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Weil-Petersson curves, β-numbers, and minimal surfaces. (English. English summary)

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[References]

1. R. A. Adams and J. J. F. Fournier (eds.), *Sobolev Spaces*, second ed., *Pure and Applied Mathematics* (*Amsterdam*) **140**, Elsevier/Academic Press, Amsterdam, 2003. MR 2424078. Zbl 1098.46001. MR2424078
2. L. Ahlfors and G. Weill, A uniqueness theorem for Beltrami equations, *Proc. Amer. Math. Soc.* **13** (1962), 975–978. MR 0148896. Zbl 0106.28504. https://doi.org/10.2307/2034099. MR0148896
3. L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, second ed., *University Lecture Series* **38**, American Mathematical Society, Providence, RI, 2006, With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J.H. Hubbard. MR 2241787. Zbl 1103.30001. https://doi.org/10.1090/ulect/038. MR2241787
4. L. V. Ahlfors, *Conformal Invariants*, AMS Chelsea Publishing, Providence, RI, 2010, Topics in geometric function theory, Reprint of the 1973 original, With a foreword by Peter Duren, F. W. Gehring and Brad Osgood. MR 2730573. Zbl 1211.30002. https://doi.org/10.1090/chel/371. MR2730573
5. S. Alexakis and R. Mazzeo, Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds, *Comm. Math. Phys.* **297** no. 3 (2010), 621–651. MR 2653898. Zbl 1193.53131. https://doi.org/10.1007/s00220-010-1054-3. MR2653898
6. M. T. Anderson, Complete minimal varieties in hyperbolic space, *Invent. Math.* **69** no. 3 (1982), 477–494. MR 0679768. Zbl 0515.53042. https://doi.org/10.1007/BF01389365. MR0679768
7. M. T. Anderson, Complete minimal hypersurfaces in hyperbolic n-manifolds, *Comment. Math. Helv.* **58** no. 2 (1983), 264–290. MR 0705537. Zbl 0549.53058. https://doi.org/10.1007/BF02564636. MR0705537
8. K. Astala and M. Zinsmeister, Teichmüller spaces and BMOA, *Math. Ann.* **289** no. 4 (1991), 613–625. MR 1103039. Zbl 0896.30028. https://doi.org/10.1007/BF01446592. MR1103039
9. M. Badger and S. McCurdy, Subsets of rectifiable curves in Banach spaces I: Sharp exponents in traveling salesman theorems, *Illinois J. Math.* **67** no. 2 (2023), 203–274. MR 4593892. Zbl 07724273. https://doi.org/10.1215/00192082-10592363. MR4593892
10. M. Badger and S. McCurdy, Subsets of rectifiable curves in Banach spaces II: Universal estimates for almost flat arcs, *Illinois J. Math.* **67** no. 2 (2023), 275–331. MR 4593893. Zbl 07724274. https://doi.org/10.1215/00192082-10592390. MR4593893
11. M. Bauer, P. Harms, and P. W. Michor, Sobolev metrics on shape space of surfaces, *J. Geom. Mech.* **3** no. 4 (2011), 389–438. MR 2888014. Zbl 1262.58004. https://doi.org/10.3934/jgm.2011.3.389. MR2888014
12. A. Beurling and L. Ahlfors, The boundary correspondence under quasiconformal mappings, *Acta Math.* **96** (1956), 125–142. MR 0086869. Zbl 0072. 29602. https://doi.org/10.1007/BF02392360. MR0086869
13. A. Beurling, *Études sur un problème de majoration*, Ph.D. thesis, Uppsala University, 1933. Zbl 0008.31802.

14. C. J. Bishop, Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture, *Ark. Mat.* **40** no. 1 (2002), 1–26. MR 1948883. Zbl 1034.30013. https://doi.org/10.1007/BF02384499. MR1948883
15. C. J. Bishop, An explicit constant for Sullivan's convex hull theorem, in *In the Tradition of Ahlfors and Bers, III, Contemp. Math.* **355**, Amer. Math. Soc., Providence, RI, 2004, pp. 41–69. MR 2145055. Zbl 1069.30030. https://doi.org/10.1090/conm/355/06444. MR2145055
16. C. J. Bishop, Conformal welding and Koebe's theorem, *Ann. of Math.* (2) **166** no. 3 (2007), 613–656. MR 2373370. Zbl 1144.30007. https://doi.org/10.4007/annals.2007.166.613. MR2373370
17. C. J. Bishop, Function theoretic characterizations of Weil-Petersson curves, *Rev. Mat. Iberoam.* **38** no. 7 (2022), 2355–2384. MR 4526317. Zbl 1516.30028. https://doi.org/10.4171/rmi/1398. MR4526317
18. C. J. Bishop, The traveling salesman theorem for Jordan curves, *Adv. Math.* **404** no. part A (2022), Paper No. 108443, 27. MR 4420442. Zbl 1492.28002. https://doi.org/10.1016/j.aim.2022.108443. MR4420442
19. C. J. Bishop and P. W. Jones, Harmonic measure, L^2 estimates and the Schwarzian derivative, *J. Anal. Math.* **62** (1994), 77–113. MR 1269200. Zbl 0801.30024. https://doi.org/10.1007/BF02835949. MR1269200
20. C. J. Bishop and Y. Peres, *Fractals in Probability and Analysis, Cambridge Studies in Advanced Mathematics* **162**, Cambridge University Press, Cambridge, 2017. MR 3616046. Zbl 1390.28012. https://doi.org/10.1017/9781316460238. MR3616046
21. S. Blatt, Boundedness and regularizing effects of O'Hara's knot energies, *J. Knot Theory Ramifications* **21** no. 1 (2012), 1250010, 9. MR 2887901. Zbl 1238. 57007. https://doi.org/10.1142/S0218216511009704. MR2887901
22. G. Bourdaud, Changes of variable in Besov spaces. II, *Forum Math.* **12** no. 5 (2000), 545–563. MR 1779495. Zbl 0971.46022. https://doi.org/10.1515/form.2000.018. MR1779495
23. M. J. Bowick and S. G. Rajeev, The holomorphic geometry of closed bosonic string theory and Diff S^1/S^1, *Nuclear Phys. B* **293** no. 2 (1987), 348–384. MR 0908048. https://doi.org/10.1016/0550-3213(87)90076-9. MR0908048
24. M. J. Bowick and S. G. Rajeev, String theory as the Kähler geometry of loop space, *Phys. Rev. Lett.* **58** no. 6 (1987), 535–538. MR 0873068. https://doi.org/10.1103/PhysRevLett.58.535. MR0873068
25. M. Bridgeman, R. Canary, and A. Yarmola, An improved bound for Sullivan's convex hull theorem, *Proc. Lond. Math. Soc.* (3) **112** no. 1 (2016), 146–168. MR 3458148. Zbl 1344.30018. https://doi.org/10.1112/plms/pdv064. MR3458148
26. M. Bridgeman, K. Bromberg, F. V. Pallete, and Y. Wang, Universal Liouville action as a renormalized volume and its gradient flow, 2023, *Duke Math. J.*, to appear. arXiv 2311.18767.
27. M. Bridgeman and R. D. Canary, The Thurston metric on hyperbolic domains and boundaries of convex hulls, *Geom. Funct. Anal.* **20** no. 6 (2010), 1317–1353. MR 2738995. Zbl 1218.30123. https://doi.org/10.1007/s00039-010-0102-7. MR2738995
28. M. Bridgeman and R. D. Canary, Uniformly perfect domains and convex hulls: improved bounds in a generalization of a theorem of Sullivan, *Pure Appl. Math. Q.* **9** no. 1 (2013), 49–71. MR 3126500. Zbl 1291.30124. https://doi.org/10.4310/PAMQ.2013.v9.n1.a2. MR3126500
29. M. Bridgeman and R. D. Canary, Renormalized volume and the volume of the convex core, *Ann. Inst. Fourier* (*Grenoble*) **67** no. 5 (2017), 2083–2098. MR 3732685. Zbl 1404.57031. https://doi.org/10.5802/aif.3130. MR3732685
30. J. F. Brock, The Weil-Petersson metric and volumes of 3-dimensional hyperbolic

convex cores, *J. Amer. Math. Soc.* **16** no. 3 (2003), 495–535. MR 1969203. Zbl 1059.30036. https://doi.org/10.1090/S0894-0347-03-00424-7. MR1969203

31. M. Bruveris and F.-X. Vialard, On completeness of groups of diffeomorphisms, *J. Eur. Math. Soc.* (*JEMS*) **19** no. 5 (2017), 1507–1544. MR 3635359. Zbl 1370.58003. https://doi.org/10.4171/JEMS/698. MR3635359
32. S.-Y. A. Chang and D. E. Marshall, On a sharp inequality concerning the Dirichlet integral, *Amer. J. Math.* **107** no. 5 (1985), 1015–1033. MR 0805803. Zbl 0578.30010. https://doi.org/10.2307/2374345. MR0805803
33. I. Chavel and E. A. Feldman, Isoperimetric inequalities on curved surfaces, *Adv. in Math.* **37** no. 2 (1980), 83–98. MR 0591721. Zbl 0461.53012. https://doi.org/10.1016/0001-8708(80)90028-6. MR0591721
34. Q. Chen and Y. Cheng, Chern-Osserman inequality for minimal surfaces in H^n, *Proc. Amer. Math. Soc.* **128** no. 8 (2000), 2445–2450. MR 1664325. Zbl 0955.53036. https://doi.org/10.1090/S0002-9939-00-05635-5. MR1664325
35. M. Chuaqui and B. Osgood, Ahlfors-Weill extensions of conformal mappings and critical points of the Poincaré metric, *Comment. Math. Helv.* **69** no. 4 (1994), 659–668. MR 1303231. Zbl 0826.30013. https://doi.org/10.1007/BF02564508. MR1303231
36. M. Lanza de Cristoforis and L. Preciso, Differentiability properties of some nonlinear operators associated to the conformal welding of Jordan curves in Schauder spaces, *Hiroshima Math. J.* **33** no. 1 (2003), 59–86. MR 1966652. Zbl 1031.30006. https://doi.org/10.32917/hmj/1150997868. MR1966652
37. G. Cui, Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces, *Sci. China Ser. A* **43** no. 3 (2000), 267–279. MR 1766456. Zbl 0965.30018. https://doi.org/10.1007/BF02897849. MR1766456
38. G. C. David and R. Schul, The analyst's traveling salesman theorem in graph inverse limits, *Ann. Acad. Sci. Fenn. Math.* **42** no. 2 (2017), 649–692. MR 3701642. Zbl 1376.28003. https://doi.org/10.5186/aasfm.2017.4260. MR3701642
39. E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** no. 5 (2012), 521–573. MR 2944369. Zbl 1252.46023. https://doi.org/10.1016/j.bulsci.2011.12.004. MR2944369
40. J. R. Dorronsoro, Mean oscillation and Besov spaces, *Canad. Math. Bull.* **28** no. 4 (1985), 474–480. MR 0812124. Zbl 0648.46032. https://doi.org/10.4153/CMB-1985-058-3. MR0812124
41. J. Douglas, Solution of the problem of Plateau, *Trans. Amer. Math. Soc.* **33** no. 1 (1931), 263–321. MR 1501590. Zbl 0001.14102. https://doi.org/10.2307/1989472. MR1501590
42. C. L. Epstein, The hyperbolic Gauss map and quasiconformal reflections, *J. Reine Angew. Math.* **372** (1986), 96–135. MR 0863521. Zbl 0591.30018. https://doi.org/10.1515/crll.1986.372.96. MR0863521
43. D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, in *Analytical and Geometric Aspects of Hyperbolic Space* (*Coventry/Durham*, 1984), *London Math. Soc. Lecture Note Ser.* **111**, Cambridge Univ. Press, Cambridge, 1987, pp. 113–253. MR 0903852. Zbl 0612.57010. MR0903852
44. D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, in *Fundamentals of Hyperbolic Geometry: Selected Expositions, London Math. Soc. Lecture Note Ser.* **328**, Cambridge Univ. Press, Cambridge, 2006, pp. 117–266. MR 2235711. MR2235711
45. D. B. A. Epstein and V. Markovic, The logarithmic spiral: a counterexample to the $K = 2$ conjecture, *Ann. of Math.* (2) **161** no. 2 (2005), 925–957. MR 2153403.

Zbl 1076.30048. https://doi.org/10.4007/annals.2005.161.925. MR2153403

46. K. J. Falconer and D. T. Marsh, Classification of quasi-circles by Hausdorff dimension, *Nonlinearity* **2** no. 3 (1989), 489–493. MR 1005062. Zbl 0684.58023. Available at http://stacks.iop.org/0951-7715/2/489. MR1005062
47. H. Federer, *Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, Band* **153**, Springer-Verlag New York, Inc., New York, 1969. MR 0257325. Zbl 0176.00801. MR0257325
48. M. Feiszli, Extremal distance, hyperbolic distance, and convex hulls over domains with smooth boundary, *Ann. Acad. Sci. Fenn. Math.* **36** no. 1 (2011), 195–214. MR 2797691. Zbl 1222.30017. https://doi.org/10.5186/aasfm.2011.3612. MR2797691
49. M. Feiszli, S. Kushnarev, and K. Leonard, Metric spaces of shapes and applications: compression, curve matching and low-dimensional representation, *Geom. Imaging Comput.* **1** no. 2 (2014), 173–221. MR 3396622. Zbl 1314.30122. https://doi.org/10.4310/GIC.2014.v1.n2.a1. MR3396622
50. M. Feiszli and A. Narayan, Numerical computation of Weil-Peterson geodesics in the universal Teichmüller space, *SIAM J. Imaging Sci.* **10** no. 3 (2017), 1322–1345. MR 3686793. Zbl 1383.30012. https://doi.org/10.1137/15M1043947. MR3686793
51. F. Ferrari, B. Franchi, and H. Pajot, The geometric traveling salesman problem in the Heisenberg group, *Rev. Mat. Iberoam.* **23** no. 2 (2007), 437–480. MR 2371434. Zbl 1142.28004. https://doi.org/10.4171/RMI/502. MR2371434
52. F. Fiala, Le problème des isopérimètres sur les surfaces ouvertes à courbure positive, *Comment. Math. Helv.* **13** (1941), 293–346. MR 0006422. Zbl 0025. 23003. https://doi.org/10.1007/BF01378068. MR0006422
53. M. H. Freedman, Z.-X. He, and Z. Wang, Möbius energy of knots and unknots, *Ann. of Math.* (2) **139** no. 1 (1994), 1–50. MR 1259363. Zbl 0817. 57011. https://doi.org/10.2307/2946626. MR1259363
54. D. H. Fremlin, Skeletons and central sets, *Proc. London Math. Soc.* (3) **74** no. 3 (1997), 701–720. MR 1434446. Zbl 0949.54050. https://doi.org/10.1112/S0024611597000233. MR1434446
55. P. K. Friz and A. Shekhar, On the existence of SLE trace: finite energy drivers and non-constant κ, *Probab. Theory Related Fields* **169** no. 1–2 (2017), 353–376. MR 3704771. Zbl 1407.60113. https://doi.org/10.1007/s00440-016-0731-3. MR3704771
56. E. A. Gallardo-Gutiérrez, M. J. González, F. Pérez-González, C. Pommerenke, and J. Rättyä, Locally univalent functions, VMOA and the Dirichlet space, *Proc. Lond. Math. Soc.* (3) **106** no. 3 (2013), 565–588. MR 3048550. Zbl 1276.30026. https://doi.org/10.1112/plms/pds040. MR3048550
57. F. P. Gardiner and D. P. Sullivan, Symmetric structures on a closed curve, *Amer. J. Math.* **114** no. 4 (1992), 683–736. MR 1175689. Zbl 0778.30045. https://doi.org/10.2307/2374795. MR1175689
58. J. B. Garnett, *Bounded Analytic Functions, Pure and Applied Mathematics* **96**, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981. MR 0628971. Zbl 0469.30024. MR0628971
59. J. B. Garnett and D. E. Marshall, *Harmonic Measure, New Mathematical Monographs* **2**, Cambridge University Press, Cambridge, 2008, Reprint of the 2005 original. MR 2450237. Zbl 1139.31001. MR2450237
60. F. Gay-Balmaz and T. S. Ratiu, The geometry of the universal Teichmüller space and the Euler-Weil-Petersson equation, *Adv. Math.* **279** (2015), 717–778. MR 3345193. Zbl 1320.32018. https://doi.org/10.1016/j.aim.2015.04.005. MR3345193
61. F. W. Gehring, The definitions and exceptional sets for quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A I* **281** (1960), 28. MR 0124488. Zbl 0090.05303. MR0124488

62. F. W. Gehring and K. Hag, *The Ubiquitous Quasidisk, Mathematical Surveys and Monographs* **184**, American Mathematical Society, Providence, RI, 2012, With contributions by Ole Jacob Broch. MR 2933660. Zbl 1267.30003. https://doi.org/10.1090/surv/184. MR2933660
63. Z. Geng and F. Lin, The two-dimensional liquid crystal droplet problem with a tangential boundary condition, *Arch. Ration. Mech. Anal.* **243** no. 3 (2022), 1181–1221. MR 4381139. Zbl 1487.76007. https://doi.org/10.1007/s00205-021-01733-5. MR4381139
64. C. R. Graham and E. Witten, Conformal anomaly of submanifold observables in AdS/CFT correspondence, *Nuclear Phys. B* **546** no. 1–2 (1999), 52–64. MR 1682674. Zbl 0944.81046. https://doi.org/10.1016/S0550-3213(99)00055-3. MR1682674
65. H. Guo, Integrable Teichmüller spaces, *Sci. China Ser. A* **43** no. 1 (2000), 47–58. MR 1766239. Zbl 0948.30052. https://doi.org/10.1007/BF02903847. MR1766239
66. Z.-X. He, The Euler-Lagrange equation and heat flow for the Möbius energy, *Comm. Pure Appl. Math.* **53** no. 4 (2000), 399–431. MR 1733697. MR1733697
67. J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** no. 1 (1998), 1–61. MR 1654771. Zbl 0915.30018. https://doi.org/10.1007/BF02392747. MR1654771
68. M. Henningson and K. Skenderis, The holographic Weyl anomaly, *J. High Energy Phys.* no. 7 (1998), Paper 23, 12. MR 1644988. Zbl 0958.81083. https://doi.org/10.1088/1126-6708/1998/07/023. MR1644988
69. N. J. Hicks, *Notes on Differential Geometry, Van Nostrand Mathematical Studies, No.* 3, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965. MR 0179691. Zbl 0132.15104. MR0179691
70. Y. Hu and Y. Shen, On quasisymmetric homeomorphisms, *Israel J. Math.* **191** no. 1 (2012), 209–226. MR 2970868. Zbl 1258.30007. https://doi.org/10.1007/s11856-011-0204-4. MR2970868
71. H. Inci, T. Kappeler, and P. Topalov, On the regularity of the composition of diffeomorphisms, *Mem. Amer. Math. Soc.* **226** no. 1062 (2013), vi+60. MR 3135704. Zbl 1293.58004. https://doi.org/10.1090/S0065-9266-2013-00676-4. MR3135704
72. B. Jaye, X. Tolsa, and M. Villa, A proof of Carleson's ε^2–conjecture, *Ann. of Math. (2)* **194** no. 1 (2021), 97–161. MR 4276285. Zbl 1472.28005. https://doi.org/10.4007/annals.2021.194.1.2. MR4276285
73. Y. Jiang, *Renormalization and Geometry in One-Dimensional and Complex Dynamics, Advanced Series in Nonlinear Dynamics* **10**, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. MR 1442953. Zbl 0864.58018. https://doi.org/10.1142/9789814350105. MR1442953
74. P. W. Jones, Rectifiable sets and the traveling salesman problem, *Invent. Math.* **102** no. 1 (1990), 1–15. MR 1069238. Zbl 0731.30018. https://doi.org/10.1007/BF01233418. MR1069238
75. J. Krandel, The traveling salesman theorem for Jordan curves in Hilbert space, 2023, Advance Publication, pp. 1–73. https://doi.org/10.1307/mmj/20226254. MR4929112
76. K. Krasnov and J.-M. Schlenker, The Weil-Petersson metric and the renormalized volume of hyperbolic 3-manifolds, in *Handbook of Teichmüller Theory. Volume III, IRMA Lect. Math. Theor. Phys.* **17**, Eur. Math. Soc., Zürich, 2012, pp. 779–819. MR 2952776. Zbl 1256.30001. https://doi.org/10.4171/103-1/15. MR2952776
77. G. F. Lawler and W. Werner, The Brownian loop soup, *Probab. Theory Related Fields* **128** no. 4 (2004), 565–588. MR 2045953. Zbl 1049.60072. https://doi.org/10.1007/s00440-003-0319-6. MR2045953
78. O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, second ed.,

Die Grundlehren der mathematischen Wissenschaften, Band **126**, Springer-Verlag, New York-Heidelberg, 1973, Translated from the German by K. W. Lucas. MR 0344463. Zbl 0267.30016. MR0344463

79. O. Lehto, *Univalent Functions and Teichmüller Spaces, Graduate Texts in Mathematics* **109**, Springer-Verlag, New York, 1987. MR 0867407. Zbl 0606.30001. https://doi.org/10.1007/978-1-4613-8652-0. MR0867407
80. S. Li, Stratified β-numbers and traveling salesman in Carnot groups, *J. Lond. Math. Soc.* (2) **106** no. 2 (2022), 662–703. MR 4477201. Zbl 1538.28012. https://doi.org/10.1112/jlms.12582. MR4477201
81. S. Li and R. Schul, The traveling salesman problem in the Heisenberg group: upper bounding curvature, *Trans. Amer. Math. Soc.* **368** no. 7 (2016), 4585–4620. MR 3456155. Zbl 1350.53044. https://doi.org/10.1090/tran/6501. MR3456155
82. S. Li and R. Schul, An upper bound for the length of a traveling salesman path in the Heisenberg group, *Rev. Mat. Iberoam.* **32** no. 2 (2016), 391–417. MR 3512421. Zbl 1355.28005. https://doi.org/10.4171/RMI/889. MR3512421
83. F.-H. Lin, Asymptotic behavior of area-minimizing currents in hyperbolic space, *Comm. Pure Appl. Math.* **42** no. 3 (1989), 229–242. MR 0982349. Zbl 0688.49042. https://doi.org/10.1002/cpa.3160420302. MR0982349
84. J. Maldacena, Wilson loops in large N field theories, *Phys. Rev. Lett.* **80** no. 22 (1998), 4859–4862. MR 1732582. Zbl 0947.81128. https://doi.org/10.1103/PhysRevLett.80.4859. MR1732582
85. A. Marden, *Outer Circles*, Cambridge University Press, Cambridge, 2007, An Introduction to Hyperbolic 3-Manifolds. MR 2355387. Zbl 1149.57030. https://doi.org/10.1017/CBO9780511618918. MR2355387
86. A. Marden, The view from above, *Pure Appl. Math. Q.* **7** no. 2, Special Issue: In honor of Frederick W. Gehring, Part 2 (2011), 383–394. MR 2815381. Zbl 1291.30244. https://doi.org/10.4310/PAMQ.2011.v7.n2.a6. MR2815381
87. D. E. Marshall, A new proof of a sharp inequality concerning the Dirichlet integral, *Ark. Mat.* **27** no. 1 (1989), 131–137. MR 1004727. Zbl 0692.30028. https://doi.org/10.1007/BF02386365. MR1004727
88. O. Martio and J. Sarvas, Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4** no. 2 (1979), 383–401. MR 0565886. Zbl 0406.30013. https://doi.org/10.5186/aasfm.1978-79.0413. MR0565886
89. P. W. Michor and D. Mumford, Riemannian geometries on spaces of plane curves, *J. Eur. Math. Soc.* (*JEMS*) **8** no. 1 (2006), 1–48. MR 2201275. Zbl 1101.58005. https://doi.org/10.4171/JEMS/37. MR2201275
90. P. W. Michor and D. Mumford, An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach, *Appl. Comput. Harmon. Anal.* **23** no. 1 (2007), 74–113. MR 2333829. Zbl 1116.58007. https://doi.org/10.1016/j.acha.2006.07.004. MR2333829
91. F. Morgan, A regularity theorem for minimizing hypersurfaces modulo ν, *Trans. Amer. Math. Soc.* **297** no. 1 (1986), 243–253. MR 0849477. Zbl 0641.49027. https://doi.org/10.2307/2000466. MR0849477
92. J. W. Morgan, The Smith conjecture, in *The Smith Conjecture* (*New York*, 1979), *Pure Appl. Math.* **112**, Academic Press, Orlando, FL, 1984, pp. 3–6. MR 0758460. https://doi.org/10.1016/S0079-8169(08)61632-3. MR0758460
93. S. Nag and D. Sullivan, Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle, *Osaka J. Math.* **32** no. 1 (1995), 1–34. MR 1323099. Zbl 0820.30027. Available at http://projecteuclid.org/euclid.ojm/1200785862. MR1323099
94. T. Nishioka, S. Ryu, and T. Takayanagi, Holographic entanglement entropy: an

overview, *J. Phys. A* **42** no. 50 (2009), 504008, 35. MR 2566335. Zbl 1179.81138. https://doi.org/10.1088/1751-8113/42/50/504008. MR2566335

95. J. O'Hara, Energy of a knot, *Topology* **30** no. 2 (1991), 241–247. MR 1098918. Zbl 0733.57005. https://doi.org/10.1016/0040-9383(91)90010-2. MR1098918
96. J. O'Hara, Energy functionals of knots, in *Topology Hawaii* (*Honolulu, HI*, 1990), World Sci. Publ., River Edge, NJ, 1992, pp. 201–214. MR 1181493. Zbl 1039.58500. MR1181493
97. J. O'Hara, Family of energy functionals of knots, *Topology Appl.* **48** no. 2 (1992), 147–161. MR 1195506. Zbl 0769.57006. https://doi.org/10.1016/0166-8641(92)90023-S. MR1195506
98. K. Okikiolu, Characterization of subsets of rectifiable curves in $\mathbf{R}^n$, *J. London Math. Soc.* (2) **46** no. 2 (1992), 336–348. MR 1182488. Zbl 0758.57020. https://doi.org/10.1112/jlms/s2-46.2.336. MR1182488
99. G. de Oliveira Filho, Compactification of minimal submanifolds of hyperbolic space, *Comm. Anal. Geom.* **1** no. 1 (1993), 1–29. MR 1230271. Zbl 0794.53038. https://doi.org/10.4310/CAG.1993.v1.n1.a1. MR1230271
100. B. Osgood, Old and new on the Schwarzian derivative, in *Quasiconformal Mappings and Analysis* (*Ann Arbor, MI*, 1995), Springer, New York, 1998, pp. 275–308. MR 1488455. Zbl 0894.30001. https://doi.org/10.1007/978-1-4612-0605-7_16. MR1488455
101. R. Osserman, The isoperimetric inequality, *Bull. Amer. Math. Soc.* **84** no. 6 (1978), 1182–1238. MR 500557. Zbl 0411.52006. https://doi.org/10.1090/S0002-9904-1978-14553-4. MR0500557
102. H. Pajot, *Analytic Capacity, Rectifiability, Menger Curvature and the Cauchy Integral, Lecture Notes in Mathematics* **1799**, Springer-Verlag, Berlin, 2002. MR 1952175. Zbl 1043.28002. https://doi.org/10.1007/b84244. MR1952175
103. F. Pérez-González and J. Rättyä, Dirichlet and VMOA domains via Schwarzian derivative, *J. Math. Anal. Appl.* **359** no. 2 (2009), 543–546. MR 2546769. Zbl 1178.30005. https://doi.org/10.1016/j.jmaa.2009.06.023. MR2546769
104. C. Pommerenke, On univalent functions, Bloch functions and VMOA, *Math. Ann.* **236** no. 3 (1978), 199–208. MR 0492206. Zbl 0385.30013. https://doi.org/10.1007/BF01351365. MR0492206
105. D. Radnell, E. Schippers, and W. Staubach, Dirichlet problem and Sokhotski-Plemelj jump formula on Weil-Petersson class quasidisks, *Ann. Acad. Sci. Fenn. Math.* **41** no. 1 (2016), 119–127. MR 3467700. Zbl 1338.31006. https://doi.org/10.5186/aasfm.2016.4108. MR3467700
106. D. Radnell, E. Schippers, and W. Staubach, Quasiconformal Teichmüller theory as an analytical foundation for two-dimensional conformal field theory, in *Lie Algebras, Vertex Operator Algebras, and Related Topics, Contemp. Math.* **695**, Amer. Math. Soc., Providence, RI, 2017, pp. 205–238. MR 3709713. Zbl 1393.30034. https://doi.org/10.1090/conm/695/14003. MR3709713
107. E. Reich, Quasiconformal mappings of the disk with given boundary values, in *Advances in Complex Function Theory* (*Proc. Sem., Univ. Maryland, College Park, Md.*, 1973–1974), 1976, pp. 101–137. Lecture Notes in Math., Vol. 505. MR 0419758. Zbl 0328.30018. https://doi.org/10.1007/BFb0081102. MR0419758
108. S. Rohde, Quasicircles modulo bilipschitz maps, *Rev. Mat. Iberoamericana* **17** no. 3 (2001), 643–659. MR 1900898. Zbl 1003.30013. https://doi.org/10.4171/RMI/307. MR1900898
109. S. Rohde and Y. Wang, The Loewner energy of loops and regularity of driving functions, *Int. Math. Res. Not. IMRN* no. 10 (2021), 7715–7763. MR 4259153. Zbl 1482.30068. https://doi.org/10.1093/imrn/rnz071. MR4259153

110. W. T. Ross, The classical Dirichlet space, in *Recent Advances in Operator-related Function Theory, Contemp. Math.* **393**, Amer. Math. Soc., Providence, RI, 2006, pp. 171–197. MR 2198379. Zbl 1135.31007. https://doi.org/10.1090/conm/393/07378. MR2198379
111. S. Ryu and T. Takayanagi, Aspects of holographic entanglement entropy, *J. High Energy Phys.* no. 8 (2006), 045, 48. MR 2249925. https://doi.org/10.1088/1126-6708/2006/08/045. MR2249925
112. L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976, With a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1. MR 0433364. Zbl 0342.53049. MR0433364
113. J.-M. Schlenker, The renormalized volume and the volume of the convex core of quasifuchsian manifolds, *Math. Res. Lett.* **20** no. 4 (2013), 773–786. MR 3188032. Zbl 1295.30101. https://doi.org/10.4310/MRL.2013.v20.n4.a12. MR3188032
114. R. Schul, Subsets of rectifiable curves in Hilbert space—the analyst's TSP, *J. Anal. Math.* **103** (2007), 331–375. MR 2373273. Zbl 1152.28006. https://doi.org/10.1007/s11854-008-0011-y. MR2373273
115. A. Seppi, Minimal discs in hyperbolic space bounded by a quasicircle at infinity, *Comment. Math. Helv.* **91** no. 4 (2016), 807–839. MR 3566524. Zbl 1356.53063. https://doi.org/10.4171/CMH/403. MR3566524
116. E. Sharon and D. Mumford, 2D-Shape analysis using conformal mapping, *Int. J. Comput. Vis.* **70** (2006), 55–75. Zbl 1477.68492. https://doi.org/10.1007/s1263-006-6121-z.
117. Y. Shen, Weil-Petersson Teichmüller space, *Amer. J. Math.* **140** no. 4 (2018), 1041–1074. MR 3828040. Zbl 1421.30059. https://doi.org/10.1353/ajm.2018.0023. MR3828040
118. Y. Shen and L. Wu, Weil-Petersson Teichmüller space III: dependence of Riemann mappings for Weil-Petersson curves, *Math. Ann.* **381** no. 1–2 (2021), 875–904. MR 4322630. Zbl 1554.30022. https://doi.org/10.1007/s00208-020-02067-5. MR4322630
119. L. Simon, *Lectures on Geometric Measure Theory, Proceedings of the Centre for Mathematical Analysis, Australian National University* **3**, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR 0756417. Zbl 0546.49019. MR0756417
120. K. Strebel, On the existence of extremal Teichmueller mappings, *J. Analyse Math.* **30** (1976), 464–480. MR 0440031. Zbl 0334.30013. https://doi.org/10.1007/BF02786734. MR0440031
121. P. Strzelecki and H. von der Mosel, Menger curvature as a knot energy, *Phys. Rep.* **530** no. 3 (2013), 257–290. MR 3105400. Zbl 1358.57019. https://doi.org/10.1016/j.physrep.2013.05.003. MR3105400
122. T. Takayanagi, Entanglement entropy from a holographic viewpoint, *Classical Quantum Gravity* **29** no. 15 (2012), 153001, 25. MR 2960962. Zbl 1247.83005. https://doi.org/10.1088/0264-9381/29/15/153001. MR2960962
123. L. A. Takhtajan and L.-P. Teo, Weil-Petersson metric on the universal Teichmüller space, *Mem. Amer. Math. Soc.* **183** no. 861 (2006), viii+119. MR 2251887. Zbl 1243.32010. https://doi.org/10.1090/memo/0861. MR2251887
124. F. Viklund and Y. Wang, Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines, *Geom. Funct. Anal.* **30** no. 1 (2020), 289–321. MR 4080509. Zbl 1436.30009. https://doi.org/10.1007/s00039-020-00521-9. MR4080509
125. F. Viklund and Y. Wang, The Loewner-Kufarev energy and foliations by Weil-Petersson quasicircles, *Proc. Lond. Math. Soc.* (3) **128** no. 2 (2024), e12582, 62 pp. MR 4704164. Zbl 1542.30007. https://doi.org/10.1112/plms.12582. MR4704164

126. Y. Wang, The energy of a deterministic Loewner chain: reversibility and interpretation via SLE_{0+}, *J. Eur. Math. Soc.* (*JEMS*) **21** no. 7 (2019), 1915–1941. MR 3959854. Zbl 1422.30031. https://doi.org/10.4171/JEMS/876. MR3959854
127. Y. Wang, Equivalent descriptions of the Loewner energy, *Invent. Math.* **218** no. 2 (2019), 573–621. MR 4011706. Zbl 1435.30074. https://doi.org/10.1007/s00222-019-00887-0. MR4011706
128. Y. Wang, A note on Loewner energy, conformal restriction and Werner's measure on self-avoiding loops, *Ann. Inst. Fourier* (*Grenoble*) **71** no. 4 (2021), 1791–1805. MR 4398248. Zbl 1486.30063. https://doi.org/10.5802/aif.3427. MR4398248
129. Y. Wang, From the random geometry of conformally invariant systems to the Kähler geometry of universal Teichmüller space, *Notices Amer. Math. Soc.* **71** no. 6 (2024), 732–739. MR 4767165. Zbl 1546.60033. MR4767165
130. H. Wei and K. Matsuzaki, Parametrization of the p-Weil-Petersson curves: holomorphic dependence, *J. Geom. Anal.* **33** no. 9 (2023), Paper No. 292, 32. MR 4610351. Zbl 1522.32034. https://doi.org/10.1007/s12220-023-01338-2. MR4610351
131. B. White, On the compactness theorem for embedded minimal surfaces in 3-manifolds with locally bounded area and genus, *Comm. Anal. Geom.* **26** no. 3 (2018), 659–678. MR 3844118. Zbl 1394.53065. https://doi.org/10.4310/CAG.2018.v26.n3.a7. MR3844118

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Bishop, Christopher J. (1-SUNYS); **Eremenko, Alexandre** (1-PURD); **Lazebnik, Kirill** (1-TXD-M)

On the shapes of rational lemniscates. (English. English summary)

Geom. Funct. Anal. **35** (2025), *no.* 2, 359–407.

This paper deals with the shapes of rational lemniscates. A *rational lemniscate* is a level set of the form

$$L_r(c) := \{z \in \widehat{\mathbb{C}} : |r(z)| = c\},$$

where $0 < c < \infty$, $\widehat{\mathbb{C}}$ is the Riemann sphere and r is a rational map. After rescaling if necessary, we may take $c = 1$, and write $L_r := L_r(1)$. The topology of rational lemniscates is described using the notion of *lemniscate graph*, which is a set $G \subset \widehat{\mathbb{C}}$ so that there is a finite set $V \subset G$ called the *vertices* of G such that

(1) $G \setminus V$ has finitely many components (these are called the *edges* of G), each of which is either a (closed) Jordan curve, or else a (open) simple arc γ satisfying $\overline{\gamma} \setminus \gamma \subset V$;
(2) the degree of each vertex is even and at least four, where the *degree* of a vertex v is defined as the number of edges γ satisfying $v \in \overline{\gamma} \setminus \gamma$, and we count an edge γ twice if $\{v\} = \overline{\gamma} \setminus \gamma$.

It is not difficult to prove that every rational lemniscate is a lemniscate graph. The main result of the paper under review states on the other hand that every lemniscate graph can be approximated in a strong sense by a rational lemniscate. More precisely, if G is a lemniscate graph, then for every $\epsilon > 0$ there exists a rational map r and a homeomorphism $\varphi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi(G) = L_r$ and $\sup_{z\in\widehat{\mathbb{C}}} d(\varphi(z), z) < \epsilon$, where $d(\cdot,\cdot)$ denotes the spherical metric on $\widehat{\mathbb{C}}$. One can also prescribe the poles of r in some precise sense. This generalizes the classical Hilbert lemniscate theorem.

As shown in the paper, the fact that every lemniscate graph can be approximated by a rational lemniscate has important consequences, such as a sharp quantitative version of the classical Runge theorem on rational approximation as well as a generalization of a result from [K. A. Lindsey and M. Younsi, Trans. Amer. Math. Soc. **371** (2019), no. 12, 8489–8511; MR3955554] on the approximation of planar continua by Julia sets of rational maps.

This is a very nice paper on a fundamental topic, and the results should be interesting to a broad range of complex analysts. *Malik Younsi*

[References]

1. Ahlfors, L.V.: Lectures on Quasiconformal Mappings, 2nd edn. University Lecture Series, vol. 38. Am. Math. Soc., Providence (2006). With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. MR2241787
2. Andrievskii, V.: On the approximation of a continuum by lemniscates. *J. Approx. Theory* **105**(2), 292–304 (2000). MR1775150
3. Andrievskii, V.V.: On Hilbert lemniscate theorem for a system of quasidisks. *Jaen J. Approx.* **10**(1–2), 133–145 (2018). MR3978145
4. Andrei, D., Nevanlinna, O., Vesanen, T.: Rational functions as new variables. *Banach J. Math. Anal.* **16**(3):Paper No. 37, 22 (2022). MR4416767
5. Bloom, T., Levenberg, N., Lyubarskii, Yu.: A Hilbert lemniscate theorem in $\mathbb{C}^2$. *Ann. Inst. Fourier (Grenoble)* **58**(6), 2191–2220 (2008). MR2473634
6. Borwein, P.: The arc length of the lemniscate $\{|p(z)| = 1\}$. *Proc. Amer. Math. Soc.* **123**(3), 797–799 (1995). MR1223265
7. Bishop, C.J., Pilgrim, K.M.: Dynamical dessins are dense. *Rev. Mat. Iberoam.* **31**(3), 1033–1040 (2015). MR3420484
8. Brouwer, L.E.J.: Über Abbildung von Mannigfaltigkeiten. *Math. Ann.* **71**(1), 97–115 (1911). MR1511644
9. Boc Thaler, L.: On the geometry of simply connected wandering domains. *Bull. Lond. Math. Soc.* **53**(6), 1663–1673 (2021). MR4375923
10. Carleson, L., Gamelin, T.W.: Complex Dynamics. Universitext: Tracts in Mathematics. Springer, New York (1993). MR1230383
11. Catanese, F., Paluszny, M.: Polynomial-lemniscates, trees and braids. *Topology* **30**(4), 623–640 (1991). MR1133876
12. Eremenko, A., Gabrielov, A.: Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry. *Ann. of Math. (2)* **155**(1), 105–129 (2002). MR1888795
13. Eremenko, A., Hayman, W.: On the length of lemniscates. *Michigan Math. J.* **46**(2), 409–415 (1999). MR1704189
14. Epstein, M., Hanin, B., Lundberg, E.: The lemniscate tree of a random polynomial. *Ann. Inst. Fourier (Grenoble)* **70**(4), 1663–1687 (2020). MR4245584
15. Erdős, P., Herzog, F., Piranian, G.: Metric properties of polynomials. *J. Anal. Math.* **6**, 125–148 (1958). MR0101311
16. Ebenfelt, P., Khavinson, D., Shapiro, H.S.: Two-dimensional shapes and lemniscates. In: Complex Analysis and Dynamical Systems IV. Part 1. Contemp. Math., vol. 553, pp. 45–59. Am. Math. Soc., Providence (2011). MR2868587
17. Fortier Bourque, M., Younsi, M.: Rational Ahlfors functions. *Constr. Approx.* **41**(1), 157–183 (2015). MR3296178
18. Frolova, A., Khavinson, D., Vasil'ev, A.: Polynomial lemniscates and their finger-prints: from geometry to topology. In: Complex Analysis and Dynamical Systems, Trends Math, pp. 103–128. Springer, Cham (2018). MR3784168
19. Fryntov, A., Nazarov, F.: New estimates for the length of the Erdős-Herzog-Piranian lemniscate. In: Linear and Complex Analysis. Amer. Math. Soc. Transl. Ser. 2, vol. 226, pp. 49–60. Am. Math. Soc., Providence (2009). MR2500509

20. Garnett, J.B., Marshall, D.E.: Harmonic Measure. New Mathematical Monographs, vol. 2. Cambridge University Press, Cambridge (2008). Reprint of the 2005 original. MR2450237
21. Guth, L.: Unexpected applications of polynomials in combinatorics. In: The Mathematics of Paul Erdős. I, pp. 493–522. Springer, New York (2013). MR3202645
22. Hilbert, D.: über die Entwicklung einer beliebigen analytischen Funktion einer variablen in eine unendliche nach ganzen rationalen Funktionen fortschreitende Reihe. Gött. Nachr., 63–70 (1897).
23. Jamison, R.E., Ruckle, W.H.: Factoring absolutely convergent series. *Math. Ann.* **224**(2), 143–148 (1976). MR0435791
24. Kirillov, A.A.: Kähler structure on the K-orbits of a group of diffeomorphisms of the circle. *Funktsional. Anal. i Prilozhen.* **21**(2), 42–45 (1987). MR0902292
25. Koch, S., Lei, T.: On balanced planar graphs, following W. Thurston. In: What's Next?—the Mathematical Legacy of William P. Thurston. Ann. of Math. Stud., vol. 205, pp. 215–232. Princeton Univ. Press, Princeton (2020). MR4205641
26. Korda, M., Lasserre, J.-B., Lazarev, A., Magron, V., Naldi, S.: Urysohn in action: separating semialgebraic sets by polynomials. Preprint. arXiv:2207.00570v1.
27. Kosukhin, O.N.: On the rate of approximation of closed Jordan curves by lemniscates. *Mat. Zametki* **77**(6), 861–876 (2005). MR2246962
28. Lazebnik, K.: Analytic and topological nets. *Advances in Mathematics* **461**, (2025). MR4837125
29. Lindsey, K.A.: Shapes of polynomial Julia sets. *Ergodic Theory Dynam. Systems* **35**(6), 1913–1924 (2015). MR3377290
30. Lerario, A., Lundberg, E.: Statistics on Hilbert's 16th problem. *Int. Math. Res. Not. IMRN* **12**, 4293–4321 (2015). MR3356754
31. Lerario, A., Lundberg, E.: On the geometry of random lemniscates. *Proc. Lond. Math. Soc. (3)* **113**(5), 649–673 (2016). MR3570241
32. Lundberg, E., Ramachandran, K.: The arc length and topology of a random lemniscate. *J. Lond. Math. Soc. (2)* **96**(3), 621–641 (2017). MR3742436
33. Lehto, O., Virtanen, K.I.: Quasiconformal Mappings in the Plane, 2nd edn. Die Grundlehren der Mathematischen Wissenschaften, vol. 126. Springer, New York (1973). Translated from the German by K. W. Lucas. MR0344463
34. Lindsey, K.A., Younsi, M.: Fekete polynomials and shapes of Julia sets. *Trans. Amer. Math. Soc.* **371**(12), 8489–8511 (2019). MR3955554
35. Marshall, D.: Complex Analysis. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge (2019). MR4321146
36. Milnor, J.W.: Topology from the Differentiable Viewpoint. Princeton Landmarks in Mathematics. Princeton University Press, Princeton (1997). Based on notes by David W. Weaver, Revised reprint of the 1965 original. MR1487640
37. Milnor, J.: Dynamics in One Complex Variable, 3rd edn. Annals of Mathematics Studies., vol. 160. Princeton University Press, Princeton (2006). MR2193309
38. Malozemov, V.N., Plotkin, A.V.: Strict polynomial separation of two sets. *Vestn. St.-Peterbg. Univ. Mat. Mekh. Astron.* **6**(64)(2), 232–240 (2019). MR3964612
39. Martí-Pete, D., Rempe, L., Waterman, J.: Bounded Fatou and Julia components of meromorphic functions. (2022). arXiv:2204.11781. MR4846778
40. Nivoche, S.: Polynomial convexity, special polynomial polyhedra and the pluricomplex Green function for a compact set in $\mathbb{C}^n$. *J. Math. Pures Appl. (9)* **91**(4), 364–383 (2009). MR2518003
41. Nagy, B., Totik, V.: Sharpening of Hilbert's lemniscate theorem. *J. Anal. Math.* **96**, 191–223 (2005). MR2177185
42. Pommerenke, Ch.: Boundary Behaviour of Conformal Maps. Grundlehren der

Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 299. Springer, Berlin (1992). MR1217706

43. Putinar, M.: Notes on generalized lemniscates. In: Operator Theory, Systems Theory and Scattering Theory: Multidimensional Generalizations. Oper. Theory Adv. Appl., vol. 157, pp. 243–266. Birkhäuser, Basel (2005). MR2129650
44. Richards, T.: Conformal equivalence of analytic functions on compact sets. *Comput. Methods Funct. Theory* **16**(4), 585–608 (2016). MR3558373
45. Richards, T.J.: Some recent results on the geometry of complex polynomials: the Gauss-Lucas theorem, polynomial lemniscates, shape analysis, and conformal equivalence. *Complex Anal. Synergies* **7**(2), Article ID 20 (2021). MR4261771
46. Royden, H.L.: The interpolation problem for Schlicht functions. *Ann. of Math. (2)* **60**, 326–344 (1954). MR0064147
47. Richards, T., Younsi, M.: Conformal models and fingerprints of pseudo-lemniscates. *Constr. Approx.* **45**(1), 129–141 (2017). MR3590700
48. Richards, T., Younsi, M.: Computing polynomial conformal models for low-degree Blaschke products. *Comput. Methods Funct. Theory* **19**(1), 173–182 (2019). MR3922299
49. Rashkovskii, A., Zakharyuta, V.: Special polyhedra for Reinhardt domains. *C. R. Math. Acad. Sci. Paris* **349**(17–18), 965–968 (2011). MR2838245
50. Sharon, E., Mumford, D.: 2D-shape analysis using conformal mapping. *Int. J. Comput. Vis.* **70**, 55–75 (2006).
51. Stone, A.H., Tukey, J.W.: Generalized “sandwich” theorems. *Duke Math. J.* **9**, 356–359 (1942). MR0007036
52. Stafney, J.D.: Set approximation by lemniscates and the spectrum of an operator on an interpolation space. *Pacific J. Math.* **60**(2), 253–265 (1975). MR0428074
53. Stephenson, K.: Analytic functions sharing level curves and tracts. *Ann. of Math. (2)* **123**(1), 107–144 (1986). MR0825840
54. Sullivan, D.: Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. *Ann. of Math. (2)* **122**(3), 401–418 (1985). MR0819553
55. Thurston, B.: What are the shapes of rational functions? MathOverflow (2010). https://mathoverflow.net/q/38274 (version: 2017-04-13).
56. Tomasini, J.: Realizations of branched self-coverings of the 2-sphere. *Topology Appl.* **196**(A), 31–53 (2015). MR3422731
57. Walsh, J.L., Russell, H.G.: x On the convergence and overconvergence of sequences of polynomials of best simultaneous approximation to several functions analytic in distinct regions. *Trans. Amer. Math. Soc.* **36**(1), 13–28 (1934). MR1501732
58. Younsi, M.: Shapes, fingerprints and rational lemniscates. *Proc. Amer. Math. Soc.* **144**(3), 1087–1093 (2016). MR3447662
59. Zakeri, S.: A Course in Complex Analysis. Princeton University Press, Princeton (2021). MR4422101

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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Bishop, Christopher J. (1-SUNYS)
BiLipschitz homogeneous hyperbolic nets. (English. English, Finnish summary)
Ann. Fenn. Math. **49** (2024), *no.* 2, 685–694.

In the paper under review, the author studies discrete subsets of the hyperbolic disk that are homogeneous with respect to uniformly bi-Lipschitz self-mappings of the disk. To state this more formally, let $K \geq 1$ and $\varepsilon > 0$. A discrete set $X \subset \mathbb{D}$ is an *ε-net* if every point of $\mathbb{D}$ is within hyperbolic distance ε of X, and X is a *(K, ε)-net* if it is an ε-net that is homogeneous with respect to a collection (not necessarily a group) of K-bi-Lipschitz self-mappings of $\mathbb{D}$. The author then defines

$$\varepsilon(K) := \inf\{\varepsilon : (K, \varepsilon)\text{-nets exist}\}.$$

After explaining why $\varepsilon(K) < \infty$ for all $K \geq 1$, the author proceeds to define

$$K_c := \inf\{K : \varepsilon(K) = 0\} = \sup\{K : \varepsilon(K) > 0\}.$$

The main result of the paper under review (Theorem 1.1) is that $1 < K_c < \infty$. The author explains that $\varepsilon(1) > 0$ due to well-known results of Kazhdan and Margulis pertaining to Fuchsian groups, and attributes the question of whether or not $K_c > 1$ to Itai Benjamini.

To prove $K_c > 1$, the author argues by contradiction, assuming there exist sets $\{X_n\}$ and sequences $K_n \to 1$, $\varepsilon_n \to 0$ such that each X_n is a (K_n, ε_n)-net. Under this assumption, he demonstrates that, for large n, the local structure of X_n is incompatible with the global exponential growth of hyperbolic area in $\mathbb{D}$.

To prove $K_c < \infty$, given any $\varepsilon > 0$, the author constructs a (K, ε)-net in $\mathbb{D}$ for some $K < \infty$ independent of ε. These (K, ε)-nets are obtained as refined tessellations of $\mathbb{D}$ by right pentagons (as constructed in [C. J. Bishop, Discrete Comput. Geom. **44** (2010), no. 2, 308–329; MR2671014]). *David Matthew Freeman*

[References]

1. Ahlfors, L. V.: Lectures on quasiconformal mappings. - Univ. Lecture Ser. 38, Amer. Math. Soc., Providence, RI, second edition, 2006. MR2241787
2. Beardon, A. F.: The geometry of discrete groups. - Grad. Texts in Math. 91, Springer-Verlag, New York, 1983. MR0698777
3. Bishop, C. J.: Bi-Lipschitz homogeneous curves in $\mathbb{R}^2$ are quasicircles. - Trans. Amer. Math. Soc. 353:7, 2001, 2655–2663. MR1828465
4. Bishop, C. J.: Optimal angle bounds for quadrilateral meshes. - Discrete Comput. Geom. 44:2, 2010, 308–329. MR2671014
5. Garnett, J. B., and D. E. Marshall: Harmonic measure. - New Math. Monogr. 2, Cambridge Univ. Press, Cambridge, 2008. MR2450237
6. Hoehn, L. C., and L. G. Oversteegen: A complete classification of homogeneous plane continua. - Acta Math. 216:2, 2016, 177–216. MR3573330
7. Každan, D. A., and G. A. Margulis: A proof of Selberg's hypothesis. - Mat. Sb. (N.S.) 75 (117), 1968, 163–168. MR0223487
8. Marden, A.: Universal properties of Fuchsian groups in the Poincaré metric: - In: Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Ann. of Math. Stud. 79, 1974, 315–339. MR0379837
9. Mori, A.: On an absolute constant in the theory of quasi-conformal mappings. - J. Math. Soc. Japan 8, 1956, 156–166. MR0079091
10. Stepanyants, H., A. F. Beardon, J. Paton, and D. Krioukov: Diameter of compact Riemann surfaces. - arXiv:2301.10844v3 [math.GT], 2024.

11. Sturm, J., and M. Shinnar: The maximal inscribed ball of a Fuchsian group. - In: Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Ann. of Math. Stud. 79, 1974, 439–443. MR0349994
12. Tukia, P.: On two-dimensional quasiconformal groups. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5:1, 1980, 73–78. MR0595178
13. Yamada, A.: On Marden's universal constant of Fuchsian groups. - Kodai Math. J. 4:2, 1981, 266–277. MR0630246

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MR4761785 30E10 30C62 37F10

Bishop, Christopher J. (1-SUNYS-NDM); **Lazebnik, Kirill** (1-NTXS-NDM)

A geometric approach to polynomial and rational approximation. (English. English summary)

Int. Math. Res. Not. IMRN **2024,** *no.* 12, 9936–9961.

In the paper under review, the authors revisit classic approximation theorems of analytic functions defined on domains of the Riemann sphere by polynomials or rational functions that they improve by controlling the locus of critical points and critical values (and poles). These include Runge's theorem, as well as Mergelyan and Weierstrass' theorems.

These approximations are constructed using the method of quasiconformal foldings that is based on first extending the given map by a quasiregular mapping before straightening it thanks to the measurable Riemann mapping theorem. The starting point is the approximation by proper mappings provided by Grunsky. One of the issues of this approach is to obtain a good control of the distortion of the quasiregular extension that will ensure the proximity of the rational function with prescribed data that is constructed. Further geometric properties of these approximations may be achieved by suitable choices made during the construction. *Peter Haïssinsky*

[References]

1. Alexander, H. and J. Wermer. *Several Complex Variables and Banach Algebras*, Vol. 35 of *Graduate Texts in Mathematics*, 3rd ed. New York: Springer, 1998. MR1482798
2. Benini, A. M., V. Evdoridou, N. Fagella, P. J. Rippon, and G. M. Stallard. "Classifying simply connected wandering domains." *Math. Ann.* **383**, no. 3–4 (2022): 1127–78. https://doi.org/10.1007/s00208-021-02252-0. MR4458398
3. Branner, B. and N. Fagella. *Quasiconformal surgery in holomorphic dynamics*. Vol. 141 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press, 2014. With contributions by Xavier Buff, Shaun Bullett, Adam L. Epstein, Peter Haïssinsky, Christian Henriksen, Carsten L. Petersen, Kevin M. Pilgrim, Tan Lei and Michael Yampolsky. MR3445628
4. Bishop, C. J. "Constructing entire functions by quasiconformal folding." *Acta Math.* **214**, no. 1 (2015): 1–60. https://doi.org/10.1007/s11511-015-0122-0. MR3316755
5. Bishop, C. J. and K. Lazebnik. "Prescribing the postsingular dynamics of meromorphic functions." *Math. Ann.* **375**, no. 3–4 (2019): 1761–82. https://doi.org/10.1007/s00208-019-01869-6. MR4023391
6. Burkart, J. and K. Lazebnik. "Interpolation of power mappings." *Rev. Mat. Iberoamericana* **39**, no. 3 (2023): 1181–200. https://doi.org/10.4171/rmi/1359. MR4603648
7. Bishop, C. J., K. Lazebnik, and M. Urbański. "Equilateral triangulations and the postcritical dynamics of meromorphic functions." *Math. Ann.* **387**, no. 3–4 (2023): 1777–818. https://doi.org/10.1007/s00208-022-02507-4. MR4657437

8. Thaler, L. B. "On the geometry of simply connected wandering domains." *Bull. Lond. Math. Soc.* **53**, no. 6 (2021): 1663–73. https://doi.org/10.1112/blms.12518. MR4375923
9. Caratheodory, C. *Theory of Functions of a Complex Variable*, Vol. 2. New York: Chelsea Publishing Co., 1954. Translated by F. Steinhardt. MR0064861
10. DeMarco, L. G., S. C. Koch, and C. T. McMullen. "On the postcritical set of a rational map." *Math. Ann.* **377**, no. 1–2 (2020): 1–18. https://doi.org/10.1007/s00208-018-1732-6. MR4099617
11. Erëmenko, A. È. and M. Y. Ljubich. "Examples of entire functions with pathological dynamics." *J. London Math. Soc. (2)* **36**, no. 3 (1987): 458–68. MR0918638
12. Evdoridou, V., P. J. Rippon, and G. M. Stallard. "Oscillating simply connected wandering domains." arXiv e-prints, arXiv:2011.14736, November 2020. MR4555827
13. Forstnerič, F. "Noncritical holomorphic functions on stein manifolds." *Acta Math.* **191**, no. 2 (2003): 143–89. https://doi.org/10.1007/BF02392963. MR2051397
14. Garnett, J. B. *Bounded Analytic Functions*, Vol. 96 of *Pure and Applied Mathematics*. New York-London: Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], 1981. MR0628971
15. Garcia, S. R., J. Mashreghi, and W. T. Ross. "Finite Blaschke products: a survey." *Harmonic Analysis, Function Theory, Operator Theory, and Their Applications*, Vol. 19 of *Theta Ser. Adv. Math.*,. 133–58. Theta, Bucharest, 2017. MR3753897
16. Grunsky, H. *Lectures on Theory of Functions in Multiply Connected Domains*, Vol. 4 of *Studia Mathematica: Skript*. Göttingen: Vandenhoeck & Ruprecht, 1978. MR0463413
17. Khavinson, S. Y. *Two Papers on Extremal Problems in Complex Analysis*, Vol. 129. American Mathematical Society Translations: Series 2, 1986. Translated by D. Khavinson.
18. Martí-Pete, D. and M. Shishikura. "Wandering domains for entire functions of finite order in the Eremenko-Lyubich class." *Proc. Lond. Math. Soc. (3)* **120**, no. 2 (2020): 155–91. MR4008367
19. Martí-Pete, D., L. Rempe, and J. Waterman. "Eremenko's conjecture, wandering Lakes of Wada, and maverick points." *arXiv e-prints*, arXiv:2108.10256, August 2021.
20. Martí-Pete, D., L. Rempe, and J. Waterman. "Bounded Fatou and Julia components of meromorphic functions." *arXiv e-prints*, arXiv:2204.11781, April 2022.
21. Runge, C. "Zur Theorie der Eindeutigen Analytischen Functionen." *Acta Math.* **6**, no. 1 (1885): 229–44. https://doi.org/10.1007/BF02400416. MR1554664
22. Totik, V. "The gauss-Lucas theorem in an asymptotic sense." *Bull. Lond. Math. Soc.* **48**, no. 5 (2016): 848–54. https://doi.org/10.1112/blms/bdw047. MR3556367
23. Totik, V. "A quantitative gauss-Lucas theorem." *Ark. Mat.* **60**, no. 1 (2022): 195–212. https://doi.org/10.4310/ARKIV.2022.v60.n1.a9. MR4423276

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MR4657437 30D05 30D30 37F10

Bishop, Christopher J. (1-SUNYS); **Lazebnik, Kirill** (1-NTXS-NDM);
Urbański, Mariusz (1-NTXS-NDM)

Equilateral triangulations and the postcritical dynamics of meromorphic functions. (English. English summary)

Math. Ann. **387** (2023), *no.* 3-4, 1777–1818.

The paper under review studies post-critical dynamics of meromorphic functions. Let $f\colon D \to \widehat{\mathbb{C}}$ be holomorphic. One can study iterative behaviours of f, and the global dynamics depends in a sense on the dynamics of its singular values. By a singular value is meant a critical or asymptotic value. Then the post-singular set is defined as the closure of the forward orbits of all singular values. The paper is concerned with the following question: For which set $S \subset D$ and which function $\phi\colon S \to S$ can one find a holomorphic function $f\colon D \to \widehat{\mathbb{C}}$ such that the post-singular dynamics is conjugate to ϕ?

The authors give a partial answer to the above question when S is discrete with at least three points and when the conjugacy is replaced by ε-conjugacy (a notion introduced in the paper; see Definition 1.1). Similar results were previously obtained by C. J. Bishop and K. Y. Lazebnik [Math. Ann. **375** (2019), no. 3-4, 1761–1782; MR4023391] and by L. G. DeMarco, S. C. Koch and C. T. McMullen [Math. Ann. **377** (2020), no. 1-2, 1–18; MR4099617].

The proof of the result relies on the use of quasiconformal mappings and equilateral triangulation of Riemann surfaces. The latter is related to the existence of Belyi functions, which is classical when the surface is compact and was obtained recently by Bishop and Rempe when the surface is non-compact. To prove their result, the authors need additional information on the diameter of the triangles in the equilateral triangulation (Theorem B). *Weiwei Cui*

[References for MR4657437]

1. Ahlfors, L.V.: Lectures on Quasiconformal Mappings, Volume 38 of University Lecture Series, 2nd edn. American Mathematical Society, Providence (2006).. (**With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard**) MR2241787
2. Barański, K.: On realizability of branched coverings of the sphere. Topol. Appl. **116**(3), 279–291 (2001) MR1857667
3. Bergweiler, W.: Iteration of meromorphic functions. Bull. Am. Math. Soc. (N.S.) **29**(2), 151–188 (1993) MR1216719
4. Bishop, C.J.: True trees are dense. Invent. Math. **197**(2), 433–452 (2014) MR3232011
5. Bishop, C.J., Lazebnik, K.: Prescribing the postsingular dynamics of meromorphic functions. Math. Ann. **375**(3–4), 1761–1782 (2019) MR4023391
6. Bishop, C.J., Rempe, L.: Non-compact Riemann surfaces are equilaterally triangulable. arXiv:2103.16702 (arXiv e-prints) (2021)
7. DeMarco, L.G., Koch, S.C., McMullen, C.T.: On the postcritical set of a rational map. Math. Ann. **377**(1–2), 1–18 (2020) MR4099617
8. Epstein, D.B.A., Marden, A., Markovic, V.: Quasiconformal homeomorphisms and the convex hull boundary. Ann. Math. (2) **159**(1), 305–336 (2004) MR2052356
9. Garnett, J., Marshall, D.: Harmonic Measure, New Mathematical Monographs, vol. 2. Cambridge University Press, Cambridge (2005) MR2150803
10. Lazebnik, K.: Oscillating wandering domains for functions with escaping singular values. J. Lond. Math. Soc. (2) **103**(4), 1643–1665 (2021) MR4273483
11. Lehto, O., Virtanen, K.I.: Quasiconformal Mappings in the Plane, 2nd edn. Springer, New York (1973).. (**Translated from the German by K, p. 126. W. Lucas, Die**

Grundlehren der mathematischen Wissenschaften, Band) MR0344463
12. Lando, S.K., Zvonkin, A.K.: Graphs on Surfaces and Their Applications, Volume 141 of Encyclopaedia of Mathematical Sciences. Springer, Berlin (2004) MR2036721
13. MacManus, P.: Bi-Lipschitz extensions in the plane. J. Anal. Math. **66**, 85–115 (1995) MR1370347
14. Milnor, J.: Dynamics in One Complex Variable, Volume 160 of Annals of Mathematics Studies, 3rd edn. Princeton University Press, Princeton (2006) MR2193309
15. Martí-Pete, D., Shishikura, M.: Wandering domains for entire functions of finite order in the Eremenko-Lyubich class. Proc. Lond. Math. Soc. (3) **120**(2), 155–191 (2020) MR4008367
16. Nicks, D.A., Sixsmith, D.J.: Which sequences are orbits? Anal. Math. Phys. **11**(2), 14 (2021) MR4216362
17. Rudin, W.: Functional Analysis. International Series in Pure and Applied Mathematics, 2nd edn. McGraw-Hill Inc, New York (1991) MR1157815
18. Tukia, P.: Extension of quasisymmetric and Lipschitz embeddings of the real line into the plane. Ann. Acad. Sci. Fenn. Ser. A I Math. **6**(1), 89–94 (1981) MR0639966
19. Voevodskiĭ, V.A., Shabat, G.B.: Equilateral triangulations of Riemann surfaces, and curves over algebraic number fields. Dokl. Akad. Nauk SSSR **304**(2), 265–268 (1989) MR0988486

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Bishop, Christopher J. (1-SUNYS-NDM); **Lazebnik, Kirill** (1-NTXS-NDM); **Urbański, Mariusz** (1-NTXS-NDM)

Correction to: Equilateral triangulations and the postcritical dynamics of meromorphic functions. (English. English summary)

Math. Ann. **388** (2024), *no.* 1, 1117.

MR4670369 52B70 52B55 68U05

Bishop, Christopher J. (1-SUNYS)

Uniformly acute triangulations of polygons. (English. English summary)

Discrete Comput. Geom. **70** (2023), *no.* 4, 1571–1592.

The primary result of this paper is to establish the existence of triangulations with angles between 30 and 75 degrees for polygons without an interior angle less than 30 degrees. It largely builds off the author's previous work [“Optimal triangulation of polygons”, preprint, 2021; revised 2023, 2025]. Despite the wider angle range this does represent an improvement on his earlier result, due to the tighter restriction on what happens when a polygon has an interior angle of less than 30 degrees. Unlike “Optimal triangulation of polygons”, where proof relies on a conformal mapping of the original polygon's interior to another polygon, here the interior of the original polygon is mapped to an unbounded space with “infinite ends” corresponding to each of the original vertices. The bulk of the text is spent establishing an appropriate triangulation of this space. A working understanding of the author's previous methods and results is beneficial for a thorough understanding of what is presented here. *Lindsay C. Piechnik*

[References]

1. Baker, B.S., Grosse, E., Rafferty, C.S.: Nonobtuse triangulation of polygons. Discrete Comput. Geom. **3**(2), 147–168 (1988) MR0920700
2. Bern, M., Eppstein, D.: Mesh generation and optimal triangulation. In: Computing

in Euclidean Geometry. Lecture Notes Ser. Comput., vol. 1, pp. 23–90. World Sci. Publ., River Edge (1992) MR1239190
3. Bern, M., Eppstein, D., Gilbert, J.: Provably good mesh generation. J. Comput. Syst. Sci. **48**(3), 384–409 (1994) MR1279408
4. Bern, M., Mitchell, S., Ruppert, J.: Linear-size nonobtuse triangulation of polygons. Discrete Comput. Geom. **14**(4), 411–428 (1995) MR1360945
5. Bern, M., Plassmann, P.: Mesh generation. In: Handbook of Computational Geometry, pp. 291–332. North-Holland, Amsterdam (2000) MR1746679
6. Bishop, C.J.: Optimal angle bounds for quadrilateral meshes. Discrete Comput. Geom. **44**(2), 308–329 (2010) MR2671014
7. Bishop, C.J.: Nonobtuse triangulations of PSLGs. Discrete Comput. Geom. **56**(1), 43–92 (2016) MR3509031
8. Bishop, C.J.: Optimal triangulation of polygons (2021). https://www.math.stonybrook.edu/~bishop/papers/opttri.pdf
9. Bishop, C.J.: Uniformly acute triangulations for PSLGs. Discrete Comput. Geom. (2023). https://doi.org/10.1007/s00454-023-00524-x MR4650037
10. Brandts, J., Korotov, S., Křížek, M., Šolc, J.: On nonobtuse simplicial partitions. SIAM Rev. **51**(2), 317–335 (2009) MR2505583
11. Burago, Yu.D., Zalgaller, V.A.: Polyhedral embedding of a net. Vestnik Leningrad. Univ. **15**(7), 66–80 (1960). (in Russian) MR0116317
12. Burago, Yu.D., Zalgaller, V.A.: Isometric piecewise-linear embeddings of two-dimensional manifolds with a polyhedral metric into $\mathbb{R}^3$. Algebra i Analiz **7**(3), 76–95 (1995). (in Russian) MR1353490
13. Cassidy, Ch., Lord, G.: A square acutely triangulated. J. Recreational Math. **13**(4), 263–268 (1980/1981) MR0625260
14. Driscoll, T.A.: Algorithm 843: improvements to the Schwarz–Christoffel toolbox for MATLAB. ACM Trans. Math. Softw. **31**(2), 239–251 (2005) MR2266791
15. Driscoll, T.A., Trefethen, L.N.: Schwarz–Christoffel Mapping. Cambridge Monographs on Applied and Computational Mathematics, vol. 8. Cambridge University Press, Cambridge (2002) MR1908657
16. Edelsbrunner, H.: Triangulations and meshes in computational geometry. Acta Numer. **9**, 133–213 (2000) MR1883628
17. Eppstein, D.: Acute square triangulation (2021). https://www.ics.uci.edu/~eppstein/junkyard/acute-square/
18. Erten, H., Üngör, A.: Computing acute and non-obtuse triangulations. In: 19th Canadian Conference on Computational Geometry (Ottawa 2007), pp. 205–208. http://cccg.ca/proceedings/2007/09a2.pdf
19. Gerver, J.L.: The dissection of a polygon into nearly equilateral triangles. Geom. Dedicata **16**(1), 93–106 (1984) MR0757798
20. Hangan, Th., Itoh, J., Zamfirescu, T.: Acute triangulations. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **43**(**91**)(3–4), 279–285 (2000) MR1837482
21. Itoh, J.: Acute triangulations of sphere and icosahedron. In: 1st International Symposium on Differential Geometry (Sakado 2001). Josai Math. Monogr., vol. 3, pp. 53–62. Josai University, Sakado (2001) MR1824599
22. Itoh, J., Yuan, L.: Acute triangulations of flat tori. Eur. J. Comb. **30**(1), 1–4 (2009) MR2460210
23. Itoh, J., Zamfirescu, T.: Acute triangulations of the regular icosahedral surface. Discrete Comput. Geom. **31**(2), 197–206 (2004) MR2060635
24. Itoh, J., Zamfirescu, T.: Acute triangulations of the regular dodecahedral surface. Eur. J. Comb. **28**(4), 1072–1086 (2007) MR2305575
25. Kopczyński, E., Pak, I., Przytycki, P.: Acute triangulations of polyhedra and $\mathbb{R}^N$.

Combinatorica **32**(1), 85–110 (2012) MR2927633
26. Křížek, M.: There is no face-to-face partition of $\mathbb{R}^5$ into acute simplices. Discrete Comput. Geom. **36**(2), 381–390 (2006) MR2252110
27. Li, J.Y.S., Zhang, H.: Nonobtuse remeshing and mesh decimation. In: 4th Eurographics Symposium on Geometry Processing (Cagliari 2006), pp. 235–238. Eurographics Association, Goslar (2006)
28. Lindgren, H.: Dividing a square into acute-angled triangles. Aust. Math. Teach. **18**, 14–15 (1962)
29. Maehara, H.: On acute triangulations of quadrilaterals. In: Japanese Conference on Discrete and Computational Geometry (Tokyo 2000). Lecture Notes in Comput. Sci., vol. 2098, pp. 237–243. Springer, Berlin (2001) MR2043655
30. Maehara, H.: Acute triangulations of polygons. Eur. J. Comb. **23**(1), 45–55 (2002) MR1878775
31. Manheimer, W.: Dissecting an obtuse triangle into acute triangles (solution to problem E1406). Am. Math. Mon. **67**(9), 923 (1960)
32. Melissaratos, E.A., Souvaine, D.L.: Coping with inconsistencies: a new approach to produce quality triangulations of polygonal domains with holes. In: 8th Annual Symposium on Computational Geometry (Berlin 1992), pp. 202–211. ACM, New York (1992)
33. Nehari, Z.: Conformal Mapping. Dover, New York (1975) MR0377031
34. Ruppert, J.: A new and simple algorithm for quality 2-dimensional mesh generation. In: 4th Annual ACM-SIAM Symposium on Discrete Algorithms (Austin 1993), pp. 83–92. ACM, New York (1993) MR1213221
35. Saraf, S.: Acute and nonobtuse triangulations of polyhedral surfaces. Eur. J. Comb. **30**(4), 833–840 (2009) MR2504642
36. Trefethen, L.N., Driscoll, T.A.: Schwarz–Christoffel mapping in the computer era. In: International Congress of Mathematicians (Berlin 1998), vol. 3. Doc. Math., pp. 533–542. Deutsche Mathematiker-Vereinigung, Berlin (1998) MR1648186
37. Yuan, L.: Acute triangulations of polygons. Discrete Comput. Geom. **34**(4), 697–706 (2005) MR2173934
38. Yuan, L., Zamfirescu, C.T.: Acute triangulations of doubly covered convex quadrilaterals. Boll. Unione Mat. Ital. **10-B**(3), 933–938 (2007) MR2507906
39. Yuan, L., Zamfirescu, T.: Acute triangulations of flat Möbius strips. Discrete Comput. Geom. **37**(4), 671–676 (2007) MR2321748
40. Zamfirescu, C.: Acute triangulations of the double triangle. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **47**(**95**)(3-4), 189–193 (2004) MR2121985
41. Zamfirescu, C.T.: Survey of two-dimensional acute triangulations. Discrete Math. **313**(1), 35–49 (2013) MR3016971

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MR4650037 68U05 05C10 52B55

Bishop, Christopher J. (1-SUNYS)

Uniformly acute triangulations of PSLGs. (English. English summary)

Discrete Comput. Geom. **70** (2023), *no.* 3, 1090–1120.

Summary: "We show that any PSLG Γ has an acute conforming triangulation $\mathcal{T}$ with an upper angle bound that is strictly less than $90°$ and that depends only on the minimal angle occurring in Γ. In fact, all angles are inside $[\theta_0, 90° - \theta_0/2]$ for some fixed $\theta_0 > 0$ independent of Γ, except for triangles T containing a vertex v of Γ where Γ has an interior angle $\theta_v < \theta_0$; then T is an isosceles triangle with angles in the sharpest possible interval $[\theta_v, 90° - \theta_v/2]$."

[References]

1. Bern, M., Mitchell, S., Ruppert, J.: Linear-size nonobtuse triangulation of polygons. Discrete Comput. Geom. **14**(4), 411–428 (1995) MR1360945
2. Bern, M., Shewchuk, J.R., Amenta N.: Triangulations and mesh generation. In: Handbook of Discrete and Computational Geometry, 3rd ed., pp. 763–785. CRC Press, Boca Raton (2017) MR3190484
3. Bishop, C.J.: Optimal angle bounds for quadrilateral meshes. Discrete Comput. Geom. **44**(2), 308–329 (2010) MR2671014
4. Bishop, C.J.: Conformal mapping in linear time. Discrete Comput. Geom. **44**(2), 330–428 (2010) MR2671015
5. Bishop, C.J.: Quadrilateral meshes for PSLGs. Discrete Comput. Geom. **56**(1), 1–42 (2016) MR3509030
6. Bishop, C.J.: Nonobtuse triangulations of PSLGs. Discrete Comput. Geom. **56**(1), 43–92 (2016) MR3509031
7. Bishop, C.J.: Optimal triangulation of polygons (2021). https://www.math.stonybrook.edu/~bishop/papers/opttri.pdf
8. Bishop, C.J.: Uniformly acute triangulations of polygons. Discrete Comput. Geom. (2023). https://doi.org/10.1007/s00454-023-00525-w MR4670369
9. Brunck, F.: Acute triangulations of spherical and hyperbolic triangle complexes. Preprint (2022)
10. Brunck, F.: Iterated medial triangle subdivision in surfaces of constant curvature. Discrete Comput. Geom. (2023). https://doi.org/10.1007/s00454-023-00500-5 MR4650036
11. Burago, Yu.D., Zalgaller, V.A.: Polyhedral embedding of a net. Vestnik Leningrad. Univ. **15**(7), 66–80 (1960). (in Russian) MR0116317
12. Gabriel, K.R., Sokal, R.R.: A new statistical approach to geographic variation analysis. Syst. Zool. **18**(3), 259–278 (1969)
13. Maehara, H.: Acute triangulations of polygons. Eur. J. Combin. **23**(1), 45–55 (2002) MR1878775
14. Mumford, D.: A remark on Mahler's compactness theorem. Proc. Am. Math. Soc. **28**, 289–294 (1971) MR0276410
15. Saraf, Sh.: Acute and nonobtuse triangulations of polyhedral surfaces. Eur. J. Combin. **30**(4), 833–840 (2009) MR2504642
16. Yuan, L.: Acute triangulations of polygons. Discrete Comput. Geom. **34**(4), 697–706 (2005) MR2173934
17. Zamfirescu, C.T.: Survey of two-dimensional acute triangulations. Discrete Math. **313**(1), 35–49 (2013) MR3016971

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MR4526317 30F60 30C62

Bishop, Christopher J. (1-SUNYS)

Function theoretic characterizations of Weil-Petersson curves. (English. English summary)

Rev. Mat. Iberoam. **38** (2022), *no.* 7, 2355–2384.

The paper under review studies the class of Weil-Petersson curves, that is, the closure of the closed smooth curves in the Weil-Petersson metric on universal Teichmüller space as introduced by L. A. Takhtajan and L.-P. Teo [Mem. Amer. Math. Soc. **183** (2006), no. 861, viii+119 pp.; MR2251887]. The author of the current paper has already, in a preprint ["Weil-Petersson curves, conformal energies, β-numbers, and minimal surfaces", 2020] at the time of writing this review, collated 20 (!) different characterizations on Weil-Petersson curves.

Here, the author collects seven characterizations of Weil-Petersson curves, some of which are new and some of which were previously known, but new proofs are provided. To highlight just one of these, a characterization is given in terms of P. W. Jones' β-numbers used in the proof of the analyst's traveling salesman problem [Invent. Math. **102** (1990), no. 1, 1–15; MR1069238]. Jones proved that the length of a curve Γ can be approximated by $\mathrm{diam}(\Gamma)+\sum\beta_\Gamma(Q)^2\mathrm{diam}(Q)$, where the sum is over all dyadic cubes. In this paper, it is proved that Γ is a Weil-Petersson curve if and only if $\sum\beta_\Gamma(Q)^2$ is finite, showing that Weil-Petersson curves have a particularly strong form of rectifiability. *Alastair N. Fletcher*

[References]

1. Adams, R. A. and Fournier, J. J. F.: *Sobolev spaces.* Second edition. Pure and Applied Mathematics (Amsterdam) 140, Elsevier/Academic Press, Amsterdam, 2003. MR2424078
2. Ahlfors, L. V.: *Lectures on quasiconformal mappings.* Second edition. University Lecture Series 38, American Mathematical Society, Providence, RI, 2006. MR2241787
3. Ahlfors, L. V.: *Conformal invariants. Topics in geometric function theory.* Reprint of the 1973 original. AMS Chelsea Publishing, Providence, RI, 2010. MR2730573
4. Ahlfors, L. and Weill, G.: A uniqueness theorem for Beltrami equations. *Proc. Amer. Math. Soc.* **13** (1962), 975–978. MR0148896
5. Alexakis, S. and Mazzeo, R.: Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds. *Comm. Math. Phys.* **297** (2010), no. 3, 621–651. MR2653898
6. Astala, K. and Zinsmeister, M.: Teichmüller spaces and BMOA. *Math. Ann.* **289** (1991), no. 4, 613–625. MR1103039
7. Becker, J. and Pommerenke, C.: Über die quasikonforme Fortsetzung schlichter Funktionen. *Math. Z.* **161** (1978), no. 1, 69–80. MR0507822
8. Beurling, A.: *Études sur un problème de majoration.* Ph.D. Thesis, Uppsala, 1933.
9. Beurling, A. and Ahlfors, L.: The boundary correspondence under quasiconformal mappings. *Acta Math.* **96** (1956), 125–142. MR0086869
10. Bishop, C. J.: Conformal welding and Koebe's theorem. *Ann. of Math. (2)* **166** (2007), no. 3, 613–656. MR2373370
11. Bishop, C. J.: Weil–Petersson curves, conformal energies, β-numbers and minimal surfaces. Preprint 2020. Available at: www.math.stonybrook.edu/bishop/papers/wpbeta.pdf.
12. Bishop, C. J.: The traveling salesman theorem for Jordan curves. *Adv. Math.* **404** (2022), Paper no. 108443, 27 pp. MR4420442
13. Bishop, C. J. and Jones, P. W.: Harmonic measure, L^2 estimates and the Schwarzian

derivative. *J. Anal. Math.* **62** (1994), 77–113. MR1269200
14. Bowick, M. J. and Rajeev, S. G.: The holomorphic geometry of closed bosonic string theory and Diff S^1/S^1. *Nuclear Phys. B* **293** (1987), no. 2, 348–384. MR0908048
15. Bowick, M. J. and Rajeev, S. G.: String theory as the Kähler geometry of loop space. *Phys. Rev. Lett.* **58** (1987), no. 6, 535–538. MR0873068
16. Chang, S.-Y. A. and Marshall, D. E.: On a sharp inequality concerning the Dirichlet integral. *Amer. J. Math.* **107** (1985), no. 5, 1015–1033. MR0805803
17. Chuaqui, M. and Osgood, B.: Ahlfors–Weill extensions of conformal mappings and critical points of the Poincaré metric. *Comment. Math. Helv.* **69** (1994), no. 4, 659–668. MR1303231
18. Cui, G.: Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces. *Sci. China Ser. A* **43** (2000), no. 3, 267–279. MR1766456
19. Di Nezza, E., Palatucci, G. and Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), no. 5, 521–573. MR2944369
20. Douglas, J.: Solution of the problem of Plateau. *Trans. Amer. Math. Soc.* **33** (1931), no. 1, 263–321. MR1501590
21. Dyn'kin, E.: Estimates for asymptotically conformal mappings. *Ann. Acad. Sci. Fenn. Math.* **22** (1997), no. 2, 275–304. MR1469792
22. Fefferman, R. A., Kenig, C. E. and Pipher, J.: The theory of weights and the Dirichlet problem for elliptic equations. *Ann. of Math. (2)* **134** (1991), no. 1, 65–124. MR1114608
23. Feiszli, M., Kushnarev, S. and Leonard, K.: Metric spaces of shapes and applications: compression, curve matching and low-dimensional representation. *Geom. Imaging Comput.* **1** (2014), no. 2, 173–221. MR3396622
24. Feiszli, M. and Narayan, A.: Numerical computation of Weil–Peterson geodesics in the universal Teichmüller space. *SIAM J. Imaging Sci.* **10** (2017), no. 3, 1322–1345. MR3686793
25. Gallardo-Gutiérrez, E. A., González, M. J., Pérez-González, F., Pommerenke, C. and Rättyä, J.: Locally univalent functions, VMOA and the Dirichlet space. *Proc. Lond. Math. Soc. (3)* **106** (2013), no. 3, 565–588. MR3048550
26. Garnett, J. B.: *Bounded analytic functions.* Pure and Applied Mathematics 96, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1981. MR0628971
27. Garnett, J. B. and Marshall, D. E.: *Harmonic measure.* New Mathematical Monographs 2, Cambridge University Press, Cambridge, 2008. MR2450237
28. Jones, P. W.: Rectifiable sets and the traveling salesman problem. *Invent. Math.* **102** (1990), no. 1, 1–15. MR1069238
29. Koosis, P.: *Introduction to H_p spaces.* Second edition. Cambridge Tracts in Mathematics 115, Cambridge University Press, Cambridge, 1998. MR1669574
30. Lehto, O.: *Univalent functions and Teichmüller spaces.* Graduate Texts in Mathematics 109, Springer-Verlag, New York, 1987. MR0867407
31. Marshall, D. E.: A new proof of a sharp inequality concerning the Dirichlet integral. *Ark. Mat.* **27** (1989), no. 1, 131–137. MR1004727
32. Mesikepp, T.: *How to weld: energies, welding and driving functions.* Ph.D. Thesis, Univ. of Washington, 2021. MR4315314
33. Nag, S. and Sullivan, D.: Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle. *Osaka J. Math.* **32** (1995), no. 1, 1–34. MR1323099
34. Nishioka, T., Ryu, S. and Takayanagi, T.: Holographic entanglement entropy: an overview. *J. Phys. A* **42** (2009), no. 50, 504008, 35 pp. MR2566335
35. Okikiolu, K.: Characterization of subsets of rectifiable curves in $\mathbb{R}^n$. *J. London*

Math. Soc. (2) **46** (1992), no. 2, 336–348. MR1182488
36. Osgood, B.: Old and new on the Schwarzian derivative. In *Quasiconformal mappings and analysis (Ann Arbor, MI, 1995)*, pp. 275–308. Springer, New York, 1998. MR1488455
37. Pau, J. and Peláez, J. A.: Logarithms of the derivative of univalent functions in Q_p spaces. *J. Math. Anal. Appl.* **350** (2009), no. 1, 184–194. MR2476901
38. Pavlović, M. and Vukotić, D.: The weak Chang–Marshall inequality via Green's formula. *Rocky Mountain J. Math.* **36** (2006), no. 5, 1631–1636. MR2285305
39. Pérez-González, F. and Rättyä, J.: Dirichlet and VMOA domains via Schwarzian derivative. *J. Math. Anal. Appl.* **359** (2009), no. 2, 543–546. MR2546769
40. Pommerenke, C.: On univalent functions, Bloch functions and VMOA. *Math. Ann.* **236** (1978), no. 3, 199–208. MR0492206
41. Radnell, D., Schippers, E. and Staubach, W.: Quasiconformal Teichmüller theory as an analytical foundation for two-dimensional conformal field theory. In *Lie algebras, vertex operator algebras, and related topics*, pp. 205–238. Contemp. Math. 695, Amer. Math. Soc., Providence, RI, 2017. MR3709713
42. Reich, E.: Quasiconformal mappings of the disk with given boundary values. In *Advances in complex function theory (Proc. Sem., Univ. Maryland, College Park, Md., 1973–1974)*, pp. 101–137. Lecture Notes in Math. 505, Springer, Berlin, 1976. MR0419758
43. Rohde, S. and Wang, Y.: The Loewner energy of loops and regularity of driving functions. *Int. Math. Res. Not. IMRN* (2021), no. 10, 7715–7763. MR4259153
44. Ross, W. T.: The classical Dirichlet space. In *Recent advances in operator-related function theory*, pp. 171–197. Contemp. Math. 393, Amer. Math. Soc., Providence, RI, 2006. MR2198379
45. Ryu, S. and Takayanagi, T.: Aspects of holographic entanglement entropy. *J. High Energy Phys.* **2006** (2006), no. 8, 045, 48 pp. MR2249925
46. Sharon, E. and Mumford, D.: 2D-Shape analysis using conformal mapping. *Int. J. Comput. Vis.* **70** (2006), 55–75.
47. Shekhar, A., Tran, H. and Wang, Y.: Remarks on Loewner chains driven by finite variation functions. *Ann. Acad. Sci. Fenn. Math.* **44** (2019), no. 1, 311–327. MR3919140
48. Shen, Y.: Weil–Petersson Teichmüller space. *Amer. J. Math.* **140** (2018), no. 4, 1041–1074. MR3828040
49. Shen, Y. and Wei, H.: Universal Teichmüller space and BMO. *Adv. Math.* **234** (2013), 129–148. MR3003927
50. Takayanagi, T.: Entanglement entropy from a holographic viewpoint. *Classical Quantum Gravity* **29** (2012), no. 15, 153001, 25 pp. MR2960962
51. Takhtajan, L. A. and Teo, L.-P.: Weil–Petersson metric on the universal Teichmüller space. *Mem. Amer. Math. Soc.* **183** (2006), no. 861, viii+119 pp. MR2251887
52. Viklund, F. and Wang, Y.: Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines. *Geom. Funct. Anal.* **30** (2020), no. 1, 289–321. MR4080509
53. Wang, Y.: The energy of a deterministic Loewner chain: reversibility and interpretation via SLE0C. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 7, 1915–1941. MR3959854
54. Wang, Y.: Equivalent descriptions of the Loewner energy. *Invent. Math.* **218** (2019), no. 2, 573–621. MR4011706
55. Wang, Y.: A note on Loewner energy, conformal restriction and Werner's measure on selfavoiding loops. *Ann. Inst. Fourier (Grenoble)* **71** (2021), no. 4, 1791–1805. MR4398248

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Bishop, Christopher J. (1-SUNYS)

Quasiconformal maps with thin dilatations. (English. English summary)

Publ. Mat. **66** (2022), *no.* 2, 715–727.

A quasiconformal mapping (homeomorphism) $F\colon \mathbb{C}\to\mathbb{C}$ can be regarded as a solution of the Beltrami equation

$$\frac{\partial F}{\partial \overline{z}} = \mu \frac{\partial F}{\partial z}$$

where μ is a complex-valued measurable function on $\mathbb{C}$ that satisfies $\|\mu\|_\infty < 1$. Such a μ is called the complex dilatation of F. In the paper under review, the author gives some estimates of normalized quasiconformal mappings when the set

$$E := \{z \in \mathbb{C} \mid \mu(z) \neq 0\}$$

has small area. More precisely, the author assumes that the set E is (ϵ, h)-*thin*, that is, $\epsilon > 0$ and

$$\mathrm{area}(E \cap D(z,1)) \leq \epsilon\, h(|z|)$$

for any $z \in \mathbb{C}$, where $D(z,1) = \{w \in \mathbb{C} \mid |w-z| < 1\}$ and $h\colon [0,\infty) \to [0,\pi]$ is a bounded decreasing function with

$$\int_0^\infty h(r)\, r^n\, dr < \infty$$

for every $n > 1$.

If E is a bounded set, then F is conformal near ∞ and we can normalize F such that $|F(z)-z| = O(1/z)$ as $|z| \to \infty$ by composing a complex affine mapping. Then the author shows that for all $z \in \mathbb{C}$,

$$|F(z)-z| = \frac{\epsilon^\beta}{|z|+1}$$

for some $\beta > 0$ depending only on $\|\mu\|_\infty$ and h (Theorem 1.1). This implies that F converges uniformly to the identity on the whole plane when $\epsilon \to 0$.

If E happens to be unbounded, we can normalize F so that $F(0) = 0$ and $F(1) = 1$ by composing with another complex affine mapping. Then the author shows that for all z and $w \in \mathbb{C}$,

$$(1 - C\epsilon^\beta)|z-w| - C\epsilon^\beta \leq |f(z) - f(w)| \leq (1 + C\epsilon^\beta)|z-w| + C\epsilon^\beta$$

for some C, $\beta > 0$ depending only on $\|\mu\|_\infty$ and h (Corollary 1.2). This result was previously used in [N. Fagella, S. Godillon and X. Jarque, J. Math. Anal. Appl. **429** (2015), no. 1, 478–496; MR3339086] to construct wandering domains of transcendental entire functions. *Tomoki Kawahira*

[References]

1. L. V. Ahlfors, *"Lectures on Quasiconformal Mappings"*, Second edition, With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard, University Lecture Series **38**, American Mathematical Society, Providence, RI, 2006. `DOI: 10.1090/ulect/038`. MR2241787
2. S. Albrecht and C. J. Bishop, Speiser class Julia sets with dimension near one, *J. Anal. Math.* **141(1)** (2020), 49–98. `DOI: 10.1007/s11854-020-0128-1`. MR4174037

3. C. J. Bishop, Constructing entire functions by quasiconformal folding *Acta Math.* **214(1)** (2015), 1–60. `DOI: 10.1007/s11511-015-0122-0`. MR3316755
4. C. J. Bishop and K. Lazebnik, Prescribing the postsingular dynamics of meromorphic functions, *Math. Ann.* **375(3-4)** (2019), 1761–1782. `DOI: 10.1007/s00208-019-01869-6`. MR4023391
5. C. J. Bishop and L. Rempe, Non-compact Riemann surfaces are equilaterally triangulable, Preprint (2021). `arXiv:2103.16702v2`.
6. E. M. Dyn'kin, Smoothness of a quasiconformal mapping at a point (Russian) *Algebra i Analiz* **9(3)** (1997), 205–210; translation in *St. Petersburg Math. J.* **9(3)** (1998), 601–605. MR1466801
7. N. Fagella, S. Godillon, and X. Jarque, Wandering domains for composition of entire functions, *J. Math. Anal. Appl.* **429(1)** (2015), 478–496. `DOI: 10.1016/j.jmaa.2015.04.020`. MR3339086
8. N. Fagella, X. Jarque, and K. Lazebnik, Univalent wandering domains in the Eremenko–Lyubich class, *J. Anal. Math.* **139(1)** (2019), 369–395. `DOI: 10.1007/s11854-027-0079-x`. MR4041106
9. A. A. Goldberg and I. V. Ostrovskii, *"Value Distribution of Meromorphic Functions"*, Translated from the 1970 Russian original by Mikhail Ostrovskii, With an appendix by Alexandre Eremenko and James K. Langley, Translations of Mathematical Monographs **236**, American Mathematical Society, Providence, RI, 2008. `DOI: 10.1090/mmono/236`. MR2435270
10. K. Lazebnik, Several constructions in the Eremenko–Lyubich class *J. Math. Anal. Appl.* **448(1)** (2017), 611–632. `DOI: 10.1016/j.jmaa.2016.11.007`. MR3579902
11. J. W. Osborne and D. J. Sixsmith, On the set where the iterates of an entire function are neither escaping nor bounded, *Ann. Acad. Sci. Fenn. Math.* **41(2)** (2016), 561–578. `DOI: 10.5186/aasfm.2016.4134`. MR3525384
12. L. Rempe-Gillen, Arc-like continua, Julia sets of entire functions, and Eremenko's Conjecture, Preprint (2016). `arXiv:1610.06278`.
13. O. Teichmüller, Eine Umkehrung des zweiten Hauptsatzes der Wertverteilungslehre *Deutsche Math.* **2** (1937), 96–107.
14. H. Wittich, Zum Beweis eines Satzes über quasikonforme Abbildungen *Math. Z.* **51** (1948), 278–288. `DOI: 10.1007/BF01181593`. MR0027057
15. Y. Zhang and G. Zhang, Constructing entire functions with non-locally connected Julia set by quasiconformal surgery, *Sci. China Math.* **61(9)** (2018), 1637–1646. `DOI: 10.1007/s11425-017-93330-0`. MR3845329

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Bishop, Christopher J. (1-SUNYS)

The traveling salesman theorem for Jordan curves. (English. English summary)

Adv. Math. **404** (2022), *part A*, *Paper No.* 108443, 27 *pp.*

The main result of this paper sharpens the classical Analyst's Traveling Salesman Theorem for Jordan arcs in $\mathbb{R}^n$. As an application, the author provides new metric characterizations for the n-dimensional analog of the Weil-Petersson class of planar curves.

The Analyst's Traveling Salesman Theorem was first established by P. W. Jones [Invent. Math. **102** (1990), no. 1, 1–15; MR1069238] and later extended by K. Okikiolu to higher dimensions [J. London Math. Soc. (2) **46** (1992), no. 2, 336–348; MR1182488] and by R. Schul to Hilbert spaces [J. Anal. Math. **103** (2007), 331–375; MR2373273]. This theorem provides an intrinsic and quantitative characterization for the collection of sets which lie within rectifiable curves. To state the result precisely, we introduce the

Jones β numbers. For a set $E \subset \mathbb{R}^n$ and a dyadic cube Q, set

$$\beta_E(Q) := \frac{1}{\operatorname{diam}(Q)} \inf_L \sup\{\operatorname{dist}(x, L) : x \in 3Q \cap E\},$$

where the infimum is taken over all lines L that intersect Q, and $3Q$ denotes the cube concentric with Q whose diameter is equal to $3\operatorname{diam}(Q)$. The Analyst's Traveling Salesman Theorem states that for any given set $E \subset \mathbb{R}^n$, the shortest curve Γ containing E has length $\ell(\Gamma)$ which satisfies

$$\ell(\Gamma) \simeq \operatorname{diam}(E) + \sum_Q \beta_E^2(Q) \operatorname{diam}(Q).$$

Here, the sum is taken over all dyadic cubes Q in $\mathbb{R}^n$, and the notation $A \simeq B$ means that $C^{-1}A \le B \le CA$ for some universal constant C. Jones' original proof was formulated for $n = 2$; Okikiolu generalized this to general n with constant $C = C(n)$ and Schul's extension to Hilbert spaces showed (in particular) that the n-dimensional version holds with constant C independent of n.

The first main result of this paper can be formulated as follows: Let Γ be a Jordan arc in $\mathbb{R}^n$ with endpoints x and y. Then

$$\operatorname{diam}(\Gamma) - \frac{1}{C}\sum_Q \beta_E^2(Q)\operatorname{diam}(Q) \le \ell(\Gamma) \le \operatorname{diam}(\Gamma) + C\sum_Q \beta_E^2(Q)\operatorname{diam}(Q), \tag{1}$$

and

$$|x-y| - \frac{1}{C}\sum_Q \beta_E^2(Q)\operatorname{diam}(Q) \le \ell(\Gamma) \le |x-y| + C\sum_Q \beta_E^2(Q)\operatorname{diam}(Q),$$

for some constant $C = C(n) > 0$. This sharpening of the classical Analyst's Traveling Salesman Theorem has several intriguing consequences. For instance, the replacement of $\operatorname{diam}(\Gamma)$ with the 'chordal length' $|x - y|$ implies that, for closed Jordan curves Γ, the length of Γ is comparable to the β number sum (with no additional term).

The second main result of the paper characterizes the aforementioned class of closed Jordan curves, i.e., the set of curves for which the corresponding β number sum (without the diameter summand) remains finite. When $n = 2$, the class of closed curves so obtained is the Weil-Petersson class, which arises in connection with string theory, pattern recognition, quasiconformal mapping theory, and in the study of Schramm-Loewner evolution. The n-dimensional analog of these curves features in Theorem 1.4 of the paper, which asserts the following: For a closed Jordan curve Γ in $\mathbb{R}^n$, the following are equivalent:

- $\sum_Q \beta_\Gamma^2(Q) < \infty$;
- Γ is a chord-arc curve and, denoting by Γ_m the (polygonal) level m dyadic decomposition of Γ, it holds true that $\sum_{m\ge1} 2^m(\ell(\Gamma) - \ell(\Gamma_m)) < \infty$;
- Γ has finite Möbius energy, i.e., $\int_\Gamma\int_\Gamma (|x-y|^{-2} - \ell(\Gamma_{x,y})^{-2})\,dx\,dy < \infty$.

Here, $\Gamma_{x,y}$ denotes the shorter of the two subarcs of Γ connecting x to y.

To conclude this review, we briefly sketch the proof of the upper bound in (1). Without loss of generality, assume that Γ is bounded. The proof consists of an inductive construction of a sequence of nested compact sets $\{\Gamma_m : m \ge 0\}$ shrinking down to Γ. The first set Γ_0 is the convex hull of Γ, while Γ_m is a union of 2^m compact, convex sets that cover Γ. The constituent sets in Γ_m are obtained by splitting each of the sets in Γ_{m-1} into two pieces. The splitting operation is as follows: If R is a compact, convex set, choose a suitable diameter I of R (i.e., a closed interval $I \subset R$ so that $\operatorname{diam}(I) = \operatorname{diam}(R)$), divide I into equal halves I_1 and I_2, and let R_1 and R_2 (respectively) be the convex hulls of the subsets of $R \cap \Gamma$ given by the preimages of I_1 and I_2 under orthogonal projection to the line through I. The key lemma (Lemma 2.1) asserts the following: If a given compact set R is split into R_1 and R_2 by the above procedure, then $\operatorname{diam}(R_1)+\operatorname{diam}(R_2)\le \operatorname{diam}(R)+C\beta^2(R)\operatorname{diam}(R)$. Here, $\beta(R)=\inf_I \sup_x \operatorname{dist}(x,I)/\operatorname{diam}(R)$, where the supremum is over all $x\in R$ and the infimum is over all diameters I of R. By induction, it follows that

$$\sum_{R:R\subset\Gamma_{m+1}} \operatorname{diam}(R) \le \sum_{R:R\subset\Gamma_m} \operatorname{diam}(R) + C\sum_{R:R\subset\Gamma_m} \beta^2(R)\operatorname{diam}(R) \le \operatorname{diam}(\Gamma) + C\sum_{R:R\subset\Gamma_{\le m}} \beta^2(R)\operatorname{diam}(R),$$

where $\Gamma_{\le m}:=\Gamma_1\cup\cdots\cup\Gamma_m$. It is a standard fact of geometric measure theory that the left-hand side converges to $\ell(\Gamma)$ as m tends to infinity. Lemma 2.4 of the paper asserts uniform boundedness for the number of such dyadic descendants of a fixed size which meet a given ball of comparable size. This boundedness condition implies that the sum on the right-hand side (over constituent elements in the mth level cumulative approximant $\Gamma_{\le m}$) is bounded above by a constant—depending on the dimension n—times the full Jones β number sum over all dyadic cubes. This completes the proof.

Jeremy T. Tyson

[References]

1. M. Aigner, G.M. Ziegler, Proofs from The Book, sixth edition, Springer, Berlin, 2018, See corrected reprint of the 1998 original [MR1723092], Including illustrations by Karl H. Hofmann. MR3823190
2. S. Arora, Polynomial time approximation schemes for Euclidean traveling salesman and other geometric problems, J. ACM 45 (5) (1998) 753–782. MR1668147
3. J. Azzam, R. Schul, An analyst's traveling salesman theorem for sets of dimension larger than one, Math. Ann. 370 (3–4) (2018) 1389–1476. MR3770170
4. M. Badger, L. Naples, V. Vellis, Holder curves and parameterizations in the Analyst's Traveling Salesman theorem, Adv. Math. 349 (2019) 564–647. MR3941391
5. Christopher J. Bishop, Weil-Petersson curves, conformal energies, β-numbers, and minimal surfaces, 2020. Preprint.
6. C.J. Bishop, P.W. Jones, Harmonic measure, L^2 estimates and the Schwarzian derivative, J. Anal. Math. 62 (1994) 77–113. MR1269200
7. C.J. Bishop, Y. Peres, Fractals in Probability and Analysis, Cambridge Studies in Advanced Mathematics, vol. 162, Cambridge University Press, Cambridge, 2017. MR3616046
8. K. Borsuk, Drei sätze über die n-dimensionale euklidische sphäre, Fundam. Math. 20 (1933) 177–190.
9. G. David, S. Semmes, Singular integrals and rectifiable sets in $\mathbf{R}^n$: beyond Lipschitz graphs, Astérisque 152 (193) (1991). MR1113517
10. G.C. David, R. Schul, The analyst's traveling salesman theorem in graph inverse limits, Ann. Acad. Sci. Fenn., Math. 42 (2) (2017) 649–692. MR3701642
11. M. Feiszli, S. Kushnarev, K. Leonard, Metric spaces of shapes and applications: compression, curve matching and low-dimensional representation, Geom. Imaging Comput. 1 (2) (2014) 173–221. MR3396622
12. M. Feiszli, A. Narayan, Numerical computation of Weil-Peterson geodesics in the universal Teichmüller space, SIAM J. Imaging Sci. 10 (3) (2017) 1322–1345. MR3686793
13. F. Ferrari, B. Franchi, H. Pajot, The geometric traveling salesman problem in the Heisenberg group, Rev. Mat. Iberoam. 23 (2) (2007) 437–480. MR2371434

14. J.B. Garnett, D.E. Marshall, Harmonic Measure, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2008, Reprint of the 2005 original. MR2450237
15. Z.-X. He, The Euler-Lagrange equation and heat flow for the Möbius energy, Commun. Pure Appl. Math. 53 (4) (2000) 399–431. MR1733697
16. P.W. Jones, Rectifiable sets and the traveling salesman problem, Invent. Math. 102 (1) (1990) 1–15. MR1069238
17. J. Kahn, G. Kalai, A counterexample to Borsuk's conjecture, Bull. Am. Math. Soc. (N.S.) 29 (1) (1993) 60–62. MR1193538
18. D.A. Klain, G.-C. Rota, Introduction to Geometric Probability, Lezioni Lincee (Lincei Lectures), Cambridge University Press, Cambridge, 1997. MR1608265
19. J. Krandel, The traveling salesman theorem for Jordan curves in Hilbert space, preprint, arXiv:2107.07017v2 [math.CA], 2022.
20. G. Lerman, Quantifying curvelike structures of measures by using L_2 Jones quantities, Commun. Pure Appl. Math. 56 (9) (2003) 1294–1365. MR1980856
21. S. Li, R. Schul, The traveling salesman problem in the Heisenberg group: upper bounding curvature, Trans. Am. Math. Soc. 368 (7) (2016) 4585–4620. MR3456155
22. S. Li, R. Schul, An upper bound for the length of a traveling salesman path in the Heisenberg group, Rev. Mat. Iberoam. 32 (2) (2016) 391–417. MR3512421
23. J.S.B. Mitchell, Guillotine subdivisions approximate polygonal subdivisions: a simple polynomial-time approximation scheme for geometric TSP, k-MST, and related problems, SIAM J. Comput. 28 (4) (1999) 1298–1309. MR1681022
24. J. O'Hara, Energy Knot Topol. 30 (2) (1991) 241–247. MR1098918
25. J. O'Hara, Energy functionals of knots, in: Topology Hawaii, Honolulu, HI, 1990, World Sci. Publ., River Edge, NJ, 1992, pp. 201–214. MR1181493
26. J. O'Hara, Family of energy functionals of knots, Topol. Appl. 48 (2) (1992) 147–161. MR1195506
27. K. Okikiolu, Characterization of subsets of rectifiable curves in $\mathbf{R}^n$, J. Lond. Math. Soc. (2) 46 (2) (1992) 336–348. MR1182488
28. H. Pajot, Analytic Capacity, Rectifiability, Menger Curvature and the Cauchy Integral, Lecture Notes in Mathematics, vol. 1799, Springer-Verlag, Berlin, 2002. MR1952175
29. S. Rohde, Y. Wang, The Loewner energy of loops and regularity of driving functions, Int. Math. Res. Notes 04, 201, arXiv:1601.05297v2 [math.CV]. MR4259153
30. L.A. Santaló, Integral Geometry and Geometric Probability, Encyclopedia of Mathematics and Its Applications, vol. 1, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976, With a foreword by Mark Kac. MR0433364
31. R. Schul, Subsets of rectifiable curves in Hilbert space—the analyst's TSP, J. Anal. Math. 103 (2007) 331–375. MR2373273
32. E. Sharon, D. Mumford, 2D-shape analysis using conformal mapping, Int. J. Comput. Vis. 70 (2006) 55–75.
33. L.A. Takhtajan, L.-P. Teo, Weil-Petersson metric on the universal Teichmüller space, Mem. Am. Math. Soc. 183 (861) (2006), viii+119. MR2251887
34. X. Tolsa, Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality, Proc. Lond. Math. Soc. (3) 98 (2) (2009) 393–426. MR2481953
35. Y. Wang, A note on Loewner energy, conformal restriction and Werner's measure on self-avoiding loops, Ann. Inst. Fourier 71 (4) (2021) 1791–1805. MR4398248
36. Y. Wang, The energy of a deterministic Loewner chain: reversibility and interpretation via SLE_{0+}, J. Eur. Math. Soc. 21 (7) (2019) 1915–1941. MR3959854
37. Y. Wang, Equivalent descriptions of the Loewner energy, Invent. Math. 218 (2) (2019) 573–621. MR4011706

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Bishop, Christopher J. (1-SUNYS)

★**Optimal angle bounds for Steiner triangulations of polygons. (English. English summary)**

Proceedings of the 2022 *Annual ACM-SIAM Symposium on Discrete Algorithms* (*SODA*), 3127–3143, [*Society for Industrial and Applied Mathematics* (*SIAM*)], *Philadelphia*, *PA*, 2022.

Summary: "For any simple polygon P we compute the optimal upper and lower angle bounds for triangulating P with Steiner points, and show that these bounds can be attained (except in one special case). The sharp angle bounds for an N-gon are computable in time $O(N)$, even though the number of triangles needed to attain these bounds has no bound in terms of N alone. In general, the sharp upper and lower bounds cannot both be attained by a single triangulation, although this does happen in some cases. For example, we show that any polygon with minimal interior angle θ has a triangulation with all angles in the interval $I=[\theta,90^\circ-\min(36^\circ,\theta)/2]$, and for $\theta\le 36^\circ$ both bounds are best possible. Surprisingly, we prove the optimal angle bounds for polygonal triangulations are the same as for triangular dissections. The proof of this verifies, in a stronger form, a 1984 conjecture of Gerver."

MR4402047 30H10 28A75 30C20

Bishop, Christopher J. (1-SUNYS)

Conformal images of Carleson curves. (English. English summary)

Proc. Amer. Math. Soc. Ser. B **9** (2022), 90–94.

In the article under review the author provides a quite interesting characterization: given a curve γ in the unit disk, the arclength on γ is a Carleson measure if, and only if, the image of γ under every conformal map onto a bounded domain with rectifiable boundary has finite length.

One direction of this equivalence is quite standard (it is a consequence of the F. and M. Riesz theorem), while the other direction is obtained through a non-trivial construction of an explicit conformal mapping. The proof in fact provides much more than stated in the main theorem: what the author really proves is that a positive measure μ on the disk is Carleson if, and only if, $\int_{\mathbb{D}}|f'|d\mu<\infty$ for any conformal map f onto a rectifiable domain.

The author takes care to motivate and explain his construction in detail, in order to make the work of the reader as easy as possible; it is a clearly written article and should be of interest to people working on conformal mappings, fractal geometry, geometric function theory and related topics. *Lucas da Silva Oliveira*

[References]

1. John B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, vol. 96, Academic Press, Inc. Harcourt Brace Jovanovich, Publishers], New York-London, 1981. MR628971 MR0628971
2. John B. Garnett and Donald E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2008. Reprint of the 2005 original. MR2450237 MR2450237

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR4276301 28A78 52A20

Bishop, Christopher J. (1-SUNYS); **Drillick, Hindy** (1-CLMB);
Ntalampekos, Dimitrios (1-SUNYS-IM)

Falconer's (K, d) distance set conjecture can fail for strictly convex sets K in $\mathbb{R}^d$. (English. English summary)

Rev. Mat. Iberoam. **37** (2021), *no.* 5, 1953–1968.

Summary: "For any norm on $\mathbb{R}^d$ with countably many extreme points, we prove that there is a set $E \subset \mathbb{R}^d$ of Hausdorff dimension d whose distance set with respect to this norm has zero linear measure. This was previously known only for norms associated to certain finite polygons in $\mathbb{R}^2$. Similar examples exist for norms that are very well approximated by polyhedral norms, including some examples where the unit ball is strictly convex and has C^1 boundary."

[References]

1. Bishop, C. J. and Peres, Y.: *Fractals in probability and analysis.* Cambridge Studies in Advanced Mathematics 162, Cambridge Univ. Press, Cambridge, 2017. MR3616046
2. Bourgain, J.: Hausdorff dimension and distance sets. *Israel J. Math.* **87** (1994), no. 1-3, 193–201. MR1286826
3. Du, X., Guth, L., Wang, H., Wilson, B. and Zhang, R.: Weighted restriction estimates and application to Falconer distance set problem. *Amer. J. Math.* **143** (2021), no. 1, 175–211. MR4201782
4. Du, X. and Zhang, R.: Sharp L^2 estimate of Schrödinger maximal function in higher dimensions. *Ann. of Math. (2)* **189** (2019), no. 3, 837–861. MR3961084
5. Erdoğan, M. B.: On Falconer's distance set conjecture. *Rev. Mat. Iberoam.* **22** (2006), no. 2, 649–662. MR2294792
6. Falconer, K. J.: On the Hausdorff dimensions of distance sets. *Mathematika* **32** (1985), no. 2, 206–212. MR0834490
7. Falconer, K. J.: Dimensions of intersections and distance sets for polyhedral norms. *Real Anal. Exchange* **30** (2004/05), no. 2, 719–726. MR2177429
8. Falconer, K. J.: *Fractal geometry. Mathematical foundations and applications.* Third edition. John Wiley and Sons, Ltd., Chichester, 2014. MR3236784
9. Ghomi, M.: Optimal smoothing for convex polytopes. *Bull. London Math. Soc.* **36** (2004), no. 4, 483–492. MR2069010
10. Guth, L., Iosevich, A., Ou, Y. and Wang, H.: On Falconer's distance set problem in the plane. *Invent. Math.* **219** (2020), no. 3, 779–830. MR4055179
11. Guth, L. and Katz, N. H.: On the Erdős distinct distances problem in the plane. *Ann. of Math. (2)* **181** (2015), no. 1, 155–190. MR3272924
12. Iosevich, A. and Łaba, I.: K-distance sets, Falconer conjecture, and discrete analogs. *Integers* **5** (2005), no. 2, A8, 11pp. MR2192086
13. Iosevich, A.: What is Falconer's conjecture? *Notices Amer. Math. Soc.* **66** (2019), no. 4, 552–555. MR3889529
14. Kahane, J.-P.: Sur la dimension des intersections. In *Aspects of mathematics and its applications*, 419–430. North-Holland Math. Library 34, North-Holland, Amsterdam, 1986. MR0849569
15. Konyagin, S. and Łaba, I.: Falconer's distance set conjecture for polygonal norms. In *Proceedings of harmonic analysis and its applications at Sapporo*, 43–55. Hokkaido University Technical Report Series in Mathematics 103, Hokkaido University, Sapporo, 2005. Electronically available at arXiv: `math/0407503`.
16. Konyagin, S. and Łaba, I.: Distance sets of well-distributed planar sets for

polygonal norms. *Israel J. Math.* **152** (2006), 157–179. MR2214458
17. Mattila, P.: *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge, 1995. MR1333890
18. Rockafellar, R. T.: *Convex analysis.* Princeton Mathematical Series 28, Princeton University Press, Princeton, NJ, 1970. MR0274683
19. Wolff, T.: Decay of circular means of Fourier transforms of measures. *Internat. Math. Res. Notices* 1999, no. 10, 547–567. MR1692851

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR4174037 37F10 37F35

Albrecht, Simon (4-LVRP); **Bishop, Christopher J.** (1-SUNYS)

Speiser class Julia sets with dimension near one. (English. English summary)

J. Anal. Math. **141** (2020), *no.* 1, 49–98.

Transcendental entire functions with a finite singular set (the closure of its critical values and finite asymptotic values) form the Speiser class. The authors present a novel construction of functions from this class with non-locally connected Cantor bouquet Julia sets whose Hausdorff dimensions are arbitrarily close to 1. Such fractals were known to have Hausdorff dimension greater than 1 by results of G. M. Stallard [Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 3, 513–536; MR1357062] and full packing dimension by P. J. Rippon and Stallard [Ergodic Theory Dynam. Systems **26** (2006), no. 2, 525–538; MR2218773].

The construction involves delicate handling of C. J. Bishop's flexible *quasiconformal folding* construction [Acta Math. **214** (2015), no. 1, 1–60; MR3316755] and several new ideas to obtain control of the dimension. The examples constructed all have infinite order of growth and are of disjoint type (they possess exactly three singular values $(0, -1, 1)$ that are contained in the basin of attraction of the fixed point 0).

The ideas in this well-written and beautifully illustrated paper are deep yet clearly visible from the surface. That said, this reviewer would like to have seen a more careful proof demonstrating that the Julia sets constructed are in fact Cantor bouquets, but the paper was almost at fifty pages before the proof sketch of this delicacy. The authors believe that the dimensions of their Speiser Julia sets must be concentrated on the endpoints of the hairs on the respective bouquets.

There exist a range of challenging open problems in the dimension theory of various fractal phenomena arising from transcendental dynamics; one should follow the burgeoning literature closely. For instance, the authors state that it is open whether there exist Speiser Julia sets of every dimension between 1 and 2, yet there already appears to be recent progress by W. Bergweiler and W. Cui ["The Hausdorff dimension of Julia sets of meromorphic functions in the Speiser class", preprint, arXiv:2105.00938]. The 2020 online conference On Geometric Complexity of Julia Sets—II, as well as its hybrid follow-up in 2021, On Geometric Complexity of Julia Sets—III, delineates several allied directions that await resolution at the time of writing. *Tushar Das*

[References]

1. J. Aarts and L. Oversteegen, *The geometry of Julia sets*, Trans. Amer. Math. Soc. **338** (1993), 897–918. MR1182980
2. I. N. Baker, *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), 277–283. MR0402044
3. K. Barański, *Hausdorff dimension of hairs and ends for entire maps of finite order*,

Math. Proc. Cambridge Philos. Soc. **145** (2008), 719–737. MR2464786
4. K. Barański, X. Jarque and L. Rempe, *Brushing the hairs of transcendental entire functions.* Topology Appl. **159** (2012), 2102–2114. MR2902745
5. K. Barański, B. Karpińska and A. Zdunik, *Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts*, Int. Math. Res. Not. IMRN **2009** (2009), 615–624. MR2480096
6. A. F. Beardon, *The hyperbolic metric in a rectangle. II*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 143–152. MR1976836
7. P. P. Belinskii, *Obshchie Svoistva Kvazikonformnykh Otobrazhenii*, Izdat. "Nauka" Sibirsk. Otdel., Novosibirsk, 1974.
8. W. Bergweiler, P. J. Rippon and G. M. Stallard, *Dynamics of meromorphic functions with direct or logarithmic singularities*, Proc. Lond. Math. Soc. (3) **97** (2008), 368–400. MR2439666
9. C. J. Bishop, *Constructing entire functions by quasiconformal folding*, Acta Math. **214** (2015), 1–60. MR3316755
10. C. J. Bishop, *Models for the Eremenko–Lyubich class*, J. Lond. Math. Soc. (2) **92** (2015), 202–221. MR3384512
11. C. J. Bishop, *Models for the Speiser class*, Proc. Lond. Math. Soc. (3) **114** (2017), 765–797. MR3653246
12. C. J. Bishop, *A transcendental Julia set of dimension 1*, Invent. Math. **212** (2018), 407–460. MR3787831
13. C. Bishop, V. Y. Gutlyanskiĭ, O. Martio and M. Vuorinen, *On conformal dilatation in space*, Int. J. Math. Math. Sci. **2003** (2003). 1397–1420. MR1980177
14. C. J. Bishop and K. Lazebnik, *Prescribing the postsingular dynamics of meromorphic functions*, Math. Ann. **375** (2019), 1761–1782. MR4023391
15. C. Bishop and Y. Peres, *Fractals in Probability and Analysis*, Cambridge University Press, Cambridge, 2017. MR3616046
16. X. Buff and A. Chéritat, *Quadratic Julia sets with positive area*, Ann. of Math. (2) **176** (2012), 673–746. MR2950763
17. W. Bula and L. Oversteegen, *A characterization of smooth Cantor bouquets*, Proc. Amer. Math. Soc. **108** (1990), 529–534. MR0991691
18. A. È. Erëmenko and M. Y. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), 989–1020. MR1196102
19. N. Fagella, S. Godillon and X. Jarque, *Wandering domains for composition of entire functions*, J. Math. Anal. Appl. **429** (2015), 478–496. MR3339086
20. J. Garnett and D. Marshall, *Harmonic Measure*, Cambridge University Press, Cambridge, 2008. MR2450237
21. F. W. Gehring, *The definitions and exceptional sets for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A. I. **281** (1960), 1–28. MR0124488
22. L. Goldberg and L. Keen, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynam. Systems **6** (1986), 183–192. MR0857196
23. J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61. MR1654771
24. B. Karpińska, *Hausdorff dimension of the hairs without endpoints for* $\lambda \exp z$, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 1039–1044. MR1696203
25. K. Lazebnik, *Several constructions in the Eremenko–Lyubich class*, J. Math. Anal. Appl. **448** (2017), 611–632. MR3579902
26. O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer, New York–Heidelberg, 1973. MR0344463
27. C. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. **300** (1987), 329–342. MR0871679

28. H. Mihaljević-Brandt and L. Rempe-Gillen, *Absence of wandering domains for some real entire functions with bounded singular sets*, Math. Ann. **357** (2013), 1577–1604. MR3124942
29. M. Misiurewicz, *On iterates of* e^z, Ergodic Theory Dynam. Systems **1** (1981), 103–106. MR0627790
30. J. W. Osborne and D. J. Sixsmith, *On the set where the iterates of an entire function are neither escaping nor bounded*, Ann. Acad. Sci. Fenn. Math. **41** (2016), 561–578. MR3525384
31. L. Rempe, P. J. Rippon and G. M. Stallard. *Are Devaney hairs fast escaping?* J. Difference Equ. Appl. **16** (2010), 739–762. MR2675603
32. L. Rempe-Gillen, *Arc-like continua, Julia sets of entire functions, and Eremenko's Conjecture*, arXiv:1610.06278 [math.DS]
33. P. J. Rippon and G. M. Stallard, *Dimensions of Julia sets of meromorphic functions with finitely many poles*, Ergodic Theory Dynam. Systems **26** (2006), 525–538. MR2218773
34. G. Rottenfusser, J. Rückert, L. Rempe and D. Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) **173** (2011), 77–125. MR2753600
35. H. Schubert, *Über die Hausdorff-Dimension der Juliamenge von Funktionen endlicher Ordnun*, PhD thesis, Christian-Albrechts-Universität zu Kiel, 2007.
36. M. Shishikura, *The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets*, Ann. of Math. (2) **147** (1998), 225–267. MR1626737
37. G. Stallard, *The Hausdorff dimension of Julia sets of entire functions. III.* Math. Proc. Cambridge Philos. Soc. **122** (1997), 223–244. MR1458228
38. G. Stallard, *The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions. II*, Ergodic Theory Dynam. Systems **20** (2000), 895–910. MR1764934
39. G. Stallard, *The Hausdorff dimension of Julia sets of entire functions. IV*, J. London Math. Soc. (2) **61** (2000), 471–488. MR1760674
40. O. Teichmüller, *Untersuchungen über konforme and quasikonforme Abbildung*, Deutsche Math. **3** (1938), 621–678.
41. J. Waterman, *Wiman–Valiron disks and the dimension of Julia sets*, Int. Math. Res. Not. IMRN **3** (2020), doi 10.1093/imrn/rnaa029
42. H. Wittich, *Zum Beweis eines Satzes über Quasikonforme Abbildungen*, Math. Z. **51** (1948), 278–288. MR0027057
43. M. Zinsmeister, *Fleur de Leau-Fatou et dimension de Hausdorff*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), 1227–1232. MR1650223

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR4165309 53C55 57K32 57K41

Bishop, Christopher J. (1-SUNYS); **LeBrun, Claude** (1-SUNYS)

Anti-self-dual 4-manifolds, quasi-Fuchsian groups, and almost-Kähler geometry. (English. English summary)

Comm. Anal. Geom. **28** (2020), *no.* 4, 745–780.

An anti-self-dual 4-manifold is an oriented Riemannian manifold of dimension 4, (M^4, g), such that $W_+ = 0$, where W_+ denotes the self-dual Weyl curvature, which is the orthogonal projection of the Riemann curvature tensor R into the trace-free symmetric square $\bigodot_0^2 \bigwedge^+$ of the bundle of self-dual 2-forms.

An oriented Riemannian manifold (M, g) equipped with a closed 2-form ω is said to be almost-Kähler if there is an orientation compatible almost-complex structure $J\colon TM \to TM$, $J^2 = -\mathbf{1}$, such that $g = \omega(\cdot, J\cdot)$. If, in addition, J is integrable, then M is said to be a Kähler manifold. A Kähler manifold (M^4, g, J) of complex dimension two is anti-self-dual if and only if its scalar curvature vanishes [C. R. LeBrun, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, 498–507, Birkhäuser, Basel, 1995; MR1403950].

The almost-Kähler anti-self-dual metrics on a given 4-manifold sweep out an open subset $\mathcal{O}$ in the moduli space of anti-self-dual metrics. Nevertheless, the authors present examples of 4-manifolds that admit locally conformally flat conformal classes $[g]$ that cannot be represented by almost-Kähler metrics. Moreover, they infer that the subset $\mathcal{O}$ is not closed in general, and so need not sweep out entire connected components of the moduli space. Their construction uses quasi-Fuchsian groups.

The final part of the paper raises some interesting questions regarding the method used, and possible further research on the subject. *Mario A. Fioravanti*

[References]

1. L. V. Ahlfors, Lectures on Quasiconformal Mappings, Vol. 38 of University Lecture Series, American Mathematical Society, Providence, RI, second ed., (2006). MR2241787
2. L. V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404. MR0115006
3. M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in fourdimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461. MR0506229
4. L. Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc. **66** (1960), 94–97. MR0111834
5. C. J. Bishop, *Divergence groups have the Bowen property*, Ann. of Math. (2) **154** (2001), 205–217. MR1847593
6. R. Bowen, *Hausdorff dimension of quasicircles*, Inst. Hautes Études Sci. Publ. Math. (1979), 11–25. MR0556580
7. D. Burns and P. De Bartolomeis, *Stability of vector bundles and extremal metrics*, Invent. Math. **92** (1988), 403–407. MR0936089
8. E. B. Davies, Heat Kernels and Spectral Theory, Vol. 92 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, (1989). MR1103113
9. F. W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. (2) **6** (1973), 504–512. MR0324028
10. C. R. Graham, *The Dirichlet problem for the Bergman Laplacian. II*, Comm. Partial Differential Equations **8** (1983), 563–641. MR0700730
11. I. Kim, *Almost-Kähler anti-self-dual metrics*, Ann. Global Anal. Geom. **49** (2016), 369–391. MR3510523
12. J. Kim, *On the scalar curvature of self-dual manifolds*, Math. Ann., **297** (1993), 235–251. MR1241804
13. J. Kim, C. LeBrun, and M. Pontecorvo, *Scalar-flat Kähler surfaces of all genera*, J. Reine Angew. Math. **486** (1997), 69–95. MR1450751
14. C. LeBrun, *On the topology of self-dual* 4-*manifolds*, Proc. Amer. Math. Soc. **98** (1986), 637–640. MR0861766
15. C. LeBrun, *Explicit self-dual metrics on* $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$, J. Differential Geom., **34** (1991), 223–253. MR1114461
16. C. LeBrun, *Scalar-flat Kähler metrics on blown-up ruled surfaces*, J. Reine Angew. Math. **420** (1991), 161–177. MR1124569
17. C. LeBrun, *Self-dual manifolds and hyperbolic geometry*, in: Einstein Metrics and Yang-Mills Connections (Sanda, 1990), Vol. 145 of Lecture Notes in Pure and Appl. Math., Dekker, New York, (1993), 99–131. MR1215284
18. C. LeBrun, *Anti-self-dual metrics and Kähler geometry*, in: Proceedings of the

International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser Basel, (1995), 498–507. MR1403950
19. C. LeBrun, *Weyl curvature, Del Pezzo surfaces, and almost-Kähler geometry*, J. Geom. Anal. **25** (2015), 1744–1772. MR3358072
20. B. Maskit, Kleinian Groups, Springer-Verlag, Berlin, (1988). MR0959135
21. R. Penrose, *Nonlinear gravitons and curved twistor theory*, General Relativity and Gravitation **7** (1976), 31–52. MR0439004
22. Y. Rollin and M. Singer, *Construction of Kähler surfaces with constant scalar curvature*, J. Eur. Math. Soc. (JEMS) **11** (2009), 979–997. MR2538497
23. R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), 47–71. MR0931204
24. D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. (1979), 171–202. MR0556586
25. D. Sullivan and W. P. Thurston, *Manifolds with canonical coordinate charts: some examples*, Enseign. Math. (2) **29** (1983), 15–25. MR0702731
26. K. K. Uhlenbeck, *Closed minimal surfaces in hyperbolic* 3*-manifolds*, in: Seminar on Minimal Submanifolds, Vol. 103 of Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, (1983), 147–168. MR0795233

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR4023391 37F10 30D05 30D30

Bishop, Christopher J. (1-SUNYS-NDM); **Lazebnik, Kirill** (1-CAIT-NDM)

Prescribing the postsingular dynamics of meromorphic functions. (English. English summary)

Math. Ann. **375** (2019), *no.* 3-4, 1761–1782.

In the paper under review, the authors construct transcendental meromorphic functions with prescribed dynamics on the postsingular set.

In more detail, let $f\colon \mathbb{C}\to\widehat{\mathbb{C}}$ be a meromorphic function, and let S_f be the set of singular values of f. The set $P_f=\overline{\bigcup_{n=0}^{\infty} f^{\circ n}(S_f)}$ is called the *postsingular* set f.

A sequence $S\subset\mathbb{C}$ is called *discrete* if S has no accumulation points in $\mathbb{C}$. Denote by $|S|$ the number of elements in S. If $|S|=\infty$, then the elements of S accumulate only at ∞.

The main result of the paper is the following theorem:

Theorem 1. If $S\subset\mathbb{C}$ is a discrete sequence with $4\leqslant|S|\leqslant\infty$, and $h\colon S\to S$ is any map, then for every $\varepsilon>0$ there exist a transcendental meromorphic function $f\colon\mathbb{C}\to\widehat{\mathbb{C}}$ and a bijection $\psi\colon S\to P_f$ such that

- h models the dynamics of f on the postsingular set, i.e., $f|_{P_f}=\psi\circ h\circ\psi^{-1}$;
- ψ is an ε-perturbation of S, i.e., $|\psi(s)-s|\leqslant\varepsilon$ for every $s\in S$;
- in the case $|S|=\infty$, the map ψ is asymptotically the identity, i.e., $|\psi(s)-s|\to 0$ as $s\to\infty$.

The proof is based on the quasiconformal folding technique introduced by C. J. Bishop [Acta Math. **214** (2015), no. 1, 1–60; MR3316755]. *Kostiantyn Drach*

[References]

1. Albrecht, S., Bishop, C.J.: Spieser class Julia sets with dimension near one (2017) **(preprint)** MR4174037
2. Bishop, C.J.: Constructing entire functions by quasiconformal folding. Acta Math. **214**(1), 1–60 (2015). https://doi.org/10.1007/s11511-015-0122-0 MR3316755

3. Bishop, C.J.: Models for the Speiser class. Proc. Lond. Math. Soc. (3) **114**(5), 765–797 (2017). https://doi.org/10.1112/plms.12025 MR3653246
4. Carleson, L., Gamelin, T.W.: Complex dynamics. Universitext: Tracts in Mathematics. Springer, New York (1993). https://doi.org/10.1007/978-1-4612-4364-9 MR1230383
5. DeMarco, L.G., Koch, S.C., McMullen, C.T.: On the postcritical set of a rational map. Mathematische Annalen (2018) MR1981611
6. Dyn' kin, E.M.: Smoothness of a quasiconformal mapping at a point. Algebra i Analiz **9**(3), 205–210 (1997)
7. Fagella, N., Jarque, X., Lazebnik, K.: Univalent wandering domains in the Eremenko–Lyubich Class. arXiv:1711.10629 (2017) MR4041106
8. Fagella, N., Godillon, S., Jarque, X.: Wandering domains for composition of entire functions. J. Math. Anal. Appl. **429**(1), 478–496 (2015). https://doi.org/10.1016/j.jmaa.2015.04.020 MR3339086
9. Garnett, J., Marshall, D.: Harmonic Measure, New Mathematical Monographs, vol. 2. Cambridge University Press, Cambridge (2005) MR2150803
10. Lazebnik, K.: Oscillating Wandering Domains for Functions with Escaping Singular Values. arXiv e-prints (2019)
11. Lazebnik, K.: Several constructions in the Eremenko–Lyubich class. J. Math. Anal. Appl. **448**(1), 611–632 (2017). https://doi.org/10.1016/j.jmaa.2016.11.007 MR3579902
12. Martí-Pete, D., Shishikura, M.: Wandering domains for entire functions of finite order in the Eremenko–Lyubich class (2018) MR4008367
13. Munkres, J.R.: Topology: A First Course. Prentice-Hall Inc, Englewood Cliffs (1975) MR0464128
14. Osborne, J.W., Sixsmith, D.J.: On the set where the iterates of an entire function are neither escaping nor bounded. Ann. Acad. Sci. Fenn. Math. **41**(2), 561–578 (2016) MR3525384
15. Pommerenke, Ch.: Boundary behaviour of conformal maps, volume 299 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin (1992) MR1217706
16. Rempe-Gillen, L.: Arc-like continua, Julia sets of entire functions, and Eremenko's conjecture (2016) **(preprint)**
17. Sixsmith, D.J.: Dynamics in the Eremenko–Lyubich class. ArXiv e-prints (2017) MR3852466
18. Tychonoff, A.: Ein Fixpunktsatz. Math. Ann. **111**(1), 767–776 (1935). https://doi.org/10.1007/BF01472256 MR1513031

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3966818 30C85 30C30 37F30 65D99

Bishop, Christopher J. (1-SUNYS)

★**Harmonic measure: algorithms and applications. (English. English summary)**

Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. *Vol. III. Invited lectures*, 1511–1537, *World Sci. Publ.*, *Hackensack*, *NJ*, 2018.

This paper is a survey of various results related to harmonic measure in the complex plane. The first part of the paper deals with the computational aspects of harmonic measure and conformal maps, including applications to computational geometry. The second part describes applications to the study of hyperbolic 3-manifolds and limit sets of Kleinian groups. The third part consists of a survey of results related to Makarov's law of iterated logarithm (LIL) and the connections among harmonic measure, random walks and Hausdorff dimension. This includes a discussion of results related to harmonic measure and rectifiability as well as conformal welding. Finally, the last part of the paper deals with applications of harmonic measure and the author's quasiconformal folding technique to the study of true trees and transcendental entire functions.

This is a very interesting and well-written survey from a renowned expert on harmonic measure. *Malik Younsi*

MR3787831 37F10 30D05 37F35
Bishop, Christopher J. (1-SUNYS)
A transcendental Julia set of dimension 1. (English. English summary)
Invent. Math. **212** (2018), *no.* 2, 407–460.

The Julia set $J(f)$ of an entire function f is the set where the family of iterates of f fails to be normal. By a result of Baker, the Julia set of a transcendental entire function contains continua and thus has Hausdorff dimension at least 1. A result of C. T. McMullen says that $J(\lambda e^z) = 2$ for $\lambda \neq 0$, and we have $J(e^z) = \mathbb{C}$ by a result of M. Misiurewicz. G. M. Stallard showed that for any $d \in (1, 2)$ there exists a transcendental entire function f such that $J(f)$ has Hausdorff dimension d.

In the present paper the author completes the picture by constructing a transcendental entire function f for which the Julia set has Hausdorff dimension 1. In fact, $H^1(J(f) \cap D(x, r)) = O(r)$ for every $x \in \mathbb{C}$, where $H^1(\cdot)$ denotes the one-dimensional Hausdorff measure and $D(x, r)$ is the disk of radius r centered at x. Moreover, $J(f)$ has packing dimension 1, and given any function ψ satisfying $\lim_{t\to\infty} \psi(t)t^{-n} = \infty$ for every n, one may choose f such that $|f(z)| = O(\psi(|z|))$ as $|z| \to \infty$.

The function constructed has multiply connected wandering domains. These are domains U_k containing annuli around 0 such that $f(U_k) \subset U_{k+1}$ and $U_k \to \infty$. They are constructed in such a way that the "outer" boundary of U_k coincides with the "inner" boundary of U_{k+1}, and is a rectifiable Jordan curve. The Julia set consists of these Jordan curves, together with their preimages and limit points thereof.

Walter Bergweiler

[References]

1. Agol, I.: Tameness of hyperbolic 3-manifolds. 2004. Preprint available at arXiv:math/0405568 [math.GT]
2. Avila, A., Buff, X., Chéritat, A.: Siegel disks with smooth boundaries. Acta Math. **193**(1), 1–30 (2004) MR2155030
3. Avila, A., Lyubich, M.: Lebesgue measure of Feigenbaum Julia sets. 2015. Preprint available at arXiv:1504.02986 [math.DS]
4. Baker, I.N.: The domains of normality of an entire function. Ann. Acad. Sci. Fenn. Ser. A I Math **1**(2), 277–283 (1975) MR0402044
5. Baker, I.N.: An entire function which has wandering domains. J. Aust. Math. Soc. Ser. A **22**(2), 173–176 (1976) MR0419759
6. Baker, I.N.: Some entire functions with multiply-connected wandering domains. Ergod. Theory Dyn. Syst. **5**(2), 163–169 (1985) MR0796748
7. Barański, K.: Hausdorff dimension of hairs and ends for entire maps of finite order. Math. Proc. Camb. Philos. Soc. **145**(3), 719–737 (2008) MR2464786
8. Baumgartner, M.: Üer Ränder von mehrfach zusammenhängenden wandernden Gebieten. PhD thesis, Christian-Albrechts-Universität zu Kiel (2015)
9. Bergweiler, W.: On the packing dimension of the Julia set and the escaping set of an entire function. Israel J. Math. **192**(1), 449–472 (2012a) MR3004090
10. Bergweiler, W.: On the set where the iterates of an entire function are bounded.

Proc. Am. Math. Soc. **140**(3), 847–853 (2012b) MR2869069

11. Bergweiler, W., Hinkkanen, A.: On semiconjugation of entire functions. Math. Proc. Camb. Philos. Soc. **126**(3), 565–574 (1999) MR1684251
12. Bergweiler, W., Zheng, J.-H.: On the uniform perfectness of the boundary of multiply connected wandering domains. J. Aust. Math. Soc. **91**(3), 289–311 (2011) MR2900609
13. Bers, L.: On boundaries of Teichmüller spaces and on Kleinian groups. I. Ann. Math. **2**(91), 570–600 (1970) MR0297992
14. Bishop, C.J.: True trees are dense. Invent. Math. **197**(2), 433–452 (2014) MR3232011
15. Bishop, C.J.: Constructing entire functions by quasiconformal folding. Acta Math. **214**(1), 1–60 (2015a) MR3316755
16. Bishop, C.J.: Models for the Eremenko–Lyubich class. J. Lond. Math. Soc. (2) **92**(1), 202–221 (2015b) MR3384512
17. Bishop, C.J.: The order conjecture fails in S. J. Anal. Math. **127**, 283–302 (2015) MR3421995
18. Bishop, C.J.: Models for the Speiser class. Proc. Lond. Math. Soc. (3) **114**(5), 765–797 (2017) MR3653246
19. Bishop, C.J., Albrecht, S.: Spesier class Julia sets with dimension near one. (2017). preprint MR4174037
20. Bishop, C.J., Jones, P.W.: Hausdorff dimension and Kleinian groups. Acta Math. **179**(1), 1–39 (1997) MR1484767
21. Bishop, C.J., Peres, Y.: Fractals in Probability and Analysis. Cambridge Studies in Advanced Mathematics, vol. 162. Cambridge University Press, Cambridge (2017) MR3616046
22. Buff, X., Chéritat, A.: Quadratic Julia sets with positive area. Ann. Math. (2) **176**(2), 673–746 (2012) MR2950763
23. Calegari, D., Gabai, D.: Shrinkwrapping and the taming of hyperbolic 3-manifolds. J. Am. Math. Soc. **19**(2), 385–446 (2006) MR2188131
24. Eremenko, A.E.: On the iteration of entire functions. In: Dynamical Systems and Ergodic Theory (Warsaw, 1986), vol. 23 of Banach Center Publication, pp. 339–345. PWN, Warsaw (1989) MR1102727
25. Garnett, J.B., Marshall, D.E.: Harmonic Measure, volume 2 of New Mathematical Monographs. Cambridge University Press, Cambridge (2005) MR2150803
26. Greenberg, L.: Fundamental polyhedra for Keinian groups. Ann. Math. **2**(84), 433–441 (1966) MR0200446
27. Jørgensen, T.: Compact 3-manifolds of constant negative curvature fibering over the circle. Ann. Math. (2) **106**(1), 61–72 (1977) MR0450546
28. Karpińska, B.: Hausdorff dimension of the hairs without endpoints for $\lambda \exp z$. C. R. Acad. Sci. Paris Sér. I Math. **328**(11), 1039–1044 (1999) MR1696203
29. Kisaka, M., Shishikura, M.: On multiply connected wandering domains of entire functions. In: Transcendental dynamics and complex analysis, volume 348 of London Mathematical Society Lecture Note Series, pp. 217–250. Cambridge University Press, Cambridge (2008) MR2458806
30. Marden, A.: The geometry of finitely generated Kleinian groups. Ann. Math. **2**(99), 383–462 (1974) MR0349992
31. McMullen, C.T.: Area and Hausdorff dimension of Julia sets of entire functions. Trans. Am. Math. Soc. **300**(1), 329–342 (1987) MR0871679
32. McMullen, C.T.: Renormalization and 3-manifolds which fiber over the circle, volume 142 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ (1996) MR1401347
33. Mihaljević-Brandt, H., Rempe-Gillen, L.: Absence of wandering domains for some

real entire functions with bounded singular sets. Math. Ann. **357**(4), 1577–1604 (2013) MR3124942

34. Misiurewicz, M.: On iterates of e^z. Ergod. Theory Dyn. Syst. **1**(1), 103–106 (1981) MR0627790
35. Osborne, J.W., Sixsmith, D.J.: On the set where the iterates of an entire function are neither escaping nor bounded. Ann. Acad. Sci. Fenn. Math. **41**(2), 561–578 (2016) MR3525384
36. Przytycki, F.: Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. Invent. Math. **80**(1), 161–179 (1985) MR0784535
37. Rempe, L.: Hyperbolic dimension and radial Julia sets of transcendental functions. Proc. Am. Math. Soc. **137**(4), 1411–1420 (2009) MR2465667
38. Rempe-Gillen, L., Sixsmith, D.: Hyperbolic entire functions and the Eremenko–Lyubich class: class B or not class B. (2016). Preprint available at arXiv:1502.00492v2 [math.DS] MR3671560
39. Rippon, P.: Obituary: Irvine Noel Baker 1932–2001. Bull. Lond. Math. Soc. **37**(2), 301–315 (2005) MR2119030
40. Rippon, P.J., Stallard, G.M.: Dimensions of Julia sets of meromorphic functions. J. Lond. Math. Soc. (2) **71**(3), 669–683 (2005a) MR2132377
41. Rippon, P.J., Stallard, G.M.: On questions of Fatou and Eremenko. Proc. Am. Math. Soc **133**(4), 1119–1126 (2005b). (electronic) MR2117213
42. Schleicher, D.: Dynamics of entire functions. In: Holomorphic Dynamical Systems, volume 1998 of Lecture Notes in Mathematics, pp. 295–339. Springer, Berlin (2010) MR2648691
43. Shen, Z., Rempe-Gillen, L.: The exponential map is chaotic: an invitation to transcendental dynamics. Am. Math. Mon. **122**(10), 919–940 (2015) MR3447747
44. Shishikura, M.: The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. Ann. Math. (2) **147**(2), 225–267 (1998) MR1626737
45. Stallard, G.M.: The Hausdorff dimension of Julia sets of meromorphic functions. J. Lond. Math. Soc. (2) **49**(2), 281–295 (1994) MR1260113
46. Stallard, G.M.: The Hausdorff dimension of Julia sets of entire functions II. Math. Proc. Camb. Philos. Soc. **119**(3), 513–536 (1996) MR1357062
47. Stallard, G.M.: The Hausdorff dimension of Julia sets of entire functions III. Math. Proc. Camb. Philos. Soc. **122**(2), 223–244 (1997) MR1458228
48. Stallard, G.M.: TheHausdorff dimension of Julia sets of entire functions. IV. J. Lond. Math. Soc. (2) **61**(2), 471–488 (2000) MR1760674
49. Stallard, G.M.: Dimensions of Julia sets of transcendental meromorphic functions. In: Transcendental dynamics and complex analysis, volume 348 of London Mathematical Society Lecture Note Series, pp. 425–446. Cambridge University Press, Cambridge (2008) MR2458811
50. Sullivan, D.: Growth of positive harmonic functions and Kleinian group limit sets of zero planar measure and Hausdorff dimension two. In: Geometry Symposium, Utrecht 1980 (Utrecht, 1980), volume 894 of Lecture Notes in Mathematics, pp. 127–144. Springer, Berlin (1981) MR0655423
51. Zdunik, A.: Parabolic orbifolds and the dimension of the maximal measure for rational maps. Invent. Math. **99**(3), 627–649 (1990) MR1032883

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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Bishop, Christopher J. (1-SUNYS)

Models for the Speiser class. (English. English summary)

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The Eremenko-Lyubich class, $\mathcal{B}$, consists of those transcendental entire functions for which the set of singular values, $S(f)$, is bounded. The Speiser class, $\mathcal{S} \subset \mathcal{B}$, consists of those functions for which $S(f)$ is finite. These important classes have been widely studied, particularly in complex dynamics. The paper under review is, with [C. J. Bishop, Acta Math. **214** (2015), no. 1, 1–60; MR3316755] and [C. J. Bishop, J. Lond. Math. Soc. (2) **92** (2015), no. 1, 202–221; MR3384512], one of a remarkable trilogy of papers which together give new techniques to construct functions in these classes, as well as answering a number of open questions.

Assume that $f \in \mathcal{B}$ and $R > 0$ are such that $S(f) \subset \{z\colon |z| < R\}$. Then $\Omega = f^{-1}(W)$, where $W = \{z\colon |z| > R\}$, is a disjoint union of analytic, unbounded, simply connected domains (called tracts), and f is a covering map from each tract to W.

Roughly speaking, in [op. cit.; MR3384512] Bishop considered the reverse question: Given a suitable set of tracts and covering maps (known as a model), is there a function $f \in \mathcal{B}$ which approximates the functions in the original model? This question is answered in the affirmative, and the sense of the approximation is made precise.

In the present paper, Bishop answers the same question, but with the additional restriction that f must lie in $\mathcal{S}$. Once again, the answer is in the affirmative, although the approximation is weaker than that possible in the class $\mathcal{B}$. In particular, the function f may have additional tracts which are not in the initial model.

The construction in [C. J. Bishop, op. cit.; MR3384512] uses a Blaschke product, and is self-contained. The construction in the present paper is more complicated, and uses the quasiconformal folding technique introduced in [C. J. Bishop, op. cit.; MR3316755].

David Jonathon Sixsmith

[References]

1. L. V. Ahlfors, *Lectures on quasiconformal mappings*, 2nd edn, University Lecture Series 38 (American Mathematical Society, Providence, RI, 2006) With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. MR2241787
2. C. J. Bishop, 'Constructing entire functions by quasiconformal folding', *Acta Math.* 214 (2015) 1–60. MR3316755
3. C. J. Bishop, 'The order conjecture fails in $\mathcal{S}$', *J. Anal. Math.* 127 (2015) 283–302. MR3421995
4. C. J. Bishop, 'Models for the Eremenko–Lyubich class', *J. Lond. Math. Soc.* 92 (2015) 202–221. MR3384512
5. C. J. Bishop, 'Quasiconformal maps with dilatations of small support', Preprint, 2016.
6. D. Drasin, 'On the Teichmüller-Wittich-Belinskiĭtheorem', *Results Math.* 10 (1986) 54–65. MR0869798
7. D. Drasin, A. A. Gol'dberg and P. Poggi-Corradini, 'Quasiconformal mappings in value-distribution theory', *Handbook of complex analysis: geometric function theory*, vol. 2 (Elsevier Science, Amsterdam, 2005) 755–808. MR2121873
8. T. A. Driscoll and L. N. Trefethen, *Schwarz-Christoffel mapping*, Cambridge Monographs on Applied and Computational Mathematics 8 (Cambridge University Press, Cambridge, 2002). MR1908657
9. A. E. Eremenko and M. Yu. Lyubich, 'Dynamical properties of some classes of entire functions', *Ann. Inst. Fourier* (*Grenoble*) 42 (1992) 989–1020. MR1196102

10. J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs 2 (Cambridge University Press, Cambridge, 2005). MR2150803
11. L. R. Goldberg and L. Keen, 'A finiteness theorem for a dynamical class of entire functions', *Ergodic Theory Dynam. Systems* 6 (1986) 183–192. MR0857196
12. A. A. Goldberg and I. V. Ostrovskii, *Value distribution of meromorphic functions*, Translations of Mathematical Monographs 236 (American Mathematical Society, Providence, RI, 2008) Translated from the 1970 Russian original by Mikhail Ostrovskii, With an appendix by A. Eremenko and J. K. Langley. MR2435270
13. J. Heinonen, *Lectures on analysis on metric spaces*, Universitext (Springer, New York, 2001). MR1800917
14. O. Lehto, 'On the differentiability of quasiconformal mappings with prescribed complex dilatation', *Ann. Acad. Sci. Fenn. Ser. A I* 275 (1960) 1–28. MR0125963
15. L. Rempe, 'Rigidity of escaping dynamics for transcendental entire functions', *Acta Math.* 203 (2009) 235–267. MR2570071
16. L. Rempe-Gillen, 'Arc-like continua, Julia sets of entire functions and Eremenko's conjecture', Preprint, 2014, arXiv:1610.06278 [math.DS].
17. G. Rottenfusser, J. Rückert, L. Rempe and D. Schleicher, 'Dynamic rays of bounded-type entire functions', *Ann. of Math.* (2) 173 (2011) 77–125. MR2753600
18. O. Teichmüller, 'Eine Umkehrung des Zweiten Hauptsatzes der Wertverteilungslehre', *Deutsche Math.* 2 (1937) 96–107.

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3616046 28A80 28A75 60G17 60G18 60J10 60J65
Bishop, Christopher J. (1-SUNYS); **Peres, Yuval** (1-MSFT)
★**Fractals in probability and analysis.**
Cambridge Studies in Advanced Mathematics, 162.
Cambridge University Press, *Cambridge*, 2017. *ix*+402 *pp. ISBN* 978-1-107-13411-9

This book provides an excellent, broad introduction to the study of fractals that arise naturally in probability and analysis. As with many texts on fractals, it starts by setting out the classical notions of dimension (i.e., Minkowski, Hausdorff, packing) and the key basic techniques applied in their study, such as the mass distribution principle, and Frostman's theory and capacity. It then proceeds to introduce some of the well-known, central examples of fractals, namely self-affine sets, the Weierstrass nowhere differentiable function, and Brownian motion. Actually, the two introductory chapters on Brownian motion are relatively extensive, incorporating not only basic definitions and properties, such as nowhere differentiability and dimension, but also the deep links between Brownian motion and potential theory, as well as conformal invariance. Following this, the book goes on to cover more novel aspects of the subject, including the relationship between capacity and the hitting probabilities of discrete Markov processes, a discussion concerning Besicovitch-Kakeya sets, and a presentation of Jones' Travelling Salesman Theorem.

Being based on lecture series of the two authors, it is perhaps natural that the material is at an appropriate level for a graduate (or possibly advanced undergraduate) course. However, it is worth underlining that the text would serve this purpose extremely well. Indeed, the writing is very clear, with a focus on exposition of the main ideas, rather than the most advanced statements of results. Nonetheless, through this accessible approach, it manages to touch on several avenues of active research. (In fact, the book even provides some elegant, original proofs itself.) Moreover, all the main results are illustrated with numerous examples, and the text includes several hundred exercises at a range of difficulties, together with hints and solutions for a number of these. The historical notes are also richly informative.

Finally, I note that a list of typos/errors appears on the homepage of the first author.

David A. Croydon

MR3513876 30C65 28A78 30L10

Bishop, Christopher J. (1-SUNYS); **Hakobyan, Hrant** (1-KSS);
Williams, Marshall (1-KSS)

Quasisymmetric dimension distortion of Ahlfors regular subsets of a metric space. (English. English summary)

Geom. Funct. Anal. **26** (2016), *no.* 2, 379–421.

Quasisymmetric maps are homeomorphisms between metric spaces that preserve relative distances between points up to a multiplicative constant but nevertheless may distort other geometric properties such as Hausdorff dimension or rectifiability of sets. A standard illustration of this phenomenon is a quasisymmetric map f from the Euclidean plane onto itself that maps the unit circle S^1 onto the Koch snowflake. On the other hand, because every quasisymmetric map from the plane onto itself is also quasiconformal, the image $f(rS^1)$ of the circle of radius r about the origin is a rectifiable curve for Lebesgue almost every r. Thus, while the map does increase the Hausdorff dimension of the unit circle, the map simultaneously preserves the Hausdorff dimension and rectifiability of generic circles about the origin. This leads us to a central question of the paper under review: How frequently can a quasisymmetric map distort the dimension of sets? The authors make an in-depth study of this question and give several novel, satisfying answers.

A special case of one of the main results is that quasisymmetric self-maps of Euclidean space preserve the Hausdorff dimension of almost every translate of an Ahlfors regular set. More precisely, let $n \geq 2$ and let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a quasisymmetric map. The authors prove that if $E \subset \mathbb{R}^n$ is a bounded, Ahlfors regular set of dimension $d \in (0,n]$ (that is, the d-dimensional Hausdorff measure satisfies $\mathcal{H}^d(E \cap B(x,r)) \sim r^d$ for all $x \in E$ and $0 < r \leq \operatorname{diam} E$), then the Hausdorff dimension of $f(x+E)$ is d (the dimension of E) for Lebesgue almost every x. This result is classical when E is a rectifiable curve, but in this generality it is original even when E is a *disconnected* Ahlfors regular set of dimension 1. Remarkably, the authors also prove that an analogous statement holds for quasisymmetric self-maps of arbitrary Carnot groups of dimension at least 2.

The authors prove a number of other interesting results, the statements of a few of which require B. Fuglede's notion of modulus of measures from [Acta Math. **98** (1957), 171–219; MR0097720]. We refer the reader to the introduction of the paper for a full account. One highlight is an extension of a theorem of Z. M. Balogh, R. Monti and J. T. Tyson [J. Math. Pures Appl. (9) **99** (2013), no. 2, 125–149; MR3007840] on the frequency of dimension distortion of leaves of a foliation in Euclidean space under quasiconformal maps to product-type sets of arbitrary (non-integral) dimension. Bishop, Hakobyan, and Williams also give a significant extension of a theorem of L. V. Kovalev and J. Onninen [Studia Math. **195** (2009), no. 3, 257–274; MR2559176] by constructing planar quasiconformal maps that send uncountably many (in fact, a set with Hausdorff dimension arbitrarily close to 1 of) parallel line segments onto purely unrectifiable curves.

The proofs of the main results in this paper ultimately involve a creative use of Fuglede's modulus of measures together with more common staples such as covering theorems and Frostman's lemma. They are essential reading for any mathematician interested in contemporary geometric function theory. *Matthew Badger*

[References]

1. P. Assouad. Plongements lipschitziens dans $\mathbb{R}^n$. *Bull. Soc. Math. Fr.*, **111** (1983), 429–448. MR0763553
2. L. V. Ahlfors. On quasiconformal mappings. *J. Anal. Math.*, **3** (1954), 1–58. MR0064875
3. L. V. Ahlfors. Lectures on quasiconformal mappings. In: *University Lecture Series*, Vol. 38. American Mathematical Society, Providence (2006). MR2241787
4. M. Badger. Beurling's criterion and extremal metrics for Fuglede modulus. *Ann. Acad. Sci. Fenn. Math.*, (2)**38** (2013), 677–689. MR3113101
5. Z. M. Balogh, P. Mattila and J. Tyson. Grassmannian frequency of Sobolev dimension distortion. *Comput. Methods Funct. Theory*, (2–3)**14** (2014), 505–523. MR3265376
6. Z. M. Balogh, R. Monti and J. Tyson. Frequency of Sobolev and quasiconformal dimension distortion. *J. Math. Pures Appl.*, (2)**99** (2013), 125–149. MR3007840
7. Z. M. Balogh, J. Tyson and K. Wildrick. Dimension distortion by Sobolev mappings in foliated metric spaces. *Anal. Geom. Metr. Spaces*, **1** (2013), 232–254. MR3108874
8. Z. M. Balogh, J. Tyson and K. Wildrick. Frequency of Sobolev dimension distortion of horizontal subgroups of Heisenberg groups. To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci* (5) (2016). MR3700380
9. C.J. Bishop. A quasisymmetric surface with no rectifiable curves. *Proc. Am. Math. Soc.*, (7)**127** (1999), 2035–2040. MR1610908
10. L. Capogna, J. Tyson, S. Wenger (editors). Mapping theory in metric spaces. American Institute of Mathematics (2012). http://aimpl.org/mappingmetric.
11. G. David and T. Toro. Reifenberg flat metric spaces, snowballs, and embeddings. *Math. Ann.*, (4)**315** (1999), 641–710. MR1731465
12. B. Fuglede. Extremal length and functional completion. *Acta Math.*, **98** (1957), 171–219. MR0097720
13. J. Garnett and D. Marshall. Harmonic measure. In: *New Mathematical Monographs*, Vol. 2. Cambridge University Press, Cambridge (2005), pp. xvi+571. MR2150803
14. F.W. Gehring. The definitions and exceptional sets for quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. AI Math.*, **281** (1960), 1–28. MR0124488
15. F.W. Gehring and J.C. Kelly. Quasi-conformal mappings and Lebesgue density. Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973). In: *Ann. of Math. Studies*, Vol. 79. Princeton Univ. Press, Princeton (1974), pp. 171–179. MR0352451
16. P. Hajlasz and P. Koskela. Sobolev Met Poincare. *Mem. Amer. Math. Soc.*, **145** (2000), 688. MR1683160
17. H. Hakobyan. Conformal dimension: Cantor sets and Fuglede Modulus. *Int. Math. Res. Not.*, (1) (2010), 87–111. MR2576285
18. J. Heinonen. Lectures on analysis in metric spaces. In: *Universitext.* Springer, New York (2001). MR1800917
19. J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, **181** (1998), 1–61. MR1654771
20. J. Heinonen and P. Koskela. Definitions of quasiconformality. *Invent. Math.*, (1)**120** (1995), 61–79. MR1323982
21. J. Heinonen, P. Koskela, N. Shanmugalingam and J. Tyson. Sobolev classes of Banach space-valued functions and quasiconformal mappings. *J. Anal. Math.*, **85** (2001), 87–139. MR1869604
22. J. Heinonen and S. Rohde. The Gehring-Hayman inequality for quasihyperbolic geodesics. *Math. Proc. Camb. Philos. Soc.*, (3)**114** (1993), 393–405. MR1235987

23. S. Hencl and P. Honzik. Dimension of images of subspaces under Sobolev mappings. *Ann. Inst. H. Poincare Anal. Non Lineaire*, **29** (2012), 401–411. MR2926241
24. S. Hencl and P. Honzik. Dimension of images of subspaces under mappings in Triebel–Lizorkin spaces. *Math. Nachr.*, (7)**287** (2014), 748–763. MR3207188
25. S. Hencl and P. Honzik. Dimension distortion of images of sets under Sobolev mappings. *Ann. Acad. Sci. Fenn. Math.*, **40** (2015), 427–442. MR3329152
26. R. Korte, N. Marola and N. Shanmugalingam. Quasiconformality, homeomorphisms between metric measure spaces preserving quasiminimizers, and uniform density property. *Ark. Mat.*, (1)**50** (2012), 111–134. MR2890347
27. J. Luukkainen and E. Saksman. Every complete doubling metric space carries a doubling measure. *Proc. Am. Math. Soc.*, (2)**126** (1998), 531–534. MR1443161
28. L.V. Kovalev and J. Onninen. Variation of quasiconformal mappings on lines. *Stud. Math.*, (3)**195** (2009), 257–274. MR2559176
29. J. Mackay. Assaud dimension of self-affine carpets. *Conform. Geom. Dyn.*, **15** (2011), 177–187. MR2846307
30. P. Mattila. Geometry of sets and measures in Euclidean spaces. In: *Cambridge Stud. Adv. Math.*, Vol. 44. Cambridge University Press, Cambridge (1995). MR1333890
31. P. Mattila and P. Saaranen. Ahlfors–David regular sets and bilipschitz maps. *Ann. Acad. Sci. Fenn. Math.*, (2)**34** (2009), 487–502. MR2553808
32. J. Randolph. Distances between points of the Cantor set. *Am. Math. Mon.*, (8)**47** (1940), 549–551. MR1524942
33. J. Tyson. Lowering the Assouad dimension by quasisymmetric mappings. *Ill. J. Math.*, (2)**45** (2001), 641–656. MR1878624
34. J. Tyson. Sets of minimal Hausdorff dimension for quasiconformal maps. *Proc. Am. Math. Soc.*, (11)**128** (2000), 3361–3367. MR1676353
35. J. Tyson. Quasiconformality and quasisymmetry in metric measure spaces. *Ann. Acad. Sci. Fenn. Math.*, **23** (1998), 525–548. MR1642158
36. J. Väisälä. Lectures on n-dimensional quasiconformal mappings. In: *Lecture Notes in Mathematics*, Vol. 229. Springer, Berlin (1971). MR0454009
37. J. Väisälä. On quasiconformal mappings in space. In: *Ann. Acad. Sci. Fenn. Ser. A I No. 298* (1961). MR0140685
38. A. L. Volberg and S. V. Konyagin. On measures with the doubling condition. *Math. USSR-Izv.*, **30** (1988), 629–638. (Russian). MR0903629
39. M. Williams. Geometric and analytic quasiconformality in metric measure spaces. *Proc. Am. Math. Soc.*, **140** (2012), 1251–1266. MR2869110
40. W. P. Ziemer. Extremal length and p-capacity. *Mich. Math. J.*, **16** (1969), 43–51. MR0247077

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3509031 68U05 52B55 68Q25

Bishop, Christopher J. (1-SUNYS)

Nonobtuse triangulations of PSLGs. (English. English summary)

Discrete Comput. Geom. **56** (2016), *no.* 1, 43–92.

The article is devoted to constructing a conforming triangulation for a planar straight line graph (PSLG) in the plane with prescribed conditions on the triangles. Let V be the set of vertices of the given PSLG Γ. Then a *conforming* triangulation for Γ is a triangulation of a point set V' that contains V and such that the union of the vertices and edges of the triangulation covers Γ. The main problem is to construct conforming triangulations that use the minimum number of triangles and have good geometry.

The author considers the problem of obtaining the best angle bounds on the triangles that allow him to polynomially bound the number of triangles needed in terms of n, the number of vertices of Γ. Some results for this problem for particular cases of PSLGs have been obtained by Y. D. Burago and V. A. Zalgaller [Vestnik Leningrad. Univ. **15** (1960), no. 7, 66–80; MR0116317], J. L. Gerver [Geom. Dedicata **16** (1984), no. 1, 93–106; MR0757798], and M. W. Bern and D. Eppstein [Internat. J. Comput. Geom. Appl. **2** (1992), no. 3, 241–255; MR1194449; errata, Internat. J. Comput. Geom. Appl. **2** (1992), no. 4, 449–450; MR1202364].

The present author proves the following theorem:

Theorem 1. Every PSLG with n vertices has an $O(n^{2.5})$ conforming nonobtuse triangulation.

Improving a bound of Bern and Eppstein [op. cit.], the author then obtains another result:

Theorem 2. Any triangulation of a simple n-gon has an $O(n^2)$ nonobtuse refinement.

Moreover, the author can also approach the quadratic lower bound by considering "almost nonobtuse" triangulations:

Theorem 3. Suppose $\theta > 0$. Every PSLG with n vertices has a conforming triangulation with $O(n^2/\theta^2)$ elements and all angles $\leq 90° + \theta$.

As corollaries, the author obtains analogous results for Delaunay and Gabriel triangulations.

The proofs are based on the properties of the mutual arrangement of circles and polygons, and on the author's idea of construction of special "P-paths" for PSLGs.

Vladimir Aleksandrovich Klyachin

MR3509030 68U05 52B55 68Q25

Bishop, Christopher J. (1-SUNYS)

Quadrilateral meshes for PSLGs. (English. English summary)

Discrete Comput. Geom. **56** (2016), *no.* 1, 1–42.

The main statement proved in the paper under review is:

Theorem 1.1. Every planar straight line graph with n vertices has a conforming quadrilateral mesh with $O(n^2)$ elements, all angles $\leq 120°$ and all new angles $\geq 60°$. Both the complexity and the angle bounds are sharp. The mesh can be taken so that all but $O(n)$ vertices have degree four (Corollary 15.3).

This paper is of technical interest for researchers in discrete and computational geometry.

Jean-Charles Pinoli

[References]

1. Ahlfors, L.V.: Lectures on Quasiconformal Mappings. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA. With the assistance of Clifford Earle, J., Jr., Reprint of the 1966 original (1987) MR0883205
2. Bern, M., Eppstein, D.: Quadrilateral meshing by circle packing. Int. J. Comput. Geom. Appl. **10**(4), 347–360 (2000). Selected papers from the Sixth International Meshing Roundtable, Part II (Park City, UT, 1997) MR1791192
3. Bern, M., Mitchell, S., Ruppert, J.: Linear-size nonobtuse triangulation of polygons. Discrete Comput. Geom. **14**(4), 411–428 (1995). ACM Symposium on Computational Geometry (Stony Brook, NY, 1994) MR1360945
4. Bishop, C.J.: Conformal mapping in linear time. Discrete Comput. Geom. **44**(2), 330–428 (2010) MR2671015
5. Bishop, C.J.: Optimal angle bounds for quadrilateral meshes. Discrete Comput.

Geom. **44**(2), 308–329 (2010) MR2671014
6. Bishop, C.J.: Nonobtuse triangulations of PSLGs. Discrete Comput. Geom. (2016). doi:10.1007/s00454-016-9772-8 MR3509031
7. Eppstein, D.: Faster circle packing with application to nonobtuse triangulation. Int. J. Comput. Geom. Appl. **7**(5), 485–491 (1997) MR1471881
8. Eppstein, D., Goodrich, M.T., Kim, E., Tamstorf, R.: Motorcycle graphs: Canonical quad mesh partitioning. In: Eurographics Symposium on Geometric Processing 2008, vol. 27, no. 5 (2009)
9. Garnett, J.B., Marshall, D.E.: Harmonic Measure. New Mathematical Monographs. Cambridge University Press, Cambridge, UK (2005) MR2150803

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3421995 30D15 30D35
Bishop, Christopher J. (1-SUNYS)
The order conjecture fails in $\mathcal{S}$. (English. English summary)
J. Anal. Math. **127** (2015), 283–302.

Let f be an entire function and $S(f)$ denote the closure of its critical values and finite asymptotic values. A. E. Eremenko and M. Yu. Lyubich [Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020; MR1196102] introduced $\mathcal{S}$, the class of entire functions for which $S(f)$ is a finite set and $\mathcal{S}_n$ is its subset having exactly n singular values. Entire functions f and F are termed equivalent if there exist quasiconformal maps ϕ and ψ of the plane such that $\psi \circ f = F \circ \phi$. Eremenko and Lyubich [op. cit.] proved that for $f \in \mathcal{S}_n$ the set of functions equivalent to f forms an $(n+2)$-complex-dimensional manifold about which it was conjectured that the order was constant on each such manifold. A. L. Epstein and L. Rempe-Gillen [Ann. Acad. Sci. Fenn. Math. **40** (2015), no. 2, 573–599; MR3409693] showed the order conjecture to be true for $n = 2$, but Theorem 1.1 of the paper under review shows that there exist equivalent functions in $\mathcal{S}_3$ with different orders. Since Epstein and Rempe-Gillen [op. cit.] proved that if f has the area property, then it also satisfies the order conjecture, Theorem 1.1 also shows that the area property does not hold in $\mathcal{S}$. A function f in $\mathcal{S}$ has the area property if

$$\iint_{f^{-1}(K)\setminus\mathbb{D}} \frac{1}{|z|^2} dxdy < \infty$$

whenever K is a compact subset of $\mathbb{C} \setminus S(f)$.

The intricate and highly creative proof of Theorem 1.1 proceeds by first describing how to construct entire functions with exactly two critical values, and then modifying this idea to give a function with three critical values. The carefully written paper includes an illuminating introduction and a number of diagrams helpful in understanding the construction of the needed function. *L. R. Sons*

[References]

1. L. Ahlfors, *Lectures on Quasiconformal Mappings*, Amer. Math. Soc., Providence, RI, 2006. MR2241787
2. W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoam. **11** (1995), 355–373. MR1344897
3. C. J. Bishop, *Constructing entire functions by quasiconformal folding*, Acta Math. **214** (2015), 1–60. MR3316755
4. A. L. Epstein and L. Rempe, *On the invariance of order for finite-type entire*

functions, Ann. Acad. Sci. Fenn., to appear. MR3409693
5. A. Eremenko, *Geometric theory of meromorphic functions*, In the tradition of Ahlfors and Bers, III, Amer. Math. Soc. Providence, RI, 2004, pp. 221–230. MR2145064
6. A. Eremenko and M. Yu Lyubich, *Dynamical properties of some classes of entire functions*, Ann Inst. Fourier (Grenoble) **42** (1992), 989–1020. MR1196102
7. J. L Fernández, J. Heinonen, and O. Martio, *Quasilines and conformal mappings*, J. Anal. Math. **52** (1989), 117–132. MR0981499
8. P. W. Jones, *On removable sets for Sobolev spaces in the plane*, Essays on Fourier Analysis in Honor of Elias M. Stein, Princeton Univ. Press, Princeton, NJ, 1995, pp. 250–267. MR1315551
9. P. W. Jones and S. K. Smirnov, *Removability theorems for Sobolev functions and quasiconformal maps*, Ark. Mat. **38** (2000), 263–279. MR1785402
10. H. P. Künzi, *Konstruktion Riemannscher flächen mit vorgegebener ordnung der erzeugenden funktionen*, Math. Ann. **128** (1955), 471–474. MR0069893
11. J. K. Langley, *On the multiple points of certain meromorphic functions*, Proc. Amer. Math. Soc. **123** (1995), 1787–1795. MR1242092
12. G. M. Stallard, *Julia sets of hyperbolic meromorphic functions*, Bull. London Math. Soc. **33** (2001), 689–694. MR1853780

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3420484 37F10 14H57 37B45

Bishop, Christopher J. (1-SUNYS); **Pilgrim, Kevin M.** (1-IN)

Dynamical dessins are dense. (English. English summary)

Rev. Mat. Iberoam. **31** (2015), *no.* 3, 1033–1040.

In this work the authors successfully apply a result of the first author [cf. Invent. Math. **197** (2014), no. 2, 433–452; MR3232011] to the problem of approximating a continuum in the complex plane in the Hausdorff topology by the Julia set of a postcritically finite polynomial (with two postcritical points) in a specific family.

In [K. M. Pilgrim, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 5, 671–693; MR1834499], a *Belyĭ* polynomial g, i.e., a non-linear polynomial whose critical values are contained in $\{0,1\}$, is called an *XDBP* (*extra-clean dynamical Belyĭ polynomial*) if its postcritical set agrees with $\{0,1\}$ and further satisfies $g(0)=g(1)=1$ and $g'(0), g'(1)\neq 0$.

The main approximation theorem is given next.

Theorem 2. Given any continuum $K\subset\mathbb{C}$ and any $\epsilon>0$, there exists an XDBP with algebraic coefficients such that $d(J(g),K)<\epsilon$ holds.

The distance is the usual Hausdorff distance and $J(g)$ stands for the Julia set of g.

As remarked above, the authors' starting point is the results given in [C. J. Bishop, op. cit.].

Alfredo Poirier

[References]

1. Ahlfors, L. V.: *Conformal invariants: topics in geometric function theory.* McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York-Düsseldorf-Johannesburg, 1973. MR0357743
2. Bishop, C. J.: True trees are dense. *Invent. Math.* **197** (2014), no. 2, 433–452. MR3232011
3. Brezin, E., Byrne, R., Levy, J., Pilgrim, K. and Plummer, K.: A census of rational maps. *Conform. Geom. Dyn.* **4** (2000), 35–74. MR1749249
4. Lindsey, K. A.: Shapes of polynomial Julia sets. *Ergodic Theory Dynam. Systems*

35 (2015), no. 6, 1913–1924. MR3377290
5. Milnor, J.: *Dynamics in one complex variable.* Third edition. Annals of Mathematics Studies 160, Princeton University, Press, NJ, 2006. MR2193309
6. Pilgraim, K.M.: Dessins d'enfants and Hubbard trees. *Ann. Sci. École Norm. Sup. (4)* **33** (2000), 671–693. MR1834499
7. Schneps, L.: Dessins d'enfants on the Riemann sphere. In *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*, 47–77. London Math. Soc. Lecture Note Ser. 200, Cambridge Univ. Press, Cambridge, 1994. MR1305393

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3384512 30D15 30C62 37F10

Bishop, Christopher J. (1-SUNYS)

Models for the Eremenko-Lyubich class. (English. English summary)

J. Lond. Math. Soc. (2) **92** (2015), *no.* 1, 202–221.

Assume that $\Omega \subseteq \mathbb{C}$ is a disjoint union of unbounded, simply connected domains Ω_j which have connected boundaries and are such that sequences of Ω_j accumulate only at infinity. Let τ be a holomorphic map of Ω into the right half-plane which, on each Ω_j, is conformal and sends ∞ to ∞. Let $F = \exp(\tau)$. The pair (Ω, F) is called a model.

The Eremenko-Lyubich class consists of entire functions F whose singular set (the set of critical values and finite asymptotic values of F) is bounded. According to a result of Eremenko and Lyubich, if the singular set of F is contained in the open disc of radius R and $\Omega = \{z\colon |F(z)| > R\}$, then (Ω, F) is a model.

This very interesting paper is concerned with the relation between general models and models which are realized by Eremenko-Lyubich functions. In particular, given a model (Ω, F), to what extent can F be approximated by an Eremenko-Lyubich function?

The main result of the paper is Theorem 1.1, the essence of which is: Assume that (Ω, F) is a model and $0 < \rho \le 1$. Then there is an Eremenko-Lyubich function f and a quasiconformal mapping $\phi\colon \mathbb{C} \to \mathbb{C}$ such that $F = f \circ \phi$ on $\Omega(2\rho) = \{z \in \Omega\colon |F(z)| > e^{2\rho}\}$. There are important additional conclusions concerning F, ϕ and f.

A related theorem (Theorem 1.2) concerns models of so-called disjoint type. As the author states, a consequence of Theorem 1.2 for such models is that 'any property of $\mathcal{J}(F)$ [the Julia set of F] that is preserved by quasiconformal mappings also holds for $\mathcal{J}(f)$; for example, every component of $\mathcal{J}(f)$ is path connected or the Julia set has positive area. Since it is generally easier to build a model with a desired property than to build an entire function directly, this result is useful in constructing Eremenko-Lyubich functions with pathological behavior.'

The paper is closely related to other recent work of the author; the links between the papers are carefully explained. *P. C. Fenton*

[References]

1. L. V. Ahlfors, *Lectures on quasiconformal mappings*, 2nd edn, University Lecture Series 38 (American Mathematical Society, Providence, RI, 2006). With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. MR2241787
2. W. Bergweiler, N. Fagella and L. Rempe, 'Hyperbolic entire functions with bounded Fatou components', Preprint, 2014, arXiv:1404.0925 [math.DS]. MR3433280
3. L. Bers, 'The moduli of Kleinian groups', *Uspekhi Mat. Nauk* 29 (1974) 86–102. Translated from the English by M. E. Novodvorskiĭ, Collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ(1901–1973), I. MR0422691

4. C. J. Bishop, 'Models for the Speiser class', Preprint, 2014, http://www.math.stonybrook.edu/~bishop/papers/S-models.pdf. MR3653246
5. C. J. Bishop, 'Constructing entire functions by quasiconformal folding', *Acta Math.* 214 (2015) 1–60. MR3316755
6. B. Branner and N. Fagella, *Quasiconformal surgery in holomorphic dynamics*, Cambridge Studies in Advanced Mathematics 141 (Cambridge University Press, Cambridge, 2014). MR3445628
7. A. Douady and J. H. Hubbard, 'On the dynamics of polynomial-like mappings', *Ann. Sci. École Norm. Sup.* (4) 18 (1985) 287–343. MR0816367
8. D. Drasin, A. A. Gol'dberg and P. Poggi-Corradini, 'Quasiconformal mappings in value-distribution theory', *Handbook of complex analysis: geometric function theory*, vol. 2 (Elsevier Science B. V., Amsterdam, 2005) 755–808. MR2121873
9. A. È. Eremenko and M. Yu. Lyubich, 'Dynamical properties of some classes of entire functions', *Ann. Inst. Fourier* (*Grenoble*) 42 (1992) 989–1020. MR1196102
10. J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics 96 (Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1981). MR0628971
11. J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs 2 (Cambridge University Press, Cambridge, 2005). MR2150803
12. A. A. Goldberg and I. V. Ostrovskii, *Value distribution of meromorphic functions*, Translations of Mathematical Monographs 236 (American Mathematical Society, Providence, RI, 2008). Translated from the 1970 Russian original by Mikhail Ostrovskii. With an appendix by Alexandre Eremenko and James K. Langley. MR2435270
13. T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs (Clarendon Press, New York, 2001). MR1859913
14. O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, 2nd edn, Die Grundlehren der mathematischen Wissenschaften 126 (Springer, New York, 1973). Translated from the German by K. W. Lucas. MR0344463
15. L. Rempe, 'Rigidity of escaping dynamics for transcendental entire functions', *Acta Math.* 203 (2009) 235–267. MR2570071
16. L. Rempe-Gillen, 'Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremenko–Lyubich functions', *Proc. London Math. Soc.* (3) 108 (2014) 1193–1225. MR3214678
17. L. Rempe-Gillen, 'Arc-like continua, Julia sets of entire functions and Eremenko's conjecture', Preprint, 2014.
18. Y. G. Reshetnyak, *Space mappings with bounded distortion*, Translations of Mathematical Monographs 73 (American Mathematical Society, Providence, RI, 1989). Translated from the Russian by H. H. McFaden. MR0994644
19. S. Rickman, 'Removability theorems for quasiconformal mappings,' *Ann. Acad. Sci. Fenn. Ser. A I No.* 449 (1969) 8. MR0254234
20. S. Rickman, *Quasiregular mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)) 26 (Springer, Berlin, 1993). MR1238941

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3316755 30D15 05C90 30C65 30D05 30D20
Bishop, Christopher J. (1-SUNYS-NDM)
Constructing entire functions by quasiconformal folding.
Acta Math. **214** (2015), *no.* 1, 1–60.
The author develops a method for constructing transcendental entire functions with good control on the singular values and on the geometry of the super-level set $\{|f|>R\}$. More precisely, he considers the following classes of functions: $\mathcal{S}$, the class of Speiser functions, that is, the entire transcendental functions f with finite singular set $S(f)$ (the closure of the critical values and finite asymptotic values of f); $\mathcal{S}_n\subset\mathcal{S}$, the class of functions with at most n singular values; $\mathcal{S}_{p,q}\subset\mathcal{S}$, the class of functions with p critical values and q finite asymptotic values (in particular, the class $\mathcal{S}_{2,0}$ contains the Shabat polynomials); $\mathcal{B}$, the class of Eremenko-Lyubich functions, that is, of transcendental entire functions with bounded (but possibly infinite) singular sets. A basic question addressed in the paper is the construction of such functions.

One basic construction of the author starts with an infinite planar tree T satisfying some mild geometric conditions, and then produces a method for constructing an entire function in $\mathcal{S}_{2,0}$ with critical values exactly ±1, so that $f^{-1}([-1,1])$ approximates T in a precise way. In fact, the author first obtains a quasiregular function g with some desired properties and then applies the measurable Riemann mapping theorem to obtain a quasiconformal mapping ϕ such that $f=g\circ\phi$ is entire.

Using this method, the author solves a certain number of open problems, in particular, the area conjecture of Eremenko and Lyubich and the existence of a function f with bounded singular set whose Fatou set contains a wandering domain.

Athanase Papadopoulos

[References]

1. Baker, I. N., Multiply connected domains of normality in iteration theory. *Math. Z.,* 81 (1963), 206–214. MR0153842
2. Baker, I. N. The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A I Math.,* 1 (1975), 277–283. MR0402044
3. Baker, I. N. An entire function which has wandering domains. *J. Austral. Math. Soc. Ser. A,* 22 (1976), 173–176. MR0419759
4. Belyĭ, G. V., Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.,* 43 (1979), 267–276, 479 (Russian); English translation in *Math. USSR–Izv.,* 14 (1980), 247–256. MR0534593
5. Bergweiler, W., Iteration of meromorphic functions. *Bull. Amer. Math. Soc.,* 29 (1993), 151–188. MR1216719
6. Bergweiler, W. & Eremenko, A., Entire functions of slow growth whose Julia set coincides with the plane. *Ergodic Theory Dynam. Systems,* 20 (2000), 1577–1582. MR1804945
7. Bergweiler, W., Haruta, M., Kriete, H., Meier, H.-G. & Terglane, N., On the limit functions of iterates in wandering domains. *Ann. Acad. Sci. Fenn. Ser. A I Math.,* 18 (1993), 369–375. MR1234740
8. Beurling, A., Some theorems on boundedness of analytic functions. *Duke Math. J.,* 16 (1949), 355–359. MR0029980
9. Bishop, C. J., A transcendental Julia set of dimension 1. Preprint, 2011. MR3787831
10. Bishop, C. J. True trees are dense. *Invent. Math.,* 197 (2014), 433–452. MR3232011
11. Bishop, C. J. Models for the Eremenko–Lyubich class. Preprint, 2014. MR3384512
12. Bishop, C. J. Models for the Speiser class. Preprint, 2014. MR3384512

13. Bishop, C. J. The order conjecture fails in class S. To appear in *J. Anal. Math.* MR3421995
14. Branner, B. & Fagella, N., *Quasiconformal Surgery in Holomorphic Dynamics.* Cambridge Studies in Advanced Mathematics, 141. Cambridge University Press, Cambridge, 2014. MR3445628
15. Cámera, G. A., On a problem of Erdős. *Acta Cient. Venezolana,* 34 (1983), 191–194. MR0910146
16. Drasin, D., Gol'dberg, A. A. & Poggi-Corradini, P., Quasiconformal mappings in value-distribution theory, in *Handbook of Complex Analysis: Geometric Function Theory.* Vol. 2, pp. 755–808. Elsevier, Amsterdam, 2005. MR2121873
17. Dyn'kin, E. M., Smoothness of a quasiconformal mapping at a point. *Algebra i Analiz,* 9 (1997), 205–210 (Russian); English translation in *St. Petersburg Math. J.,* 9 (1998), 601–605. MR1466801
18. Epstein, A., *Towers of Finite Type Complex Analytic Maps.* Ph.D. Thesis, City University of New York, New York, NY, 1995. http://pcwww.liv.ac.uk/~lrempe/adam/thesis.pdf. MR2690048
19. Eremenko, A., On the iteration of entire functions, in *Dynamical Systems and Ergodic Theory* (Warsaw, 1986), Banach Center Publ., 23, pp. 339–345. PWN–Polish Scientific Publishers, Warsaw, 1989. MR1102727
20. Eremenko, A. Transcendental meromorphic functions with three singular values. *Illinois J. Math.,* 48 (2004), 701–709. MR2085435
21. Eremenko, A. & Lyubich, M. Yu., Examples of entire functions with pathological dynamics. *J. London Math. Soc.,* 36 (1987), 458–468. MR0918638
22. Eremenko, A. & Lyubich, M. Yu. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier* (*Grenoble*), 42 (1992), 989–1020. MR1196102
23. Fagella, N., Godillion, S. & Jarque, X., Wandering domains for composition of entire functions. Preprint, 2014. arXiv:1410.3221 [math.DS]. MR3339086
24. Garnett, J. B. & Marshall, D., *Harmonic Measure.* New Mathematical Monographs, 2. Cambridge University Press, Cambridge, 2005. MR2150803
25. Gauthier, P., Unbounded holomorphic functions bounded on a spiral. *Math. Z.,* 114 (1970), 278–280. MR0267102
26. Gol'dberg, A. A., Sets on which the modulus of an entire function has a lower bound. *Sibirsk. Mat. Zh.,* 20 (1979), 512–518, 691 (Russian); English translation in *Siberian Math. J.,* 20 (1979), 360–364. MR0537357
27. Gol'dberg, A. A. & Ostrovskii, I. V., *Value Distribution of Meromorphic Functions.* Translations of Mathematical Monographs, 236. Amer. Math. Soc., Providence, RI, 2008. MR2435270
28. Goldberg, L. & Keen, L., A finiteness theorem for a dynamical class of entire functions. *Ergodic Theory Dynam. Systems,* 6 (1986), 183–192. MR0857196
29. Hayman, W. K., The minimum modulus of large integral functions. *Proc. London Math. Soc.,* 2 (1952), 469–512. MR0056083
30. Hayman, W. K. Research problems in function theory: new problems, in *Proceedings of the Symposium on Complex Analysis* (Canterbury, 1973), London Math. Soc. Lecture Note Ser., 12, pp. 155–180. Cambridge Univ. Press, London, 1974. MR0387546
31. He, Z.-X. & Schramm, O., Fixed points, Koebe uniformization and circle packings. *Ann. of Math.,* 137 (1993), 369–406. MR1207210
32. Heins, M., The set of asymptotic values of an entire function, in *Tolfte Skandinaviska Matematikerkongressen* (Lund, 1953), pp. 56–60. Lunds Universitets Matematiska Institution, Lund, 1954. MR0067989
33. Hinchliffe, J. D., The Bergweiler–Eremenko theorem for finite lower order. *Re-*

sults Math., 43 (2003), 121–128. MR1962854

34. Kochetkov, Yu. Yu., On the geometry of a class of plane trees. *Funktsional. Anal. i Prilozhen.,* 33 (1999), 78–81 (Russian); English translation in *Funct. Anal. Appl.,* 33 (1999), 304–306. MR1746434
35. Kochetkov, Yu. Yu. Geometry of planar trees. *Fundam. Prikl. Mat.,* 13 (2007), 149–158 (Russian); English translation in *J. Math. Sci.* (*N. Y.*), 158 (2009), 106–113. MR2476033
36. Langley, J. K., On differential polynomials, fixpoints and critical values of meromorphic functions. *Results Math.,* 35 (1999), 284–309. MR1694909
37. Merenkov, S., Rapidly growing entire functions with three singular values. *Illinois J. Math.,* 52 (2008), 473–491. MR2524647
38. Mihaljević-Brandt, H. & Rempe-Gillen, L., Absence of wandering domains for some real entire functions with bounded singular sets. *Math. Ann.,* 357 (2013), 1577–1604. MR3124942
39. Rempe-Gillen, L., Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremenko–Lyubich functions. *Proc. Lond. Math. Soc.,* 108 (2014), 1193–1225. MR3214678
40. Rempe-Gillen, L. & Rippon, P., Exotic Baker and wandering domains for Ahlfors islands maps. *J. Anal. Math.,* 117 (2012), 297–319. MR2944099
41. Rottenfusser, G., Rückert, J., Rempe-Gillen, L. & Schleicher, D., Dynamic rays of bounded-type entire functions. *Ann. of Math.,* 173 (2011), 77–125. MR2753600
42. Schleicher, D., Dynamics of entire functions, in *Holomorphic Dynamical Systems,* Lecture Notes in Math., 1998, pp. 295–339. Springer, Berlin, 2010. MR2648691
43. Shabat, G. & Zvonkin, A., Plane trees and algebraic numbers, in *Jerusalem Combinatorics '93,* Contemp. Math., 178, pp. 233–275. Amer. Math. Soc., Providence, RI, 1994. MR1310587
44. Stallard, G., The Hausdorff dimension of Julia sets of entire functions. II. *Math. Proc. Cambridge Philos. Soc.,* 119 (1996), 513–536. MR1357062
45. Stallard, G. The Hausdorff dimension of Julia sets of entire functions. III. *Math. Proc. Cambridge Philos. Soc.,* 122 (1997), 223–244. MR1458228
46. Sullivan, D., Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou–Julia problem on wandering domains. *Ann. of Math.,* 122 (1985), 401–418. MR0819553
47. Teichmüller, O., Eine Umkehrung des zweiten Hauptsatzes der Wertverteilungslehre. *Deutsche Math.,* 2 (1937), 96–107.
48. Wiman, A., Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem grössten Betrage bei gegebenem Argumente der Funktion. *Acta Math.,* 41 (1916), 1–28. MR1555144
49. Wittich, H., Zum Beweis eines Satzes über quasikonforme Abbildungen. *Math. Z.,* 51 (1948), 278–288. MR0027057

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3240099 40G10 90C40

Bishop, Christopher J. (1-SUNYS); **Feinberg, Eugene A.** (1-SUNYS-S);
Zhang, Junyu (PRC-ZHO-SMC)

Examples concerning Abel and Cesàro limits. (English. English summary)

J. Math. Anal. Appl. **420** (2014), *no.* 2, 1654–1661.

For a sequence $\{u_n\}_{n=0,1,\dots}$ lower and upper Cesàro and Abel limits are defined by

$$\underline{C} = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} u_i, \qquad \overline{C} = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} u_i$$

$$\underline{A} = \liminf_{\alpha\to 1-}(1-\alpha)\sum_{n=0}^{\infty} u_n a^n, \qquad \overline{A} = \limsup_{\alpha\to 1-}(1-\alpha)\sum_{n=0}^{\infty} u_n a^n$$

respectively.

In this paper, in light of the definitions given above, the authors describe examples of all possible equality and strict inequality relations between upper and lower Abel and Cesàro limits of sequences bounded above or below. They also give some propositions related to these inequalities and equalities. These phenomena provide applications to Markov decision processes. *Abdulcabbar Sönmez*

[References]

1. V.M. Abramov, Optimal control of a large dam, J. Appl. Probab. 44 (1) (2007) 249–258. MR2313000
2. P.L. Duren, Introduction to Classical Analysis, American Mathematical Society, Providence, RI, 2012. MR2933135
3. E.B. Dynkin, A.A. Yushkevich, Controlled Markov Processes, Springer-Verlag, New York, 1979. MR0554083
4. E.A. Feinberg, An ϵ-optimal control of a finite Markov chain, Theory Probab. Appl. 25 (1) (1980) 70–81. MR0560058
5. E.A. Feinberg, P.O. Kasyanov, N.V. Zadoianchuk, Average cost Markov decision processes with weakly continuous transition probabilities, Math. Oper. Res. 37 (4) (2012) 591–607. MR2997893
6. G.H. Hardy, On certain oscillating series, Q. J. Math. 38 (1907) 269–288.
7. O. Hernández-Lerma, Average optimality in dynamic programming on Borel spaces—unbounded costs and controls, Systems Control Lett. 17 (3) (1991) 237–242. MR1125975
8. J.P. Keating, J.B. Reade, Summability of alternating gap series, Proc. Edinb. Math. Soc. 43 (1) (2000) 95–101. MR1744701
9. T.M. Liggett, S.A. Lippman, Stochastic games with perfect information and time average payoff, SIAM Rev. 11 (4) (1969) 604–607. MR0260435
10. J. Neveu, Mathematical Foundations of the Calculus of Probability, Holden-Day, San Francisco, 1965. MR0198505
11. M. Schäl, Average optimality in dynamic programming with general state space, Math. Oper. Res. 18 (1) (1993) 163–172. MR1250112
12. L.I. Sennott, Stochastic Dynamic Programming and the Control of Queueing Systems, John Wiley & Sons, New York, 1999. MR1645435
13. R. Sznajder, J.A. Filar, Some comments on a theorem of Hardy and Littlewood, J. Optim. Theory Appl. 75 (1) (1992) 201–208. MR1189274
14. E.C. Titchmarsh, The Theory of Functions, 2nd ed., Oxford University Press, Oxford, 1939. MR3155290

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR3232011 30C62 31A15 54F15
Bishop, Christopher J. (1-SUNYS)
True trees are dense. (English. English summary)
Invent. Math. **197** (2014), *no.* 2, 433–452.

In this remarkable paper, the author proves that certain trees are dense in the set of plane continua, with respect to the Hausdorff distance.

First, he shows that if K is any compact connected set in the complex plane and $\varepsilon > 0$, then there is a polynomial p of one complex variable with critical values exactly at 1 and -1 such that the Hausdorff distance between K and the tree $p^{-1}([-1,1])$ (a "true tree") is less than ε.

Then, the author proves that true trees are the same as conformally balanced trees. A tree T with n edges is called conformally balanced if its complement is the image of the exterior of the unit disk under a conformal mapping f fixing infinity and such that each side of each edge corresponds to an arc of length π/n and $f(z) = f(w)$ implies that $f'(z) = f'(w)$ for almost all z, w on the unit circle. Then the harmonic measure with respect to the point at infinity is the same on the two sides of each edge.

The proof of the main theorem begins with the approximation of a given continuum by small squares, in a grid of squares, that intersect the continuum. The author then devises imaginative ways of creating various trees from this configuration. Quasiconformal mappings are used as a tool. Finally this leads us to a conformally balanced tree.

A. Hinkkanen

[References]

1. Ahlfors, L.V.: Lectures on Quasiconformal Mappings, 2nd edn. University Lecture Series, vol. 38. Am. Math. Soc., Providence (2006). With supplemental chapters by C.J. Earle, I. Kra, M. Shishikura and J.H. Hubbard MR2241787
2. Bishop, C.J.: Building entire functions by quasiconformal folding (2012, preprint) MR3316755
3. Bishop, C.J., Pilgim, K.M.: Dynamic dessins are dense (2013, preprint) MR3420484
4. Bowers, P.L., Stephenson, K.: Uniformizing dessins and Belyĭ maps via circle packing. Mem. Am. Math. Soc. **170**(805), xii+97 (2004) MR2053391
5. Erëmenko, A.È., Lyubich, M.Yu.: Dynamical properties of some classes of entire functions. Ann. Inst. Fourier (Grenoble) **42**(4), 989–1020 (1992) MR1196102
6. Goldberg, L.R., Keen, L.: A finiteness theorem for a dynamical class of entire functions. Ergod. Theory Dyn. Syst. **6**(2), 183–192 (1986) MR0857196
7. Harary, F., Prins, G., Tutte, W.T.: The number of plane trees. Nederl. Akad. Wet. Proc. Ser. A **67**, 319–329 (1964), Indag. Math. MR0166776
8. Harvey, W.J.: Teichmüller spaces, triangle groups and Grothendieck dessins. In: Handbook of Teichmüller Theory. Vol. I. IRMA Lect. Math. Theor. Phys., vol. 11, pp. 249–292. Eur. Math. Soc., Zürich (2007) MR2349672
9. Jones, P.W.: On removable sets for Sobolev spaces in the plane. In: Essays on Fourier Analysis in Honor of Elias M. Stein, Princeton, NJ, 1991. Princeton Math. Ser., vol. 42, pp. 250–267. Princeton University Press, Princeton (1995) MR1315551
10. Jones, P.W., Smirnov, S.K.: Removability theorems for Sobolev functions and quasiconformal maps. Ark. Mat. **38**(2), 263–279 (2000) MR1785402
11. Kochetkov, Yu. Yu.: On the geometry of a class of plane trees. Funkc. Anal. Prilozh. **33**(4), 78–81 (1999) MR1746434

12. Kochetkov, Yu. Yu.: Geometry of planar trees. Fundam. Prikl. Mat. **13**(6), 149–158 (2007) MR2476033
13. Kochetkov, Yu. Yu.: Planar trees with nine edges: a catalogue. Fundam. Prikl. Mat. **13**(6), 159–195 (2007) MR2476034
14. Lárusson, F., Sadykov, T.: Dessins d'enfants and differential equations. Algebra Anal. **19**(6), 184–199 (2007) MR2411966
15. Lindsey, K.A., Thurston, W.P.: Shapes of polynomial Julia sets (2012, preprint). arXiv:1209.0143 MR3377290
16. Marshall, D.E., Rohde, S.: The zipper algorithm for conformal maps and the computation of Shabat polynomials and dessins (manuscript in preparation)
17. Pakovitch, F.: Combinatoire des arbres planaires et arithmétique des courbes hyperelliptiques. Ann. Inst. Fourier (Grenoble) **48**(2), 323–351 (1998) MR1625545
18. Pilgrim, K.M.: Dessins d'enfants and Hubbard trees. Ann. Sci. Éc. Norm. Super. **33**(5), 671–693 (2000) MR1834499
19. Schneps, L. (ed.): The Grothendieck Theory of Dessins D'enfants. London Mathematical Society Lecture Note Series, vol. 200. Cambridge University Press, Cambridge (1994). Papers from the Conference on Dessins d'Enfant held in Luminy, April 19–24, 1993 MR1305393
20. Shabat, G., Zvonkin, A.: Plane trees and algebraic numbers. In: Jerusalem Combinatorics '93. Contemp. Math., vol. 178, pp. 233–275. Am. Math. Soc., Providence (1994) MR1310587
21. Stahl, H.R.: Sets of minimal capacity and extremal domains (2012, preprint). arXiv:1205.3811
22. Sullivan, D.: Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. Ann. Math. (2) **122**(3), 401–418 (1985) MR0819553
23. Walkup, D.W.: The number of plane trees. Mathematika **19**, 200–204 (1972) MR0376411
24. Wolfart, J.: *ABC* for polynomials, dessins d'enfants and uniformization—a survey. In: Elementare und Analytische Zahlentheorie. Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt Am Main, vol. 20, pp. 313–345. Franz Steiner, Stuttgart (2006) MR2310190

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2904137 28A75 26B15 68U05

Bishop, Christopher J. (1-SUNYS)

Tree-like decompositions of simply connected domains. (English. English summary)

Rev. Mat. Iberoam. **28** (2012), *no.* 1, 179–200.

The paper deals with the problem of decomposing a simply connected domain into nice subdomains. A circular arc crosscut is a circular arc in Ω with distinct endpoints on $\partial\Omega$, and a domain Ω is a Lipschitz crescent if there are $\epsilon > 0$ and $\theta \in (0, \pi/2)$ so that $\partial\Omega$ consists of two arcs connecting -1 to $+1$; the first is a circular arc in the upper half-plane that makes an angle θ with the real line at ± 1, and the second is a Lipschitz graph for which the slopes are bounded above by $\theta - \epsilon$ and below by $-\epsilon$. Also, every Möbius image of such a domain is a Lipschitz crescent, and an M-Lipschitz crescent is a Lipschitz crescent which is the image of an M-Lipschitz function. Furthermore, the 1-dimensional measure $l(E)$ of a set E in the plane is

$$l(E) = \liminf_{\delta \to 0} \left\{ \sum 2r_j : E \subset \cup B(x_j, r_j), r_j \leq \delta \right\}.$$

The author proves the following.

Theorem. There is an $M < \infty$ such that every simply connected domain Ω has a collection of disjoint circular arc crosscuts $\Gamma = \bigcup \gamma_k$ with $\sum_k l(\gamma_k) \leq M\, l(\partial\Omega)$ and such that each connected component of $\Omega \setminus \Gamma$ is an M-Lipschitz crescent.

The proof is based on the concepts of medial axis and medial axis flow from computational geometry, and the theorem contains a theorem of P. Jones from 1990 which was proved using only the conformal mapping from the disk onto Ω. *Bodo Dittmar*

[References]

1. Aurenhammer, F. and Klein, R.: Voronoi diagrams. In *Handbook of computational geometry,* 201–290. North-Holland, Amsterdam, 2000. MR1746678
2. Azzam, J. and Schul, R.: How to take shortcuts in Euclidean space: making a given set into a short quasi-convex set. To appear in *Proc. London Math. Soc.* MR2959930
3. Bishop, C. J.: Treelike decompositions and conformal maps. *Ann. Acad. Sci. Fenn.* **35** (2010), no. 2, 389–404. MR2731698
4. Bishop, C. J.: Estimates for harmonic conjugation. Preprint, 2009.
5. Bishop, C. J. and Hakobyan, H.: A central set of dimension 2. *Proc. Amer. Math. Soc.* **136** (2008), no. 7, 2453–2461. MR2390513
6. Chin, F., Snoeyink, J. and Wang, C. A.: Finding the medial axis of a simple polygon in linear time. *Discrete Comput. Geom.* **21** (1999), no. 3, 405–420. MR1672988
7. Choi, H. I., Choi, S. W. and Moon, H. P.: Mathematical theory of medial axis transform. *Pacific J. Math.* **181** (1997), no. 1, 57–88. MR1491036
8. Erdös, P.: Some remarks on the measurability of certain sets. *Bull. Amer. Math. Soc.* **51** (1945), 728–731. MR0013776
9. Fremlin, D. H.: Skeletons and central sets. *Proc. London Math. Soc. (3)* **74** (1997), no. 3, 701–720. MR1434446
10. Jones, P. W.: Rectifiable sets and the traveling salesman problem. *Invent. Math.* **102** (1990), no. 1, 1–15. MR1069238
11. Kenyon, C. and Kenyon, R.: How to take short cuts. "ACM Symposium on Computational Geometry (North Conway, NH, 1991)". *Discrete Comput. Geom.* **8** (1992), no. 3, 251–264. MR1174357
12. Preparata, F. P.: The medial axis of a simple polygon. In *Mathematical foundations of computer science (Proc. Sixth Sympos., Tatranská Lomnica, 1977),* 443–450. Lecture Notes in Comput. Sci. 53, Springer, Berlin, 1977. MR0464625

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2900167 30C62 28A80 31A05

Bishop, Christopher J. (1-SUNYS)

A set containing rectifiable arcs QC-locally but not QC-globally. (English. English summary)

Pure Appl. Math. Q. **7** (2011), *no.* 1, 121–138.

The quasiconformal Jacobian problem asks for a characterization of weights w on $\mathbb{R}^n$ for which there exists a quasiconformal map f such that $C^{-1}w \leq J_f \leq Cf$ almost everywhere for some constant C. This question goes back to G. David and S. W. Semmes [in *Analysis and partial differential equations*, 101–111, Lecture Notes in Pure and Appl. Math., 122, Dekker, New York, 1990; MR1044784] and is further motivated by its direct relation to the problem of bi-Lipschitz parametrization of metric spaces;

see the paper by M. Bonk, J. Heinonen and E. Saksman [in *In the tradition of Ahlfors and Bers, III*, 77–96, Contemp. Math., 355, Amer. Math. Soc., Providence, RI, 2004; MR2145057].

At present it is not known whether the condition $C^{-1}w \leq J_f \leq Cf$ restricts the size of the set on which w blows up to ∞. Specifically, it is not known whether there exists a compact null set E such that no weight w with $w(x) \to \infty$ as $x \to E$ can be comparable to the Jacobian of a quasiconformal map. A stronger form of this question is to ask for a compact null set E such that every quasiconformal image of E contains a rectifiable curve. The existence of such a set is not known either.

However, the author constructs a compact null set $E \subset \mathbb{C}$ for which there exists a constant $K_0 > 1$ such that the image of E under any K_0-quasiconformal map contains a rectifiable curve. He also proves that there exist quasiconformal maps f with distortion about K_0 such that $f(E)$ contains no rectifiable curves. This is the meaning of "QC-locally but not QC-globally" in the title. The construction involves a number of elements of independent interest, such as a "low visibility forest" in the plane.

The paper can be considered as a sequel to the author's previous work [in *In the tradition of Ahlfors-Bers. IV*, 7–18, Contemp. Math., 432, Amer. Math. Soc., Providence, RI, 2007; MR2342802], where an A_1 weight which is not comparable to any quasiconformal Jacobian was constructed. *Leonid V. Kovalev*

[References]

1. C. J. Bishop. An A_1 weight not comparable to any quasiconformal Jacobian. *In the tradition of Ahlfors-Bers. IV, Contemp. Math.,* Amer. Math. Soc. 432:7–18, 2007. MR2342802
2. C. J. Bishop and P. W. Jones. Harmonic measure and arclength. *Ann. of Math. (2),* 132(3):511–547, 1990. MR1078268
3. C. J. Bishop and P. W. Jones. Harmonic measure, L^2 estimates and the Schwarzian derivative. *J. Anal. Math.,* 62:77–113, 1994. MR1269200
4. M. Bonk, J. Heinonen, and E. Saksman. The quasiconformal Jacobian problem. *Contemp. Math.,* 355:77–96, 2004. MR2145057
5. J. W. S. Cassels. *An introduction to Diophantine approximation.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 45. Cambridge University Press, New York, 1957. MR0087708
6. J. B. Garnett and D.E. Marshall. *Harmonic measure.* New Mathematical Monographs, 2. Cambridge University Press, 2005. MR2150803
7. J. Heinonen. The branch set of a quasiregular mapping. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002),* pages 691–700, Beijing, 2002. Higher Ed. Press. MR1957076
8. J. Heinonen. Geometric embeddings of metric spaces. 2003. Jyväskylä Mathematics Department Reports. MR2014506
9. J. Heinonen and S. Semmes. Thirty-three yes or no questions about mappings, measures, and metrics. *Conform. Geom. Dyn.,* 1:1–12 (electronic), 1997. MR1452413
10. P. W. Jones. Rectifiable sets and the traveling salesman problem. *Invent. Math.,* 102(1):1–15, 1990. MR1069238
11. P. W. Jones. The traveling salesman problem and harmonic analysis. *Publ. Mat.,* 35(1):259–267, 1991. Conference on Mathematical Analysis (El Escorial, 1989). MR1103619
12. K. Okikiolu. Characterization of subsets of rectifiable curves in $\mathbf{R}^n$. *J. London Math. Soc. (2),* 46(2):336–348, 1992. MR1182488
13. S. Semmes. Bi-Lipschitz mappings and strong A_∞ weights. *Ann. Acad. Sci. Fenn.*

Ser. A I Math., 18(2):211–248, 1993. MR1234732

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2731698 (2011m:30009) 30C35 30C30 30C62 65E05
Bishop, Christopher J. (1-SUNYS)
Tree-like decompositions and conformal maps. (English. English summary)
Ann. Acad. Sci. Fenn. Math. **35** (2010), *no.* 2, 389–404.
Let D denote the unit disc and let Ω be a Jordan domain with rectifiable boundary $\partial\Omega$, where we denote the length of a boundary arc I by $l(I)$. The region Ω is said to be *chord-arc* if there is a number $M<\infty$ such that $l(\sigma(x,y))\le M|x-y|$ for all $x,y\in\partial\Omega$, where $\sigma(x,y)$ denotes the shorter arc on $\partial\Omega$ between x and y. Any such region Ω has a collection of cross-cuts that divide it into uniformly chord-arc subdomains, that is, each of the subdomains is chord-arc with the same constant M. Such a division is called a *tree-like decomposition of* Ω. For a Jordan region Ω with rectifiable boundary and for a tree-like decomposition of Ω, the author constructs a map from $\partial\Omega$ onto ∂D that has a quasiconformal extension from Ω onto D, where the constant K of quasi-conformality depends only on the uniform chord-arc constant M. The mapping of boundary to boundary is obtained by piecing together functions on sections of the boundary of Ω that are chosen to correspond to the individual boundaries of the subdomains of the tree-like decomposition. The result answers a question of S. A. Vavasis. This paper is related to two other yet-to-appear papers of the author dealing with related questions due to Vavasis. *P. Lappan*

[References]

1. Ahlfors, L. V.: Lectures on quasiconformal mappings. - Univ. Lecture Ser. 38, with supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J.H. Hubbard, Amer. Math. Soc., Providence, RI, second edition, 2006. MR2241787
2. Banjai, L., and L. N. Trefethen: A multipole method for Schwarz-Christoffel macping of polygons with thousands of sides. - SIA.Sei. Comput. 25:3, 2003, 1042–1065 (electronic). MR2046124
3. Beardon, A. F.: The geometry o iscrete groups. - Springer-Verlag, New York, 1983. MR0698777
4. Bishop, C. J.: Bounds for the CRDT conformal mapping algorithm.-Comput. Methods Funct. Theory 10:1, 2010, 325–366. MR2676459
5. Bishop, C. J.: Conformal mapping nlinear time. -Discrete Comput. Geom. (to appear). MR2671015
6. Bishop, C. J.: simates for harmonic conjugation. - Preprint, 2009.
7. Bishop, C. J.: A fatQC-mapping theorem for polygons. - Preprint, 2009.
8. Bishop, C. J.: Treelike decompostons of simplyconnected domains. - Preprint, 2009.
9. Davis, R. T.: Numerical methods for coordinate generation based on Schwarz-Christoffel transformations. - In: Proceedings of the 4th AIAA Computational Flud ynamics Conference, Williamsburg, VA, 1979, 1–15.
10. Dennis, J. E., Jr., and R. B. schnabel: Numerical methods for unconstrained optimization and nonlinear equations. - Prentice Hall Series in Computational Maheatics, Prentice Hall, Englewood Cliffs, NJ, 1983. MR0702023
11. Driscoll, T. A., and S.A. Vavasis: Numerical conformal mapping usinc cross-ratios and Delaunay triangulation. - SIA J Sei. Comput. 19:6, 1988, 1783–1803 (electronic). MR1638056

12. Epstein, D. B. A., A. marden, and V. markovic: Quasiconformal homeomorphisms and the convex hul buNDARY. - ANN. OF MATH. (2) 159:1, 2004, 305–336. MR2052356
13. Garnett, J. B., and D. E. Marshall: Harmonic measure. - New Math.MooGR. 2, CAMBRIDGE UNIV. PRESS, Cambridge, 2005. MR2150803
14. Gehring, F. W., and W. K. hayman: An inequality in the theory of conformal mppnG. J. MATH. Pures Appl. (9) 41, 1962, 353–361. MR0148884
15. Ghamsari, M.: ExtesinDOMAINS. - PHDthesis, Univ. of Michigan, 1990.
16. Pommerenke, Ch.: One-sided smoothness conditions and conformal mppng. -a%v. Londo%v Math.%vSoc. (2) 26:1, 1982, 77–88. MR0667246
17. Väisälä, J.: Homeocorphisms of bounded length distortion. - Ann.Acd. Sei%v Fenn. %ver. A %v Math. 12:2, 1987, 303–312. MR0951979
18. väisälä, J.: Free cuasiconformality in Banach spaces. II. - Ann. Acad. Sei. Fenn. Ser. A I Math. 16:2, 1991, 255–310. MR1139798

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2676459 (2012b:30010) 30C30 65E05

Bishop, Christopher J. (1-SUNYS)

Bounds for the CRDT conformal mapping algorithm. (English. English summary)

Comput. Methods Funct. Theory **10** (2010), *no.* 1, 325–366.

The Cross Ratios and Delaunay Triangulations (CRDT) algorithm numerically computes the classical Schwarz-Christoffel map from the unit disk $\mathbb{D}$ to the interior of a polygon P (an n-gon), with the following steps:

(1) add extra vertices to P so that the resulting polygon P' has edges that are "well separated";
(2) use the Delaunay Triangulation of P' to construct an initial guess for the images of the vertices on the unit circle $\mathbb{T}$;
(3) compute the conformal map with an existing numerical iterative process using the initial guess determined at step 2.

The authors of the algorithm suspected that the so-produced initial guess might be within a bounded distance from the actual mapping parameters (to be determined) when measured in terms of a metric derived from cross ratios (the notion of cross ratio is defined in complex analysis). The main goal of the paper under review is to show that the conjecture is true if the cross ratio is replaced by the corresponding conformal modulus.

More specifically, given two n-tuples $\{w_1, w_2, \ldots, w_n\}$ and $\{z_1, z_2, \ldots, z_n\}$ of $\mathbb{T} = \partial\mathbb{D}$, the distance between the two tuples is defined by

$$d_{\rm QC}(w,z) = \inf\{\log K\colon\ \exists\ K\text{-quasiconformal } h\colon D \to D \text{ such that } h(z) = w\}.$$

With this definition, the main result of the paper is the following theorem, which shows that the initial guess on the unit circle is uniformly close to the actual mapping parameters in a quasiconformal sense.

Independent of the polygon P, there is a constant $C < \infty$ such that the initial guess w of the CRDT algorithm satisfies $d_{\rm QC}(w,z) \le C$, where z is the actual pre-vertex of the conformal map.

Furthermore, let $f\colon \Omega \to R$ be a conformal map, where Ω is a generalized quadrilateral with vertices $\{z_1, z_2, z_3, z_4\}$ mapped to the four corners of the rectangle R, unique up to

Euclidean similarities, and define

$$\mathrm{Mod}_\Omega(z) = \frac{|f(z_2) - f(z_1)|}{|f(z_2) - f(z_3)|},$$

which measures the eccentricity of the rectangle. Then, the above theorem leads to the estimate

$$|\log \mathrm{Mod}_D(z') - \log \mathrm{Mod}_D(w')| \leq \log K$$

for any $z' = \{z_{j1}, z_{j2}, z_{j3}, z_{j4}\} \subset z$, $w' = \{w_{j1}, w_{j2}, w_{j3}, w_{j4}\} \subset w$.

In addition, the author gives counterexamples that show that the conjecture by the authors of the CRDT algorithm is false. He also gives a few examples for which bounds for QC distance are explicitly computed, that in turn shows the sharpness of the above estimate. A discussion is also provided to show that adding extra vertices, as the CRDT algorithm normally does, may not make much difference in improving an initial guess using only original vertices. In some cases, adding extra vertices can in fact make an initial guess worse. Finally, open questions are proposed. *Chenglie Hu*

[References]

1. L. V. Ahlfors, Lectures *on Quasiconformal Mappings*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987 (with the assistance of Clifford J. Earle, Jr., Reprint of the 1966 original). MR0883205
2. F. Aurenhammer, Voronoi diagrams — a survey of a fundamental geometric data structure, *ACM Comp. Surveys* **23** (1991), 345–405.
3. T. Bagby, The modulus of a plane condenser, *J. Math. Mech.* **17** (1967), 315–329. MR0218545
4. L. Banjai and L. N. Trefethen, A multipole method for Schwarz-Christoffel mapping of polygons with thousands of sides. *SIAM J. Sci. Comput.* **25** (3) (2003), 1042–1065 (electronic). MR2046124
5. M. Bern and D. Eppstein, Mesh generation and optimal triangulation, in: Computing in Euclidean Geometry, volume 1 of Lecture *Notes Ser. Comput.*, pp. 23–90, World Sci. Publishing, River Edge, NJ, 1992. MR1239190
6. C. J. Bishop. Conformal mapping in linear time, to appear in *Discrete Comput. Geom.* MR2671015
7. C. J. Bishop, A fast QC mapping theorem for polygons, preprint, 2008.
8. C. J. Bishop, Divergence groups have the Bowen property, *Ann. of Math. (2)* **154** no.1 (2001), 205–217. MR1847593
9. C. J. Bishop, Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture, *Ark. Mat.* **40** no.1 (2002), 1–26. MR1948883
10. C. J. Bishop, An explicit constant for Sullivan's convex hull theorem, in: *In the tradition of Ahlfors and Bers, III*, volume 355 of *Contemp. Math.*, pp. 41–69, Amer. Math. Soc, Providence, RI, 2004. MR2145055
11. E. B. Christoffle, Sul problema della tempurature stazonaire e la rappresetazione di una data superficie, *Ann. Mat. Pura Appl. Serie II* (1867), 89–103.
12. E. F. D'Azevedo and R. B. Simpson, On optimal interpolation triangle incidences, *SIAM J. Sci. Statist. Comput.* **10** no.6 (1986), 1063–1075. MR1025475
13. J. E. Dennis, Jr. and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice Hall Series in Computational Mathematics, Prentice Hall Inc., Englewood Cliffs, NJ, 1983. MR0702023
14. D. P. Dobkin, S. J. Friedman and K. J. Supowit, Delaunay graphs are almost as good as complete graphs, Discrete *Comput. Geom.* **5** no.4 (1990), 399–407. MR1043722

15. T. A. Driscoll and L. N. Trefethen, *Schwarz-Christoffel mapping*, volume 8 of *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge University Press, Cambridge, 2002. MR1908657
16. T. A. Driscoll and S. A. Vavasis, Numerical conformal mapping using cross-ratios and Delaunay triangulation, *SIAM J. Sci. Comput.* **19** no.6 (1998), 1783–1803 (electronic). MR1638056
17. D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, in: *Analytical and Geometric Aspects of Hyperbolic Space (Coventry/Durham, 1984)*, volume 111 of *London Math. Soc. Lecture Note Ser.*, pp. 113–253, Cambridge Univ. Press, Cambridge, 1987. MR0903852
18. D. B. A. Epstein, A. Marden, and V. Markovic. Complex angle scaling, in: *Kleinian Groups and Hyperbolic 3-Manifolds (Warwick, 2001)*, volume 299 of *London Math. Soc. Lecture Note Ser.*, pp. 343–362, Cambridge Univ. Press, Cambridge, 2003. MR2044557
19. D. B. A. Epstein, A. Marden, and V. Markovic, Quasiconformal homeomorphisms and the convex hull boundary. *Ann. of Math. (2)* **159** no.1 (2004), 305–336. MR2052356
20. D. B. A. Epstein, A. Marden, and V. Markovic, Complex earthquakes and deformations of the unit disk, *J. Differential Geom.* **73** no.l (2006), 119–166. MR2217521
21. D. B. A. Epstein and V. Markovic, The logarithmic spiral: a counterexample to the $K = 2$ conjecture, *Ann. of Math. (2)* **161** no.2 (2005), 925–957. MR2153403
22. S. Fortune, Voronoi diagrams and Delaunay triangulations, in: *Handbook of Discrete and Computational Geometry*, CRC Press Ser. Discrete Math. Appl., pp. 377–388, CRC, Boca Raton, FL, 1997. MR1730176
23. J. B. Garnett and D. E. Marshall, *Harmonic Measure*, volume 2 of *New Mathematical Monographs*, Cambridge University Press, Cambridge, 2005. MR2150803
24. J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** no.l (1998), 1–61. MR1654771
25. J. M. Keil and C. A. Gutwin, Classes of graphs which approximate the complete Euclidean graph, *Discrete Comput. Geom.* **7** no.l (1992), 13–28. MR1134449
26. C. L. Lawson. *Software for C^1 surface interpolation*, ix+388 pages. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1977, Publication of the Mathematics Research Center, No. 39.
27. Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Grundlehren der Math. Wissenschaften, 299, Springer-Verlag, 1992. MR1217706
28. V. T. Rajan, Optimality of the Delaunay triangulation in $\mathbb{R}^d$, *Discrete Comput. Geom.* **12** no.2 (1994), 189–202. MR1283887
29. H. A. Schwarz, Conforme abbildung der Oberfläche eines Tetraeders auf die Oberfläche einer Kugel, *J. Reine Ange. Math.* (1869), 121–136; also in: collected works, [30], pp. 84–101. MR1579436
30. H. A. Schwarz, *Gesammelte Mathematische Abhandlungen*, Springer, Berlin, 1890.
31. D. Sullivan, Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur S^1, In *Bourbaki Seminar, Vol. 1979/80*, pp. 196–214. Springer, Berlin, 1981. MR0636524
32. J. Väisälä, Free quasiconformality in Banach spaces, II. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **16** no.2 (1991), 255–310. MR1139798

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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Conformal mapping in linear time. (English. English summary)

Discrete Comput. Geom. **44** (2010), *no.* 2, 330–428.

This is a very interesting paper, and a technical masterpiece containing many original ideas, on the explicit computation of a quasiconformal approximation to a conformal mapping of the unit disk $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$ in the complex plane $\mathbb{C}$ to a polygon Ω with n sides. Suppose that $\varepsilon>0$ is small. The author proves that there is a $(1+\varepsilon)$-quasiconformal mapping $f\colon\mathbb{D}\to\Omega$ that can be computed in $O(np\log p)$ steps, where $p=O(\log(1/\varepsilon))$. This might be viewed as the computation of a conformal map up to a small quasiconformal error, in linear time with respect to n.

The Schwarz–Christoffel formula provides the desired conformal mapping, but requires the knowledge of the prevertices, the points on the unit circle that will be mapped onto the vertices of the polygon. The author finds an algorithm that can be used to approximate the prevertices as accurately as one likes, with an estimate on the number of steps required to get to a preassigned level of accuracy. The true set of prevertices and the set that one finds by the algorithm are compared by means of the smallest possible dilatation of a quasiconformal self-map of $\mathbb{D}$ taking one set to the other.

How is this done? A given polygon Ω is approximated by a subdomain G that is the union of finitely many suitably chosen disks. One finds the dome over G in hyperbolic 3-space. It consists of finitely many geodesic faces. The dome is deformed, giving rise to a finite sequence of domains, varying slowly, and going from G to $\mathbb{D}$.

There is a known construction of a quasiconformal mapping, by means of the dome, of $\mathbb{D}$ onto G, which is then varied, using the intermediate domains as a tool, to get a map of $\mathbb{D}$ onto G with dilatation smaller than a certain absolute constant. After this, further procedures can be used to improve the dilatation to be as close to 1 as desired.

From a quasiconformal mapping of $\mathbb{D}$ onto G one gets by approximation a quasiconformal mapping onto Ω, with small dilatation. The quasiconformal prevertices so obtained are close to what the true but unknown prevertices would be, which should be used with the Schwarz–Christoffel formula. *A. Hinkkanen*

[References]

1. Aggarwal, A., Guibas, L.J., Saxe, J., Shor, P.W.: A linear-time algorithm for computing the Voronoĭ diagram of a convex polygon. Discrete Comput. Geom. **4**(6), 591–604 (1989) MR1006080
2. Ahlfors, L.V.: Lectures on Quasiconformal Mappings. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey (1987). With the assistance of Clifford J. Earle, Jr., reprint of the 1966 original MR0883205
3. Aho, A.V., Steiglitz, K., Ullman, J.D.: Evaluating polynomials at fixed sets of points. SIAM J. Comput. **4**(4), 533–539 (1975) MR0398151
4. Aurenhammer, F.: Voronoi diagrams—a survey of a fundamental geometric data structure. ACM Comput. Surv. **23,** 345–405 (1991)
5. Aurenhammer, F., Klein, R.: Voronoi Diagrams. In: Handbook of Computational Geometry, pp. 201–290. North-Holland, Amsterdam (2000) MR1746678
6. Banchoff, T.F., Giblin, P.J.: Global theorems for symmetry sets of smooth curves and polygons in the plane. Proc. R. Soc. Edinb. A **106**(3–4), 221–231 (1987) MR0906207
7. Banjai, L., Trefethen, L.N.: A multipole method for Schwarz–Christoffel mapping of polygons with thousands of sides. SIAM J. Sci. Comput. **25**(3), 1042–1065 (2003)

(electronic) MR2046124
8. Bern, M., Eppstein, D.: Mesh generation and optimal triangulation. In: Computing in Euclidean Geometry. Lecture Notes Ser. Comput, vol. 1, pp. 23–90. World Scientific, River Edge (1992) MR1239190
9. Beurling, A., Ahlfors, L.: The boundary correspondence under quasiconformal mappings. Acta Math. **96,** 125–142 (1956) MR0086869
10. Binder, I., Braverman, M., Yampolsky, M.: On the computational complexity of the Riemann mapping. Ark. Mat. **45**(2), 221–239 (2007) MR2342601
11. Bishop, C.J.: Divergence groups have the Bowen property. Ann. Math. (2) **154**(1), 205–217 (2001) MR1847593
12. Bishop, C.J.: BiLipschitz approximations of quasiconformal maps. Ann. Acad. Sci. Fenn. Math. **27**(1), 97–108 (2002) MR1884352
13. Bishop, C.J.: Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture. Ark. Mat. **40**(1), 1–26 (2002) MR1948883
14. Bishop, C.J.: An explicit constant for Sullivan's convex hull theorem. In: In the Tradition of Ahlfors and Bers, III. Contemp. Math., vol. 355, pp. 41–69. Am. Math. Soc., Providence (2004) MR2145055
15. Bishop, C.J., Hakobyan, H.: A central set of dimension 2. Proc. Am. Math. Soc. **136**(7), 2453–2461 (2008) MR2390513
16. Bishop, C.J.: Bounds for the CRDT conformal mapping algorithm. Comput. Methods Funct. Theory **10**(1), 325–366 (2010) MR2676459
17. Bishop, C.J.: A fast QC-mapping theorem for polygons. Preprint (2009)
18. Bishop, E., Bridges, D.: *Constructive Analysis.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 279. Springer, Berlin (1985) MR0804042
19. Blum, H.: A transformation for extracting new descriptors of shape. In: Dunn, W.W. (ed.) Proc. Sympos. Models for the Perception of Speech and Visual Form, pp. 362–380. MIT Press, Cambridge (1967)
20. Blum, H.: Biological shape and visual science. J. Theoret. Biol. **38,** 205–287 (1973)
21. Blum, H., Nagel, R.N.: Shape descriptors using weighted symmetric axis functions. Pattern Recogn. **10**(3), 167–180 (1978)
22. Bridgeman, M.: Average bending of convex pleated planes in hyperbolic three-space. Invent. Math. **132**(2), 381–391 (1998) MR1621436
23. Bridgeman, M.: Average curvature of convex curves in H^2. Proc. Am. Math. Soc. **126**(1), 221–224 (1998) MR1415576
24. Bridgeman, M., Canary, R.D.: From the boundary of the convex core to the conformal boundary. Geom. Dedicata **96,** 211–240 (2003) MR1956842
25. Bruce, J.W., Giblin, P.J., Gibson, C.G.: Symmetry sets. Proc. R. Soc. Edinb. A **101**(1–2), 163–186 (1985) MR0824218
26. Canary, R.D.: The conformal boundary and the boundary of the convex core. Duke Math. J. **106**(1), 193–207 (2001) MR1810370
27. Carleson, L.: Interpolations by bounded analytic functions and the corona problem. Ann. Math. (2) **76,** 547–559 (1962) MR0141789
28. Chazai, F., Lieutier, A.: Stability and homotopy of a subset of the medial axis. In: Proc. 9th ACM Sympos. Solid Modeling Appl. (2004)
29. Chazal, F., Soufflet, R.: Stability and finiteness properties of medial axis and skeleton. J. Dynam. Control Syst. **10**(2), 149–170 (2004) MR2051964
30. Chazelle, B.: Triangulating a simple polygon in linear time. Discrete Comput. Geom. **6**(5), 485–524 (1991) MR1115104
31. Cheng, H.: A constructive Riemann mapping theorem. Pac. J. Math. **44,** 435–454 (1973) MR0323539

32. Chiang, C.-S., Hoffmann, C.M.: The medial axis transform for 2d regions. ACM Transactions on graphics (1982)
33. Chin, F., Snoeyink, J., Wang, C.A.: Finding the medial axis of a simple polygon in linear time. Discrete Comput. Geom. **21**(3), 405–420 (1999) MR1672988
34. Choi, H.I., Choi, S.W., Moon, H.P.: Mathematical theory of medial axis transform. Pac. J. Math. **181**(1), 57–88 (1997) MR1491036
35. Choi, S.W., Seidel, H.-P.: Hyperbolic Hausdorff distance for medial axis transformation. Graph. Models **63,** 369–384 (2001)
36. Choi, S.W., Seidel, H.-P.: Linear one-sided stability of MAT for weakly injective domain. J. Math. Imaging Vis. **17**(3), 237–247 (2002) MR1945473
37. Choi, S.W., Lee, S.-W.: Stability analysis of medial axis transform. In Proc. 15th ICPR Barcelona, Spain, vol. 3, pp. 139–142 (2000)
38. Christoffle, E.B.: Sul problema della tempurature stazonaire e la rappresetazione di una data superficie. Ann. Mat. Pura Appl. Ser. II, pp. 89–103 (1867)
39. Cipra, B.: The best of the 20th century: Editors name top 10 algorithms. SIAM News **33**(4), 1 (2000)
40. Cooley, J.W., Tukey, J.W.: An algorithm for the machine calculation of complex Fourier series. Math. Comput. **19,** 297–301 (1965) MR0178586
41. Culver, T., Keyser, J., Manocha, D.: Accurate computation of the medial axis of a polyhedron. In: Proceedings of the Fifth ACM Symposium on Solid Modeling and Applications, June 8–11, 1999, Ann Arbor, MI, USA, pp. 179–190 (1999)
42. Daripa, P.: A fast algorithm to solve nonhomogeneous Cauchy–Riemann equations in the complex plane. SIAM J. Sci. Stat. Comput. **13**(6), 1418–1432 (1992) MR1185654
43. Daripa, P.: A fast algorithm to solve the Beltrami equation with applications to quasiconformal mappings. J. Comput. Phys. **106**(2), 355–365 (1993) MR1218735
44. Daripa, P., Mashat, D.: An efficient and novel numerical method for quasiconformal mappings of doubly connected domains. Numer. Algorithms **18**(2), 159–175 (1998) MR1663506
45. Davis, R.T.: Numerical methods for coordinate generation based on Schwarz–Christoffel transformations. In: 4th AIAA Comput. Fluid Dynamics Conf., Williamsburg, VA, pp. 1–15 (1979)
46. DeLillo, T.K.: The accuracy of numerical conformal mapping methods: a survey of examples and results. SIAM J. Numer. Anal. **31**(3), 788–812 (1994) MR1275114
47. Dirichlet, G.L.: Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen. J. Reine Angew. Math. **40,** 209–227 (1850) MR1578693
48. Driscoll, T.A., Trefethen, L.N.: Schwarz–Christoffel Mapping. Cambridge Monographs on Applied and Computational Mathematics, vol. 8. Cambridge University Press, Cambridge (2002) MR1908657
49. Driscoll, T.A., Vavasis, S.A.: Numerical conformal mapping using cross-ratios and Delaunay triangulation. SIAM J. Sci. Comput. **19**(6), 1783–1803 (1998) (electronic), MR1638056
50. Duan, H.B., Rees, E.: The existence of bitangent spheres. Proc. R. Soc. Edinb. A **111**(1–2), 85–87 (1989) MR0985991
51. Dutt, A., Gu, M., Rokhlin, V.: Fast algorithms for polynomial interpolation, integration, and differentiation. SIAM J. Numer. Anal. **33**(5), 1689–1711 (1996) MR1411845
52. Epstein, D.B.A., Marden, A.: Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In: Analytical and Geometric Aspects of Hyperbolic Space, Coventry/Durham, 1984. London Math. Soc. Lecture Note Ser., vol. 111, pp. 113–253. Cambridge Univ. Press, Cambridge (1987) MR0903852

53. Epstein, D.B.A., Marden, A., Markovic, V.: Quasiconformal homeomorphisms and the convex hull boundary. Ann. Math. (2) **159**(1), 305–336 (2004) MR2052356
54. Epstein, D.B.A., Marden, A., Markovic, V.: Complex earthquakes and deformations of the unit disk. J. Differ. Geom. **73**(1), 119–166 (2006) MR2217521
55. Epstein, D.B.A., Marden, A., Markovic, V.: Convex regions in the plane and their domes. Proc. Lond. Math. Soc. (3) **92**(3), 624–654 (2006) MR2223539
56. Epstein, D.B.A., Markovic, V.: The logarithmic spiral: a counterexample to the $K = 2$ conjecture. Ann. Math. (2) **161**(2), 925–957 (2005) MR2153403
57. Erdős, P.: On the Hausdorff dimension of some sets in Euclidean space. Bull. Am. Math. Soc. **52,** 107–109 (1946) MR0015144
58. Evans, G., Middleditch, A., Miles, N.: Stable computation of the 2D medial axis transform. Int. J. Comput. Geom. Appl. **8**(5–6), 577–598 (1998) MR1646187
59. Evans, W.D., Harris, D.J.: Sobolev embeddings for generalized ridged domains. Proc. Lond. Math. Soc. (3) **54**(1), 141–175 (1987) MR0872254
60. Fortune, S.: Voronoĭ diagrams and Delaunay triangulations. In: Computing in Euclidean Geometry. Lecture Notes Ser. Comput, vol. 1, pp. 193–233. World Scientific, River Edge (1992) MR1239194
61. Fortune, S.: Voronoi diagrams and Delaunay triangulations. In: Handbook of Discrete and Computational Geometry. CRC Press Ser. Discrete Math. Appl., pp. 377–388. CRC, Boca Raton (1997) MR1730176
62. Fremlin, D.H.: Skeletons and central sets. Proc. Lond. Math. Soc. (3) **74**(3), 701–720 (1997) MR1434446
63. Gaier, D.: Konstruktive Methoden der konformen Abbildung. Springer Tracts in Natural Philosophy, vol. 3. Springer, Berlin (1964) MR0199360
64. Garnett, J.B.: Bounded Analytic Functions. Pure and Applied Mathematics, vol. 96. Academic Press [Harcourt Brace Jovanovich Publishers], New York (1981) MR0628971
65. Gaudeau, C., Boiron, M., Thouvenot, J.: Squelettisation et anamorphose dans l'étude de la dynamique des déformations des structures: application à l'analyse de la motricité gastrique. In: Recognition of Shapes and Artificial Intelligence (Second AFCET-IRIA Cong., Toulouse, 1979), vol. III, pp. 57–63. IRIA, Rocquencourt (1979) (in French) MR0594442
66. Gehring, F.W.: The definitions and exceptional sets for quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A I **281,** 28 (1960) MR0124488
67. Gehring, F.W.: Symmetrization of rings in space. Trans. Am. Math. Soc. **101,** 499–519 (1961) MR0132841
68. Giblin, P.: Symmetry sets and medial axes in two and three dimensions. In: The Mathematics of Surfaces, IX, Cambridge, 2000, pp. 306–321. Springer, London (2000) MR1846302
69. Giblin, P.J., O'Shea, D.B.: The bitangent sphere problem. Am. Math. Mon. **97**(1), 5–23 (1990) MR1034346
70. Greengard, L., Rokhlin, V.: A fast algorithm for particle simulations. J. Comput. Phys. **73**(2), 325–348 (1987) MR0918448
71. Gursoy, H.N., Patrikalakis, N.M.: Automated interrogation and adaptive subdivision of shape using medial axis transform. Adv. Eng. Softw. Workstations **13**(5/6), 287–302 (1991)
72. He, Z.-X., Schramm, O.: On the convergence of circle packings to the Riemann map. Invent. Math. **125**(2), 285–305 (1996) MR1395721
73. He, Z.-X., Schramm, O.: The C^∞-convergence of hexagonal disk packings to the Riemann map. Acta Math. **180**(2), 219–245 (1998) MR1638772
74. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled

geometry. Acta Math. **181**(1), 1–61 (1998) MR1654771
75. Henrici, P.: Applied and Computational Complex Analysis. Pure and Applied Mathematics, vol. 3. Wiley, New York (1986). Discrete Fourier analysis—Cauchy integrals—construction of conformal maps—univalent functions, A Wiley-Interscience Publication MR0822470
76. Hertling, P.: An effective Riemann mapping theorem. Theoret. Comput. Sci. **219**(1–2), 225–265 (1999); Computability and complexity in analysis (Castle Dagstuhl, 1997) MR1694433
77. Hoffmann, C.M.: Computer vision, descriptive geometry and classical mechanics. In: Computer Graphics and Mathematics. Eurographics Series, pp. 229–244. Springer, Berlin (1992)
78. Hoffmann, C.M., Dutta, D.: On the skeleton of simple CSG objects. Trans. ASME **115,** 87–94 (1993)
79. Holopainen, I.: Rough isometries and p-harmonic functions with finite Dirichlet integral. Rev. Mat. Iberoam. **10**(1), 143–176 (1994) MR1271760
80. Hörmander, L.: The Analysis of Linear Partial Differential Operators. I. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, 2nd edn. Springer, Berlin (1990). Distribution theory and Fourier analysis MR1065993
81. Howell, L.H.: Numerical conformal mapping of circular arc polygons. J. Comput. Appl. Math. **46**(1–2), 7–28 (1993) MR1222470
82. Hu, C.: Algorithm 785: a software package for computing Schwarz–Christoffel conformal transformation for doubly connected polygonal regions. ACM Trans. Math. Softw. **24**(3), 317–333 (1998)
83. Ivanov, V.I., Trubetskov, M.K.: Handbook of Conformal Mapping with Computer-Aided Visualization. CRC Press, Boca Raton (1995). With 1 IBM-PC floppy disk (5.25 inch; HD) MR1400664
84. Jinkerson, R.A., Abrams, S.L., Bardis, L., Chryssostomidis, C., Clement, A., Patrikalakis, N.M., Wolter, F.E.: Inspection and feature extraction of marine propellers. J. Ship Product. **9**(2), 88–106 (1993)
85. Karatsuba, A., Ofman, Yu.: Multiplication of many-digital numbers by automatic computers. Dokl. Akad. Nauk SSSR **145,** 293–294 (1962)
86. Klein, R., Lingas, A.: A linear-time randomized algorithm for the bounded Voronoi diagram of a simple polygon. Int. J. Comput. Geom. Appl. **6**(3), 263–278 (1996). ACM Symposium on Computational Geometry (San Diego, CA, 1993) MR1409647
87. Kythe, P.K.: Computational Conformal Mapping. Birkhäuser, Boston (1998) MR1651941
88. Lee, D.-T.: The medial axis transform of a planar shape. IEEE Trans. Pattern Anal. Mach. Intell. **4**(4), 363–369 (1982)
89. Lee, D.T., Drysdale, R.L. III: Generalization of Voronoĭ diagrams in the plane. SIAM J. Comput. **10**(1), 73–87 (1981) MR0605604
90. Lee, Y.-H., Horng, S.-J.: The equivalence of the chessboard distance transform and the medial axis transform. Int. J. Comput. Math. **65**(3–4), 165–177 (1997) MR1669507
91. Maekawa, T., Patrikalakis, N.M.: Computation of singularities and intersections of offsets of planar curves. Comput. Aided Geom. Design **10**(5), 407–429 (1993) MR1245546
92. Maekawa, T., Patrikalakis, N.M.: Interrogation of differential geometry properties for design and manufacture. Vis. Comput. **10**(4), 216–237 (1994)
93. Marshall, D.E., Rhode, S.: Convergence of a variant of the zipper algorithm for conformal mapping. SIAM J. Numer. Anal. **45**(6), 2577–2609 (2007) (electronic)

MR2361903
94. Milman, D.: The central function of the boundary of a domain and its differentiable properties. J. Geom. **14**(2), 182–202 (1980) MR0593216
95. Milman, D., Waksman, Z.: On topological properties of the central set of a bounded domain in $\mathbf{R}^m$. J. Geom. **15**(1), 1–7 (1981) MR0605061
96. Mohar, B.: A polynomial time circle packing algorithm. Discrete Math. **117**(1–3), 257–263 (1993) MR1226147
97. Mohar, B.: Circle packings of maps in polynomial time. Eur. J. Combin. **18**(7), 785–805 (1997) MR1478825
98. Nehari, Z.: Conformal Mapping. Dover, New York (1975). Reprinting of the 1952 edition MR0377031
99. O'Donnell, S.T., Rokhlin, V.: A fast algorithm for the numerical evaluation of conformal mappings. SIAM J. Sci. Stat. Comput. **10**(3), 475–487 (1989) MR0991737
100. O'Rourke, J.: Computational Geometry in C, 2nd edn. Cambridge University Press, Cambridge (1998) MR1649008
101. Papamichael, N., Saff, E.B. (eds.): Computational complex analysis. J. Comput. Appl. Math. **46**(1–2) (1993) MR1222469
102. Patrikalakis, N.M., Maekawa, T.: Shape Interrogation for Computer Aided Design and Manufacturing. Springer, Berlin (2002) MR1891533
103. Pottmann, H., Wallner, J.: Computational Line Geometry. Mathematics+Visualization. Springer, Berlin (2001) MR1849803
104. Preparata, F.P.: The medial axis of a simple polygon. In: Mathematical Foundations of Computer Science, Proc. Sixth Sympos., Tatranská Lomnica, 1977. Lecture Notes in Comput. Sci., vol. 53, pp. 443–450. Springer, Berlin (1977) MR0464625
105. Preparata, F.P., Shamos, M.I.: Computational Geometry. Texts and Monographs in Computer Science. Springer, New York (1985). An introduction MR0805539
106. Rajan, V.T.: Optimality of the Delaunay triangulation in $\mathbf{R}^d$. Discrete Comput. Geom. **12**(2), 189–202 (1994) MR1283887
107. Reif, J.H.: Approximate complex polynomial evaluation in near constant work per point. SIAM J. Comput. **28**(6), 2059–2089 (1999) (electronic) MR1699001
108. Rodin, B., Sullivan, D.: The convergence of circle packings to the Riemann mapping. J. Differ. Geom. **26**(2), 349–360 (1987) MR0906396
109. Schwarz, H.A.: Conforme Abbildung der Oberfläche eines Tetraeders auf die Oberfläche einer Kugel. J. Reine Ange. Math., pp. 121–136, 1869; Also in collected works, Gesammelte Mathematische Abhandlungen, pp. 84–101. Springer, Berlin (1890) MR1579436
110. Schwarz, H.A.: Gesammelte Mathematische Abhandlungen. Springer, Berlin (1890)
111. Shamos, M.I., Hoey, D.: Closest-point problems. In: 16th Annual Symposium on Foundations of Computer Science (Berkeley, CA, 1975), pp. 151–162. IEEE Computer Society, Long Beach (1975) MR0426498
112. Sherbrooke, E.C., Patrikalakis, N.M., Brisson, E.: Computation of the medial axis transform of 3-d. In: Symposium on Solid Modeling and Applications, pp. 187–200 (1995)
113. Sherbrooke, E.C., Patrikalakis, N.M., Brisson, E.: An algorithm for the medial axis transform of 3d polyhedral solids. IEEE Trans. Vis. Comput. Graph. **2**(1), 44–61 (1996)
114. Sherbrooke, E.C., Patrikalakis, N.M., Wolter, F.-E.: Differential and topological properties of medial axis transforms. CVGIP: Graph. Model Image Process. **58**(6), 574–592 (1996)
115. Smith, W.D.: Accurate circle configurations and numerical conformal mapping in polynomial time. Unpublished technical memorandum, NEC Research Institute,

Princeton, NJ (1991)
116. Stephenson, K.: Circlepack. Software available from http://www.math.utk.edu/∼kens/
117. Stephenson, K.: The approximation of conformal structures via circle packing. In: Computational Methods and Function Theory 1997 (Nicosia). Ser. Approx. Decompos., vol. 11, pp. 551–582. World Scientific, River Edge (1999) MR1700374
118. Stephenson, K.: Circle packing and discrete analytic function theory. In: Handbook of Complex Analysis: Geometric Function Theory, vol. 1, pp. 333–370. North-Holland, Amsterdam (2002) MR1966198
119. Stephenson, K.: Circle packing: a mathematical tale. Not. Am. Math. Soc. **50**(11), 1376–1388 (2003) MR2011604
120. Strebel, K.: On the existence of extremal Teichmueller mappings. J. Anal. Math. **30,** 464–480 (1976) MR0440031
121. Sullivan, D.: Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur S^1. In: Bourbaki Seminar, vol. 1979/80, pp. 196–214. Springer, Berlin (1981) MR0636524
122. Tang, Z.: Fast Transformations Based on Structured Matrices with Applications to the Fast Multipole Method. PhD thesis, University of Maryland, College Park, Maryland (2004) MR2705481
123. Thom, R.: Sur le cut-locus d'une variété plongée. J. Differ. Geom. **6,** 577–586 (1972). Collection of articles dedicated to S.S. Chern and D.C. Spencer on their sixtieth birthdays MR0391131
124. Thurston, W.P.: The Geometry and Topology of 3-Manifolds. The Geometry Center. University of Minnesota (1979)
125. Trefethen, L.N. (ed.): Numerical Conformal Mapping. North-Holland, Amsterdam (1986). Reprint of J. Comput. Appl. Math. **14**(1–2) (1986) MR0874989
126. Trefethen, L.N., Driscoll, T.A.: Schwarz–Christoffel mapping in the computer era. In: Proceedings of the International Congress of Mathematicians, vol. III, pp. 533–542 Berlin (1998) (electronic) MR1648186
127. van der Hoeven, J.: Fast evaluation of holonomic functions. Theoret. Comput. Sci. **210**(1), 199–215 (1999) MR1650888
128. van der Hoeven, J.: Relax, but don't be too lazy. J. Symb. Comput. **34**(6), 479–542 (2002) MR1943041
129. von Koppenfels, W., Stallmann, F.: Praxis der konformen Abbildung. Die Grundlehren der mathematischen Wissenschaften, vol. 100. Springer, Berlin (1959) MR0107698
130. Voronoi, G.M.: Nouvelles applications des paramètres continus à la théorie des formes quadratiques. recherches sur les parallélloèdres primitifs. J. Reine Angew. Math. **134,** 198–287 (1908) MR1580754
131. Wang, J.: Medial axis and optimal locations for min-max sphere packing. J. Comput. Optim. **4**(4), 487–503 (2000) MR1796705
132. Wegmann, R.: Methods for numerical conformal mapping. In: Handbook of Complex Analysis: Geometric Function Theory, vol. 2, pp. 351–477. Elsevier, Amsterdam (2005) MR2121864
133. Woess, W.: Random Walks on Infinite Graphs and Groups. Cambridge Tracts in Mathematics, vol. 138. Cambridge University Press, Cambridge (2000) MR1743100
134. Wolter, E.-F.: Cut locus and the medial axis in global shape interrogation and representation. MIT, Dept. of Ocean Engineering, Design Laboratory Memorandum 92–2 (1993)
135. Wu, Q.J.: Sphere packing using morphological analysis. In: Discrete Mathematical Problems with Medical Applications, New Brunswick, NJ, 1999. DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 55, pp. 45–54. Am. Math. Soc., Providence

(2000) MR1802397

136. Yao, C., Rokne, J.G.: A straightforward algorithm for computing the medial axis of a simple polygon. Int. J. Comput. Math **39,** 51–60 (1991)
137. Yap, C.-K.: An $O(n \log n)$ algorithm for the Voronoĭ diagram of a set of simple curve segments. Discrete Comput. Geom. **2**(4), 365–393 (1987) MR0911190
138. Zhou, Q.: Computable real-valued functions on recursive open and closed subsets of Euclidean space. Math. Logic Q. **42**(3), 379–409 (1996) MR1398144

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2671014 (2011f:30011) 30C20 30C30 65D17

Bishop, Christopher J. (1-SUNYS)

Optimal angle bounds for quadrilateral meshes. (English. English summary)

Discrete Comput. Geom. **44** (2010), *no.* 2, 308–329.

The main result of this paper is the following theorem: "Any simply-connected planar domain whose boundary is a simple n-gon has a quadrilateral mesh with $O(n)$ pieces so that all angles are between 60° and 120°, except that original $< 60^\circ$ angles of the polygon remain. The mesh can be constructed in time $O(n)$." The theorem extends and strengthens earlier results by M. W. Bern and D. Eppstein [Internat. J. Comput. Geom. Appl. **10** (2000), no. 4, 347–360; MR1791192]. For its proof the author employs various function theoretic tools (such as conformal mappings) and proves several intermediate (but interesting in their own right) results concerning the subdivision of the unit disc into hyperbolic pentagons, quadrilaterals and triangles and the meshing of each of these regions into quadrilaterals with angles in the interval $[60^\circ, 120^\circ]$. *N. Papamichael*

[References]

1. Beardon, A.F.: The Geometry of Discrete Groups. Springer, New York (1983) MR0698777
2. Bern, M., Eppstein, D.: Quadrilateral meshing by circle packing. Int. J. Comput. Geom. Appl. **10**(4), 347–360 (2000). Selected papers from the Sixth International Meshing Roundtable, Part II (Park City, UT, 1997) MR1791192
3. Bishop, C.J.: Divergence groups have the Bowen property. Ann. Math. (2) **154**(1), 205–217 (2001) MR1847593
4. Bishop, C.J.: Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture. Ark. Mat. **40**(1), 1–26 (2002) MR1948883
5. Bishop, C.J.: An explicit constant for Sullivan's convex hull theorem. In: In the Tradition of Ahlfors and Bers, III. Contemp. Math., vol. 355, pp. 41–69. Am. Math. Soc., Providence (2004) MR2145055
6. Bishop, C.J.: Conformal mapping in linear time. Discrete Comput. Geom. doi:10.1007/s00454–010-9269–9 (2010) MR2671015
7. Christoffel, E.B.: Sul problema della tempurature stazonaire e la rappresetazione di una data superficie. Ann. Mat. Pura Appl., Ser. II **1,** 89–103 (1867)
8. Driscoll, T.A., Trefethen, L.N.: Schwarz–Christoffel Mapping. Cambridge Monographs on Applied and Computational Mathematics, vol. 8. Cambridge University Press, Cambridge (2002) MR1908657
9. Epstein, D.B.A., Marden, A.: Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In: Analytical and Geometric Aspects of Hyperbolic Space, Conventry/Durham, 1984. London Math. Soc. Lecture Note Ser., vol. 111, pp. 113–253. Cambridge University Press, Cambridge (1987) MR0903852
10. Garnett, J.B., Marshall, D.E.: Harmonic Measure. New Mathematical Monographs,

vol. 2. Cambridge University Press, Cambridge (2005) MR2150803
11. Gerver, J.L.: The dissection of a polygon into nearly equilateral triangles. Geom. Dedic. **16**(1), 93–106 (1984) MR0757798
12. Nehari, Z.: Conformal Mapping. Dover, New York (1975). Reprinting of the 1952 edition MR0377031
13. Schwarz, H.A.: Confome Abbildung der Oberfläche eines Tetraeders auf die Oberfläche einer Kugel. J. Reine Angew. Math., 121–136 (1869); Also in collected works, [14], pp. 84–101 MR1579436
14. Schwarz, H.A.: Gesammelte Mathematische Abhandlungen. Springer, Berlin (1890)
15. Sullivan, D.: Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur S^1. In: Bourbaki Seminar, vol. 1979/80, pp. 196–214. Springer, Berlin (1981) MR0636524
16. Trefethen, L.N., Driscoll, T.A.: Schwarz–Christoffel mapping in the computer era. In: Proceedings of the International Congress of Mathematicians (Berlin, 1998), Vol. III, pp. 533–542 (electronic) (1998) MR1648186

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MR2390513 (2009e:28021) 28A78 28A75 51M25

Bishop, Christopher J. (1-SUNYS); **Hakobyan, Hrant** (3-TRNT)

A central set of dimension 2. (English. English summary)

Proc. Amer. Math. Soc. **136** (2008), *no.* 7, 2453–2461.

The central set $C(D)$ of a plane domain D is the set of points $z \in D$ for which the disc $D(z, \mathrm{dist}(z, \partial D))$ is not strictly contained in any larger disc contained in D. The authors resolve a question of Fremlin by constructing a domain $D \subset \mathbb{R}^2$ for which $C(D)$ has Hausdorff dimension two. In addition, they show that the domain in question can be chosen to be arbitrarily close to the unit disc (in a certain technical sense whose precise statement we omit) and so that $C(D)$ has positive H_φ measure for any gauge function φ for which $\lim_{t \to 0} \varphi(t)/t^2 = +\infty$.

By way of contrast, the medial axis $M(D)$ of D, defined as the set of points $z \in D$ for which $\mathrm{dist}(z, \partial D)$ is realized by at least two distinct points of ∂D, always has Hausdorff dimension one, by a theorem of Erdős. A simple example of a domain D for which $C(D) \neq M(D)$ is any noncircular ellipse.

The domain D which the authors construct has medial axis $M(D)$, which is a tree whose closure $\overline{M(D)}$ is contained in $C(D)$. The authors construct a probability measure μ on $\partial M(D) := \overline{M(D)} \setminus M(D)$ with controlled volume decay by equidistributing measure along the limbs of $M(D)$. An appeal to the Mass Distribution Principle yields the desired estimate $H_\varphi(C(D)) \geq H_\varphi(\partial M(D)) > 0$. *Jeremy T. Tyson*

[References]

1. C. J. Bishop, *Conformal mapping in linear time*, preprint, 2006. MR2671015
2. L.A. Caffarelli, A. Friedman, *The free boundary for elastic-plastic torsion problems*, Trans. Amer. Math. Soc. **252** (1979), 65–97. MR0534111
3. H. I. Choi, S. W. Choi and H. P. Moon. *Mathematical theory of medial axis transform*, Pacific J. Math. (1) **181** (1997), 57–88. MR1491036
4. P. Erdős, *Some remarks on measurability of certain sets*, Bull. Amer. Math. Soc. **51** (1945), 728-731.) MR0013776
5. W. D. Evans, D. J. Harris, *Sobolev embeddings for generalized ridged domains*, Proc. London Math. Soc. (3) **54** (1987), 141–175. MR0872254

6. D. H. Fremlin, *Skeletons and central sets*, Proc. London Math. Soc. (3) **74** (1997), no. 3, 701–720. MR1434446
7. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability,* Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995. MR1333890
8. T. W. Ting, *The ridge of a Jordan domain and completely plastic torsion*, J. Math. Mech. **15** (1966), 15–47. MR0184503

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2417424 (2009d:30048) 30C62 30F35
Bishop, Christopher J. (1-SUNYS)
Decreasing dilatation can increase dimension. (English. English summary)
Illinois J. Math. **51** (2007), *no.* 4, 1243–1248.
Let D be the unit disk and let

$$\mathrm{CM}(D)=\left\{\mu\in L^\infty(D)\colon \frac{|\mu(z)|^2dxdy}{1-|z|} \text{ is a Carleson measure}\right\}.$$

For a Fuchsian group G acting on D let

$$M(G)=\left\{\mu\in L^\infty(D)\colon \|\mu\|_\infty<1, \forall g\in G, \mu=\frac{\overline{g'}}{g'}\mu\circ g\right\},$$

$$\mathcal{M}(G)=M(G)\cap CM(D).$$

When G is the identity we will write $M(1)$ and $\mathcal{M}(1)$, correspondingly. For $\mu\in M(1)$ there exists a quasiconformal map f_μ of D to itself with dilatation μ. A question of G. Z. Cui and M. Zinsmeister [Illinois J. Math. **48** (2004), no. 4, 1223–1233; MR2114154] is: Let $\mu\in\mathcal{M}(1)$ be such that $f_\mu(\partial D)$ is a bi-Lipschitz image of a circle or a line. Is the same true for $f_{t\mu}(\partial D)$, $0<t<1$? The author shows that this is false even if $f_\mu(\partial D)$ is a circle. *Bodo Dittmar*

[References]

1. L. V. Ahlfors, *Lectures on quasiconformal mappings*, Nostrand Mathematical Studies, No. 10, Van Nostrand Co., Inc., Toronto, 1966. MR 0200442(34 #336) MR0200442
2. K. Astala and M. Zinsmeister, *Mostow rigidity and Fuchsian groups*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 301–306. MR 1071631 (92c:30021) MR1071631
3. C. J. Bishop, *Divergence groups have the Bowen property*, Ann. of Math. (2) **154** (2001), 205–217. MR 1847593 (2003b:30052) MR1847593
4. C. J. Bishop and P. W. Jones, *Harmonic measure, L^2 estimates and the Schwarzian derivative*, J. Anal. Math. **62** (1994), 77–113. MR 1269200 (95f:30034) MR1269200
5. C. J. Bishop and P. W. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math. **179** (1997), 1–39. MR 1484767 (98k:22043) MR1484767
6. C. J. Bishop and P. W. Jones, *Wiggly sets and limit sets*, Ark. Mat. **35** (1997), 201–224. MR 1478778 (99f:30066) MR1478778
7. C. J. Bishop and P. W. Jones, *Compact deformations of Fuchsian groups*, J. Anal. Math. **87** (2002), 5–36, Dedicated to the memory of Thomas H. Wolff. MR 1945276 (2003m:20065) MR1945276
8. R. Bowen, *Hausdorff dimension of quasicircles*, Inst. Hautes Études Sci. Publ. Math. (1979), 11–25. MR 556580 (81g:57023) MR0556580

9. M. Bridgeman and E. C. Taylor, *Length distortion and the Hausdorff dimension of limit sets*, Amer. J. Math. **122** (2000), 465–482. MR 1759885 (2001e:37058) MR1759885
10. G. Cui and M. Zinsmeister, *BMO-Teichmüller spaces*, Illinois J. Math. **48** (2004), 1223–1233. MR 2114154 (2005h:30038) MR2114154
11. D. Sullivan, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. **153** (1984), 259–277. MR 766265 (86c:58093) MR0766265

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MR2373370 (2009d:30010) 30C35 30C62 37E10 37F99

Bishop, Christopher J. (1-SUNYS)

Conformal welding and Koebe's theorem. (English. English summary)

Ann. of Math. (2) **166** (2007), *no.* 3, 613–656.

Let $\mathbb{D}$ be the open unit disk in the plane $\mathbb{R}^2$, let $\mathbb{T}=\partial\mathbb{D}$, and let $\mathbb{D}^*=\mathbb{R}^2\setminus(\mathbb{D}\cup\mathbb{T})$. For any Jordan curve $\Gamma\subset\mathbb{R}^2$, let Ω and Ω^* respectively be its bounded and unbounded complementary components. If $f\colon\mathbb{D}\to\Omega$ and $g\colon\mathbb{D}^*\to\Omega^*$ are conformal homeomorphisms, then $h=g^{-1}\circ f\colon\mathbb{T}\to\mathbb{T}$ is said to be a conformal welding.

It is known that there exist orientation-preserving homeomorphisms of the circle that are not conformal weldings.

In the paper under review, the author proves, using Koebe's circle domain theorem, several results that show that every homeomorphism of the circle is close to a conformal welding in a precise sense. In particular, he obtains the following:

Theorem. For any orientation-preserving homeomorphism $h\colon\mathbb{T}\to\mathbb{T}$ and any $\epsilon>0$, there exist a set $E\subset\mathbb{T}$ such that $|E|+|h(E)|<\epsilon$ (where $|E|$ denotes Lebesgue measure) and a conformal welding homeomorphism $H\colon\mathbb{T}\to\mathbb{T}$ such that $h(x)=H(x)$ for all $x\in\mathbb{T}\setminus E$.

Theorem. An orientation-preserving homeomorphism $h\colon\mathbb{T}\to\mathbb{T}$ is the conformal welding of a flexible curve if and only if there is a Borel set E such that both E and its complement have zero logarithmic capacity.

The paper also contains results on generalized conformal welding, which is defined as follows. A map h is said to be a generalized conformal welding on the set $E\subset\mathbb{T}$ if there are conformal maps $f\colon\mathbb{D}\to\Omega$ and $g\colon\mathbb{D}^*\to\Omega^*$ onto disjoint domains such that f has radial limits on E, g has radial limits on $h(E)$, and these limits satisfy $f=g\circ h$ on E. Generalized conformal welding was introduced by D. Hamilton, who used it in the study of Kleinian groups and of Julia sets. In the paper under review, the author obtains results concerning generalized welding that were conjectured by Hamilton.

The author explains the relation of his results with the theorem of R. L. Moore, stating that given a decomposition of the plane satisfying certain conditions (and called a Moore decomposition) then the quotient space obtained by identifying each subset to a point is homeomorphic to the plane. He relates the question of which Moore decompositions are conformal, and the question of when is the quotient map unique up to a Möbius transformation, to his approach of conformal welding by collapsing arcs of a foliation.

The author also provides a new and elementary proof of the fact that quasisymmetric maps are conformal weldings, and he states the following conjecture that generalizes Koebe's circle conjecture: For any orientation-preserving homeomorphism $h\colon\mathbb{T}\to\mathbb{T}$, there exists a countable subset $E\subset\mathbb{T}$ such that h is a generalized conformal welding on $\mathbb{T}\setminus E$.

Athanase Papadopoulos

[References]

1. L. Ahlfors and A. Beurling, Conformal invariants and function-theoretic null-sets, *Acta Math.* **83** (1950), 101–129. MR0036841
2. Lars V. Ahlfors, *Lectures on Quasiconformal Mappings,* D. Van Nostrand Co., Inc., Toronto, Ontario-New York-London, 1966. MR0200442
3. Z. Balogh and M. Bonk, Lengths of radii under conformal maps of the unit disc, *Proc. Amer. Math. Soc.* **127** (1999), 801–804. MR1469396
4. A. Beurling and L. Ahlfors, The boundary correspondence under quasiconformal mappings, *Acta Math.* **96** (1956), 125–142. MR0086869
5. C. J. Bishop, Constructing continuous functions holomorphic off a curve, *J. Funct. Anal.* **82** (1989), 113–137. MR0976315
6. C. J. Bishop, Boundary interpolation sets for conformal maps, *Bull. London Math. Soc.* **38** (2006), 607–616. MR2250753
7. C. J. Bishop, Some homeomorphisms of the sphere conformal off a curve, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **19** (1994), 323–338. MR1274085
8. C. J. Bishop, L. Carleson, J. B. Garnett, and P. W. Jones, Harmonic measures supported on curves, *Pacific J. Math.* **138** (1989), 233–236. MR0996199
9. A. Browder, *Introduction to Function Algebras,* W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0246125
10. A. Browder and J. Wermer, Some algebras of functions on an arc, *J. Math. Mech.* **12** (1963), 119–130. MR0144223
11. A. Browder and J. Wermer, A method for constructing Dirichlet algebras, *Proc. Amer. Math. Soc.* **15** (1964), 546–552. MR0165385
12. L. Carleson, Representations of continuous functions, *Math. Z.* **66** (1957), 447–451. MR0084035
13. L. Carleson, *Selected Problems on Exceptional Sets,* Van Nostrand, 1967. MR0225986
14. R. J. Daverman, *Decompositions of Manifolds,* Volume 124 of *Pure and Applied Mathematics,* Academic Press Inc., Orlando, FL, 1986. MR0872468
15. G. David, Solutions de l'équation de Beltrami avec $|\mu|_\infty = 1$, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **13** (1988), 25–70. MR0975566
16. Peter Duren, Harold M. Edwards, and Uta C. Merzbach, editors, *A Century of Mathematics in America. Part III,* American Mathematical Society, Providence, RI, 1989. MR1025365
17. B. Fitzpatrick, Jr, Some aspects of the work and influence of R. L. Moore, in *Handbook of the History of General Topology* **1** (1997), 41–61, Kluwer Acad. Publ., Dordrecht. MR1617585
18. F. W. Gehring and W. K. Hayman, An inequality in the theory of conformal mapping, *J. Math. Pures Appl.* **41** (1962) 353–361. MR0148884
19. D. H. Hamilton, Generalized conformal welding, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **16** (1991), 333–343. MR1139801
20. D. H. Hamilton, Simultaneous uniformisation, *J. Reine Angew. Math.* **455** (1994), 105–122. MR1293875
21. D. H. Hamilton, Length of Julia curves, *Pacific J. Math.* **169** (1995), 75–93. MR1346247
22. D. H. Hamilton, Conformal welding, in *The Handbook of Geometric Function Theory.* North Holland, 2002. MR1966191
23. Z.-X. He and O. Schramm, Fixed points, Koebe uniformization and circle packings, *Ann. of Math.* **137** (1993), 369–406. MR1207210
24. Z.-X. He and O. Schramm, Koebe uniformization for "almost circle domains",

Amer. J. Math. **117** (1995), 653–667. MR1333941
25. R. Kaufman, Fourier-Stieltjes coefficients and continuation of functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), 27–31. MR0752389
26. R. Kaufman, Plane curves and removable sets, *Pacific J. Math.* **125** (1986), 409–413. MR0863535
27. P. Koebe, Uber die uniformisierung beliebiger analytisher kurven, III, *Nachr. Ges. Wiss. Gott* (1908), 337–358.
28. P. Koebe, Abhandlungen zur theorie der konformen abbildung: Vi abbildung mehrfach zusammenhängender bereiche auf kreisbereiche etc., *Math. Z.* **7** (1920), 235–301. MR1544421
29. O. Lehto, Homeomorphisms with a given dilatation, In *Proceedings of the Fifteenth Scandinavian Congress* (Oslo, 1968, 58–73, Berlin, 1970, Springer. MR0260997
30. O. Lehto and K. I. Virtanen, On the existence of quasiconformal mappings with prescribed complex dilatation, *Ann. Acad. Sci. Fenn. Ser. A I No.* **274** (1960), 24. MR0125962
31. O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane,* Springer-Verlag, New York, second edition, 1973, Translated from the German by K. W. Lucas, Die Grundlehren der mathematischen Wissenschaften, Band 126. MR0344463
32. K. Lundberg, A theorem on the boundary behavior of a uniformly convergent sequence of conformal maps on the disk, PhD thesis, State University of New York at Stony Brook, Stony Brook, NY, USA, August 2005. MR2707849
33. R. L. Moore, Concerning upper semi-continuous collections of continua, *Trans. Amer. Math. Soc.* **27** (1925), 416–428. MR1501320
34. R. L. Moore, Concerning triods in the plane and the junction points of plane continua, *Proc. Nat. Acad. Sci.* **14** (1928), 85–88.
35. K. Oikawa, Welding of polygons and the type of Riemann surfaces, *Kōdai Math. Sem. Rep.* **13** (1961), 37–52. MR0125956
36. Albert Pfluger, Ueber die Konstruktion Riemannscher Flächen durch Verheftung, *J. Indian Math. Soc (N.S.)* **24** (1960), 401–412. MR0132827
37. C. R. Pittman, An elementary proof of the triod theorem, *Proc. Amer. Math. Soc.* **25** (1970), 919. MR0263049
38. Ch. Pommerenke, On the boundary continuity of conformal maps, *Pacific J. Math.* **120** (1985), 423–430. MR0810781
39. Ch. Pommerenke, *Boundary Behaviour of Conformal Maps,* Springer-Verlag, Berlin, 1992. MR1217706
40. W. Rudin, Boundary values of continuous analytic functions, *Proc. Amer. Math. Soc.* **7** (1956), 808–811. MR0081948
41. L. Sario and K. Oikawa, *Capacity Functions,* Springer-Verlag New York Inc., New York, 1969. MR0254232
42. J. V. Vainio, Conditions for the possibility of conformal sewing, *Ann. Acad. Sci. Fenn. Ser.* A I *Math. Dissertationes* **53** (1985), 43. MR0779328
43. J. V. Vainio, On the type of sewing functions with a singularity, *Ann. Acad. Sci. Fenn. Ser.* A I *Math.* **14** (1989), 161–167. MR0997980
44. J. V. Vainio, Properties of real sewing functions, *Ann. Acad. Sci. Fenn. Ser.* A I *Math.* **20** (1995), 87–95. MR1304108
45. R. L. Wilder, The mathematical work of R. L. Moore: its background, nature and influence, *Arch. Hist. Exact Sci.* **26** (1982), 73–97. MR0664470
46. G. B. Williams, Approximation of quasisymmetries using circle packings, *Discrete Comput. Geom.* **25** (2001), 103–124. MR1797299
47. G. B. Williams, Discrete conformal welding, *Indiana Univ. Math. J.* **53** (2004), 765–804. MR2086700

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MR2342802 (2008i:30021) 30C62 46E30

Bishop, Christopher J. (1-SUNYS)

★**An A_1 weight not comparable with any quasiconformal Jacobian. (English. English summary)**

In the tradition of Ahlfors-Bers. IV, 7–18, *Contemp. Math.*, 432, *Amer. Math. Soc.*, *Providence*, *RI*, 2007.

The quasiconformal Jacobian problem of G. David and S. W. Semmes [in *Analysis and partial differential equations*, 101–111, Dekker, New York, 1990; MR1044784] asks for a characterization of those nonnegative functions ω in $\mathbb{R}^n$, $n \geq 2$, for which there exist a constant $M \geq 1$ and a quasiconformal map f so that $M^{-1}\omega \leq J_f \leq M\omega$ almost everywhere. A related problem is to find out which metric spaces are bi-Lipschitz equivalent to Euclidean spaces.

In order to understand these problems, one looks for natural sufficient conditions. A well-known question along these lines was asked by Semmes [Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), no. 2, 211–248; MR1234732] and J. Heinonen and Semmes [Conform. Geom. Dyn. **1** (1997), 1–12 (electronic); MR1452413]: Is every A_1-weight comparable to a quasiconformal Jacobian in the above sense? The paper under review gives a negative answer to this question. The interesting counterexample consists of a Sierpinski carpet $E \subset \mathbb{R}^2$ and an A_1-weight ω which blows up on E. The author shows that careful constructions of E and ω imply that a quasiconformal map f with Jacobian comparable to ω must have the property that fE contains a rectifiable curve. This gives the desired contradiction, since the preimage of such a curve under f should be a single point under these circumstances. By combining the construction with earlier results, the author also shows that there exists a geometrically well-behaved surface inside $\mathbb{R}^3$ which is not bi-Lipschitz equivalent to the plane. To conclude the paper, the author presents related open problems.

Kai Rajala

[References]

1. C. J. Bishop. A quasisymmetric surface with no rectifiable curves. *Proc. Amer. Math. Soc.*, 127(7):2035–2040, 1999. MR1610908
2. C. J. Bishop. A set containing rectifiable arcs QC-locally but not QC-globally. 2006. preprint. MR2900167
3. M. Bonk, J. Heinonen, and S. Rohde. Doubling conformal densities. *J. Reine Angew. Math.*, 541:117–141, 2001. MR1876287
4. M. Bonk, J. Heinonen, and E. Saksman. The quasiconformal Jacobian problem. *Contemp. Math.*, 355:77–96, 2004. MR2145057
5. M. Bonk and B. Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.*, 150(1):127–183. 2002. MR1930885
6. G. David and S. Semmes. Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings. In *Analysis and partial differential equations*, volume 122 of *Lecture Notes in Pure and Appl. Math.*, pages 101–111. Dekker, New York, 1990. MR1044784
7. G. David and T. Toro. Reifenberg flat, metric spaces, snowballs, and embeddings. *Math. Ann.*, 315(4):641–710, 1999. MR1731465
8. J. Heinonen. The branch set of a quasiregular mapping. In *Proceedings of the International Congress of Mathematicians. Vol. II (Beijing, 2002)*, pages 691–700, Beijing, 2002. Higher Ed. Press. MR1957076
9. J. Heinonen. Geometric embeddings of metric spaces. 2003. Jyväskylä Mathematics

Department Reports. MR2014506

10. J. Heinonen and S. Semmes. Thirty-three yes or no questions about mappings, measures, and metrics. *Conform. Geom. Dyn.*, 1:1–12 (electronic), 1997. MR1452413
11. L. V. Kovalev and D. Maldonado. Mappings with convex potentials and the quasiconformal Jacobian problem. 2005. preprint. MR2210351
12. T. J. Laakso. Plane with A_∞-weighted metric not bi-Lipschitz embeddable to $\mathbb{R}^N$. *Bull. London Math. Soc.*, 34(6):667–676, 2002. MR1924353
13. S. Semmes, Bi-Lipschitz mappings and strong A_∞ weights. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 18(2):211–248, 1993. MR1234732
14. S. Semmes. On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights, *Rev. Mat. Iberoamericana*, 12(2):337–410, 1996. MR1402671

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2279263 (2008j:37097) 37F30 30F35 37F35

Bishop, Christopher J. (1-SUNYS)

A criterion for the failure of Ruelle's property. (English. English summary)

Ergodic Theory Dynam. Systems **26** (2006), *no.* 6, 1733–1748.

Summary: "D. Ruelle [Ergodic Theory Dynamical Systems **2** (1982), no. 1, 99–107; MR0684247] proved that for quasiconformal deformations of cocompact Fuchsian groups, the Hausdorff dimension of the limit set is an analytic function of the deformation. In this paper, we give a criterion for the failure of analyticity for certain infinitely generated groups. In particular, we show that it fails for any infinite abelian cover of a compact surface, answering a question posed by K. Astala and M. Zinsmeister in [Ann. Acad. Sci. Fenn. Ser. A I Math. **20** (1995), no. 1, 81–86; MR1304107]."

[References]

1. L. V. Ahlfors. *Lectures on Quasiconformal Mappings (Mathematical Studies, 10).* Van Nostrand, New York, 1966. MR0200442
2. K. Astala and M. Zinsmeister. Mostow rigidity and Fuchsian groups. *C. R. Acad. Sci. Paris Sér I Math.* **311** (1990), 301–306. MR1071631
3. K. Astala and M. Zinsmeister. Holomorphic families of quasi-Fuchsian groups. *Ergod. Th. & Dynam. Sys.* **14** (1994), 207–212. MR1279468
4. K. Astala and M. Zinsmeister. Abelian coverings, Poincaré exponent of convergence and holomorphic deformations. *Ann. Acad. Sci. Fenn.* **20** (1995), 81–86. MR1304107
5. C. J. Bishop. Big deformations near infinity. *Illinois J. Math.* **47** (2003), 977–996. MR2036986
6. C. J. Bishop and P. W. Jones. Hausdorff dimension and Kleinian groups. *Acta. Math.* **179** (1997), 1–39. MR1484767
7. C. J. Bishop and P. W. Jones. Wiggly sets and limit sets. *Arkiv Mat.* **35** (1997), 201–224. MR1478778
8. C. J. Bishop and P. W. Jones. Compact deformations of Fuchsian groups. *J. Anal. Math.* **87** (2002), 5–36. MR1945276
9. A. Douady and C. J. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.* **157**(1–2) (1986), 23–48. MR0857678
10. F. W. Gehring and J. Väisälä. Hausdorff dimension and quasiconformal mappings. *J. London Math. Soc.* (2) **6** (1973), 504–512. MR0324028
11. P. J. Nicholls. *The Ergodic Theory of Discrete Groups (LMS Lecture Notes, 143).* Cambridge University Press, Cambridge, 1989. MR1041575

12. D. Ruelle. Repellers for real analytic maps. *Ergod. Th. & Dynam. Sys.* **2** (1982), 99–107. MR0684247
13. D. Sullivan. Related aspects of positivity in Riemannian geometry. *J. Differential Geom.* **25** (1987), 327–351. MR0882827

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2250753 (2007f:30014) 30C35 30C62 30C85
Bishop, Christopher J. (1-SUNYS)
Boundary interpolation sets for conformal maps. (English. English summary)
Bull. London Math. Soc. **38** (2006), *no.* 4, 607–616.

A compact subset E of the unit circle $\mathbb{T}$ is said to be an interpolation set for conformal mappings if given any homeomorphism $g\colon \mathbb{D}\to\Omega\subset\mathbb{R}^2$ of the unit disk which extends continuously to $\mathbb{T}$, there is a conformal map $f\colon\mathbb{D}\to\Omega$ which extends continuously to $\mathbb{T}$ such that $f|_E=g|_E$. The main result of this paper states that a compact subset of the unit circle is an interpolation set for conformal mappings if and only if it has logarithmic capacity zero. For the proof of this result, the author establishes the following striking theorem.

Theorem. Suppose that $E\subset\mathbb{T}$ is a compact set of zero logarithmic capacity and $h\colon\mathbb{T}\to\mathbb{T}$ is an orientation-preserving homeomorphism. Then there is a conformal map $f\colon\mathbb{D}\to\Omega\subset\mathbb{D}$ onto a Jordan domain Ω such that $f|_E=h|_E$.

The proof of this theorem, which occupies a large portion of this paper, follows from Evans' theorem in potential theory and an explicit geometric construction. In this process, one constructs first a quasiconformal map that does the interpolation. To obtain a conformal map, then one combines this with an iterative construction that solves a Beltrami equation at each step to keep the map conformal. As indicated in the paper, this theorem can be used in solving conformal interpolation problems as well as conformal welding problems. *Shan Shuang Yang*

[References]

1. C. J. Bishop, 'Conformal welding and Koebe's theorem', Ann. of Math., to appear. MR2373370
2. R. J. Daverman, Decompositions of manifolds, Pure and Applied Mathematics 124 (Academic Press, Orlando, FL, 1986). MR0872468
3. P. Duren, H. M. Edwards and U. C. Merzbach, eds, A century of mathematics in America, Part III (Amer. Math. Soc., Providence, RI, 1989). MR1025365
4. B. Fitzpatrick, Jr, 'Some aspects of the work and influence of R. L. Moore', Handbook of the history of general topology, Vol. 1 (Kluwer, Dordrecht, 1997) 41–61. MR1617585
5. J. B. Garnett and D. Marshall, Harmonic measure (Cambridge University Press, Cambridge, 2005). MR2150803
6. D. H. Hamilton, 'Conformal welding', The handbook of geometric function theory (North- Holland, 2002). MR1966191
7. L. L. Helms, Introduction to potential theory (Wiley-Interscience, New York/London/ Sydney, 1969). MR0261018
8. R. L. Moore, 'Concerning upper semi-continuous collections of continua', Trans. Amer. Math. Soc. 27 (1925) 416–428. MR1501320
9. Ch. Pommerenke, Univalent functions, with a chapter on quadratic differentials by G. Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV (Vandenhoeck Ruprecht, Göttingen, 1975). MR0507768

10. Ch. Pommerenke, 'On the boundary continuity of conformal maps', Pacific J. Math. 120 (1985) 423–430. MR0810781
11. T. Ransford, Potential theory in the complex plane (Cambridge University Press, Cambridge, 1995). MR1334766
12. R. L. Wilder, 'The mathematical work of R. L. Moore: its background, nature and influence', Arch. Hist. Exact Sci. 26 (1982) 73–97. MR0664470

Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

MR2195060 (2006j:30092) 30H05 30D55 46J15

Bishop, Christopher J. (1-SUNYS)

Orthogonal functions in H^∞. (English. English summary)

Pacific J. Math. **220** (2005), *no.* 1, 1–31.

Let H^∞ denote the Banach algebra of bounded holomorphic functions on the unit disk D. If $\psi \in H^\infty$ is an inner function with $\psi(0)=0$, it is easy to see that $\{\psi^n\}$, $n = 0,1,2,\ldots$, is orthogonal, that is, $\int_{\mathbb{T}} \psi^n \overline{\psi^m} d\theta = 0$ whenever $n \neq m$. In 1988, W. Rudin asked if the converse is true; this is called Rudin's "orthogonality conjecture". The conjecture was disproved by C. Sundberg [J. Amer. Math. Soc. **16** (2003), no. 1, 69–90 (electronic); MR1937200], and by the author, independently [Publ. Mat. **37** (1993), no. 1, 95–109; MR1240926].

For a function $f \in H^\infty$ with $\|f\|_\infty \le 1$ and a measurable set E in $\overline{D}$, let $\mu_f(E) = |f^{-1}(E)|$, where $|\cdot|$ denotes the normalized Lebesgue measure on $\mathbb{T}$. A measure μ on $\overline{D}$ is called radial if $\mu(E) = \mu(e^{i\theta}E)$ for every θ and measurable set E. The author proves that $\{f^n\}$, $n = 0,1,2,\ldots$, is orthogonal if and only if μ_f is a radial probability measure on $\overline{D}$ such that $\int_{\overline{D}} \log(1/|z|) d\mu_f(z) < \infty$. Moreover, given any measure satisfying these conditions, then there is $f \in H^\infty$, $\|f\|_\infty \le 1$, such that $\mu = \mu_f$.

As an application, it is proved that there is $f \in H^\infty$ with $\|f\|_\infty \le 1$ such that for any analytic g on D, g is in the Bergman space A^p if and only if $g \circ f$ is in the Hardy space H^p and their norms are equal. Also, it is proved that there is an orthogonal f such that $f(z)/z$ is a nonconstant outer function. *Keiji Izuchi*

[References]

1. A. B. Aleksandrov, J. M. Anderson, and A. Nicolau, "Inner functions, Bloch spaces and symmetric measures", *Proceedings London Math. Soc.* (3) **79**:2 (1999), 318–352. MR 2000g:46029 Zbl 01463593 MR1702245
2. C. J. Bishop, "An indestructible Blaschke product in the little Bloch space", *Publ. Mat.* **37**:1 (1993), 95–109. MR 94j:30032 Zbl 0810.30024 MR1240926
3. P. S. Bourdon, "Rudin's orthogonality problem and the Nevanlinna counting function", *Proc. Amer. Math. Soc.* **125**:4 (1997), 1187–1192. MR 98b:30034 Zbl 0866.30028 MR1363413
4. P. S. Bourdon, "Rudin's orthogonality problem and the Nevanlinna counting function, II", 1997. Unpublished notes. MR1363413
5. A. Cantón, "Singular measures and the little Bloch space", *Publ. Mat.* **42**:1 (1998), 211–222. MR 99g:30046 Zbl 0916.30032 MR1628174
6. J. A. Cima and L. J. Hansen, "Space-preserving composition operators", *Michigan Math. J.* **37**:2 (1990), 227–234. MR 91m:47042 Zbl 0715.30028 MR1058395
7. J. A. Cima, J. Thomson, and W. Wogen, "On some properties of composition operators", *Indiana Univ. Math. J.* **24** (1974/75), 215–220. MR 50 #2979 Zbl 0276.47038 MR0350487
8. J. A. Cima, B. Korenblum, and M. Stessin, "Composition isometries and Rudin's

problems", preprint, 1993. MR1542349

9. J. L. Fernández, D. Pestana, and J. M. Rodríguez, "Distortion of boundary sets under inner functions, II", *Pacific J. Math.* **172**:1 (1996), 49–81. MR 97b:30035 Zbl 0847.32005 MR1379286
10. J. B. Garnett, *Bounded analytic functions,* Pure and Applied Mathematics **96,** Academic Press Inc., New York, 1981. MR 83g:30037 Zbl 0469.30024 MR0628971
11. K. Löwner, "Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I", *Math. Ann.* **89** (1923), 103–121. JFM 49.0714.01 MR1512136
12. E. A. Nordgren, "Composition operators", *Canad. J. Math.* **20** (1968), 442–449. MR 36 #6961 Zbl 0161.34703 MR0223914
13. W. Rudin, "A generalization of a theorem of Frostman", *Math. Scand* **21** (1967), 136–143 (1968). MR 38 #3463 Zbl 0185.33301 MR0235151
14. W. Rudin, *Functional analysis,* McGraw-Hill, New York, 1973. MR 51 #1315 Zbl 0185.33301 MR0365062
15. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Grundlehren der Mathematischen Wissenschaften **241,** Springer, New York, 1980. MR 82i:32002 Zbl 0495.32001 MR0601594
16. J. V. Ryff, "Subordinate H^p functions", *Duke Math. J.* **33** (1966), 347–354. MR 33 #289 Zbl 0148.30205 MR0192062
17. J. H. Shapiro, "The essential norm of a composition operator", *Ann. of Math.* (2) **125**:2 (1987), 375–404. MR 88c:47058 Zbl 0642.47027 MR0881273
18. W. Smith, "Composition operators between Bergman and Hardy spaces", *Trans. Amer. Math. Soc.* **348**:6 (1996), 2331–2348. MR 96i:47056 Zbl 0857.47020 MR1357404
19. W. Smith, "Inner functions in the hyperbolic little Bloch class", *Michigan Math. J.* **45**:1 (1998), 103–114. MR 2000e:30070 Zbl 0976.30018 MR1617418
20. W. Smith and L. Yang, "Composition operators that improve integrability on weighted Bergman spaces", *Proc. Amer. Math. Soc.* **126**:2 (1998), 411–420. MR 98d:47070 Zbl 0892.47031 MR1443167
21. K. Stephenson, "Construction of an inner function in the little Bloch space", *Trans. Amer. Math. Soc.* **308**:2 (1988), 713–720. MR 89k:30031 Zbl 0654.30024 MR0951624
22. C. Sundberg, "Measures induced by analytic functions and a problem of Walter Rudin", *J. Amer. Math. Soc.* **16**:1 (2003), 69–90. MR 2003i:30049 Zbl 1012.30022 MR1937200
23. M. Tsuji, *Potential theory in modern function theory,* Maruzen Co. Ltd., Tokyo, 1959. MR 22 #5712 Zbl 0322.30001 MR0114894
24. R. L. Wheeden and A. Zygmund, *Measure and integral,* Marcel Dekker Inc., New York, 1977. MR 58 #11295 Zbl 0362.26004 MR0492146
25. N. Zorboska, "Composition operators with closed range", *Trans. Amer. Math. Soc.* **344**:2 (1994), 791–801. MR 94k:47050 Zbl 0813.47037 MR1236226

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2145055 (2006c:30022) 30C62

Bishop, Christopher J. (1-SUNYS)

★**An explicit constant for Sullivan's convex hull theorem. (English. English summary)**

In the tradition of Ahlfors and Bers, *III*, 41–69, *Contemp. Math.*, 355, *Amer. Math. Soc.*, *Providence*, *RI*, 2004.

Let Ω be a simply connected domain in the complex plane, which we identify in the usual way as a plane in 3-space. We identify the upper half space with the hyperbolic 3-space and denote by $C(\partial\Omega)$ the hyperbolic convex hull of $\partial\Omega$, that is, the hyperbolic convex hull of the set of all hyperbolic geodesics whose endpoints lie in $\partial\Omega$. Let S be the boundary component that separates Ω from $C(\partial\Omega)$.

Let ρ_S denote the intrinsic metric of S, using the hyperbolic arc length, and let ρ denote the hyperbolic metric in the unit disk D or in Ω. Thurston observed that there is an isometry ι of (S,ρ_S) onto (D,ρ). Further, there is an absolute constant K_0 and a K_0-bi-Lipschitz map σ of (Ω,ρ) to (S,ρ_S) which extends continuously to the identity map on $\partial\Omega$. Thus σ is K-quasiconformal for an absolute constant K. D. B. A. Epstein and A. Marden [in *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, 113–253, Cambridge Univ. Press, Cambridge, 1987; MR0903852] proved that one can take $K_0 \approx 88.2$ and $K \approx 82.6$.

If Ω is invariant under a group of Möbius transformations, one can ask what happens to K if one chooses σ, as one then may, to commute with the group action. We do not discuss that in greater detail here but refer to the paper by Epstein and Marden.

In this paper, the author develops a method to construct such maps σ without group invariance, and proves that one can take $K = 7.82$ and $K_0 = 13.3$, hence improving previously known dilatation bounds.

To prove the result, the author develops some very clever geometric procedures to define a map. These are too complicated for us to explain here, but the author gives a nice overview of the construction in Section 2 of the paper. Roughly speaking, he first constructs the map for domains that can be expressed as the union of finitely many nice pieces (crescents and hyperbolic triangles) and then uses approximation to extend the result to general simply connected domains Ω.

A corollary is that if f is a conformal map of D onto Ω, then one can write $f = g \circ h$, where h is a 7.82-quasiconformal self-map of D while $|g'|$ is bounded away from zero. The author proved earlier [Ark. Mat. **40** (2002), no. 1, 1–26; MR1948883] that if this were to hold with 7.82 replaced by 2, then the Brennan conjecture [J. E. Brennan, J. London Math. Soc. (2) **18** (1978), no. 2, 261–272; MR0509942] would follow, but recently D. B. A. Epstein and V. Marković [Ann. of Math. (2) **161** (2005), no. 2, 925–957; MR2153403] showed that sometimes the dilatation must be > 2.1 here. *A. Hinkkanen*

MR2053343 (2005c:30045) 30F40

Bishop, Christopher J. (1-SUNYS)

The linear escape limit set. (English. English summary)

Proc. Amer. Math. Soc. **132** (2004), *no.* 5, 1385–1388.

Let G be a discrete group of isometries, acting on hyperbolic space $\mathbb{B}^n$, $n \geq 2$, and let Λ be its limit set. For a point $x \in \Lambda$, the radial segment $[0,x)$ projects to a geodesic ray γ in the quotient $M = \mathbb{B}^n/G$. The bounded limit set Λ_b consists of all those $x \in \Lambda$ for which the corresponding ray γ remains bounded for all time. On the other hand, the linear escape limit set Λ_l is the set of all $x \in \Lambda$ for which the corresponding ray γ escapes to ∞ at the fastest possible speed. Precisely, parameterizing γ by arclength, $x \in$

Λ_l if

$$\liminf_{t\to\infty} \frac{\mathrm{dist}_M(\gamma(t),\gamma(0))}{t} > 0.$$

Clearly, the bounded limit set is a subset of the conical limit set, and the linear escape limit set is a subset of the escaping limit set.

The main theorem in the current paper says that the dimension of the limit set Λ is equal to either the dimension of the bounded limit set Λ_b or to the dimension of the linear escape limit set Λ_l. *Petra Bonfert-Taylor*

[References]

1. C. J. Bishop and P. W. Jones. Hausdorff dimension and Kleinian groups. *Acta. Math.* 179:1–39, 1997. MR1484767
2. T. Lundh. Geodesics on quotient manifolds and their corresponding limit points. *Michigan Math. J.* 51:279-304, 2003. MR1992947
3. P. Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics 44. Cambridge University Press, Cambridge, 1995. MR1333890
4. C. T. McMullen. *Renormalization and* 3*-manifolds which fiber over the circle.* Annals of Math- ematics Studies 142. Princeton University Press, Princeton, NJ, 1996. MR1401347
5. P. J. Nicholls. *The ergodic theory of discrete groups.* London Mathematical Society Lecture Note Series 143. Cambridge University Press, 1989. MR1041575

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MR2036986 (2005e:30072) 30F35 30F25
Bishop, Christopher J. (1-SUNYS)
Big deformations near infinity. (English. English summary)
Illinois J. Math. **47** (2003), *no.* 4, 977–996.

Suppose G is a Fuchsian group acting in the unit disk, $\mathbb{D}$, and μ is a Beltrami coefficient for G with support in $\mathbb{D}$. Let f be a (suitably normalized) solution to the Beltrami equation $(f)_{\overline{z}}=\mu f_z$. Then $G_\mu := fGf^{-1}$ is a quasi-Fuchsian group. Let $\delta(\mu)=\delta(G_\mu)$ be the critical exponent for G_μ, with Λ_μ being the limit set of G_μ and $\dim(\bullet)$ denoting Hausdorff dimension, $\dim(\mu)=\dim(\Lambda_\mu)$. Call μ big if $\delta(\mu)>1$. G has big deformations near ∞ if there exist $\epsilon,\delta>0$ such that, for each compact $K\subset S:=\mathbb{D}/G$, there exists a μ, supported on the complement of K, satisfying (1) $\|\mu\|_\infty<1-\epsilon$, and (2) $\delta(G_\mu)\geq 1+\delta$.

This paper is devoted to finding conditions for the existence of big deformations of G (or equivalently of S) near ∞. The conditions relate to injectivity radius and whether G has divergence type. Without details and sometimes using a slightly different language, the following infinitely generated, torsion free groups are shown to admit big deformations:

(1) The injectivity radius of G is bounded above and below away from zero.
(2) G is of divergence type and the injectivity radius is bounded below away from zero.
(3) G satisfies a technical geometric condition (Theorem 1.2 in the paper).

It is also shown that groups admitting infinite pants decompositions, where the hyperbolic lengths of the border curves of the individual pants form a null sequence, do not have big deformations near infinity.

A related notion is defined as follows. G has the Ruelle property if, whenever there is a real analytic family $\{G_t\}$ of quasiconformal deformations of G, $\dim(\Lambda(G_t))$ is a real analytic function of t. The Ruelle property holds for cocompact families.

The author has previously shown ["A criterion for the failure of Ruelle's property", preprint, SUNY Stony Brook, Stony Brook, NY, 1999, available at www.math.sunysb.edu/~bishop/classes/math626.F00/math626.html] that, under rather natural side conditions, the existence of big deformations implies the failure of the Ruelle condition. Here, other examples are given where the Ruelle property fails. *William Abikoff*

[References]

1. V. Alvarez and J.M. Rodríguez, *Structure theorems for topological and Riemann surfaces*, J. London Math. Soc., to appear. MR2025333
2. K. Astala and M. Zinsmeister, *Holomorphic families of quasi-Fuchsian groups*, Ergodic Theory Dynam. Sys. **14** (1994), 207–212. MR **95k:**30095 MR1279468
3. K. Astala and M. Zinsmeister, *Abelian coverings, Poincaré exponent of convergence and holomorphic deformations*, Ann. Acad. Sci. Fenn. **20** (1995), 81–86. MR **96a:**30047 MR1304107
4. C. J. Bishop, *Divergence groups have the Bowen property*, Ann. of Math. (2) **154** (2001), 205–217. MR **2003b:**30052 MR1847593
5. C. J. Bishop, *Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture*, Ark. Mat. **40** (2002), 1–26. MR **2003i:**30063 MR1948883
6. C. J. Bishop, *Quasiconformal mappings of Y-pieces*, Rev. Mat. Iberoamericana **18** (2002), 627–652. MR **2003j:**30070 MR1954866
7. C. J. Bishop, *A criterion for the failure of Ruelle's property*, preprint, 1999. MR2279263
8. C.J. Bishop and P.W. Jones, *Hausdorff dimension and Kleinian groups*, Acta. Math **179** (1997), 1–39. MR **98k:**22043 MR1484767
9. C.J. Bishop and P.W. Jones, *Wiggly sets and limit sets*, Arkiv Mat. **35** (1997), 201–224. MR **99f:**30066 MR1478778
10. C.J. Bishop and P.W. Jones, *Compact deformations of Fuchsian groups*, Dedicated to the memory of Thomas H. Wolff, J. Anal. Math. **87** (2002), 5–36. MR **2003m:**20065 MR1945276
11. A. Epstein and A. Marden, *Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces*, Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984), Cambridge Univ. Press, Cambridge, 1987, pp. 113–253. MR **89c:**52014 MR0903852
12. H. M. Farkas and I. Kra, *Riemann surfaces*, Springer-Verlag, New York, 1980. MR **82c:**30067 MR0583745
13. P.J. Nicholls, *The ergodic theory of discrete groups*, London Mathematical Society Lecture Note Series, vol. 143, Cambridge University Press, Cambridge, 1989. MR **91i:**58104 MR1041575
14. A. Pfluger, *Sur une propriété de l'application quasi-conforme d'une surface de Riemann ouverte*, C.R. Acad. Sci. Paris **227** (1948), 25–26. MR 10,28f MR0025576
15. D. Ruelle, *Repellers for real analytic maps*, Ergodic Theory Dynam. Systems **2** (1982), 99–107. MR **84f:**58095 MR0684247
16. D. Sullivan, *Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension* 3 *fibrées sur* S^1, Bourbaki Seminar, Vol. 1979/80, Lecture

Notes in Math., vol. 842, Springer, Berlin, 1981, pp. 196–214. MR **83h:**58079 MR0636524
17. W.P. Thurston, *The geometry and topology of 3-manifolds*, The Geometry Center, University of Minnesota, 1979.

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1980177 (2004c:30038) 30C65 30C62

Bishop, Christopher J. (1-SUNYS); **Gutlyanskiĭ, Vladimir Ya.** (UKR-AOS-A1); **Martio, Olli** (FIN-HELS); **Vuorinen, Matti** (FIN-HELS)

On conformal dilatation in space. (English. English summary)

Int. J. Math. Math. Sci. **2003,** *no.* 22, 1397–1420.

The problem of showing that a quasiconformal mapping with dilatation suitably tending to 1 near a point resembles a conformal mapping at that point has received much attention over the years. A few similar results for quasiregular mappings have also been proved. This paper proves some nice results of this type.

Suppose $f\colon G\to\mathbf{R}^n$ is a nonconstant quasiregular map, with inner dilatation L_f. For $y\in G$ and U a neighborhood of y in G, let

$$I(y,U)=\frac{1}{\omega_{n-1}}\int_U\frac{L_f(x)-1}{|x-y|^n}\,dx.$$

One of the main results says that if $G=\mathbf{R}^n$, $n\ge 3$, $f(0)=0$, and $I(r)=I(0,B(0,r))<\infty$ for some fixed $r>0$, then f has injectivity radius $R_f(0)>0$ and there is a constant C such that

$$\min_{|x|=R}|f(x)|\frac{e^{-I(R)}}{R}\le C\le\max_{|x|=R}|f(x)|\frac{e^{I(R)}}{R},\qquad 0<R\le R_f(0),$$

and $|f(x)|/|x|\to C$ as $x\to 0$. Moreover the same result holds for quasiconformal mappings in the plane. A uniform variant of this result implies that if f is quasiconformal, $n\ge 2$, and $I(y,U)$ is uniformly convergent on a compact rectifiable curve γ in G, then $f(\gamma)$ is rectifiable. The proofs are based on the concept of the infinitesimal space and new Grötsch-type modulus estimates. *Stephen Buckley*

MR1976837 (2004a:30040) 30F35 30F40

Bishop, Christopher J. (1-SUNYS)

δ-stable Fuchsian groups. (English. English summary)

Ann. Acad. Sci. Fenn. Math. **28** (2003), *no.* 1, 153–167.

There are two non-negative real numbers associated to every Kleinian group G: the Hausdorff dimension $\dim(\Lambda(G))$ of its limit set $\Lambda(G)$ and the critical exponent of its Poincaré series $\delta(G)$. It is known that $\delta(G)\le\dim(\Lambda(G))$. It is natural to ask when equality holds. It is known that equality holds for geometrically finite groups as a result of work of the author available in preprints. A Fuchsian group G is called δ-stable if $\delta(G')=\dim(\Lambda(G'))$ for every quasiconformal deformation G' of G. Finitely generated Fuchsian groups have this property because finitely generated quasi-Fuchsian groups are geometrically finite. The author gives examples of infinitely generated Fuchsian groups that are δ-stable and other examples that are not. *I. Kra*

[References]

1. Ahlfors, L.V.: Lectures on Quasiconformal Mappings. - Math. Studies 10, Van Nostrand, 1966. MR0200442

2. Astala, K.: Area distortion of quasiconformal mappings. - Acta Math. 173, 1994, 37–60. MR1294669
3. Astala, K., and M. Zinsmeister: Mostow rigidity and Fuchsian groups. - C. R. Acad. Sci. Paris 311, 1990, 301–306. MR1071631
4. Beardon, A.F., and B. Maskit: Limit points of Kleinian groups and finite sided fundamental polyhedra. - Acta Math. 132, 1974, 1–12. MR0333164
5. Bishop, C.J.: Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture. - Ark. Mat. 40, 2002, 1–26 MR1948883
6. Bishop, C.J.: Quasiconformal mappings of Y-pieces. - Rev. Mat. Iberoamericana (to appear). MR1954866
7. Bishop, C.J.: On a theorem of Beardon and Maskit. - Ann. Acad. Sci. Fenn. Math. 21, 1996, 383–388. MR1404092
8. Bishop, C.J.: A criterion for the failure of Ruelle's property. - Preprint, 1999. MR2279263
9. Bishop, C.J.: Divergence groups have the Bowen property. - Ann. of Math. 154, 2001, 205–217. MR1847593
10. Bishop, C.J., V. Ya. Gutlyanskii, O. Martio, and M. Vuorinen: On conformal dilatation in space. - Internat. J. Math. Math. Sci. (to appear). MR1980177
11. Bishop, C.J., and P.W. Jones: Compact deformations of Fuchsian groups. - J. Anal. (to appear). MR1945276
12. Bishop, C.J., and P.W. Jones: Hausdorff dimension and Kleinian groups. - Acta. Math. 179, 1997, 1–39. MR1484767
13. Bishop, C.J., and P.W. Jones: Wiggly sets and limit sets. - Ark. Mat. 35, 1997, 201–224. MR1478778
14. Bowen, R.: Hausdorff dimension of quasicircles. - Inst. Hautes Études Sci. Publ. Math. 50, 1979, 11–25. MR0556580
15. Bridgeman, M., and E.C. Taylor: Length distortion and the Hausdorff dimension of limit sets. - Amer. J. Math. 122, 2000, 465–482. MR1759885
16. Burger, M.: Small eigenvalues of Riemann surfaces and graphs. - Math. Z. 205, 1990, 395–420. MR1082864
17. Buser, P.: A note on the isoperimetric constant. - Ann. Sci. École Norm. Sup. (4) 15, 1982, 213–230. MR0683635
18. Canary, R.D.: On the Laplacian and the geometry of hyperbolic 3-manifolds. - J. Differential Geom. 36, 1992, 349–367. MR1180387
19. Cheeger, J.: A lower bound for the smallest eigenvalue of the Laplacian. - In: Problems in Analysis (Papers Dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, N. J., 1970, 195–199. MR0402831
20. Dodziuk, J., T. Pignataro, B. Randol, and D. Sullivan: Estimating small eigenvalues of Riemann surfaces. - In: The Legacy of Sonya Kovalevskaya (Cambridge, Mass., and Amherst, Mass., 1985), Amer. Math. Soc., Providence, R.I., 1987, 93–121. MR0881458
21. Fernández, J.L., and M.V. Melián: Bounded geodesics of Riemann surfaces and hyperbolic manifolds. - Trans. Amer. Math. Soc. 347, 1995, 3533–3549. MR1297524
22. Fernández, J.L., and M.V. Melián: Escaping geodesics of Riemannian surfaces. - Acta Math. 187, 2001, 213–236. MR1879849
23. Gehring, F.W., and J. Väisälä: Hausdorff dimension and quasiconformal mappings. - J. London Math. Soc. (2) 6, 1973, 504–512. MR0324028
24. Keen, L.: Collars on Riemann surfaces. - Ann. of Math. Studies 79, 1974, 263–268. MR0379833
25. Makarov, N.G.: Conformal mapping and Hausdorff measures. - Ark. Mat. 25, 1987, 41–89. MR0918379

26. Matelski, J.P.: A compactness theorem for Fuchsian groups of the second kind. - Duke Math. J. 43, 1976, 829–840. MR0432921
27. Sullivan, D.: The density at infinity of a discrete group of hyperbolic motions. - Inst. Hautes Études Sci. Publ. Math. 50, 1979, 172–202. MR0556586
28. Sullivan, D.: On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. - In: Riemann Surfaces and Related Topics, Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Princeton Univ. Press, Princeton, N.J., 1981, 465–496. MR0624833
29. Sullivan, D.: Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. - Acta. Math. 153, 1984, 259–277. MR0766265
30. Sullivan, D.: Related aspects of positivity in Riemannian geometry. - J. Differential Geom. 25, 1987, 327–351. MR0882827

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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Bishop, Christopher J. (1-SUNYS)

Non-rectifiable limit sets of dimension one. (English. English summary)

Rev. Mat. Iberoamericana **18** (2002), *no.* 3, 653–684.

R. Bowen proved that any deformation of a cocompact Fuchsian group gives a quasi-Fuchsian Kleinian group whose limit set is either a circle or has Hausdorff dimension > 1. This was extended to all divergence type groups by the author and was shown to be false for all convergence groups (of the first kind) by K. Astala and M. Zinsmeister. They showed that all such groups have a deformation such that the limit set is a non-circular rectifiable curve. Zinsmeister asked if Bowen's property could fail in a different way, namely, are there quasi-Fuchsian groups whose limit sets are not locally rectifiable, but still have dimension 1? The author shows that there are many such groups by constructing quasiconformal deformations of convergence type Fuchsian groups such that the resulting limit set is a Jordan curve of Hausdorff dimension 1, but having tangents almost nowhere. The main tools in this construction are a characterization of tangent points in terms of Peter Jones' β's, a result of Stephen Semmes that gives a Carleson type condition on a Beltrami coefficient μ which implies rectifiability, and a construction of quasiconformal deformations of a surface which shrink a given geodesic and whose dilatations satisfy an exponential decay estimate away from the geodesic.

Vasily A. Chernecky

[References]

1. Astala, K. and Zinsmeister, M.: Mostow rigidity and Fuchsian groups C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 301–306. MR1071631
2. Bishop,C.J.:Quasiconformal mappings of Y-pieces. Rev. Mat. Iberoamericana **18** (2002), 627–652. MR1954866
3. Bishop, C.J.: Divergence groups have the Bowen property. Ann. of Math. **154** (2001), 205–217. MR1847593
4. Bishop, C.J. and Jones, P.W.: Compact deformations of Fuchsian groups. To appear in J. Anal. Math. MR1945276
5. Bishop, C.J. and Jones, P.W.: Harmonic measure, $L2$ estimates and the Schwarzian derivative. J. Anal. Math. **62** (1994), 77–113. MR1269200
6. Bowen, R.: Hausdorff dimension of quasicircles. Inst. HautesÉtudes Sci. Publ. Math. **50** (1979), 11–25. MR0556580
7. Garnett, J.B.: Bounded analytic functions. Academic Press, 1981. MR0628971

8. Jerison, D.S. and Kenig, C.E.: Hardy spaces, $A\infty$ and singular integrals on chord-arc domains. Math. Scand. **50** (1982), 221–248. MR0672926
9. Jones, P.W.: Rectifiable sets and travelling salesman problem. Invent. Math. **102** (1990), 1–15. MR1069238
10. Keen, L.: Collars on Riemann surfaces. In Discontinuous groups and Rie- mann surfaces, Ann. of Math. Stud. **79**, Princeton Univ. Press, Princeton, N.J., 1974, 263–268. MR0379833
11. Matelski, J.P.: A compactness theorem for Fuchsian groups of the sec- ond kind. Duke Math. J. **43** (1976), no. 4, 829–840. MR0432921
12. Pommerenke, Ch.: Boundary behaviour of conformal maps. Grundlehren Math. Wiss. **299**, Springer-Verlag, Berlin, 1992. MR1217706
13. Semmes, S.: Quasiconformal mappings and chord-arc curves. Trans. Amer. Math. Soc. **306** (1988), 233–263. MR0927689

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MR1954866 (2003j:30070) 30F60 30F35

Bishop, Christopher J. (1-SUNYS)

Quasiconformal mappings of Y-pieces. (English. English summary)

Rev. Mat. Iberoamericana **18** (2002), *no.* 3, 627–652.

The main purpose of the paper is to present an explicit way of deforming a Riemann surface collapsing a given closed geodesic γ. In particular, it is done by a quasiconformal deformation with the complex dilatation μ such that $|\mu|$ decays exponentially fast away from γ. The construction is used in a companion paper in the same volume to construct quasi-Fuchsian groups whose limit sets are non-rectifiable curves of dimension 1. In fact, the author gives precise estimates of $|\mu|$ for generalized Y-pieces that are Riemann surfaces bounded by three closed geodesics (or punctures) which are homeomorphic to a sphere minus three discs (or points). Every finite area Riemann surface can be written as a finite union of such pieces. *Alexander Vasil′ev*

[References]

1. Álvarez, V. and Rodríguez, J.M.: Structure theorems for topological and Riemann surfaces. To appear in J. London. Math. Soc. MR2025333
2. Beardon, A.F.: The geometry of discrete groups. Springer-Verlag, New York, 1983. MR0698777
3. Bishop, C.J.:Non-rectifiable limit sets of dimension one. Rev. Mat. Iberoamericana **18** (2002), 653–684. MR1954867
4. Bishop, C.J.: A criterion for the failure of Ruelle's property. Preprint, 1999. MR2279263
5. Bishop, C.J.: Divergence groups have the Bowen property. Ann. of Math. **154** (2001), 205–217. MR1847593
6. Bishop, C.J.: Big deformations near infinity. Preprint, 2002. MR2036986
7. Bishop, C.J.: δ-stable Fuchsian groups. Preprint, 2002. MR1976837

Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.