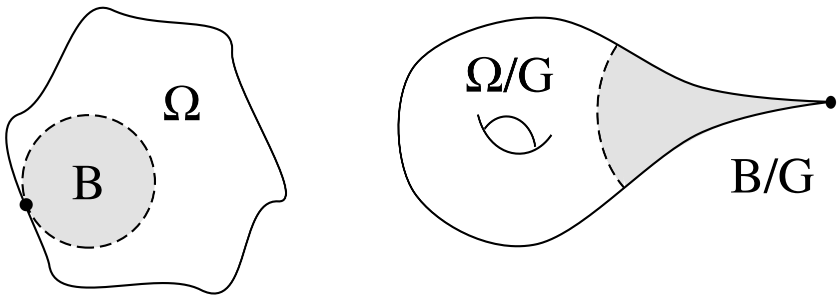

FIGURE 5.2. Horoballs and cusps

circles are bounded away from p. $\square$

A horoball in $\Omega(G)$ is a Euclidean ball $B \subset \Omega \subset S^2$ which is invariant under a rank 1 parabolic subgroup of G. The fixed point p of the parabolic element is on the boundary of the horoball and corresponds to a cusp on the surface $\Omega(G)/G$. Conversely, to each cusp on the quotient surface there is a equivalence class of horoballs in $\Omega(G)$.

We say that p is doubly cusped if there are two disjoint horoballs in $\Omega(G)$, both invariant under the parabolic subgroup fixing p. The following is due to Beardon and Maskit.

Theorem 5.2. *([3]) If G is a Kleinian group then G is geometrically finite iff every point of $\Lambda(G)$ is either a conical limit point, a rank 2 parabolic fixed point or a doubly cusped rank 1 parabolic fixed point.*

In particular, their result implies that in a geometrically finite group, every rank 1 parabolic fixed point must be doubly cusped (this particular consequence is also proven in [7], Corollary 4.6). By Lemma 5.1 this means that in a geometrically finite group there is a $\eta_0 > 0$ (depending only on G) so that for every component $\eta(\Omega_j) \geq \eta_0$.

Lemma 5.3. *Suppose G is an analytically finite Kleinian group such that $\Omega(G)$ has infinitely many components $\{\Omega_j\}$ and that η_j is bounded away from zero independently of j. Then*

$$\delta_{inrad}(\Lambda) = \delta_{area}(\Lambda) = \delta_{diam}(\Lambda) = \delta_{Whit}(\Lambda).$$

Proof. The proof of the theorem is easy because under this hypothesis, each component $\Omega_j = g_j(\Omega_0)$ is only a rescaling of Ω_0 with a bounded amount of distortion. More precisely suppose $\Omega_j = g_j(\Omega_0)$ and let c_j, r_j be the center and radius of the isometric circle of g_j. Recall that we can write $g_j = f_j \circ h_j$ where

$$h_j(z) = \frac{r_j^2}{(z-c_j)^2} + c_j,$$

is inversion though the isometric circle of g_j and f_j is a Euclidean isometry. Clearly f_j does not effect any of the quantities in question. We can further decompose h_j as the composition of four maps: translation by $-c_j$, inversion through the unit circle, a dilation (multiplication by r_j^2) and translation by c_j. The inversion is a smooth map except at the origin, so will have derivative bounded and bounded away from zero on any annulus $\{\eta_0 \leq |z| \leq \eta_0^{-1}\}$.

Thus $g_j : \Omega_0 \to \Omega_j$ can be written as a composition of three maps: the first has uniformly bounded distortion on a uniform neighborhood of $\partial\Omega_0$, the second is a Euclidean dilation and the third is a Euclidean isometry. From this we deduce that

$$\mathrm{inrad}(\Omega_j) \simeq \mathrm{diam}(\Omega_j) \simeq \sqrt{\mathrm{area}(\Omega_j)},$$

with constants that depend only on Ω_0 and η_0, but not on j. Thus

$$\delta_{\mathrm{inrad}} = \delta_{\mathrm{diam}} = \delta_{\mathrm{area}}.$$

Let $\{Q_k\}$ be the Whitney squares for Ω_0 with $\text{dist}(Q_k, \partial\Omega_0) \leq 1$. Each is mapped via g_j is to a region which satisfies

$$\text{diam}(g_j(Q_k)) \simeq \text{inrad}(g_j(Q_k)) \simeq \text{dist}(g_j(Q_k), \partial\Omega_j) \simeq \text{diam}(\Omega_j)\text{diam}(Q_j).$$

Thus each $g_j(Q_k)$ intersects a bounded number of Whitney squares $\tilde{Q}_k$ for Ω_j and only a bounded number of $\tilde{Q}_k$'s are not hit. Thus the Whitney sum for Ω_j is bounded by

$$\begin{aligned}\sum_k \text{diam}(\tilde{Q}_k)^s &\leq C\text{diam}(\Omega_j)^s + \text{diam}(\Omega_j)^s \sum_k \text{diam}(Q_k)^s \\ &\leq C\text{diam}(\Omega_j)^s.\end{aligned}$$

Thus $\delta_{\text{Whit}} \leq \delta_{\text{diam}}$. Equality holds since there is a Whitney square in Ω_j with diameter comparable to $\text{diam}(\Omega_j)$. $\square$

Lemma 5.4. *If G is geometrically finite then $\delta_{Poin} = \delta_{Whit}$.*

Proof. We already know this result if $\Omega(G)/G$ is compact (Lemma 3.1) so we may as well assume there are cusps, i.e., $\Omega(G)$ contains horoballs. Since G is analytically finite, the quotient surface has at most finitely many cusps, so there are at most finitely many equivalence classes on horoballs in $\Omega(G)$. Let $\{B_1, \dots, B_s\}$ be a choice of representative from each class. The collection of all G images of these balls will be denoted $\mathcal{B}$. It will be convenient to assume that G is normalized so that $\infty \in \Omega(G)$ and ∞ is not contained in any horoball (which we may assume by conjugating the group and taking smaller horoballs if necessary). We also assume that all the horoballs are pairwise disjoint. By making the horoballs even smaller we may assume that each horoball in our finite collection $\{B_j\}$ is contained in a horoball of twice the Euclidean diameter and so that the larger balls do not intersect the orbit $G(\infty)$. The larger balls will be denoted $\{\tilde{B}_j\}$.

Choose a finite number of points $E = \{z_1, \dots, z_t\}$, one from each equivalence class of components of $\Omega(G)$. There is $M < \infty$ so that Q is a Whitney square which is not contained in any horoball of $\mathcal{B}$, then Q is within hyperbolic distance M of some G-image of a point of E. The argument of Section 3 then implies that

$$\sum_{Q_j \cap \mathcal{B} = \emptyset} \mathrm{diam}(Q_j)^s \leq C \sum_{z \in G(E)} \mathrm{dist}(z, \Lambda)^s.$$

Thus to prove $\delta_{\mathrm{Whit}} \leq \delta_{\mathrm{Poin}}$ we only have to consider Whitney squares contained in horoballs.

Given a ball $B \subset \Omega(G)$, it is easy to see that B contains at most $C2^n$ Whitney squares for $\Omega(G)$ of size $\geq 2^{-n}\mathrm{diam}(B)$. Therefore we get

$$\sum_{Q_j \subset B} \mathrm{diam}(Q_j)^s \leq C \mathrm{diam}(B)^s.$$

Therefore it is enough to show that if $s > \delta_{\mathrm{Poin}}$ then

$$\sum_{B \in \mathcal{B}} \mathrm{diam}(B)^s < \infty.$$

By breaking the sum into a finite number of sums, we need only consider the sum over all horoballs are G-equivalent to a fixed horoball, say B_1.

Given a horoball $B = g(B_1)$, let $H_1 = \{h \in G : h(B_1) = B_1\}$ be the subgroup of G fixing B_1 and let $S = \{h \in G : h(B) = B\} = gH_1$. Let $T = \ g^{-1}(\infty) : g \in S$ be the collection of centers of the isometric circles of these elements. We want to see that T is a H_1-invariant set. To see this suppose $gh_1 \in S$ and $h_2 \in H_1$. Then

$$h_2((gh_1)^{-1}(\infty)) = (gh_1h_2^{-1})^{-1}(\infty) \in T,$$

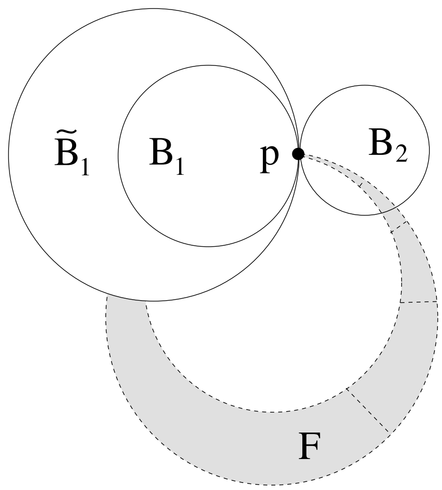

FIGURE 5.3. Fundamental domain for a parabolic group

as desired. Thus for any horoball $B \neq B_1$ which is equivalent to B_1, we may choose $g \in G$ with $B = g(B_1)$ so that $g^{-1}(\infty)$ lies in a fundamental domain F for H_1, but outside B_1. See Figure 5.3.

Since G is geometrically finite, the parabolic fixed point p is doubly cusped, so there is a second horoball (disjoint from B_1) at p. Since B_2 does not contain any orbits of ∞ the centers of any isometric circles can be chosen in F outside of both B_1 and B_2. Thus if $B = g(B_1)$ we see g can be chosen so that the center of the corresponding isometric circle is bounded away from p (independently of B). This means that B is a bounded distortion image of B_1 under g. Therefore there is point $z \in G(E)$ with

$$\text{dist}(z, B) \simeq \text{dist}(z, \Lambda) \simeq \text{diam}(B),$$

with constants that are independent of B. Thus

$$\sum_{B \in \mathcal{B}} \text{diam}(B)^s \leq C \sum_{z \in G(E)} \text{dist}(z, \Lambda)^s < \infty,$$

if $s > \delta_{\text{Poin}}$. This proves $\delta_{\text{Whit}} \leq \delta_{\text{Poin}}$, as desired. $\square$

If G has no parabolics then every component of $\Omega(G)$ has η bounded away from zero. Combining this observation with the previous two lemmas and Lemma 3.1 gives

Corollary 5.5. *Suppose G is an analytically finite Kleinian group such that $\Omega(G)$ has infinitely many components $\{\Omega_j\}$ and that either G has no parabolics or G is geometrically finite. Then*

$$\delta_{inrad}(\Lambda) = \delta_{area}(\Lambda) = \delta_{diam}(\Lambda) = \delta_{Poin}(G) = \delta_{Whit}(\Lambda).$$

6. The proof of Maskit's conjecture

Before proceeding with the proof of Theorem 1.1 in the remaining cases (i.e., geometrically infinite with parabolics) we will prove Maskit's diameter sum conjecture (Theorem 1.2). The estimates we use here are a model for the argument used to finish the proof of Theorem 1.1.

Suppose $\{\Omega_j\}$ is an enumeration of the components of $\Omega(G)$. We wish to show

$$\sum_j \operatorname{diam}(\Omega_j)^2 < \infty.$$

The components of $\Omega(G)$ belong to a finite number of equivalence classes, so it is enough to prove that the sum over each class is finite. So fix a component Ω_0 and normalize G so that $\infty \in \Omega_0$, $0 \in \Lambda$ and $\operatorname{diam}(\Lambda) = 1$. Suppose $\{\Omega_j\}$ is an enumeration of the components which are equivalent to Ω_0. If $\{\Omega_j\}$ is a finite list, there is nothing to do, so we assume that there are infinitely many such components.

Let $D_0 = \{|z| > R\}$ where R is chosen so large that all G-images of D_0 are disjoint (we may assume without loss of generality that ∞ is not an elliptic fixed point). Consider a component $\Omega_j = g_j(\Omega_0)$ and decompose $g_j = f_j \circ h_j$ as a composition of inversion through the isometric circle followed by a Euclidean isometry. As in

Section 4, $c_j = g^{-1}(\infty)$ be the center of the isometric circle, r_j its radius and $\eta_j \simeq \text{dist}(c_j, P)$ (where P is the set of parabolic fixed points on ∂F_0, the fundamental domain of $G(\Omega_0)$).

By Lemma 4.1 we know

$$\text{diam}(\Omega_j) \simeq \frac{r_j^2}{\eta_j}.$$

On the other hand, $D_j = g^{-1}(D_0)$ is a disk containing c_j of diameter $\simeq r_j^2$. These disks are disjoint from their images under the parabolics fixing various points of P, so

$$\text{diam}(D_j) \leq C\text{dist}(c_j, P)^2.$$

Thus the union of the disks $\{D_j\}$ is contained in a neighborhood F' of F which also has a cusp at p. In particular, near each parabolic point $p \in \partial F$,

$$\text{area}(F' \cap D(p,r)) \leq Cr^3.$$

Thus

$$\begin{aligned}
\sum_j \text{diam}(\Omega_j)^2 &\leq C \sum_j \frac{r_j^4}{\eta_j^2} \\
&\leq C \sum_j \text{area}(D_j)\text{dist}(D_j, P)^{-2} \\
&\leq C \sum_n 2^{2n} \sum_{j:2^{-n-1} < \text{dist}(c_j,P) \leq 2^{-n}} \text{area}(D_j) \\
&\leq C \sum_n 2^{-3n} 2^{2n} \\
&< \infty.
\end{aligned}$$

This proves Theorem 1.2

7. An estimate for the Poincaré series

To prove $\delta_{\rm diam} \leq \delta_{\rm Poin}$ and $\delta_{\rm Whit} \leq \delta_{\rm Poin}$ we will use the following estimate for the Poincaré sum which will be proven in Section 12. In what follows,

$$V(x) = \mathrm{vol}(B_\rho(x,1)/G),$$

i.e., $V(x)$ is the volume of a unit hyperbolic ball around the projection of x in $M = \mathbb{B}^3/G$.

Lemma 7.1. *Suppose $x \in S^2$ and let $y = (1-r)x \in \mathbb{B}^3$. Given any $s > \delta_{Poin}$ there is a $C_s < \infty$ so that for any (spherical) ball $B = B(x,r)$,*

$$\sum_{g(0)\in B} (1-|g(0)|)^s \leq \frac{C_s r^s e^{-s\rho(y,G(0))}}{\sqrt{V(0)V(y)}}.$$

We will only need to apply this is a very special case which we now describe in detail. According to the Margulis lemma (e.g., page 134 of [4]) there is an $\epsilon > 0$ (independent of G) so that for any $y \in \mathbb{B}$, then the group generated by

$$G_y = \{g \in G : \rho(y, g(y)) \leq \epsilon\},$$

is almost-nilpotent and hence elementary.

Suppose $p \in S^2$ is a parabolic fixed point and let $H \subset G$ be the parabolic subgroup which fixes p. Fix the H-invariant Euclidean ball $T \subset \mathbb{B}^3$ so that each element of H moves points in T (the "thin part") less than this ϵ. Then for any $y \in T$ the subgroup G_y contains H and is elementary. By the classification of elementary groups given on page 1085 of [20], G_y is either H or conjugate to one of a finite number of possibilities which contain H as a finite index subgroup. In each case the elliptics have order at most 6 and the elliptic fixed points in ∂T are uniformly separated in the hyperbolic metric. This implies that for $y \in \partial T$, $V(y)$ is bounded away from zero by a constant which depends only on the choice of T, i.e., only on ϵ and the conjugacy class of p.

Thus Lemma 7.1 implies that if $y = (1-r)x$ is on the boundary of the thin part T of some parabolic we have

$$\sum_{g(0)\in B} (1 - |g(0)|)^s \leq C_s r^s. \tag{7.1}$$

with a constant that depends on s and G, but not on y. More accurate estimates are available for geometrically finite groups. See e.g., [29].

Another result which we will need in the next two sections is

Lemma 7.2. *If G is analytically finite and $\delta_{Poin} < 1$ then G is geometrically finite.*

This is proven in [7] under the weaker hypothesis $\delta_{\text{Poin}} \leq 1$, but we will sketch a different proof of the lemma for completeness. Also see [10].

If Λ is totally disconnected then $\Omega(G)$ is connected and so G is finitely generated. In this case, a theorem of Abikoff and Maskit [1] implies G is geometrically finite. If Λ is not totally disconnected then the same is true of the boundary of any component of Ω. Choose a component Ω_0 and let $G_0 \subset G$ be the subgroup fixing Ω_0. Then Ω_0 is an invariant component for G_0 and G_0 is finitely generated, so the decomposition theorem for function groups [1] says there is a finitely generated subgroup $H \subset G_0$ with connected limit set. Let Ω_1 be a (simply connected) component of $\Omega(H)$ and $\Phi : \mathbb{D} \to \Omega_1$ be a Riemann mapping from the unit disk. If $z_0 \in \Omega_1$ and $\{z_j\} = H(z_0)$, and $\delta_{\text{Poin}}(H) < 1$ then

$$\sum_{z\in H(z_0)} \text{dist}(z, \partial\Omega_1)^s < \infty,$$

for some $s < 1$. This means that the set

$$F = \cap_j D(z_j, 2\text{dist}(z_j, \partial\Omega_j)),$$

has dimension $\leq s < 1$. On the other hand, $\Phi^{-1}(\{z_j\})$ is the orbit in $\mathbb{D}$ of a finitely generated Fuchsian group of the first kind, which implies it is non-tangentially dense almost everywhere on the unit circle, which in turn implies $\Phi^{-1}(F)$ has full measure. This contradicts Makarov's theorem [21] and proves $\delta_{\text{Poin}}(G) \geq \delta_{\text{Poin}}(H) \geq 1$, as desired.

8. The diameter sum and Poincaré series

In this section we will show that $\delta_{\text{diam}} \leq \delta_{\text{Poin}}$ (except for the proof of Lemma 7.1). The argument is almost the same as in Section 6, except that we use the estimate of the previous section instead of the fact that $\sum_k \text{area}(D_j) < \infty$.

As in Section 6, let $\{\Omega_j\}$ be an enumeration of the components equivalent to a fixed component Ω_0 which has been normalized so that $\infty \in \Omega_0$, $0 \in \Lambda$ and $\text{diam}(\Lambda) = 1$. Also, since we have equality for geometrically finite groups we may assume that G is geometrically infinite. This implies that $\delta_{\text{Poin}} \geq 1$ by Lemma 7.2.

Note,

$$\begin{aligned}
\sum_j \text{diam}(\Omega_j)^s &\leq C \sum_j \frac{r_j^{2s}}{\eta_j^s} \\
&\leq C \sum_j \text{dist}(c_j, \Lambda)^s \text{dist}(c_j, P)^{-s} \\
&\leq C \sum_n \sum_{j:2^{-n-1} < \text{dist}(c_j, P) \leq 2^{-n}} \text{dist}(c_j, \Lambda)^s 2^{sn}.
\end{aligned}$$

For each n we only have to consider the points $\{c_j\}$ which lie in the intersection of the annulus $2^{-n-1} \leq |z - p| < 2^{-n}$ with F. These points lie in a narrow rectangle of width 2^{-2n} and length 2^{-n}. This strip can be covered by approximately 2^n disks of diameter approximately 2^{-2n}. The tops of the hemispheres with these disks as bases lie may be chosen to be on the boundary of the thin part associated to p, if n is large enough. Thus equation (7.1) implies that the Poincaré sum for each disk is bounded by $C2^{-2sn}$. Since there are 2^n disks, the Poincaré sum for the whole strip is bounded by $C2^{(1-2s)n}$. Thus since we are assuming $s > 1$,

$$\sum_j \operatorname{diam}(\Omega_j)^s \le C\sum_n 2^{sn}2^{n(1-2s)} \le C\sum_n 2^{n(1-s)} < \infty.$$

This proves $\delta_{\mathrm{diam}} \le \delta_{\mathrm{Poin}}$.

9. The Whitney and Poincaré sums

In this section we will prove $\delta_{\mathrm{Whit}} \le \delta_{\mathrm{Poin}}$. The proof is almost identical to the proof of $\delta_{\mathrm{diam}} \le \delta_{\mathrm{Poin}}$, except that we apply Maskit's lemma (Lemma 4.1) to horoballs instead of components.

We already know the result if G is geometrically finite (see Section 5) so we may assume G is geometrically infinite. Thus we may assume $\delta_{\mathrm{Poin}} \ge 1$.

We use the same notation as in the proof of Lemma 5.4. From the proof of Lemma 5.4 we know that it is enough to show that

$$\sum_B \operatorname{diam}(B)^s < \infty,$$

if $s > \delta_{\mathrm{Poin}}$ and the sum is over all horoballs which are equivalent to a fixed horoball B_1. Estimate (4.1) gives us

$$\operatorname{diam}(B_j) \le C\frac{r_j^2}{\eta_j},$$

if $B_j = g_j(B_1)$.

The proof from this point is identical to the argument in Section 8. Let $\{B_j\} = \{g_j(B_1)\}$ be an enumeration of the horoballs which are equivalent to B_1 and assume g_j has been chosen so that $g^{-1}(\infty)$ lies in a fixed fundamental domain for H_1 (the parabolic subgroup fixing the point $p \in \partial B_1$). The part of the fundamental domain near the fixed point p is bounded by two circles tangent at p and so can be covered by a union of rectangles of size approximately $2^{-n} \times 2^{-2n}$. Thus exactly as in Section 8 we get

$$\begin{aligned}
\sum_j \operatorname{diam}(B_j)^s &\leq C \sum_j \frac{r_j^2}{\eta_j} \\
&\leq C \sum_j \operatorname{dist}(c_j, \Lambda)^s \operatorname{dist}(c_j, p)^s \\
&\leq \sum_{n=1}^{\infty} 2^{sn} \sum_{j: 2^{-n-1} < \operatorname{dist}(x_j, p) \leq 2^n} \operatorname{dist}(c_j, \Lambda)^s.
\end{aligned}$$

As in Section 8 we use (7.1) to show that the inner sum is bounded by $C2^{n(1-2s)}$ if $s > \delta_{\text{Poin}}$. Thus since $s > \delta_{\text{Poin}} \geq 1$, we get

$$\sum_j \operatorname{diam}(B)^s \leq C_s \sum_n 2^{sn} 2^{n(1-2s)} < \infty,$$

as desired.

10. Limit sets with zero area

We have now proven that if G is analytically finite then all the critical exponents we have defined are the same, i.e.,

$$\delta_{\text{Poin}} = \delta_{\text{inrad}} = \delta_{\text{area}} = \delta_{\text{Whit}} = \delta_{\text{diam}}.$$

Moreover, by Lemma 2.1 they all equal $\overline{\dim_M}(\Lambda)$ if Λ has zero area. By Theorem 1.1 of [7], $\delta_{\text{Poin}} \leq \dim_H(\Lambda)$, for any non-elementary Kleinian group, so with the trivial inequality $\dim_H(\Lambda) \leq \overline{\dim_M}(\Lambda)$, we deduce

$$\delta_{\text{Poin}} = \delta_{\text{inrad}} = \delta_{\text{area}} = \delta_{\text{Whit}} = \delta_{\text{diam}} = \dim_H(\Lambda) = \dim_M(\Lambda),$$

whenever G is analytically finite with infinitely many components and $\text{area}(\Lambda) = 0$. If $\text{area}(\Lambda) > 0$ then $\dim_H(\Lambda) = \dim_M(\Lambda) = 2$, so the two dimensions agree in all

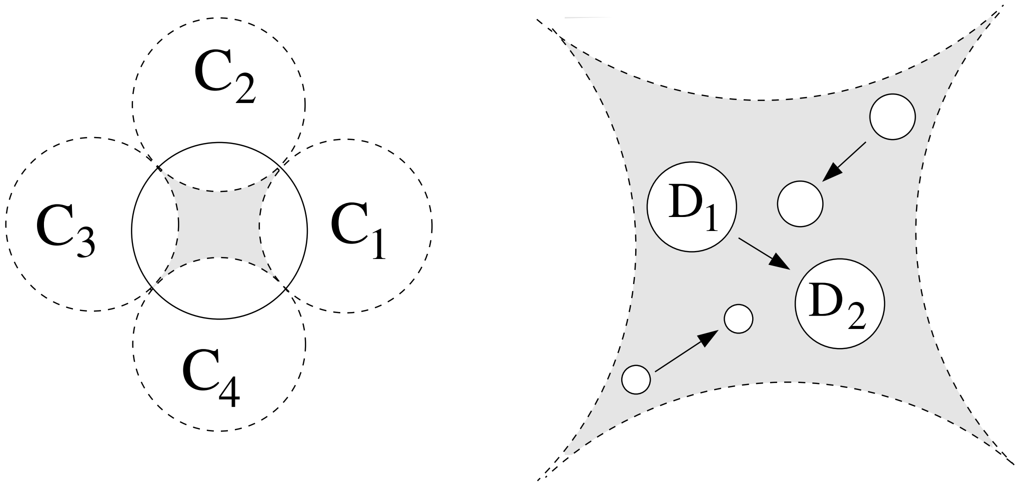

FIGURE 11.1. Constructing a limit set with positive area

cases.

11. LIMIT SETS WITH POSITIVE AREA

There exist analytically finite groups with area$(\Lambda) > 0$, but $\delta_{\mathrm{Poin}} < 2$. The construction is given in Maskit's book [23], so we will only sketch it here. Let C_1, C_2, C_3, C_4 be circles of radius $\sqrt{2}/2$ centered at the points $1, i, -1, -i$, and let G_0 be the Fuchsian group generated by mapping the exterior of C_1 to the interior of C_3 and the exterior of C_2 to the interior of C_4 and preserving the circle C_0 of radius $\sqrt{2}/2$ around the origin. Take a fundamental region for the group (say the exterior of all four circles) and let F be the component which lies inside C_0. Now choose a sequence of circles with disjoint interiors $\{D_j\}$ inside F and assume $\mathrm{diam}(K_{2j}) = \mathrm{diam}(K_{2j+1}) \simeq \mathrm{dist}(K_{2j}, K_{2j+1})$. Choose the circle so that $E = F \backslash \cup_j D_j$ has no interior. Let G be the group generated by G_0 and the transformations which map D_{2j} to the exterior of D_{2j+1}. In [23], Maskit proves that G is an analytically finite group where every component of $\Omega(G)$ is equivalent to the exterior of C_0.

However, G is a union of classical Schottky groups, so $\delta_{\text{Poin}} < 2$ by Doyle's theorem [14] and the fact that $\delta(\cup_n G_n) = \sup \delta(G_n)$ if $G = \cup_n G_n$ is a Kleinian group (see [7]). On the other hand, the accumulation set of the disks $\{D_j\}$ is clearly in the limit set and it is easy to choose the disks so that this set has positive area. Thus $\text{area}(\Lambda) > 0$ but $\delta_{\text{Poin}} < 2$.

In fact, this construction leads to sets of positive area, no matter how the disjoint disks are chosen. This is because the group is geometrically infinite, and so $\delta_{\text{Poin}} < 2$ implies $\text{area}(\Lambda) > 0$ by the results of [7].

12. The heat kernel and Poincaré sum

In this section we will use an estimate on the heat kernel to deduce a bound for the number of orbit points in a certain region. This is turn will give the estimate of the Poincaré series which was stated in Section 7.

The heat kernel is the fundamental solution for the heat equation on M, i.e.,

$$u(x,t) = \int_M u_0(y) K(x,y,t) dy,$$

is the solution of the heat equation $u_t = \Delta u$ with initial value u_0. It can also be interpreted in terms of Brownian motion: $K(x,y,t)dy$ is the distribution at time t of a Brownian motion started at the point x. For $x \in M = \mathbb{B}/G$, let $V(x) = \text{vol}(B_M(x,1))$ denote the hyperbolic volume of the unit ball centered at x. In [12] E.B. Davies proved:

Theorem 12.1. *If $\epsilon > 0$ then there is a constant $C = C_\epsilon$ such that*

$$K(x,y,t) \le C_\epsilon V(x)^{-1/2} V(y)^{-1/2} e^{(\epsilon-\lambda_0)t} e^{-\rho(x,y)^2(4+\epsilon)^{-1}t^{-1}},$$

for $1 \leq t < \infty$, where λ_0 is the base eigenvalue for the Laplacian on M and ρ is the hyperbolic metric on M.

A different proof is given by Grigor'yan in [18].

We recall that if $k(w,z,t)$ is the heat kernel on the hyperbolic ball $\mathbb{B}$, then ([13], page 178)

$$k(w,z,t) \quad = \quad (4\pi t)^{-3/2} \frac{\rho(w,z)}{\sinh(\rho(w,z))} e^{-t-\frac{\rho(w,z)^2}{4t}}.$$

Using the observation that

$$\frac{r}{\sinh(r)} \simeq (1+r)e^{-r},$$

for $r > 0$ gives,

$$k(w,z,t) \quad \simeq \quad t^{-3/2}(1+\rho(w,z)) \exp(-t-\rho(w,z) - \frac{\rho(w,z)^2}{4t}). \tag{12.1}$$

The exponent δ_{Poin} is also related to the base eigenvalue λ_0 for the manifold $M = \mathbb{B}^3/G$ by

$$\lambda_0 = \begin{cases} 1, & \delta_{\text{Poin}} \leq 1 \\ \delta_{\text{Poin}}(2-\delta_{\text{Poin}}), & 1 < \delta_{\text{Poin}} \leq 2 \end{cases}.$$

See [31]. This allows us to prove the following estimate (which was stated in [7]),

Lemma 12.2. *Let $\delta = \delta_{Poin} = 1 + \sqrt{1-\lambda_0}$. Given $x, y \in M = \mathbb{B}/G$, assume x corresponds to an orbit containing 0 via the covering map and y to an orbit $\{y_j\}$. Let $\rho = \rho(x,y)$ and*

$$N_k = \#(\{y_j : \rho + k \leq \rho(0, y_j) < \rho + k + 1\}).$$

Then for any $\epsilon > 0$,

$$\begin{aligned} N_k \quad &\leq \quad C_\epsilon \frac{\exp((1+\epsilon)\delta(\rho+k)[1 - \frac{(1-\lambda_0)^{1/2}}{\delta}(1 - \sqrt{\frac{2k\rho+k^2}{(\rho+k)^2}})])}{V(x)^{1/2}V(y)^{1/2}}, \\ &\leq \quad C_\epsilon \frac{e^{(1+\epsilon)\delta(\rho+k)}}{V(x)^{1/2}V(y)^{1/2}} \end{aligned}$$

where $\lambda_0 = \delta(2-\delta)$.

Proof. Fix $x, y \in M$ and let $w \in \mathbb{B}$ correspond to x via the covering map and let $\{z_j\} \subset \mathbb{B}$ be all the points corresponding to y. Then

$$K(x,y,t) = \sum_j k(w, z_j, t).$$

Using Theorem 12.1 and the formula for the heat kernel on the ball gives

$$N_k t^{-3/2}(1+\rho+k)e^{-t-\rho-k-\frac{(\rho+k)^2}{4t}} \quad \leq \quad \frac{C_\epsilon e^{(\epsilon-\lambda_0)t} e^{-\frac{1}{4+\epsilon}t^{-1}\rho^2}}{V(x)^{1/2}V(y)^{1/2}}$$

or

$$\begin{aligned} N_k &\leq \frac{C_\epsilon t^{3/2} e^{(\epsilon-\lambda_0)t} e^{t+\rho+k+\frac{(\rho+k)^2}{4t}} e^{-\frac{\rho^2}{(4+\epsilon)t}}}{(1+\rho+k)V(x)^{1/2}V(y)^{1/2}} \\ &\leq \frac{C_\epsilon t^{3/2} e^{(\epsilon-\lambda_0)t} e^{t+\rho+k+\frac{(\rho+k)^2}{4t}} e^{-\frac{\rho^2}{(4+\epsilon)t}}}{(1+\rho+k)V(x)^{1/2}V(y)^{1/2}} \\ &\leq \frac{C_\epsilon t^{3/2} e^{(1-\lambda_0)t} e^{\rho+k} e^{\frac{2\rho k+k^2}{4t}} e^{\epsilon t} e^{\frac{\epsilon\rho^2}{4(4+\epsilon)t}}}{(1+\rho+k)V(x)^{1/2}V(y)^{1/2}}. \end{aligned}$$

To find the optimal value of t to put into the exponential we minimize

$$at + b + ct^{-1} = \frac{1}{a}(t - \sqrt{\frac{c}{a}})^2 + b + 2\sqrt{ac},$$

where

$$a = 1-\lambda_0, \quad b = \rho + k, \quad c = \frac{2\rho k + k^2}{4t}.$$

This leads to the choice

$$t = \sqrt{\frac{c}{a}} = \frac{1}{2}\frac{\sqrt{2\rho k + k^2}}{\sqrt{1-\lambda_0}}.$$

Thus

$$N_k \leq \frac{C_\epsilon[\rho(\rho+k)]^{3/4} e^{\rho+k+\sqrt{2\rho k+k^2}\sqrt{1-\lambda_0}} e^{C\epsilon(\rho+k)}}{(1+\rho+k)V(x)^{1/2}V(y)^{1/2}}. \tag{12.2}$$

Since $1+\sqrt{1-\lambda_0}=\delta$, this gives (making C larger if necessary),

$$N_k \le C_\epsilon \frac{e^{(1+C\epsilon)\delta(\rho+k)[1-\frac{\sqrt{1-\lambda_0}}{\delta}(1-\sqrt{\frac{2\rho k+k^2}{(\rho+k)^2}})]}}{V(x)^{1/2}V(y)^{1/2}}.$$

Now use (12.2) and the facts $\delta=1+\sqrt{1-\lambda_0}$ and $2\rho k+k^2\le(\rho+k)^2$, to get

$$\begin{aligned} N_k &\le \frac{C_\epsilon e^{\rho+k+(\rho+k)\sqrt{1-\lambda_0}}e^{\epsilon(\rho+k)}}{V(x)^{1/2}V(y)^{1/2}} \\ &\le \frac{C_\epsilon e^{(1+\epsilon)\delta(\rho+k)}}{V(x)^{1/2}V(y)^{1/2}} \end{aligned} \tag{12.3}$$

□

We will now use this to prove Lemma 7.1.

Proof. This is clear from the estimate on the counting function. Let $R=\rho(y,G(0))$. More precisely, if we let $A_n=\{z\in B : e^{-n-1}r\le 1-|z|<e^{-n}r\}$, $s>\delta_{\mathrm{Poin}}$, and $\epsilon=(s-\delta_{\mathrm{Poin}})/2$, then

$$\begin{aligned} \sum_{g(0)\in B}(1-|g(0)|)^s &\le \sum_k \sum_{g(0)\in B\cap A_k}(1-|g(0)|)^s \\ &\le \sum_k N_k(re^{-R-k})^s \\ &\le C_s r^s e^{-Rs}[V(x)V(y)]^{-1/2}\sum_k e^{(s-\epsilon)k}e^{-sk} \\ &\le C_s r^s e^{-s\rho(y,G(0))}[V(x)V(y)]^{-1/2}. \end{aligned}$$

□

13. The geometric exponents for general sets

Theorem 13.1. *If $K\subset\mathbb{R}^2$ is a compact set with infinitely many complementary components then*

$$\delta_{inrad}\le\delta_{area}\le\delta_{diam},$$

$$\max(1,2-\frac{\delta_{inrad}}{\delta_{area}}(2-\delta_{area}))\le\delta_{Whit}\le 2.$$

Moreover, any four values which satisfy these constraints occur for some compact K.

In particular, the second constraint implies $\delta_{\mathrm{inrad}} \leq \delta_{\mathrm{area}} \leq \delta_{\mathrm{Whit}} \leq 2$, but δ_{diam} may be larger than 2 (even infinite).

Proof. The inequalities $\delta_{\mathrm{inrad}} \leq \delta_{\mathrm{area}} \leq \delta_{\mathrm{diam}}$ are obvious, so only the second constraint must be proven.

First, suppose K has a bounded complementary component Ω_0. Let $\{Q_j\}$ be a Whitney decomposition of Ω_0, and let Q_0 be a Whitney square of maximum size. For each $n = 1, 2, \dots$, note that the union of Whitney squares with

$$2^{n-10}\mathrm{diam}(Q_0) \leq \mathrm{diam}(Q_j) \leq 2^{-n}\mathrm{diam}(Q_0),$$

topologically separate Q_0 from K. Thus the sum of their diameters must be larger that $\mathrm{diam}(Q_0)$, i.e., the number of such squares is $\mathrm{diam}(Q_0) \geq 2^n$. Thus $\delta_{\mathrm{Whit}}(K) \geq 1$.

If $\{Q_k\}$ is a Whitney decomposition of a bounded component Ω_j then

$$\sum_k \mathrm{diam}(Q_k)^2 = 2\,\mathrm{area}(\Omega_0).$$

Thus $\delta_{\mathrm{Whit}} \leq 2$. Furthermore, for $t \leq 1$,

$$\begin{aligned} 2 &\leq \sum_k \left(\frac{\mathrm{diam}(Q_k)^2}{\mathrm{area}(\Omega_j)}\right) \\ &\leq \sum_k \left(\frac{\mathrm{diam}(Q_k)^2}{\mathrm{inrad}(\Omega_j)^2}\right)\left(\frac{\mathrm{inrad}(Q_k)^2}{\mathrm{area}(\Omega_j)}\right) \\ &\leq \left(\frac{\mathrm{inrad}(Q_k)^2}{\mathrm{area}(\Omega_j)}\right)\sum_k \left(\frac{\mathrm{diam}(Q_k)^2}{\mathrm{inrad}(\Omega_j)^2}\right)^t. \end{aligned}$$

because $\mathrm{inrad}(\Omega_j) \geq \mathrm{diam}(Q_k)$ for every Q_k. Thus for any $s \leq 2$,

$$\sum_k \mathrm{diam}(Q_k)^s \geq 2\mathrm{area}(\Omega_j)\mathrm{inrad}(\Omega_j)^{s-2}.$$

Now let $0 < t < 2$ and use Hölder's inequality with the conjugate indices

$$p = \frac{2}{t}, \quad q = \frac{2}{2-t},$$

to get

$$\text{(13.1)} \quad \begin{aligned} \sum_j \mathrm{area}(\Omega_j)^{t/2} &\leq 2(\sum_j \mathrm{area}(\Omega_j)\mathrm{inrad}(\Omega_j)^{s-2})^{\frac{t}{2}}(\sum_j \mathrm{inrad}(\Omega_j)^{t\frac{2-s}{2-t}})^{\frac{2-t}{2}} \\ &\leq 2(\sum_j \sum_{Q_k \subset \Omega_j} \mathrm{diam}(Q_k)^s)^{\frac{t}{2}}(\sum_j \mathrm{inrad}(\Omega_j)^{t\frac{2-s}{2-t}})^{\frac{2-t}{2}}. \end{aligned}$$

The two terms on the right hand side are finite if

$$s > \delta_{\mathrm{Whit}}(K), \quad t\frac{2-s}{2-t} > \delta_{\mathrm{inrad}}(K),$$

respectively. Rewriting the second inequality as

$$t > \frac{2\delta_{\mathrm{inrad}}(K)}{2 - s + \delta_{\mathrm{inrad}}(K)},$$

and letting $s = \delta_{\mathrm{Whit}}(K)$, we see that the left hand side of (13.1) is finite if

$$t > \frac{2\delta_{\mathrm{inrad}}(K)}{2 - \delta_{\mathrm{Whit}}(K) + \delta_{\mathrm{inrad}}(K)}.$$

Since the left hand side of (13.1) must be infinite if $t < \delta_{\mathrm{area}}(K)$ we deduce

$$\delta_{\mathrm{area}} \leq \frac{\delta_{\mathrm{inrad}}}{2 - \delta_{\mathrm{Whit}} + \delta_{\mathrm{inrad}}}.$$

Solving in terms of the other exponents gives

$$\delta_{\mathrm{Whit}} \geq 2 - \frac{\delta_{\mathrm{inrad}}}{\delta_{\mathrm{area}}}(2 - \delta_{\mathrm{area}}),$$

$$\delta_{\mathrm{inrad}} \geq \delta_{\mathrm{area}}(\frac{2 - \delta_{\mathrm{Whit}}}{2 - \delta_{\mathrm{area}}}),$$

as desired.

Next we give an example of a compact set K which shows that these are the only restrictions which hold for general sets. Consider the set K_1 in Figure 13.1.

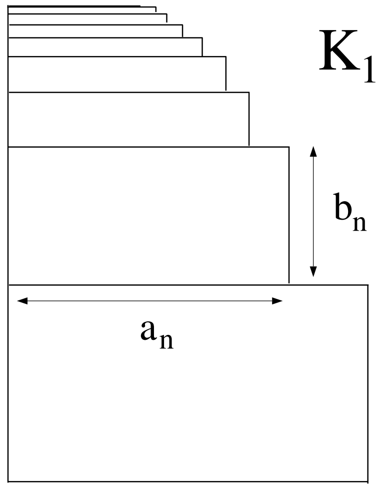

FIGURE 13.1.

The bounded complementary components are rectangles of dimension $a_n \times b_n$ where $a_n \leq b_n$, $\sum_n a_n < \infty$ and $b_n \to 0$. The various exponents become

$$\begin{aligned} \delta_{\text{inrad}} &= \inf\{s : \sum b_n^s < \infty\}, \\ \delta_{\text{area}} &= \inf\{s : \sum (a_n b_n)^{s/2} < \infty\}, \\ \delta_{\text{diam}} &= \inf\{s : \sum a_n^s < \infty\}. \end{aligned}$$

If we set $a_n = 2^{-n}$ and and $b_n = n^{-\beta}$, $\beta > 0$, then it easy to check

$$\delta_{\text{inrad}} = \delta_{\text{area}} = 0, \quad \delta_{\text{diam}} = \frac{1}{\beta}.$$

To compute the exponent δ_{Whit}, think of dividing each rectangle into b_n/a_n squares of size a_n. For $s > 1$, its easy to see that the Whitney sum corresponding to each subsquare is approximately a_n^s. Thus the sum over all Whitney squares in all the rectangles is approximately

$$\sum_n a_n^{s-1} b_n = \sum_n 2^{-n(s-1)} n^{-\beta}.$$

This converges for all $s > 1$. Finally, the sum over the Whitney squares in the

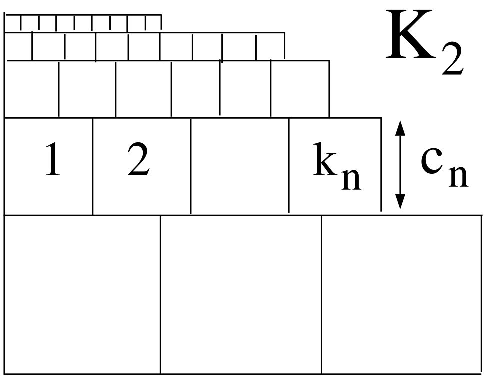

FIGURE 13.2.

unbounded component sum for any $s > 1$ since the boundary of this component is just a rectifiable curve. Thus $\delta_{\text{Whit}}(K_1) = 1$.

If we consider the set K_2 in Figure 13.2, the complementary components are squares, with k_n squares of size c_n. If $c_n = \lfloor n^{-\gamma} \log^{-2}(n) \rfloor$ with $\gamma > 1$ and $k_n = \lfloor n^\alpha \rfloor$ with $\alpha \leq \gamma$ then

$$\delta_{\text{inrad}} = \delta_{\text{area}} = \delta_{\text{diam}} = \frac{1+\alpha}{\gamma},$$

and all values ≤ 2 can be obtained.

Next modify K_2 by removing a finite number of points from each bounded component as in Figure 13.3 so that the largest remaining disk has diameter $d_n \leq c_n$. This does not alter the values of δ_{area} or δ_{diam}, but if we choose $d_n = n^{-\delta} c_n$, then

$$\delta_{\text{inrad}}(K_3) = \frac{1+\alpha}{\gamma + \delta},$$

can take any value between 0 and δ_{area}.

To compute δ_{Whit} divide each $c_n \times c_n$ square into $(c_n/d_n)^2$ equal sized subsquares. Note that the Whitney sum corresponding to each to these subsquares has size approximately d_n^s, if $s > 1$. Therefore the sum over all Whitney squares in the bounded

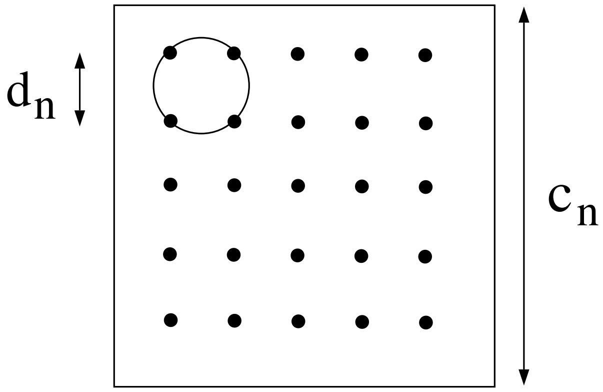

FIGURE 13.3.

components of K is approximately

$$\sum_n k_n d_n^{s-2} c_n^2 \simeq \sum_n n^\alpha n^{-(\delta+\gamma)(s-2)} n^{-2\gamma},$$

which converges if and only if

$$s > 2 - \frac{1-2\gamma+\alpha}{\delta+\gamma} = 2 - \frac{\delta_{\text{inrad}}}{\delta_{\text{area}}}(2-\delta_{\text{area}}).$$

Thus

$$\delta_{\text{Whit}}(K_3) = 2 - \delta_{\text{inrad}}(K_3)(2-\delta_{\text{area}}(K_3))/\delta_{\text{area}}(K_3).$$

Finally for any value $\delta_{\text{Whit}}(K_3) \leq \eta \leq 2$ we let K_4 be a Jordan arc with upper Minkowski dimension η (it is easy to build such sets by the usual self-similar constructions if $\eta < 2$ and only slightly more complicated if $\eta = 2$). By Lemma 2.1 we have $\delta_{\text{Whit}} = \eta$, and since K_4 has no bounded complementary components the other exponents are all 0.

It is easy to see that if we take copies of K_1, K_3, K_4 in disjoint disks and set $K = K_1 \cup K_3 \cup K_4$ then

$$\delta(K) = \max(\delta(K_1), \delta(K_3), \delta(K_4)),$$

where δ represents any on the critical exponents under consideration. Thus by appropriate choices of α, β, γ, δ and η we can make

$$\delta_{\text{diam}} = \frac{1}{\beta}, \qquad \delta_{\text{area}} = \frac{1+\alpha}{\gamma},$$

$$\delta_{\text{inrad}} = \frac{1+\alpha}{\delta+\gamma}, \qquad \delta_{\text{Whit}} = \eta,$$

take on any values consistent with the inequalities in Theorem 13.1. □

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