SOME CHARACTERIZATIONS OF $C(\mathcal{M})$

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Abstract. We show that a function f on the unit disk extends continuously to $\mathcal{M}$, the maximal ideal space of $H^\infty(\mathbb{D})$ iff it is uniformly continuous (in the hyperbolic metric) and close to constant on the complementary components of some Carleson contour.

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1. Introduction

Let $H^\infty(\mathbb{D})$ denote the algebra of bounded holomorphic functions on the unit disk, $\mathbb{D}$. Let $\mathcal{M}$ denote the maximal ideal space of $H^\infty(\mathbb{D})$ (see [3] for definitions). Hoffman proved that every bounded harmonic function on the disk extends continuously to $\mathcal{M}$ and that such functions generate $C(\mathcal{M})$ as a closed algebra (closed with respect to the sup norm). The purpose of this note is to give a more geometric characterization of the functions on $\mathbb{D}$ which extend continuously to $\mathcal{M}$. Recall that a positive measure μ on the disk is a Carleson measure if there is a $C > 0$ such that $\mu(D(a,r)) \leq Cr$ for every $a \in \mathbb{T} = \partial\mathbb{D}$ and $r > 0$. A union of curves $\Gamma \subset \mathbb{D}$ is called a Carleson contour if arclength measure on Γ is Carleson.

Theorem 1.1. *For a bounded, continuous function g on $\mathbb{D}$ the following are equivalent.*

1. *g extends continuously to $\mathcal{M}$.*
2. *For every $\epsilon > 0$ there is a smooth function φ on $\mathbb{D}$ so that $\|g - \varphi\|_\infty \leq \epsilon$, $\sup_z |\nabla\varphi(z)|(1-|z|^2) < \infty$ and $|\nabla\varphi| dxdy$ is a Carleson measure.*
3. *g is uniformly continuous with respect to the hyperbolic metric on $\mathbb{D}$ and for every $\epsilon > 0$ there is a Carleson contour Γ so that g is within ϵ of a constant on each connected component of $\mathbb{D}\backslash\Gamma$.*

Recall that a Douglas algebra is any closed algebra $H^\infty(\mathbb{T}) \subset A \subset L^\infty(\mathbb{T})$. Our first application of Theorem 1.1 is

Corollary 1.2. *Suppose f is bounded and continuous on the disk and that $H^\infty(\mathbb{T}) \subseteq A \subseteq L^\infty(\mathbb{T})$ is a Douglas algebra. Let $\mathcal{M}_A \subset \mathcal{M}$ denote the maximal ideal space of A. Then the following are equivalent.*

(1) *f extends continuously to $\mathcal{M}_A$.*

(2) *For every $\epsilon > 0$ there is an inner function I, such that $\bar{I} \in A$, and $u \in C(\mathcal{M})$ on $\mathbb{D}$ so that and $|(f(z) - u(z))I(z)| < \epsilon$ on $\mathbb{D}$.*

(3) *For every $\epsilon > 0$ there is an inner function I, $\bar{I} \in A$ and a Carleson contour Γ so that f differs from a constant by at most ϵ on every component $\mathbb{D} \backslash \Gamma$ which hits $\{|I| > 1/2\}$.*

At one extreme $A = H^\infty$ and the only allowable I's are constants. In this case, Corollary 1.2 is just Theorem 1.1. The other extreme occurs for $A = L^\infty$. Axler and Shields [1] showed that any function extending to the maximal ideal space X of L^∞ (X is also the Shilov boundary of H^∞) has nontangential limits almost everywhere and asked if the converse is true. This was proven independently by O.V. Ivanov [5].

Corollary 1.3. *A bounded, continuous function f on $\mathbb{D}$ has a continuous extension to X (the Shilov boundary of H^∞) iff f has nontangential limits a.e. on $\mathbb{T}$.*

I am grateful to Allen Shields for mentioning the problem on nontangential limits to me. This problem led me to think about the others. I also thank Sheldon Axler for bringing Ivanov's result to my attention and the referee for many helpful comments.

2. Proof of Theorem 1.1

We shall show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). The first implication is known, the second is a minor variation of a known technique, but the third seems to be new and depends on a recent result from [2].

Proof of (1) $\Rightarrow$ (2): Let $\mathcal{A}$ denote the set of smooth, bounded functions on the disk described by condition (2). $\mathcal{A}$ is an algebra and by [3], Theorem VIII.6.1 every bounded harmonic function can be uniformly approximated by functions in $\mathcal{A}$. By

Hoffman's result ([4], Lemma 4.4) bounded harmonic functions generate $C(\mathcal{M})$, so the sup norm closure of $\mathcal{A}$ contains $C(\mathcal{M})$.

Proof of (2) $\Rightarrow$ (3): This is a standard "stopping time argument", but since I do not know a reference for this precise application, I will sketch the proof.

Suppose g satisfies condition (2) of Theorem 1.1. First we show g is uniformly continuous with respect to the hyperbolic metric, ρ. Given $\epsilon > 0$ suppose $\|g - \varphi\|_\infty < \epsilon/4$ and φ satisfies the conditions in (2). The estimate $|\nabla\varphi(z)| \leq C(1-|z|)$ implies φ is uniformly continuous (in fact, Lipschitz) with respect to the ρ, so there is a $\delta = \delta(\epsilon)$ so that $\rho(z,w) \leq \delta$ implies $|\varphi(z) - \varphi(w)| \leq \epsilon/2$. Thus

$$|g(z) - g(w)| \leq |g(z) - \varphi(z)| + |\varphi(z) - \varphi(w)| + |\varphi(w) - g(w)| \leq \epsilon,$$

so g is uniformly continuous with respect to ρ.

Next we will construct a Carleson contour Γ so that φ is within ϵ of a constant on each component of $\mathbb{D} \setminus \Gamma$ (and therefore g is within 2ϵ of a constant on each component). Given $\epsilon > 0$, choose $\delta > 0$ so that φ varies by less than $\epsilon/8$ over any hyperbolic ball of diameter 10δ. It will also be convenient to take $\delta = 2^{-N}$, a power of 2. To simplify the construction we move to the upper half plane. Clearly it is enough to consider what happens in one square, say $Q_0 = [0,1] \times [0,1]$. Now divide Q_0 into rectangles of the form

$$Q = \{z = x + iy : j2^{-k} < x < (j+1)2^{-k}, 2^{-k-1} < y < 2^{-k}\}.$$

Subdivide each such Q into $2\delta^{-2}$ equal subsquares of side length $\delta 2^{-k-1}$. Each of these will then have hyperbolic diameter less than 10δ, so φ varies by at most $\epsilon/8$ on each such square.

For each square S let z_S denote the center of the square. Each square S in the

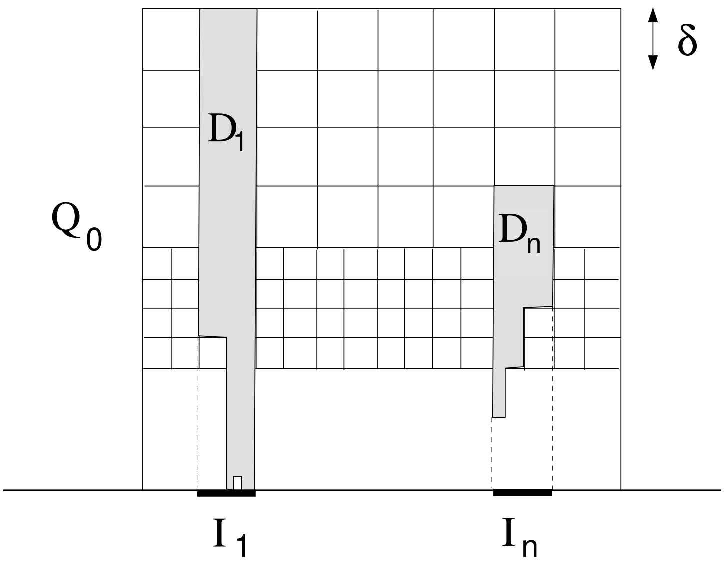

FIGURE 2.1. Stopping time regions in the proof of (2) $\Rightarrow$ (3)

partition has either 1 or 2 squares adjacent and underneath it; let $G_1(S)$ denote these squares. In general, let

$$G_n(S) = \bigcup \{S'' \in G_1(S') : S' \in G_{n-1}(Q) \text{ and } |\varphi(z_S) - \varphi(z_{S''})| \le \epsilon/2\}.$$

Let $D_S = S \cup \cup_{n=1}^{\infty} \cup_{S' \in G_n(S)} S'$.

Choose a square S on the "top row" of Q_0 and let $D_1 = D_S$. In general, let S be a square of maximal size not contained in $\cup_{j=1}^n D_j$ and let $D_{n+1} = D(S)$. This gives a countable number of disjoint (except for boundaries) regions which exhaust Q_0. For each j let I_j be the vertical projection of D_j onto $\mathbb{R}$, $F_j = \partial D_j \cap \mathbb{R} \subset I_j$ and let z_j denote the center of the top edge of D_j. Let $\Gamma = \cup_j \partial D_j$.

Note that for each such region $D = D_j$ with "base" $I = I_j$, $\ell(\partial D) \le 4(1 + \delta^{-1})|I|$, where ℓ denotes arclength. To see this, note that we can write the region D as $D = R \setminus \cup_j R_j$ where R is a $|I| \times 2\delta^{-1}|I|$ vertical rectangle and the R_j are similar rectangles with disjoint bases in I. Thus

$$\ell(\partial D) \leq \ell(\partial R_0) + \sum_j \ell(\partial R_j) \leq 4(1+\delta^{-1})|I|, \tag{2.1}$$

as desired.

By construction, $|\varphi - \varphi(z_j)| \leq \epsilon$ on D_j, so we need only check that arclength on Γ is a Carleson measure. We start by showing

$$\text{length}(\partial D_j) \leq C_1|F_j| + C_2 \iint_{D_j} |\nabla\varphi| dxdy, \tag{2.2}$$

This is proved by considering two cases. First suppose $|F_j| \geq |I_j|/2$. Then by (2.1) equation (2.2) holds with $C_1 = 8(1+\delta^{-1})$. On the other hand, if $|F_j| < |I_j|/2$, then we can find a set E of length $\geq \ell(I_j)/2$ along the top edge of D_j so that if we follow any vertical line down from E it eventually enters a different D_k. Consider the segment formed by intersecting the line with D_j. By the definition of D_j (and the fact that φ varies by less than $\epsilon/8$ over each square S) the values of φ at the endpoints of this segment differ by at least $\epsilon/4$. Thus the integral of $|\nabla\varphi|$ along this segment is at least $\epsilon/4$. Hence,

$$\iint_{D_j} |\nabla\varphi| dxdy \geq \frac{\epsilon}{4}|E| \geq \frac{\epsilon}{8}|I_j|.$$

Thus (2.2) holds with $C_2 = 8/\epsilon$.

Finally we want to deduce arclength on Γ is a Carleson measure. Consider a dyadic Carleson square

$$Q = \{z = x+iy : j2^{-k} < x < (j+1)2^{-k}, 0 < y < 2^{-k}\},$$

with base $I = [j2^{-k}, (j+1)2^{-k}]$. It is sufficient to show that $\ell(\Gamma \cap Q) \leq C|I|$, with some absolute C independent of Q.

First note that there are at most $2\delta^{-1}$ regions D which hit Q but are not contained in Q (because any such must contain a square S from the "top row" of Q and there are only $2\delta^{-1}$ such squares). For each such region, the proof of (2.1) shows $\ell(\partial D \cap Q) \leq 4(1+\delta^{-1})|I|$. On the other hand, if $D_j \subset Q$ then $F_j \subset I$, so

$$\sum_{j:D_j \subset Q} |F_j| \leq |I|.$$

Furthermore,

$$\sum_{j:D_j \subset Q} \iint_{D_j} |\nabla \varphi(z)| dxdy \leq \iint_Q |\nabla \varphi(z)| dxdy \leq C|I|,$$

by assumption. Thus

$$\begin{aligned} \ell(\Gamma \cap Q) &\leq \sum_{j:D_j \cap Q \neq \emptyset, D_j \not\subset Q} \ell(\partial D_j \cap Q) + \sum_{j:D_j \subset Q} \ell(\partial D_j) \\ &\leq 8\delta^{-1}(1+\delta^{-1})|I| + \sum_{j:D_j \subset Q} (C_1|F_j| + C_2 \iint_{D_j} |\nabla \varphi| dxdy) \\ &\leq C(\delta, \epsilon)|I|, \end{aligned}$$

as desired.

Proof of (3) $\Rightarrow$ (1): We start with some definitions. For f on the disk, $a \in \mathbb{C}$ and $\delta > 0$, let $\Omega_f(a,\delta) = f^{-1}(D(a,\delta)) = \{z : |f(z) - a| < \delta\}$. For convenience, given an open set $\Omega \subset \mathbb{D}$ we will write

$$\operatorname{dist}(g, H^\infty(\Omega)) \quad = \quad \inf_{G \in H^\infty(\Omega)} \|g|_\Omega - G\|_{L^\infty(\Omega, dxdy)}.$$

If Ω is empty, we interpret the distance as 0. Let $H^\infty(\mathbb{D})[f]$ denote the smallest algebra containing $H^\infty(\mathbb{D}) \cup f$ which is closed with respect to the sup norm on $\mathbb{D}$. If f is bounded and harmonic, then it is clear that $H^\infty(\mathbb{D})[f] \subset C(\mathcal{M})$.

The following is Theorem 1.1 of [2].

Proposition 2.1. *Suppose* $\bar{f} \in H^\infty(\mathbb{D})$ *and* $g \in L^\infty(\mathbb{D})$. *Then*

$$dist(g, H^\infty(\mathbb{D})[f]) = \inf_{\delta>0} \sup_{a\in\mathbb{C}} dist(g, H^\infty(\Omega_f(a,\delta))).$$

(Distance is in the L^∞ *norm on* $\mathbb{D}$.*)*

We will now use this to prove (3) $\Rightarrow$ (1). Suppose g satisfies condition (3) in Theorem 1.1. Fix $\epsilon > 0$ and let Γ be the Carleson contour given by part (3) of the theorem. Choose a sequence $\{z_n\} \subset \Gamma$ so that $\rho(z_n, z_m) \geq 1/4$ for all $n \neq m$ and so that every $z \in \Gamma$ is within hyperbolic distance $1/2$ of one of the z_n's. Then $\{z_n\}$ is an interpolating sequence because it is hyperbolically separated and satisfies

$$\begin{aligned} \sum_{z_n \in D(x,r)} (1-|z_n|) &\leq C \sum_{z_n\in D(x,r)} \ell(\Gamma \cap D(z_n, \frac{1}{100}(1-|z_n|))) \\ &\leq C\ell(\Gamma\cap D(x,2r)) \\ &\leq Cr. \end{aligned}$$

for every disk $D(x,r)$ centered on $\partial\mathbb{D}$. (See [3], Chapter VII). Let

$$B(z) = \prod_{n=1}^\infty \frac{z_n - z}{1-\bar{z}_n z}\frac{|z_n|}{z_n},$$

be the Blaschke product with zeros $\{z_n\}$. Note that $\Gamma \subset \{z : |B(z)| < 1/2\}$. By Exercise VII.1 of [3], we also have $|B'(z_n)|(1-|z_n|^2) > C_0$ for all n and some $C_0 > 0$ (depending only on the Carleson norm of arclength on Γ).

We claim that for any δ there is a $\nu(\delta) > 0$ such that any component of $\Omega_B(a,\delta)$ which hits Γ has hyperbolic diameter at most $\nu(\delta)$. Suppose not. Then there is a $\nu_0 > 0$ so that for any $\delta > 0$ we can choose $a \in \mathbb{C}$ and a component Ω of $\Omega_B(a,\delta)$ such that

(1) $\Omega \cap \Gamma \neq \emptyset$.

(2) The hyperbolic diameter of Ω is $\geq \nu_0$.

Choose a sequence $\delta_j \to 0$ and let $\{\Omega_j\}$ be the corresponding components. Each such Ω_j is within hyperbolic distance $1/2$ of some point z_j in our sequence. Renormalize by Möbius transformations so that $z_j = 0$ for all j. We now have a sequence of uniformly bounded functions $\{B_j\}$ on $\mathbb{D}$ such that $|B_j'(0)| > C_0$ and B_j is within δ_j of a constant on a connected set of hyperbolic diameter $\geq \nu_0$ which intersects the disk $D(0, 1/2)$. By normal families there is subsequence which converges to a constant function uniformly on compacta, but this contradicts the condition $|B_j'(0)| > C_0$. This proves the claim.

We can now apply the proposition. Since g is uniformly continuous with respect to the hyperbolic metric, there is a $\eta = \eta(g, \epsilon)$ so that g is within ϵ of a constant on any set of hyperbolic diameter $\leq \eta$. Choose δ so small that $\nu(\delta) \leq \eta(g, \epsilon)$. Suppose $a \in \mathbb{C}$ and Ω is a connected component of $\Omega_B(a, \delta)$. If Ω has hyperbolic diameter $\leq \eta$, then g is within ϵ of a constant on Ω by our choice of η. If Ω has hyperbolic diameter $\geq \eta$, then Ω cannot meet Γ, so it lies completely within a component of $\mathbb{D} \setminus \Gamma$. Thus g differs by less than ϵ from a constant by hypothesis. Since a piecewise constant function is holomorphic on $\Omega_f(a, \delta)$ we have

$$\text{dist}(g, H^\infty(\Omega_f(a, \delta))) \leq \epsilon,$$

for all a. Hence by the proposition,

$$\text{dist}(g, C(\mathcal{M})) \leq \text{dist}(g, H^\infty(\mathbb{D})[f]) \leq \epsilon.$$

Since g can be uniformly approximated by elements of $C(\mathcal{M})$, we have $g \in C(\mathcal{M})$. This completes the proof of Theorem 1.1. $\square$

Remark: The proof of the final implication shows that any function $g \in C(\mathcal{M})$ is contained in an closed subalgebra generated by $H^\infty(\mathbb{D})$ and a countable collection of anti-holomorphic functions. However, I don't know if g must be in a subalgebra generated by $H^\infty(\mathbb{D})$ and a single anti-holomorphic function (even if g itself is harmonic).

3. Proof of the corollaries

Proof of Corollary 1.2: Given Theorem 1.1, Corollary 1.2 is very easy. To prove (1) $\Rightarrow$ (2) apply the Tietze extension theorem (which we can do since $\mathcal{M}$ is a compact Hausdorff space, hence normal) to f restricted to $\mathcal{M}_A$ to get $u \in C(\mathcal{M})$ with $u = f$ on $\mathcal{M}_A$. By [3], Theorem IX.1.3 and Marshall's theorem ([3], Theorem IX.3.2)

$$\mathcal{M}_A = \cap_{I:\bar{I} \in A} \{m \in \mathcal{M} : |I(m)| = 1\},$$

which implies sets of the form $\{|I(m)| > 1 - \epsilon\}$ with $\epsilon > 0$, $\bar{I} \in A$ form a base of neighborhoods of $\mathcal{M}_A$. Since $f - u$ vanishes on $\mathcal{M}_A$ we can find an ϵ and I so that $|f - u| \leq \epsilon$ on $\{|I| > 1 - \epsilon\}$. Replacing I by a high power of itself gives (2). Conversely, (2) $\Rightarrow$ (1) because by taking $\epsilon \to 0$ we can obtain a sequence of continuous functions on $\mathcal{M}$ (and hence continuous on $\mathcal{M}_A$) which converge uniformly to f on $\mathcal{M}_A$.

Next we prove (2) $\Rightarrow$ (3). By (2) (applied with ϵ^2) we can find u and I so that $|f - u| < \epsilon$ on $\{|I| > \epsilon\}$. Take the contour corresponding to $g = u$ and ϵ in part (3) of Theorem 1.1. We also wish to insure that any component that intersects $\{|I| > 1/2\}$ is contained in $\{|I| > \epsilon\}$ as long as ϵ is less than some absolute constant ϵ_0. Carleson's construction (e.g., proof of [3], Theorem VIII.4.1) says there is an $\epsilon_0 > 0$ and a Carleson contour Γ which separates $\{|I| = 1/2\}$ from $\{|I| = \epsilon_0\}$. By adding this contour to the one for u we get the desired property. Thus we have $|f - u| \leq \epsilon$ on any component which hits $\{|I| > 1/2\}$, as required. Conversely, (3) $\Rightarrow$ (2) holds because given the piecewise constant approximation we extend it to the whole disk by setting it equal to zero outside the contours which hit $\{|I| > 1/2\}$ and smoothing it (see e.g., [3], page 357). The resulting function u satisfies (2) if we replace I by a high power of itself. This proves Corollary 1.2. $\square$

Proof of Corollary 1.3: If f extends continuously to $X = \mathcal{M}_{L^\infty(\mathbb{T})}$ then by part (2) of Corollary 1.2 for every $\epsilon > 0$ we can find some $u \in C(\mathcal{M})$ and some inner function I so that $|f - u| < 2\epsilon$ on $\{|I| > 1/2\}$. Every function in $C(\mathcal{M})$ has nontangential limits almost everywhere (they can all be uniformly approximated by products and sums of bounded harmonic functions) and for any fixed $\theta < \pi$ the set $\{|I| > 1/2\}$ contains cones with vertex angle θ at almost every point of the circle. Thus the nontangential oscillation of f in cones of angle θ is less than ϵ almost everywhere. Taking $\epsilon \to 0$ and $\theta \to \pi$ shows f has nontangential limits almost everywhere.

Conversely, suppose f has nontangential limits almost everywhere. By subtracting the harmonic function v with the same limits we may assume f has limit 0 almost everywhere. Since v extends continuously to $\mathcal{M}$ (hence to X), this does not alter the problem. Suppose $\epsilon > 0$ and fix a cone angle, say $\theta = \pi/2$ and for each $n = 1, 2, \ldots$ let K_n be the set of points x on the circle such that $|f| < \epsilon$ in the cone $W(x, \frac{\pi}{2}, n)$ with vertex x, angle $\frac{\pi}{2}$ and diameter 2^{-n}. Note that $|K_n| \to 2\pi$ as $n \to \infty$. Choose compact sets $E_n \subset K_n$ so that $|E_n| \geq |K_n| - \frac{1}{n}$.

Next, choose n_0 so that $|E_{n_0}| > 1/2$ and let $W_0 = \cup_{x \in E_{n_0}} W(x, \frac{\pi}{2}, n_0)$. For each complementary interval J of E_{n_0} we can choose a $n = n_J$ such that $|F_J| = |E_n \cap J| > \frac{1}{2}|J|$ and $2^{-n} \leq |J|$. Define $W_J = \cup_{x \in F_J} W(x, \frac{\pi}{2}, n_J)$. Then W is a sawtooth domain with boundary length at most $6|J|$.

Continuing in this way we obtain by induction a collection of intervals $\{J_k\}$, sets

$F_k \subset J_k$ and sawtooth regions W_k such that

(1) $|F_k| \geq \frac{1}{2}|J_k|$.

(2) The $\{F_k\}$ are pairwise disjoint.

(3) $\partial W_k \cap \partial \mathbb{D} = F_k$.

(4) $|\cup_k F_k| = |\partial \mathbb{D}|$.

(5) $\ell(\partial W_k) \leq C|J_k| \leq 2C|F_k|$. In particular, $\sum_k \ell(\partial W_k) < \infty$.

Define a sequence $\{z_n\}$ in the disk by placing points along each arc in each ∂W_k at hyperbolic distance ϵ from each other. The sequence is Blaschke, i.e., satisfies $\sum_n (1 - |z_n|) < \infty$ because the sum along any arc of ∂W_k is easily dominated by a multiple of the length of that arc, and we observed above that $\sum \ell(\partial W_k) < \infty$. Now let $I(z)$ be the Blaschke product corresponding to the sequence $\{z_n\}$. Then $|I| \leq \epsilon$ on each ∂W_k's (because we are never more than distance ϵ from a zero) and is therefore less than ϵ outside the union of the W_k (because any component of $\mathbb{D} \setminus \cup_k \overline{W_k}$ hits the unit circle in zero measure). Therefore f differs from zero by less than ϵ on every component of $\{|I| > \epsilon\}$. Taking u to be the zero function and applying part (2) of Corollary 1.2 completes the proof (since $A = L^\infty$ we automatically get $\bar{I} \in A$). $\square$

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