Consider a point $z \in M$ in a hyperbolic n-manifold and the geodesic rays based at this point. These are parameterized by points on S^{n-1}.

Λ_b the set of geodesics rays which remain bounded.

Λ_c the set of geodesic rays which return to some compact set infinitely often. On hyperbolic ball, these correspond to points on boundary which are non-tangential limits of orbit points.

The limit set is the whole accumulation set of the orbit points.

$\Omega = \Lambda^c$ is the ordinary set, i.e., the stable escaping geodesics.

$\Lambda_e = \Lambda \setminus \Lambda_c$; these leave every compact set eventually.

Λ_α geodesics such that

$$\liminf_{t \to \infty} \frac{1}{t} \mathrm{dist}(\gamma(\mathrm{t}), \mathrm{z}) \geq \alpha.$$

$\Lambda_\ell = \cup_\alpha \Lambda_\alpha$ is the linear escape set.

$\Lambda_s = \Lambda \setminus \Lambda_\ell$ is the sublinear limit set

Suppose G is a discrete group of isometries on hyperbolic space. The **critical exponent**, δ, is the exponential growth rate of the orbital counting function

$$\delta = \limsup_n \frac{\log N_n}{\log n}$$

where $N_n = \#\{g \in G : n \leq \rho(0, g(0)) < n+1\}$.

Equivalently

$$\begin{aligned}\delta &= \inf\{s : \sum_{g \in G} \exp(-s\rho(0, g(0))) < \infty\} \\ &= \inf\{s : \sum_{g \in G} (1 - |g(0)|)^s\} < \infty.\end{aligned}$$

Theorem: If G has more than one limit point, $\delta = \dim(\Lambda_b) = \dim(\Lambda_c) = \dim(\Lambda_s)$.

Corollary: In a hyperbolic n-manifold there is either a $n-1$ dimensional set of geodesic rays that remain bounded, or a $n-1$ dimensional set of rays that escape to infinity at linear speed.

The φ Hausdorff content of E is

$$H^{\varphi}(E) = \inf\{\sum \varphi(r_j) : E \subset \cup D(x_j, r_j), r_j \leq \delta\}.$$

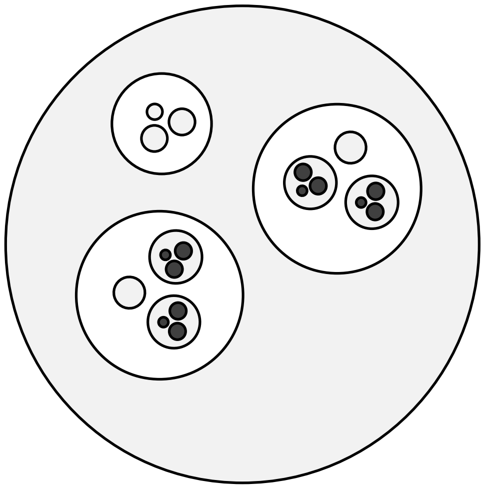
If $\varphi(t) = t^{\alpha}$ we write H^{α}. The Hausdorff dimension is

$$\dim(E) = \inf\{\alpha : H^{\alpha}(E) = 0\}.$$

Mass distribution principle + Frostman's Lemma: E has positive H^{φ} content iff E supports a measure such that $\mu(D(x,r)) \leq \varphi(r)$.

For example, suppose we build a Cantor set by inductively replacing a disk D by a disjoint collection of subdisks D_j with

$$\sum \mathrm{diam(D_j)}^{\alpha} = \mathrm{diam(D)}^{\alpha}.$$



Then the nth generation gives a cover so that

$$\sum \mathrm{diam(D_j)}^{\alpha+\epsilon} \to 0,$$

so $\dim(K) \leq \alpha$. On the other hand, setting $\mu(D_j) = \mathrm{diam(D_j)}^{\alpha}$, gives a measure which implies $\dim(K) \geq \alpha$.

Proof that $\dim(\Lambda_s) \leq \delta$**:**

It suffices to show $\delta \geq \dim(\Lambda^c_\alpha)/(1+\alpha)$ (take $\alpha \to 0$).

Fix $\beta > \alpha$ and $s < \dim(\Lambda^c_\alpha)$ and let μ be a s-Frostman measure on Λ^c_α. Cover Λ^c_α disks D so that

$$\rho(z_D, G(0)) \leq \beta\rho(z_D, 0).$$

Extract (by Vitali) a disjoint, full measure, subcollection $\{D_n\}$ and choose $w_n \in G(0)$ so that $\rho(z_n, w_n) \leq \beta\rho(z_D, 0)$. Elementary estimates show D_n is inside a ball of radius $\simeq (1-|w|)^{1/(1+\beta)}$ around w_n^*. Let $\mathcal{W}$ be the collection of orbit points chosen and for $w \in \mathcal{W}$ let $C(w)$ be the disks associated to w.

Then

$$\begin{aligned}\sum_{D_n \in \mathcal{C}(w)} \mu(D_n) &\leq \mu(D(w^*, Cr_w)) \\ &\leq C(r_w)^s \\ &\leq (1-|w|)^{s/(1+\beta)},\end{aligned}$$

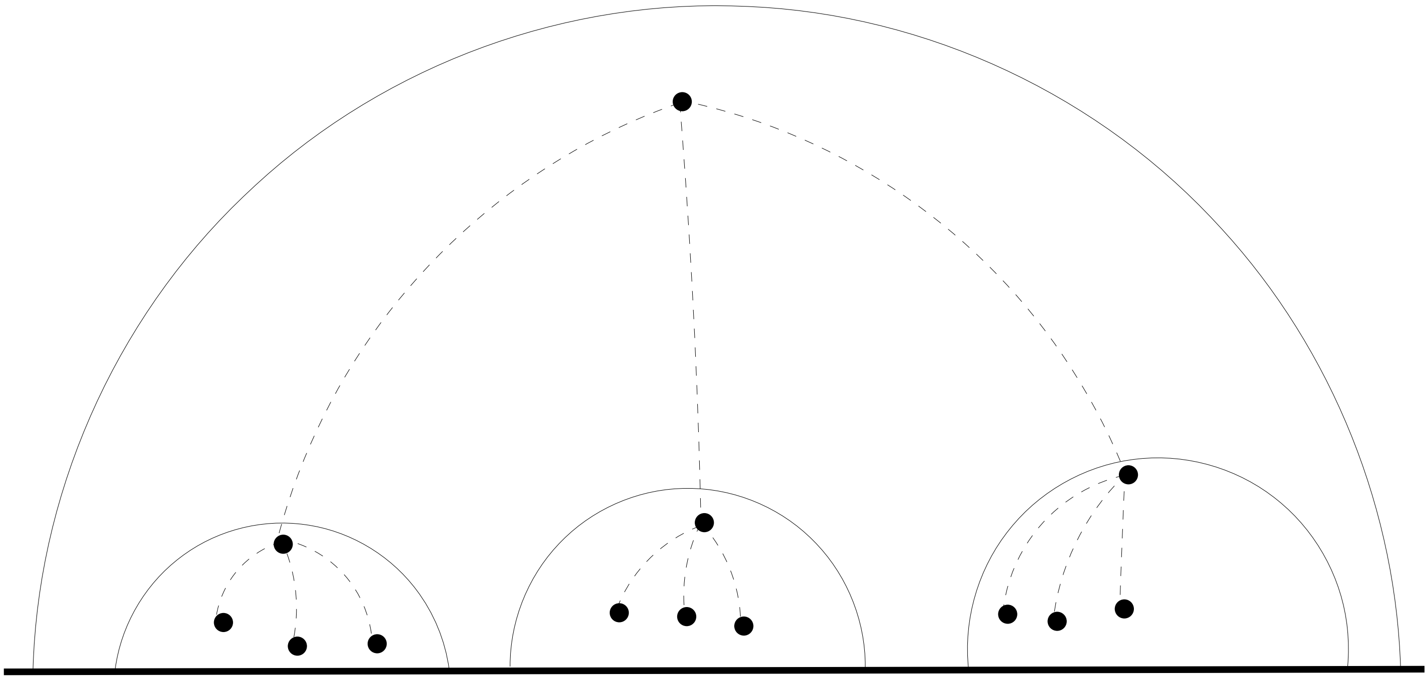
$$\sum_{w \in \mathcal{W}} (1-|w|)^{\frac{s}{1+\beta}} \geq \|\mu\|/C > 0.$$

We can inductively make infinitely many such collections which means the Poincaré series diverges at $s/(1+\beta)$.

Taking $s \nearrow \dim(\Lambda_\alpha^c)$ and $\beta \searrow \alpha$ shows $\delta \geq \dim(\Lambda_\alpha^c)/(1+\alpha)$.

Proof that $\delta \leq \dim(\Lambda_b)$:

To prove $\delta(G) \leq \dim(\Lambda)$, we fix $\epsilon > 0$ and build a Cantor K set in Λ with $\dim(K) > \delta - \epsilon$. Our disks will be the bases of "tents" with vertices at orbit points.



Let $\mathcal{C}(z)$ denote the immediate children of z in the tree. We show there are θ, N so that for any $\epsilon > 0$ we can build a tree so that if $w \in \mathcal{C}(z)$,

1. The geodesic from z to w makes a positive angle θ with the radial segment $[0, z]$.
2. $w \in B(z, N(1-|z|))$.
3. $C^{-1} \leq (1-|w|)/(1-|z|) \leq 1/2$.
4. For $w_1, w_2 \in \mathcal{C}(z)$ distinct, $B(w_1, 2N(1-|w_1|))$ and $\cap B(w_2, 2N(1-|w_2|))$ are disjoint.
5. $\sum_{w \in \mathcal{C}(z)} (1-|w|)^{\delta-2\epsilon} \geq C^2(1-|z|)^{\delta-2\epsilon}$,

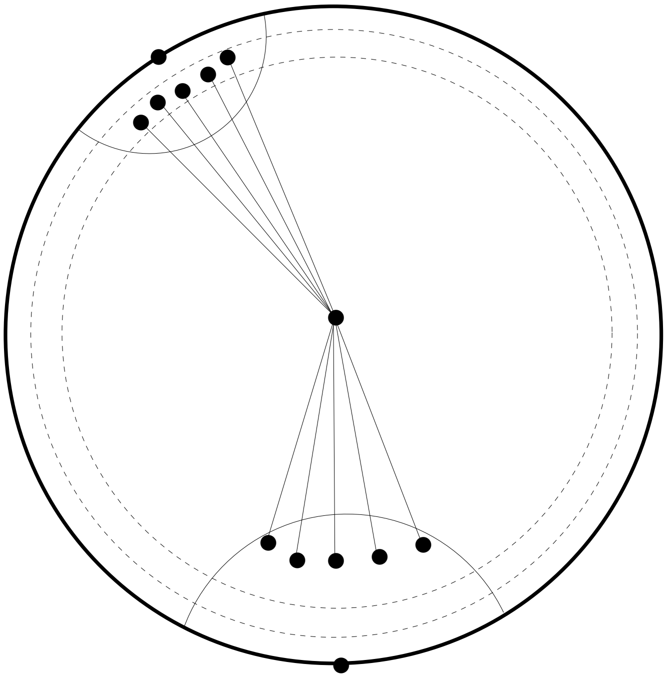
To construct tree we use fact G is non-elementary to find two disjoint Euclidean balls B_1, B_2 and $n \in \mathbb{N}$ so that

$$\sum_{g(0)\in V_i} (1 - |g(0)|)^{\delta - 2\epsilon} \geq M,$$

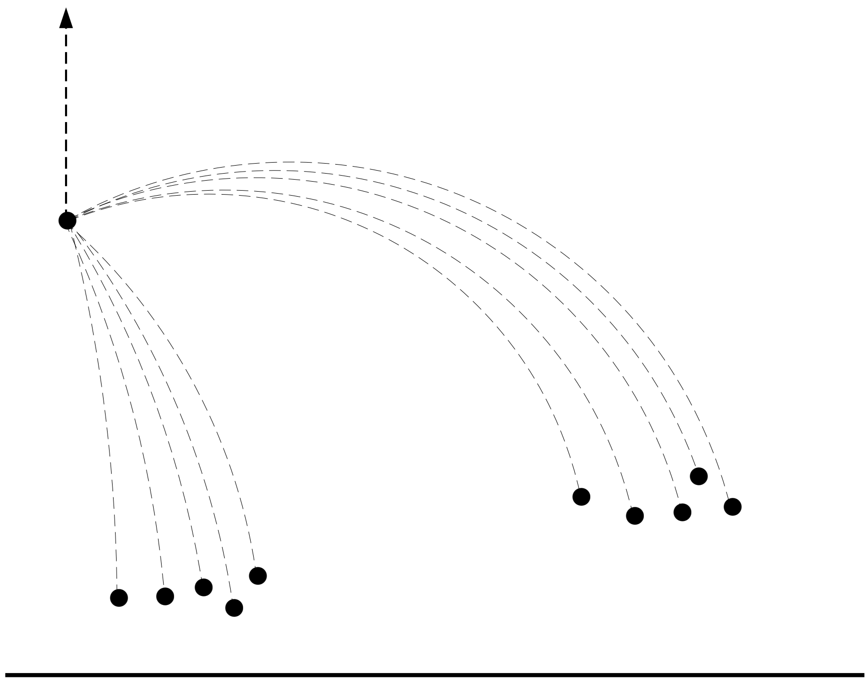
for some large M and where

$$V_i = \{g(0) : g(0) \in B_i \text{ and } 1 - |g(0)| \sim 2^{-n}\}.$$

These are first generation children.



In general, the children of $g(0)$ will be either $g(V_1)$ or $g(V_2)$, depending on which collection makes a positive angle with the radial segment $[0, g(0)]$. It is then an easy exercise in hyperbolic geometry to verfiy conditions (2)-(5).



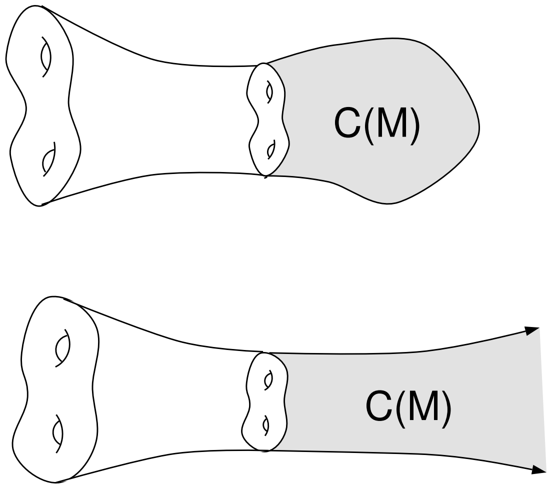
When is $\delta = \dim(\Lambda)$?

Let $C(\Lambda)$ be the convex hull of Λ in the hyperbolic metric. It is the complement of all hyperbolic half-spaces (hemispheres which miss Λ).

Λ has positive area iff a Brownian motion started in $C(\Lambda)$ stays in $C(\Lambda)$ until it hits S^2.

Let $M = \mathbb{B}/G$. The **convex core**, $C(M)$, is the image of $C(\Lambda)$ under the quotient map. Λ has positive area iff a Brownian motion started in $C(M)$ has a positive probability of never leaving $C(M)$.

G is called geometrically finite or geometrically infinite depending on whether $C(M)$ has finite or infinite volume (actually a unit neighborhood of $C(M)$).



Corollary (Ahlfors): If M has infinite volume, but $C(M)$ has finite volume then Λ has zero area.

Proof: If M has infinite volume, then for a fixed x the heat kernel $k(x, y, t)$ tends to zero pointwise at $t \to \infty$. It is bounded above by

$$C(\mathrm{vol}(B(y, 1))^{-1/2}$$

(known facts about thin parts implies this is in L^1) so the Lebesgue dominated convergence theorem implies

$$\lim_{t \to \infty} \int_{C(M)} k(x, y, t) dy = 0.$$

What happens if $C(M)$ has infinite volume, i.e., G is geometrically infinite? If $\delta = 2$ then $\dim(\Lambda) = \delta = 2$.

Theorem: If $C(M)$ has infinite volume, $\partial C(M)$ has finite area and $\delta < 2$ then Λ has positive area.

The proof has two steps:

1. If $\delta < 2$ then the heat kernel decays exponentially fast in time.

2. Exponential decay implies that if we start Brownian motion far enough inside $C(M)$ then with high probability it never leaves $C(M)$.

Ahlfors finiteness theorem says $\partial C(M)$ has finite area if G is finitely generated.

Step 1: If $n = \rho(0, y)$, then in the ball we have

$$K(0, y, t) \simeq t^{-3/2} n \exp(-t - n - \frac{n^2}{4t}).$$

The heat kernel for M is

$$\begin{aligned} k(x, x, t) &= \sum_G k(0, g(0), t) \\ &\simeq t^{-3/2} \sum_n N_n n e^{-t-n-\frac{n^2}{4t}} \end{aligned}$$

Now use $N_n \leq C_x e^{\delta n}$ and complete the square to get

$$k(x, x, t) \leq C_x e^{-\delta(2-\delta)t} . t^{-3/2} \sum_n n e^{-\frac{1}{t}((1-\delta)t - \frac{n}{2})^2}$$

This is $\leq C_x e^{-\lambda t}$, where $\lambda = \delta(2 - \delta)$. The semigroup property and Cauchy-Schwarz give

$$k(x, y, t) \leq C_x C_y e^{-\lambda t/2}.$$

Step 2: Let U be a unit neighborhood of $\partial C(M)$ and choose T so large that

$$\int_T^\infty \int_U k(y,x,t)dxdt \leq Ce^{-\lambda T/2} < 1/4$$

Choose a starting point y in $C(M)$ which is distance R from $\partial C(M)$ and has injectivity radius bounded away from zero. If R is large then probability Brownian motion hits $\partial C(M)$ in under time T is small. On other hand, probability of hitting $\partial C(M)$ at a time $> T$ is at most $1/4$.

Theorem: If G is finitely generated Kleinian group and Λ has zero area then $\delta(G) = \dim(\Lambda)$.

Marden's Conjecture: If G is a finitely generated Kleinian group then $M = \mathbb{B}^3/G$ is topologically tame (i.e., is homeomorphic to the interior of a compact manifold with boundary).

$$\Downarrow$$

Ahlfors' Conjecture: If G is a finitely generated Kleinian group then either $\Lambda = S^2$ or Λ as zero area.

$$\Downarrow$$

Conjecture: If G is a finitely generated Kleinian group then $\delta = \dim(\Lambda)$.

What can we say when $\delta = 2$?

We assume that there is a lower bound for the injectivity radius on M and that $\partial C(M)$ is compact. Then there is a positive harmonic function on M which tends to zero in the geometrically finite ends and has bounded gradient.

The lift to the ball is a positive harmonic function which must be the Poisson integral of a positive measure μ on the limit set.

Since U has bounded (hyperbolic) gradient

$$U(z) \le C\rho(0, z) \le C \log \frac{1}{1 - |z|}.$$

Thus

$$\mu(D(x, r)) \le Cr^2 \log \frac{1}{r}.$$

Thus Λ has positive measure for $\varphi(t) = t^2 \log 1/t$. This argument is due to Dennis Sullivan.

What about an upper bound? For the "half cylinder", here is a heuristic argument. The heat kernel in $C(M)$ should behave like the heat kernel on a half line, i.e.,

$$\int_{C(M)} k(x_0, y, t)dy \simeq t^{-1/2}.$$

On the ball this means that

$$\text{area}(C(\Lambda) \cap \{|z| = 1 - e^{-t}\}) \simeq t^{-1/2}.$$

Thus $e^{2n}/\sqrt{n}$ disks of size e^{-n} will cover Λ.

Thus Λ has finite measure for $\varphi(t) = t^2\sqrt{\log 1/t}$.

To be more careful let $U_n(x) = U((1-e^{-n})x)$ and

$$E_n = C(\Lambda) \cap \{|z| = 1-e^{-n}\}).$$

Note

$$U \text{ positive} \quad \Rightarrow \quad \|U_n\|_1 \simeq 1.$$

$$|\nabla_H U| \text{ bounded} \quad \Rightarrow \quad \|U_n\|_{\mathrm{BMO}} \leq Cn^{1/2}.$$

$$\|U_n\|_2 \leq (\|U_n\|_1 \|U_n\|_{\mathrm{BMO}})^{1/2} \leq Cn^{1/4}.$$

Define the **square function**

$$SU_n(x) = (\int_{\Gamma(x)} |\nabla_H U_n|^2 y dV)^{1/2}.$$

Since $|\nabla U| > c > 0$ on average in $C(M)$, $SU_n \geq n^{1/2}$ on E_n, so

$$\|U_n\|_2 \simeq \|SU_n\|_2 \geq C\mathrm{area}(E_n)^{1/2} n^{1/2}.$$

Thus

$$\mathrm{area}(E_n) \leq Cn^{1/2}.$$

Theorem: Suppose $\partial C(M)$ is compact, M has injectivity radius bounded away from zero, and G is geometrically infinite. Then Λ has positive Hausdorff measure with respect to the function

$$\varphi(t) = t^2 \sqrt{\log 1/t \log\log\log 1/t}.$$

If, in addition, G is topologically tame then Λ has finite measure for this function.

In fact the proof shows $H^{\varphi}(\Lambda \setminus \Lambda_c) > 0$.

The lower bound is fairly easy, but the upper bound it a little more complicated.

To prove the lower bound, one needs to show that if U is the Poisson integral of a positive measure μ and has bounded hyperbolic gradient then

$$\limsup_{r\to 1} \frac{U(rx)}{\sqrt{\log 1/(1-r) \log\log\log 1/(1-r)}} < \infty.$$

for μ almost every x.

This is fairly standard argument using estimate

$$\int_{S^2} \exp(C_1 U_n(x)^2/n) dx \le C_2,$$

where $U_n(x) = U((1-e^{-n})x)$. The estimate follows from Gaussian estimates on martingales with bounded steps.

Tchebyshev's inequality and Borel-Cantelli show estimate holds Lebesgue a.e. A little more care gives μ almost everywhere.

The proof of the upper bound is a stopping time construction of a good cover for Λ. For $n_0, \dots, N$ we look at dyadic squares of size 2^{-2^n} which hit Λ. If

$$\mu(Q) \geq \epsilon |Q|_2 \sqrt{2^n \log\log n} = \varphi(\mathrm{diam}(Q))$$

then we put Q in our cover. For such squares the φ-sum is bounded by $\mu(\Lambda)/\epsilon$. which is finite.

The remaining squares of size 2^{-2^N} correspond to rays along which for $2^{-2^{n_0}} \geq 1 - r \geq 2^{-2^N}$,

$$C < U(rx) \leq \epsilon \sqrt{\log 1/(1-r) \log\log\log 1/(1-r)}.$$

We show the φ sum of these tends to zero with N.

Main tools are the square function estimates for L^2, L^4 and BMO norms and the fact that $|\nabla_H U|$ is large everywhere in $C(M)$.

This is the analog for harmonic functions of the following question for a random walk on the integers: what is the probability that a random walk satisfies

$$1 \le S_k \le \epsilon \sqrt{k \log \log k}$$

for all $2^{n_0} \le k \le 2^N$.

The probability of $S_k \ge 1$ is $\simeq 2^{-N/2}$.

The probability of $S_k \le \epsilon \sqrt{k \log \log k}$ is less than N^{-4} if ϵ is small.

The two events are negatively correlated, so estimates multiply. Thus μ measure of event is

$$\le 2^{-N/2} \cdot N^{-4} \cdot \sqrt{2^N \log N}$$

which is small.