DIVERGENCE GROUPS HAVE THE BOWEN PROPERTY

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Abstract. We show that every deformation of a divergence type Fuchsian group has a limit set which is either a circle or has dimension strictly larger than 1. This is known to be false for all convergence type groups, and hence solves Bowen's problem in the general case. The proof uses a theorem of Dennis Sullivan's about convex hulls in hyperbolic 3-space and we give a new, simplier proof of this result as well.

Date: February 26, 2001.

1991 *Mathematics Subject Classification.* Primary: 30F40, Secondary: 20H10, 30C62 .

Key words and phrases. Bowen property, Hausdorff dimension, quasi-Fuchsian groups, quasiconformal deformation, critical exponent, convex core, Mostow rigidity, Sullivan's theorem .

The author is partially supported by NSF Grant DMS 9800924 .

1. Introduction

We will say that a Fuchsian group G has the Bowen property if the limit set of any quasiconformal deformation of G is either a circle or has dimension strictly larger than 1. In a 1979 paper Rufus Bowen [9] proved that this dichotomy holds for all cocompact Fuchsian groups. Dennis Sullivan [28, 29] soon extended this property to cofinite groups, and in 1990 Kari Astala and Michel Zinsmeister showed that it was false for convergence type groups of the first kind; every such group has a quasi-Fuchsian deformation whose limit set is a non-circular rectifiable curve. The only remaining interesting cases are the infinitely generated divergence type groups and the purpose of this paper is to show that these groups all have the Bowen property. So far as I know, this had not been previously proven for any such example.

We start by reviewing a few basic definitions. A Kleinian group is a discrete group of isometries of the hyperbolic three space. In this paper will will use both the upper half-space model $\mathbb{H}^3$ and the ball model $\mathbb{B}^3$. A Fuchsian group is a Kleinian group which fixes the unit disk $\mathbb{D} \subset \mathbb{R}^2 = \partial\mathbb{H}^3$. A deformation of a Fuchsian group G is a conformal map $\Phi : \mathbb{D} \to \Omega$ such that for every $g \in G$, the map $\Phi \circ g \circ \Phi^{-1}$ is a Möbius transform. If Φ has a quasiconformal extension to the whole plane then we say it is a quasi-Fuchsian deformation of G.

A Fuchsian group is cocompact if $\mathbb{D}/G$ is compact and is cofinite if the quotient has finite hyperbolic area. A Fuchsian group is called divergence type if

$$\sum_{g \in G} \exp(-\rho(0, g(0))) = \infty,$$

(where ρ is the hyperbolic metric) and otherwise it is called convergence type. The group is called first kind if the limit set is the entire circle and is second kind otherwise. It is well known that cocompact $\subset$ cofinite $\subset$ divergence type $\subset$ first kind. The critical exponent (or Poincaré exponent) of a group acting on the hyperbolic ball $\mathbb{B}^3$ is defined as

$$\delta(G) = \inf\{s : \sum_{g \in G} \exp(-s\rho(0, g(0)))\} < \infty.$$

Theorem 1. *Suppose G is a divergence type Fuchsian group and $G' = \Phi \circ G \circ \Phi^{-1}$ is a deformation. Then either $\partial\Omega$ is a circle or $\delta(G') > 1$*

It is proven in [7] that $\delta(G) \leq \dim(\Lambda)$ for any non-elementary Kleinian group, so the theorem above implies that every divergence group has the Bowen property. Astala and Zinsmeister [3] showed that the Bowen property implies divergence type, and so

Corollary 2. *A Fuchsian group is divergence type iff it has the Bowen property.*

This adds to a long list of known characteriations of divergence type Fuchsian groups (see, e.g., Theorem 8.3.4 of [19]) including

- orbits of G in the disk are non-tangentially dense a.e. on the circle,
- the geodesic flow is ergodic on $R = \mathbb{D}/G$,
- R has no Green's function (i.e., Brownian motion is recurrent on R),
- Mostow rigidity holds for G [1], [31], [3].

The key new idea in the proof of Theorem 1 is to use three dimensional hyperbolic geometry to estimate the expansion properties of conformal mappings in the plane. This is done via Sullivan's convex hull theorem: there is a universal $K < \infty$ so that for any hyperbolic, simply connected planar domain Ω, there is a K-quasiconformal map from Ω to the boundary of the convex hull of Ω^c in $\mathbb{H}^3$, with its intrinsic hyperbolic path metric (see [27], [15]). This result implies (see [5]) that a conformal map $\Phi : \mathbb{D} \to \Omega$ can be factored as $\Phi = h \circ \Psi$, where Ψ is a K-quasiconformal self-map of the disk and $h : \mathbb{D} \to \Omega$ is expanding in the sense that $|h'| > C > 0$ uniformly on $\mathbb{D}$. If Φ is a deformation of a Fuchsian group G to a group G', then $H = \Psi \circ G \circ \Psi^{-1}$ is a Fuchsian group quasiconformally conjugate to G (and hence is divergence type iff G is). Moreover, we can show the Poincaré series of H gives a term-by-term lower bound for the Poincaré series of G'. Roughly speaking, we prove Theorem 1 by comparing the Poincaré series of G' to that of G when we know $|\Phi'|$ is large (which must happen often if Φ is not Möbius) and comparing it with H otherwise. In addition, we give a new proof of Sullivan's theorem and deduce the version of the factorization theorem we will need. This connection between convex sets in $\mathbb{H}^3$ and conformal mappings should have interesting applications to other problems. A few such are described in [5], including connections to Brennan's conjecture, integral means and dimension distortion.

There are now several proofs of Bowen's original result for the cocompact case: his original proof using Markov partitions and the thermodynamic formalism [9]; Rohde's proof [25] that such limit sets are uniformly wiggly (and hence dimension > 1 by [8]), and a proof based on Schwarzian derivatives of the mapping Φ [8]. The latter also applies to the cofinite case. Recently, Bridgman and Taylor [11] have given another proof based on ergodic theory and geometric properties of the convex hull of the limit set.

Braam [10] and Canary and Taylor [12] generalized Bowen's theorem by showing that if the limit set of a geometrically finite Kleinian group is connected then it is either a circle or has dimension > 1. The same result for all analytically finite Kleinian groups is proven in [7] and [8].

In 1982 Pommerenke showed that for divergence groups, the limit set is either a circle or has tangents almost nowhere (and hence has no rectifiable subarcs). For quasi-Fuchsian deformations this also follows from the Mostow rigidity theorem for divergence groups (see e.g., Agard [1] or Tukia [31]), which states that a quasiconformal conjugacy of a divergence type is either Möbius or is singular on the unit circle. Bowen's result has also inspired analogous results in other settings. Chengbo Yue [35], [36] has studied the Bowen property for cocompact groups acting on symmetric spaces and in variable curvature. Analogous questions for Julia sets have been considered by Zdunik [37], Urbański [32] and Hamilton [16], [17]. Although these papers deal with various cases the following Bowen type problem is still open so far as I know: if J is a connected Julia set of a rational map, then is it true that J is either an analytic arc or has Hausdorff dimension strictly larger than 1?

I thank Michel Zinsmeister for reading an early draft of this paper and for his remarks which helped clarify the proof.

2. Proof of Theorem 1

In what follows, $a \simeq b$ means that a/b is bounded and bounded away from zero by uniform constants.

Given a conformal map $\Phi : \mathbb{D} \to \Omega$ which conjugates G to a Möbius group G' acting on Ω we wish to show that either Φ is Möbius or $\delta(G') > 1$. Therefore we assume from now on that Φ is not Möbius. The proof that $\delta(G') > 1$ is broken into several steps. First, we show it is enough to estimate a series on the unit disk. The following is Lemma 2.2 of [8].

Lemma 3. *Suppose G is a Kleinian group and Ω is a bounded, simple connected, invariant component of the ordinary set. Suppose $x \in \mathbb{H}^3$ and $z \in \Omega$. Then there is a constant M (depending on G, x and z, but not on g) so that for any $g \in G$,*

$$\frac{1-|g(x)|^2}{M} \leq \mathrm{dist}(g(z), \partial\Omega) \leq M(1-|g(x)|^2).$$

where distance in Ω is in the Euclidean metric.

Corollary 4. *Suppose G is a Fuchsian group and suppose $\Phi : \mathbb{D} \to \Omega$ is a deformation of G to a Kleinian group G'. Then*

$$\delta(G') \quad = \quad \inf\{s : \sum_{g \in G} |\Phi'(g(0))|^s (1-|g(0)|^2)^s < \infty\}.$$

Proof. First note that

$$\delta(G') \quad = \quad \inf\{s : \sum_{g \in G} \mathrm{dist}(g(z), \partial\Omega)^s < \infty\},$$

is obvious using the previous lemma, the definition of δ and the fact that

$$\exp(-\rho(0,x)) \simeq 1-|x|^2.$$

The corollary then follows immediately from the Koebe $\frac{1}{4}$-theorem,

$$|\Phi'(z)|(1-|z|^2) \simeq \mathrm{dist}(\Phi(z), \partial\Omega).$$

□

So we have to show that there is a $s > 1$ so that the series in Corollary 4 diverges. This will follow if we can construct a "tree" of distinct orbit points $\mathcal{C} = \cup_n \mathcal{C}_n \subset G(0) = \{g(0) : g \in G\}$ so that $\mathcal{C}_0 = \{0\}$ and any $z \in \mathcal{C}_n$ has a finite collection of children $\mathcal{C}(z) \subset \mathcal{C}_{n+1}$, so that

$$\sup_{z \in \mathcal{C}} \sup_{w \in \mathcal{C}(z)} \rho(w,z) \leq D_0 \tag{1}$$

and

$$\sum_{w \in \mathcal{C}(z)} |\Phi'(w)|(1-|w|^2) \geq M_0 |\Phi'(z)|(1-|z|^2), \tag{2}$$

for some $D_0 < \infty$ and $M_0 > 1$. Given these two conditions, it is clear that there is a $s = s(D_0, M_0) > 1$ so that

$$\sum_{w \in \mathcal{C}(z)} |\Phi'(w)|^s (1-|w|^2)^s \geq |\Phi'(z)|^s (1-|z|^2)^s,$$

and hence

$$\sum_{w \in \mathcal{C}_{n+1}} |\Phi'(w)|^s (1-|w|^2)^s \geq \sum_{w \in \mathcal{C}_n} |\Phi'(w)|^s (1-|w|^2)^s \geq 1,$$

which obviously implies the sum over the whole orbit diverges.

Given an interval $I \subset \mathbb{T}$ let

$$Q_I = \{z : z/|z| \in I, 1-|z| \leq |I|\},$$

denote the Carleson square with base I. Its "top half" is defined as

$$T(Q_I) = \{z : z/|z| \in I, \frac{1}{2}|I| \leq 1-|z| \leq |I|\}.$$

We will construct our tree of orbit points using the following two lemmas.

Lemma 5 (Growth near orbits)**.** *Suppose G is a divergence group, Q is a Carleson square and $z = g(0) \in T(Q)$ for some $g \in G$. Suppose Φ is a conformal deformation which is not a Möbius transformation. Then for any $M > 0$ there is a collection of disjoint subsquares $\{Q_j\} \subset Q$ and points $z_j \in T(Q_j)$ (not necessarily orbit points) such that*

1. $\sum_j |\Phi'(z_j)|(1-|z_j|^2) \geq M|\Phi'(z)|(1-|z|^2)$,
2. $1-|z_j|^2 \geq \delta(1-|z|^2)$ *for some $\delta > 0$ that depends only on G, Φ and M.*

Lemma 6 (Uniform renormalization)**.** *Suppose G is a divergence type Fuchsian group, $G' = \Phi \circ G \circ \Phi^{-1}$ is a quasi-Fuchsian deformation, Q is a Carleson square and $z \in T(Q)$. Then there is a collection of disjoint Carleson subsquares $\{Q_j\} \subset Q$ and points $z_j \in T(Q_j) \cap G(0)$ such that*

1. $\sum |\Phi'(z_j)|(1-|z_j|^2) \geq \epsilon|\Phi'(z)|(1-|z|^2)$ *where ϵ depends on G and Φ but not on z.*
2. $\inf_j(1-|z_j|^2) \geq \eta(1-|z|^2)$ *and $\eta > 0$ depends only on G and $\rho(z, G(0))$.*

Taking $M > 2/\epsilon$ and applying first Lemma 5 to an orbit point z and then Lemma 6 to each point of the resulting collection, gives us a collection of orbit points $\mathcal{C}(z)$ which satisfies (1) and (2). Thus in order to prove our main theorem, it suffices to prove the lemmas above.

Lemma 5 is essentially the same as Lemma 5.2 of [8]; however, we will sketch a proof in Section 3 based on the known result that the limit set is either a circle or has infinite length.

Lemma 6 for divergence groups is the new contribution of this paper. In the case of cocompact groups, this lemma is easy, since starting at any point w of the disk, one can always drop a uniformly bounded (hyperbolic) distance towards the boundary and find an orbit point z (and $|\Phi'(w)|$ and $|\Phi'(z)|$ will be comparable by the standard distortion theorems for univalent maps). When G is cofinite a little more care is needed, but the proof is very similar (see [8]).

The proof we give here for divergence groups uses a completely different approach. The idea is that condition (1) of Lemma 6 is difficult to verify because of the $|\Phi'|$ term. If we dropped it, then it would be easy to find orbits points $\{z_j\} \subset Q$ such that

$$\sum_j (1-|z_j|^2) \geq \epsilon(1-|z|^2), \tag{3}$$

because the orbit of a divergence group is non-tangentially dense almost everywhere on the circle. We shall show that the sum in (1) of Lemma 6 for the pair (G,Φ) can be bounded from below using a sum of the form (3) corresponding to a different Fuchsian group H. This new group will be quasiconformally equivalent to G and hence will also be divergence type (by a well known theorem of Pfluger [21]) and this will give us the desired lower bound.

3. Proof of Lemma 5

As we shall see, Lemma 5 follows easily from normal families and the known fact that the limit set of a quasi-Fuchsian deformation of a divergence group is either a circle or has tangents almost nowhere.

Before beginning the proof we recall that the Schwarzian derivative of a locally univalent function on the disk is defined by

$$S(f) = (f''/f')' - \frac{1}{2}(f''/f')^2.$$

We shall need only two well known facts about the Schwarzian (e.g., see Section 8.5 of [14] or Section 1.2 of [24]). First, that it vanishes identically iff f is a Möbius transformation and second that its composition law

$$S(f\circ g)(z) = S(f)(g(z))(g'(z))^2 + S(g)(z),$$

implies that if f is a deformation of a Fuchsian group then the hyperbolic Schwarzian,

$$S_H(f)(z) = |S(f)(z)|(1-|z|^2)^2$$

is constant on orbits of the group. See e.g. [8].

Now we begin the proof of the lemma. Suppose $z = g(0)$ and let $\gamma_t = \gamma_t(z)$ be the arc $\{w : \rho(z,w) = t\} \cap Q$. Let $\{z_j\}$ be a maximal set of points hyperbolic unit distance apart on γ_t. By the usual distortion estimates for univalent maps (e.g., [22] or [24]),

$$\sum_j |\Phi'(z_j)|(1-|z_j|^2) \simeq \int_{\gamma_t} |\Phi'(w)| ds,$$

where ds denotes integration with respect to Euclidean length on γ_t. Thus to prove the lemma, it suffices to show that for any M there is a t so that integral above is greater than M.

Suppose this is false. Then there is an $M < \infty$ so that for every $n = 1, 2, \dots$ there is an orbit point $z_n = g_n(0)$ so that

$$\sup_{t \leq n} \int_{\gamma_t(z_n)} |\Phi'(w)| ds \leq M.$$

By replacing Φ by $|\Phi'(z_n)|^{-1}[(\Phi \circ g_n) - \Phi(z_n)]$, and passing to the limit we may assume that we have a deformation φ such that $\varphi(0) = 0$, $\varphi'(0) = 1$, $S_H(\varphi) = S_H(\Phi)$ and

$$\sup_{r<1} \int_J |\varphi'(re^{i\theta})| d\theta \leq M,$$

for some non-degenerate interval $J \subset \mathbb{T}$. However, this easily implies that $\partial\varphi(\mathbb{D})$ contains a nontrivial rectifiable arc and hence φ' has non-tangential limits on a set of positive Lebesgue dimensional measure on the circle. On the other hand, φ is not Möbius, since $|S_H(\varphi)| = |S_H(\Phi)|$ is not identically zero.

However this contradicts known results. In particular, a theorem of Pommerenke [23] states that if φ is a deformation of a divergence type group, and φ is not Möbius, then φ' fails to have non-tangential limits almost everywhere. This contradiction completes the proof of the lemma.

Remark: The contradiction could also have been obtained from other known results. For example, in [6] it is shown that for any conformal map φ, the set where φ' has limits is almost everywhere the same as the set where

$$\int_{\Gamma_\theta} |S_H(\varphi)(z)|^2 (1-|z|^2)^{-2} dx dy < \infty,$$

where Γ_θ denotes a Stolz cone with vertex $e^{i\theta}$. Using the facts that $|S_H(\varphi(z)|$ is constant on orbits and that orbits of a divergence type group are non-tangentially dense almost everywhere, it is easy to see the integral over the Stolz cone diverges for almost every θ if φ is a deformation which is not Möbius.

If we only consider deformations with quasiconformal extensions, then yet another approach would be to use Mostow rigidity for divergence groups: if h is homeomorphism of the circle conjugating the action of two divergence type groups, then either h is Möbius or it is singular (maps a set of full Lebesgue measure to zero measure) [1], [31]. However is φ is a deformation with a quasiconformal extension to the plane then the complement of $\varphi(\mathbb{D})$ is also a topological disk (if we include ∞) and there is a conformal map g of the disk onto this region. Then $h \circ g^{-1}$ is a conjugating map for G on the circle and it is Möbius iff h and g are. Furthermore, it is singular iff the harmonic measures for the two sides of $\partial\Omega$ are mutually singular. By the F. and M. Riesz theorem (e.g., Theorem 6.8 of [24]) this cannot happen if $\partial\Omega$ contains a rectifiable subarc, giving the desired contradiction.

4. Proof of Lemma 6

Before beginning the proof of Lemma 6 we recall a few simple facts about quasiconformal mappings which we will use. The main thing we need is that a quasiconformal map f of the unit disk to itself is also quasisymmetric [2], i.e., there is a constant M (depending only on the quasi-constant of f) such that

$$|x-y| \leq |x-z| \quad \Rightarrow \quad |f(x)-f(y)| \leq M|f(x)-f(z)|.$$

From this it is easy to deduce the following lemma (which is left to the reader to prove).

Lemma 7. *Suppose f is a K-quasiconformal map of the disk to itself. Then there is a $\lambda < \infty$ (depending only on K) such that for any interval $I \subset \mathbb{T}$,*

$$Q_{f(I)/\lambda} \subset f(Q_I), \qquad f(Q_{I/\lambda}) \subset Q_{f(I)},$$

where cI denotes the concentric interval of length $c|I|$.

Now we start the proof of Lemma 6. Suppose $\Phi : \mathbb{D} \to \Omega$ is a deformation of G to G'. Take the convex hull of $\partial\Omega$ in the hyperbolic ball $\mathbb{B}^3$ and let S be the boundary component of $C(\partial\Omega)$ which faces Ω. Without loss of generality, we may also assume

$0 \in S$. It is an observation of Thurston that this surface with its intrinsic hyperbolic path metric ρ_S is isometric to the hyperbolic unit disk (e.g., [30] or Theorem 1.12.1 of [15]). Let $\iota : S \to \mathbb{D}$ denote an isometry of S to the usual hyperbolic disk. A theorem of Sullivan ([27] or Section 2 of [15]) states that there is a K-quasiconformal map $\sigma : \Omega \to S$ for some universal $K < \infty$ (in fact, σ is biLipschitz from the hyperbolic metric on Ω to the ρ_S metric on S). Moreover σ is conformally natural in the sense that if Ω is invariant under a Möbius group then σ commutes with the group action on Ω and S. Sullivan's map σ has the following useful property.

Lemma 8. *Suppose $\{0\} \in S \subset \mathbb{B}^3$. There is a $M < \infty$ so that*

$$\frac{1}{M}(1 - |\sigma(z)|) \leq \mathrm{dist}(z, \partial\Omega) \leq M(1 - |\sigma(z)|),$$

where dist *denotes spherical distance on $\Omega \subset \partial\mathbb{B}^3$.*

Proof. Let $R : \Omega \to S$ be the nearest point retraction. Geometrically, this map may be defined by expanding a horoball tangent to $\partial\mathbb{B}^3$ at $z \in \Omega$ until it first hits S at $R(z)$. If $0 \in S$ and B is the open horoball tangent at z and with $R(z)$ on its boundary, then its radial projection must lie in Ω (otherwise a smaller horoball would contain a point of S) and hence $1 - |R(z)| \leq \mathrm{diam(B)} \leq 2\mathrm{dist(z}, \partial\Omega)$ (here diam refers to Euclidean diameter). On the other hand, $R(z)$ must be on the part of ∂B visible from 0 (or else a smaller ball would hit S) and $1 - |R(z)| \geq \frac{1}{2}\mathrm{diam(B)}$. Furthermore, $\mathrm{diam(B)} \geq \frac{1}{2}\mathrm{dist(z}, \partial\Omega)$ or otherwise ∂B could not touch S. Thus we have shown $1 - |R(z)| \simeq \mathrm{dist}(z, \partial\Omega)$.

We claim the map R is a rough isometry between the hyperbolic metric ρ on Ω and ρ_S on S, i.e., it satisfies,

$$\frac{1}{A}\rho(z, w) - B \leq \rho_S(R(z), R(w)) \leq A\rho(z, w) + B,$$

for some $0 < A, B < \infty$ (this is also called a quasi-isometry by some authors). The right hand side holds with $A = 4$, $B = 0$ by Theorem 2.3.1 of [15] (i.e., R is Lipschitz). To prove the left hand side, it suffices to assume $\rho_S(R(z), R(w)) \leq 1$ and show $\rho(z, w) \leq B$. (To deduce the general case, connect $R(z)$ and $R(w)$ by a chain of $n \leq \rho_S(R(z), R(w)) + 1$ points distance ≤ 1 apart and apply the special case and the triangle inequality.) Let $r = 1 - |R(z)|$ and let γ be a path in S of length ≤ 1 between $R(z)$ and $R(w)$. The preimage of any point under R is a point or arc in Ω, so

$E = R^{-1}(\gamma)$ is compact and connected. Moreover, snce γ has hyperbolic diameter ≤ 1 in $\mathbb{B}^3$ and the estimates of the previous paragraph imply E has Euclidean diameter $\leq Cr$ and every point of E is at least distance $s = r/C$ from $\partial\Omega$. Thus its hyperbolic diameter is uniformly bounded. To prove this, choose a maximal collection of points $\{z_j\}$ in E with so that $|z_j - z_k| \geq s/4$ if $j \neq k$ and such that $\cup_j B_j = \cup_j B(z_j, s/2)$ covers E. Since the balls $B(z_j, s/8)$ are disjoint and all lie in a ball of radius $\simeq r \simeq s$ the number of points is uniformly bounded. On the other hand, the balls $\{B_j\}$ cover E and each has uniformly bounded hyperbolic diameter in Ω. Thus the same is true of E. Since $z, w \in E$, we see that $\rho(z, w) \leq B$, as desired. Thus R is a rough isometry.

Since Sullivan's map σ is biLipschitz, this implies that the composition $R \circ \sigma^{-1}$ is also a rough isometry. Since it is also the identity on the boundary a general result of Väisälä (Theorem 7.15 of [33]) implies $\rho_S(R(z), \sigma(z))$ is uniformly bounded and hence $1-|\sigma(z)| \simeq 1-|R(z)|$ (this can also be proven directly). Since $1-|R(z)| \simeq \text{dist}(z, \partial\Omega)$, this completes the proof. □

Since G' acts on S by isometries, it corresponds to some Fuchsian group H acting on the disk. Thus $\Psi = \iota \circ \sigma \circ \Phi$ is a quasiconformal map of the disk to itself which conjugates G to H. We may also assume $\Psi(0) = 0$. Since G is divergence type and divergence type is a quasiconformal invariant (e.g., Pfluger's paper [21]), H is also divergence type. Let Q and z be as in the statement of the lemma. Since Ψ is a quasiconformal map of the disk to itself, Lemma 7 implies that $\Psi(Q)$ contains a Carleson subsquare Q' of side length $\geq (1 - |w|)/\lambda$, where $w = \Psi(z)$. Since H is divergence type, the orbit of the origin is non-tangentially dense almost everywhere on the circle. To each orbit point $h(0)$, associate an interval I_h on the unit circle centered at $h(0)/|h(0)|$ and of length $\lambda(1 - |h(0)|)$. Since the orbit $H(0)$ is non-tangentially dense almost everywhere on the circle, these intervals form a Vitali covering [34] and hence we can extract a disjoint subcollection which is contained in $I = Q' \cap \mathbb{T}$ and covers almost every point of I. Let $\{w_j\}$ denote the corresponding orbit points. Then clearly

$$\sum (1 - |w_j|^2) \geq |I|/\lambda = (1 - |w|^2)/\lambda.$$

If $w_j = h(0)$, let $I_j = I_h$ be the corresponding interval. Thus if $z_j = \Psi^{-1}(w_j)$, and Q_j is the Carleson square with base $\Psi^{-1}(I_j)$, then $z_j \in Q_j \subset Q$ by Lemma 7. The squares Q_j are all subsets of Q since $I_j \subset \psi(I)$ and they are disjoint because the intervals I_j are disjoint. Thus it only remains to see that $\sum |\Phi'(z_j)|(1-|z_j|^2)$ has the correct estimate.

Now consider the points $x = \iota^{-1}(w)$ and $x_j = \iota^{-1}(w_j) \in S \subset \mathbb{H}^3$. Because ι is an isometry between $\rho_{\mathbb{D}}$ and ρ_S and since $\rho_S \geq \rho_{\mathbb{H}^3}$ by definition,

$$\sum_j \frac{1-|x_j|^2}{1-|x|^2} \geq C \sum_j e^{-\rho_{\mathbb{H}^3}(x,x_j)} \geq C \sum_j e^{-\rho_{\mathbb{D}}(w,w_j)} \geq C\lambda.$$

Lemma 8 says there is an $M < \infty$ so that

$$\frac{1}{M}(1-|\sigma(\Phi(z_j))|^2) \leq \operatorname{dist}(\Phi(z_j), \partial\Omega) \leq M(1-|\sigma(\Phi(z_j))|^2),$$
$$\frac{1}{M}(1-|\sigma(\Phi(z))|^2) \leq \operatorname{dist}(\Phi(z), \partial\Omega) \leq M(1-|\sigma(\Phi(z))|^2),$$

and hence

$$\sum_j \frac{\operatorname{dist}(\Phi(z_j), \partial\Omega)}{\operatorname{dist}(\Phi(z), \partial\Omega)} \geq \frac{1}{\lambda M^2} > 0,$$

By the Koebe distortion theorem

$$\sum_j \frac{|\Phi'(z_j)|(1-|z_j|^2)}{|\Phi'(z)|(1-|z|^2)} \geq \frac{1}{4C\lambda M^2} > 0,$$

as desired.

This gives the first claim of the lemma with $\epsilon = (4C\lambda M^2)^{-1}$. By decreasing ϵ by a factor of 2 and using a finite subcollection of points we may also assume claim (2) of the lemma is satisfied with some η which depends on z. In fact, it is easy to see that η only depends on where the point z projects to in the quotient surface R and on the direction from z the origin, i.e., η only depends on the point of the unit tangent bundle of R corresponding to the point z in the disk. By taking ϵ and η slightly smaller it is easy to see that the same numbers work for an open neighborhood of this point in the unit tangent bundle. Since the set of points within a given distance of a base point on R is compact, we can take a finite open cover by such neighborhoods and see that η can be chosen uniformly for all such points. This gives the second part of the lemma and completes the proof of Theorem 1.

Remark: In [15] Sullivan's theorem is proven by first showing the nearest point retraction defines a biLipschitz map from Ω to S_ϵ, the surface at distance ϵ from S (the retraction onto S itself is 4-Lipschitz but need not be 1 to 1). Then S_ϵ is mapped to S by averaging the retraction map over intervals in a certain foliation of the surface.

This is rather technical and is the hardest part of the paper. However, this second part can be avoided by applying other known results. As noted in the proof of Lemma 8 the nearest point retraction R from Ω to S is a rough isometry. Thus $\iota \circ R \circ \Phi$ is a rough isometry of the hyperbolic disk to itself and so extends to a quasisymmetric homeomorphism of the circle (e.g., see [20], or Theorem 7.15 [33]). Moreover, since this map conjugates G to H, so does the boundary map, and if we use the Douady-Earle [13] method to extend the boundary map to a quasiconformal mapping of the disk in a conformally natural way, we get a quasiconformal conjugacy Ψ from G to H. The Douady-Earle extension is also biLipschitz between hyperbolic metrics, giving Sullivan's theorem. For a more detailed discussion of Sullivan's theorem and its applications, see [5]. For example, it is shown there that knowing Sullivan's theorem with QC constant $K = 2$ would imply the Brennan conjecture.

5. The converse direction

As mentioned in the introduction, Astala and Zinsmeister [3] proved the converse of Theorem 1: if G has the Bowen property then it must be divergence type. For the sake of completeness, we sketch their argument.

Suppose G is convergence type and choose a deformation Φ which is conformal in $\mathbb{D}$ and quasi-conformal in the complement and whose complex dilatation $\mu = \Phi_{\bar{z}}/\Phi_z$ is supported in a disk (modulo G) and has small L^∞ norm. Since G is convergence type, a result of Jones and Marshall [18] implies (with some work) that

$$d\nu = \frac{|\mu(z)|^2 dxdy}{1 - |z|^2},$$

is a Carleson measure of small norm. One can then use a result of Semmes [26] to deduce that the image of the circle is a chord-arc curve (hence rectifiable). Finally, a result of Bers [4] is used to show that μ can be chosen so that Φ is not Möbius, which completes the proof.

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