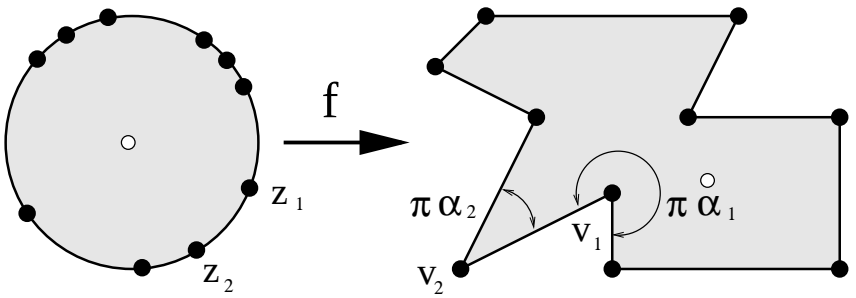
Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \mathbb{D} \to \Omega$.



The Schwarz-Christoffel formula gives a formula for the Riemann map of the disk onto a polygonal region Ω: if the interior angles of P are $\alpha\pi = \{\alpha_1\pi, \ldots, \alpha_n\pi\}$, i.e.,

$$f(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$

This maps the disk to a polygon with the correct angles but the edge lengths depend on the parameters $\mathbf{z} = \{z_1, \ldots, z_n\}$. How to find $\mathbf{z}$ in practice?

Theorem: There is a $C < \infty$ so that if Ω is bounded by a simply polygon P with n vertices we can find points $\mathbf{w} = \{w_1, \ldots, w_n\} = \iota(\mathbf{v}) \subset \mathbb{T}$ so that

1. $\mathbf{w}$ can be computed in at most Cn steps.
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \log 8$, $\mathbf{z} =$ conformal prevertices.

$d_{QC}(\mathbf{w}, \mathbf{z})$ is the infimum of $\log K$ such that there is a K-quasiconformal $h : \mathbb{D} \to \mathbb{D}$ such that $h(\mathbf{z}) = \mathbf{w}$.

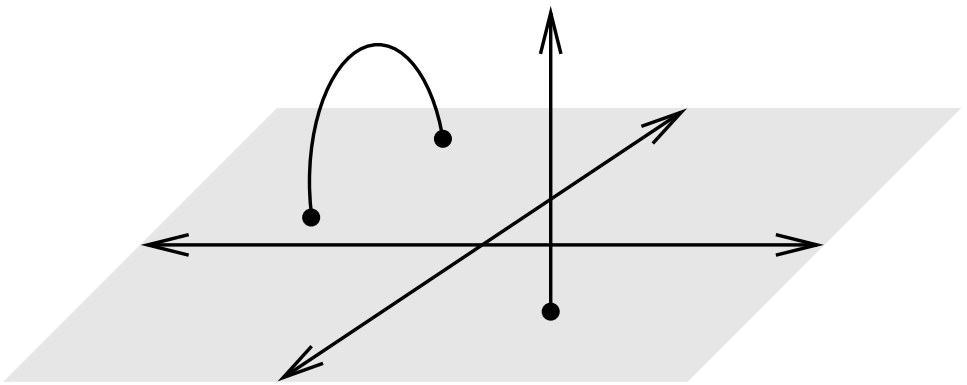
This result has its origin in a theorem about convex sets in hyperbolic 3-manifolds.

Theorem: If M is a hyperbolic 3-manifold and $C(M)$ is the convex hull of the closed geodesics in M, then there is a biLipschitz map between $\partial_\infty M$ and $\partial C(M)$.

Hyperbolic space: The hyperbolic metric on the disk or ball is

$$d\rho = 2|dz|/(1 - |z|^2).$$

Geodesics are circles orthogonal to the boundary The hyperbolic metric on a simply connected domain plane Ω is defined by transferring the metric on the disk by the Riemann map.



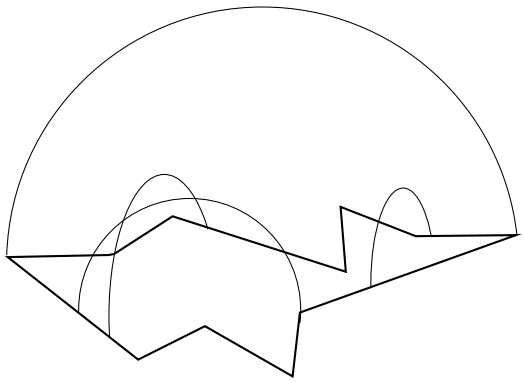
The hyperbolic metric on the upper half space

$$d\rho = |dz|/\mathrm{dist}(z, \mathbb{R}^2).$$

A **Kleinian group** is a discrete group of isometries of hyperbolic space. A hyperbolic manifold is of the form $M = \mathbb{B}/G$, G a Kleinian group.

An orbit accumulates on a set $\Lambda \subset S^2 = \partial\mathbb{B}$ called the limit set. The complement $\Omega = S^2 \setminus \Lambda$ is called the ordinary set and $\partial_\infty M = \Omega/G$ is a Riemann surface.

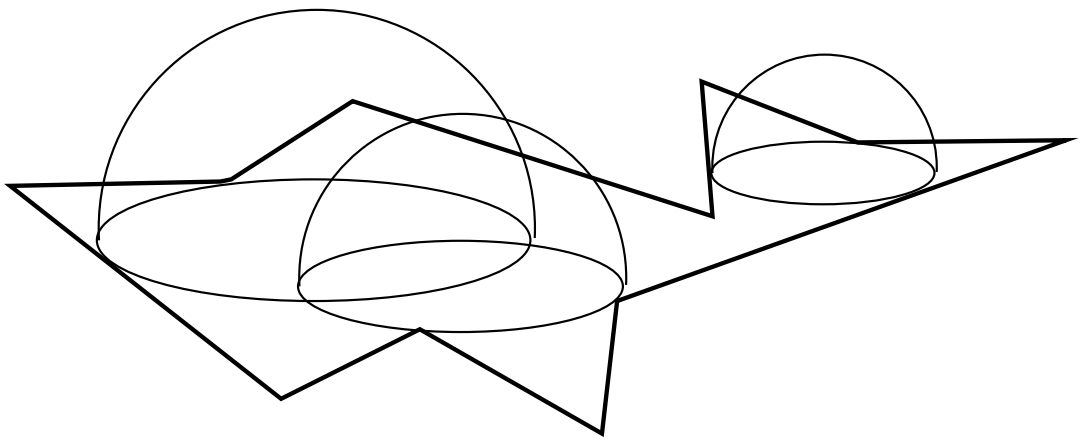
Given $E \subset \mathbb{R}^2$ let $C(E) \subset \mathbb{H}^3$ be the hyperbolic convex hull of E, i.e. smallest convex set in $\mathbb{R}^3_+$ which contains all the infinite hyperbolic geodesics with both endpoints in E.



If $E = \Lambda$ then $C(\Lambda)/G$ is **convex core** of M. Sullivan proved $\partial C(M) \simeq \partial_\infty M$. This is used in Thurston's hyperbolization theorem. $C(M)$ is of interest for other reasons, e.g., Ahlfors' conjecture is a claim about behavior heat kernels on $C(M)$.

Fortunately, $\partial C(M)$ is a very concrete object. In general $\partial C(E)$ has several boundary components, one for each component of $\mathbb{R}^2 \setminus E$. Suppose Ω is simply connected and $E = \partial\Omega$.

The **dome** of Ω is the boundary component of $C(E)$ which separates Ω from $C(E)$. It is also the boundary of the union of all hemispheres centered on $\mathbb{R}^2$ whose bases are disks contained in Ω.



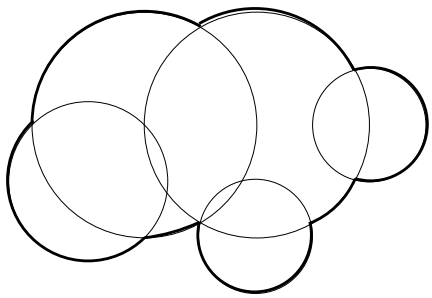
Similar to Euclidean space where complement of closed convex set is a union of half-spaces.

For each point on S there is (at least one) hyperbolic half-space H whose interior misses S, but whose boundary contains the point and an infinite geodesic in S. This is called a support plane and its intersection with S^2 is a disk in Ω. $\partial H \cap S$ is either an infinite geodesic (**bending line**) or a **geodesic face** bounded by geodesics.

Thus to compute the dome, we need only consider disks in Ω so that $I_D = \partial D \cap \partial\Omega$ has two or more points. The centers of such circles define the **medial axis** in computational geometry.

For polygons the medial axis is a tree. Disks such that $\#(I_D) > 2$ give rise to geodesic faces. Others are parts of 1-parameter families that sweep out regions on dome foliated by bending geodesics.

If Ω is simply connected and a finite union of disks we call it **finitely bent**. The dome is a finite union of geodesic faces which meet along infinite geodesics. Associated to each bending line is a non-negative number, the angle at which the two faces meet.

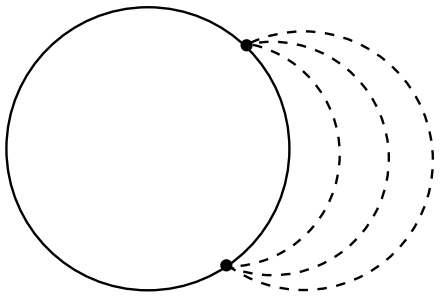


For a general simply connected domain the dome is still a union of geodesic faces (possibly empty) and bending geodesics (possibly uncountably many). There is **transverse measure** which says how much bending a transverse arc encounters as it crosses the bending lamination.

Let ρ_S be the hyperbolic path metric on S.

Theorem (Thurston): There is an isometry ι from (S, ρ_S) to the hyperbolic disk.

For finitely bent domains one can see this by simply rotating around each bending geodesic by an isometry to remove the bending. Doing this for all of them maps the surface isometrically to a hemisphere, which is isometric to the hyperbolic disk.



• Recall that a map is **biLipschitz** if

$$\frac{1}{A}d(x,y) \leq d(f(x,y)) \leq Ad(x,y).$$

• f is a **quasi-isometry** (or rough isometry) if

$$\frac{1}{A}\rho(x,y) - B \leq \rho(f(x), f(y)) \leq A\rho(x,y) + B.$$

• A homeomorphism is K**-quasiconformal** if for every $x \in \Omega$

$$\limsup_{r \to 0} \frac{\max_{y:|x-y|=r} |f(x) - f(y)|}{\min_{y:|x-y|=r} |f(x) - f(y)|} \leq K.$$

• A circle homeomorphism is **quasisymmetric** if

$$\frac{1}{M} \leq \frac{|f(I)|}{|f(J)|} \leq M$$

whenever I, J are adjacent intervals of equal length.

• BiLipschitz $\Rightarrow$ Quasiconformal

• 1-Quasiconformal $\Leftrightarrow$ conformal

• A quasi-isometry of the hyperbolic disk to itself has quasisymmetric boundary values.

• $h : \mathbb{T} \to \mathbb{T}$ is QS iff it has a QC extension iff it has a Q-I extension. Latter is usually easier to check.

• For self-maps of hyperbolic disk,
BiLipschitz $\Rightarrow$
Quasiconformal $\Rightarrow$
Quasi-isometry $\Rightarrow$
there is BiLipschitz map with same boundary values

Theorem (Sullivan, Epstein-Marden): There is a K_0 biLipschitz map $\sigma : \Omega \to S_\Omega$ so that $\sigma = \mathrm{Id}$ on $\partial\Omega = \partial S$. Thus σ is also K-QC for some K independent of Ω.

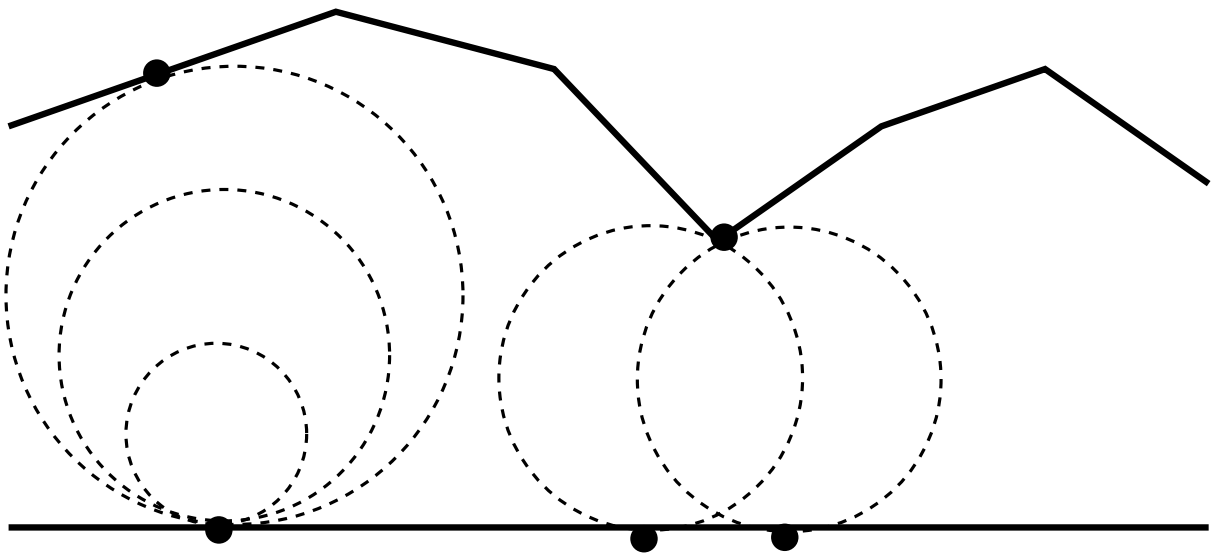
If Ω is invariant under a Kleinian group, so is σ.

Sullivan sketched proof for some invariant domains without explicit constant. Epstein-Marden proved it for all simply connected domains with $K \approx 80$. Best current bounds are $K \leq 8$ (7.82) and $K > 2.1$ (Epstein-Markovic).

I currently know of five proofs involving explicit constructions, Teichmüller theory, quasisymmetric mappings and martingales. I will give the "easy" one.

A proof of Sullivan's Theorem:

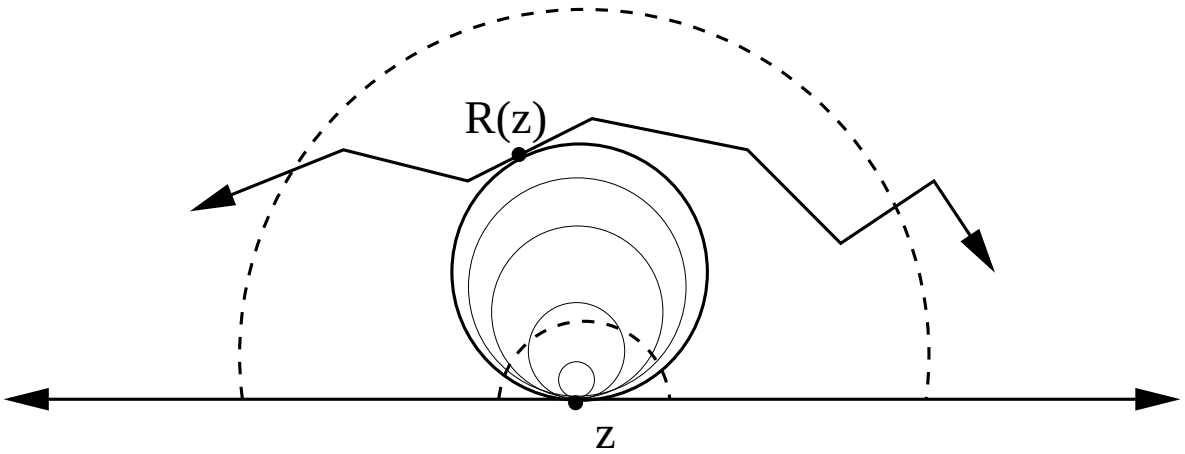
Define nearest point retraction $R : \Omega \to S$ by expanding horoball tangent at $z \in \Omega$ until it first hits S at $R(z)$. It suffices to show R is a quasi-isometry.



In general, the retraction is **not** 1-to-1 in crescents, so can't be QC.

Fact 1: If $z \in \Omega$,

$$\operatorname{dist}(z, \partial\Omega) \simeq \operatorname{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



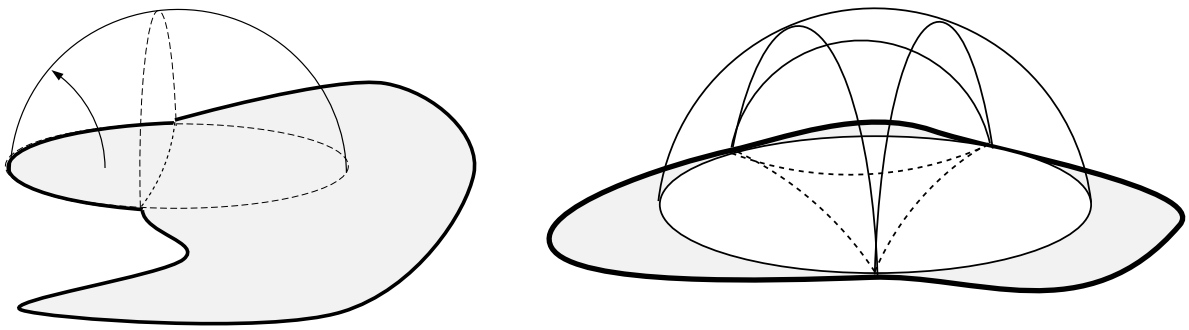
Fact 2: R is Lipschitz. For simply connected domain

$$d\rho \simeq \frac{|dz|}{\operatorname{dist}(z, \partial\Omega)},$$

and for disk $D \subset \Omega$,

$$\operatorname{dist}(z, \partial\Omega) \lesssim \operatorname{dist}(z, \partial D) \leq \operatorname{dist}(z, \partial\Omega)$$

for z in convex hull (in D) of $\partial D \cap \partial\Omega$.

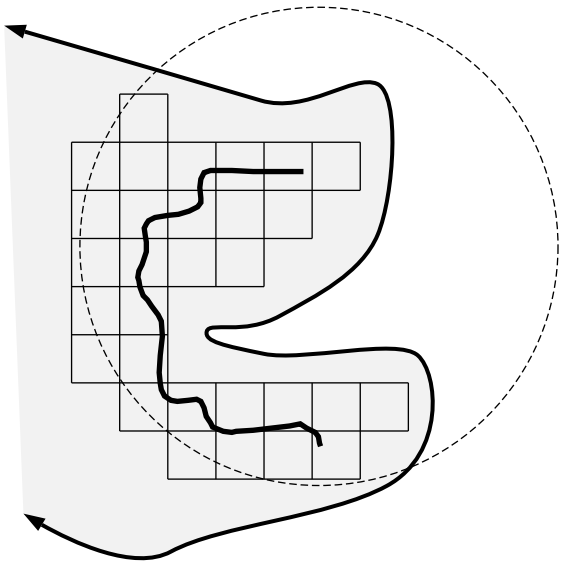


Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z, w) \leq C$.

Suppose $R(z)$ is at height r. Every point on S-geodesic from $R(z)$ to $R(w)$ has height $\simeq r$. Thus every preimage point v has

$$\text{dist}(v, \partial\Omega) \simeq r \simeq |v - z|.$$

Thus preimage of path is covered by $O(1)$ squares of hyperbolic diameter $\leq C$.

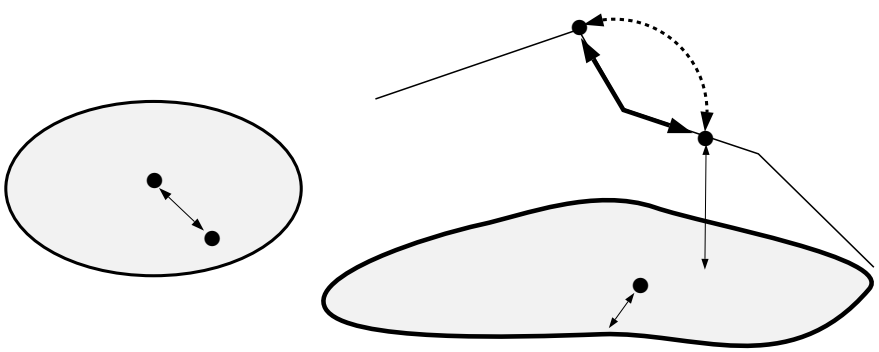


Moreover, $g = \iota \circ \sigma : \Omega \to \mathbb{D}$ is locally Lipschitz. Standard estimates show

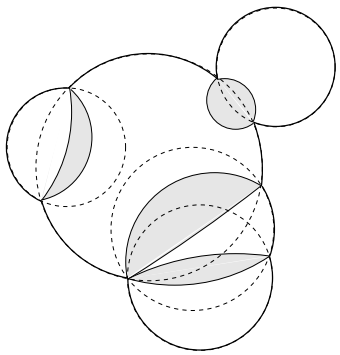
$$|g'(z)| \simeq \frac{\operatorname{dist}(g(z), \partial\mathbb{D})}{\operatorname{dist}(z, \partial\Omega)}.$$

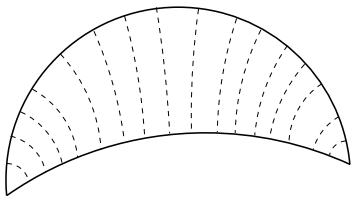
Use Fact 1

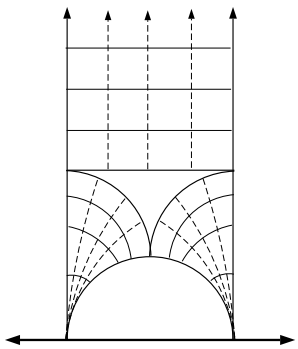
$$\begin{aligned}
\operatorname{dist}(z, \partial\Omega) &\simeq \operatorname{dist}(\sigma(z), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(\sigma(z), z_0)) \\
&\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \operatorname{dist}(g(z), \partial D)
\end{aligned}$$



Proof giving explicit constant: Divide finitely bent Ω into crescents and gaps. Foliate crescents by orthogonal circles. Triangulate gaps and extend foliation to gaps. Define explicit map of Ω into $\cup$gaps which locally QC and continuous when followed by $R : \Omega \to S$.







Requires explict upper bound of fraction of length that a foliation interval can spend in crescents.

Corollary: Every simply connected domain can be mapped to the disk by a locally Lipschitz homeomorphism.

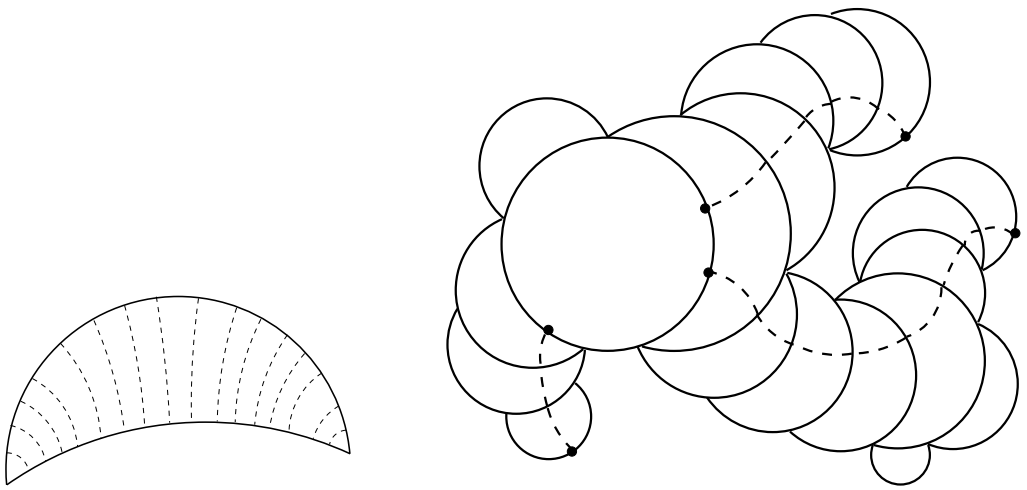
Corollary: Any conformal map $f : D \to \Omega$ can be written as $f = g \circ h$ where h is a K-QC self-map of D and $|g'|$ is bounded away from zero. Indeed $|g'(tz)| \leq C|g'(z)|$, $0 \leq t \leq 1$

Corollary: $K = 2$ in Sullivan's theorem implies Brennan's conjecture: for any conformal $f : \Omega \to \mathbb{D}$, $f' \in L^p(\Omega, dxdy)$ for all $p < 4$.

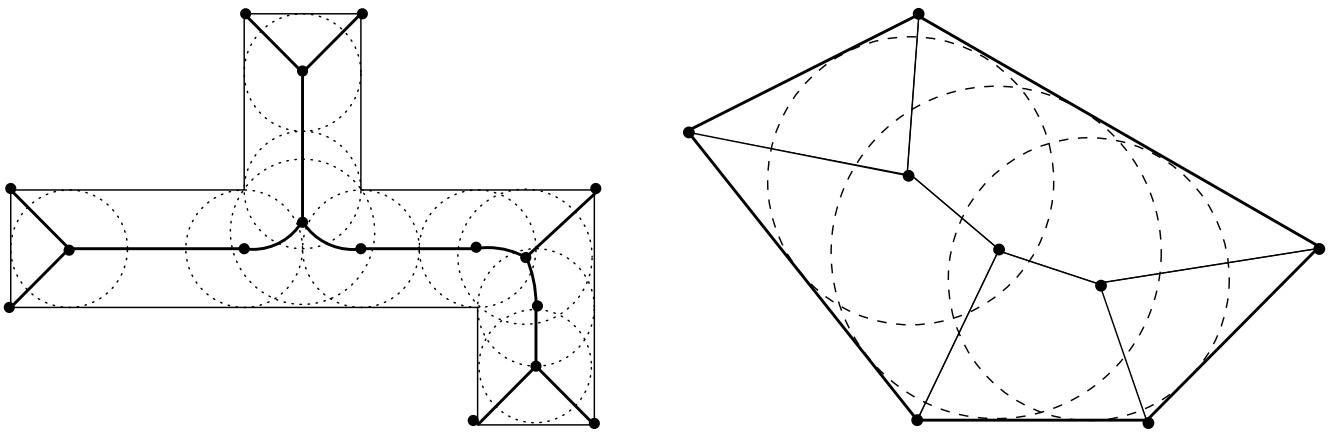
Epstein and Markovic showed $K > 2.1$.

Corollary: If Ω/G has no Greens function, then $\partial\Omega$ is either a circle or has dimension > 1.

Remember how we showed the dome is isometric to the hyperbolic disk by "unbending" one geodesic at a time? This also tells us how to compute $\iota : \partial\Omega \to \mathbb{T}$ for finitely bent domains. Write Ω as a disk and a union of crescents. Foliate crescents by circles passing though endpoints. The orthogonal foliation consists of circular arcs which are orthogonal to both boundary arcs of the crescent. Following leaves of this orthogonal foliation gives $\iota : \partial\Omega \to \partial D$.



The medial axis: The medial axis of a domain is the set of points in the interior which are equidistant from two or more boundary points.



Chin-Snoeyink-Wang proved the medial axis can be computed in time $O(n)$.

Medial axis is a type of Voronoi diagram, where sites are vertices and open edges and distance is measured along paths inside P. This constrained Voronoi diagram has extra edges at the concave vertices.

Medial axis is precisely the same as the centers of the bases of support planes for the dome of Ω. For polygons the medial axis is a finite tree. For general domains it is an $\mathbb{R}$-tree.

A $\mathbb{R}$-tree is a metric space in which any two points are joined by a unique path and this path is isometric to an interval in the real numbers.

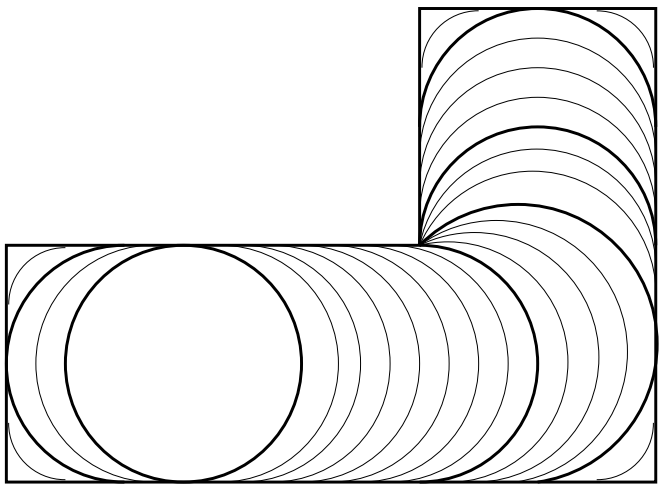
Bending measure on dome induces a metric on medial axis.

Applications of the medial axis:

- Analysis of chromosones
- Designs of type fonts
- Describe statisical features of porus materials
- Shape recognition
- Time critical collision detection
- Robotic motion
- Biological description of shape
- Mesh generation
- Computer vision
- Radiosurgery

However, medial axis is unstable under perturbations. For example, medial axis of a disk is a point, but for regular n-gon has n radial segments.

Medial axis flow: Fix a medial axis disk D_0. If D is a distinct medial axis disk $\Omega \setminus D$ has at least two components and exactly one of these intersects D_0. Take the boundary arcs of D corresponding to the components that don't hit D_0. This foliates $\Omega \setminus D_0$



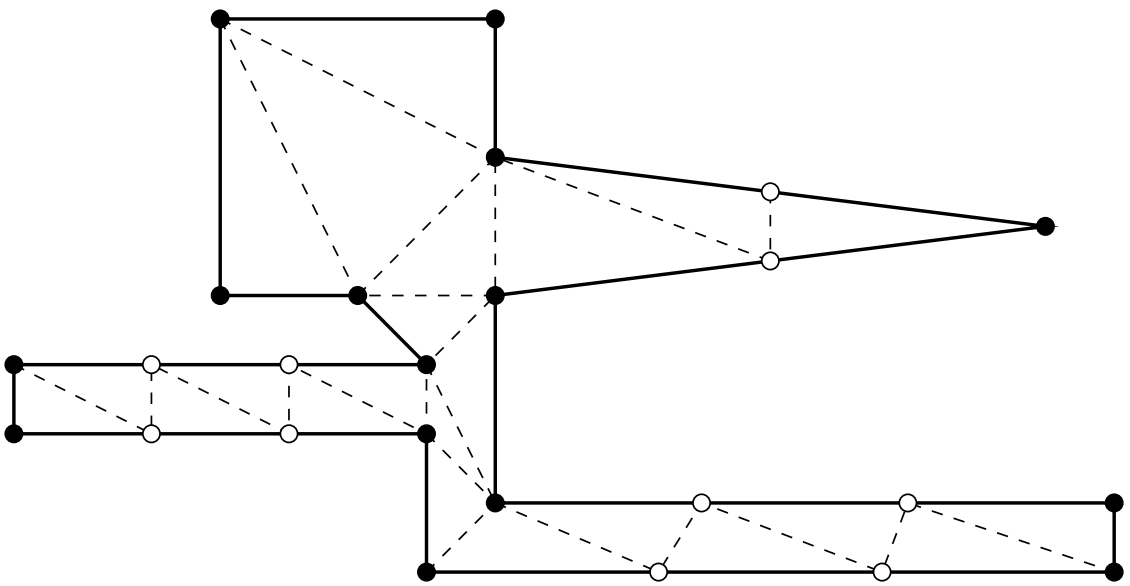
Following orthogonal flow to foliation gives $\iota : \partial\Omega \to \mathbb{T} = \partial D_0$. This map shrinks arclength and differs from Riemann map by a K-quasiconformal map (where is independent of Ω).

Theorem: There is a $C < \infty$ so that if Ω is bounded by a simply polygon P with n vertices we can find points $\mathbf{w} = \{w_1, \ldots, w_n\} = \iota(\mathbf{v}) \subset \mathbb{T}$ so that

1. $\mathbf{w}$ can be computed in at most Cn steps.
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \log 8$, $\mathbf{z} =$ conformal prevertices.

This theorem was motivated by 1998 paper of Toby Driscoll and Steve Vavasis "Numerical conformal mappings using cross ratios and Delaunay triangulations". They define an mapping which turns out to be the ι map for a domain assoicated to P.

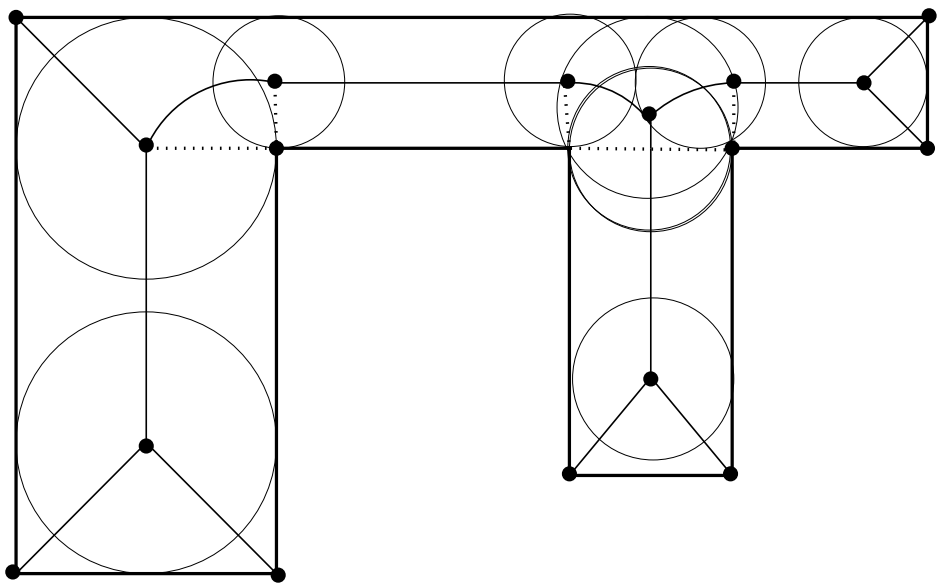
However, this approximation requires new vertices to be added to P (arbitarily many depending on the geometry), so their method is not $O(n)$.



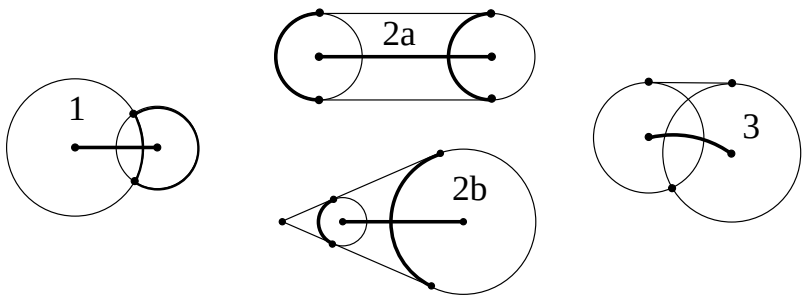
If we know the medial axis we can compute $\iota(\mathbf{v})$ in $O(n)$ steps:

• Compute the medial axis (which is a tree) and fix a root vertex.

• For each vertex of medial axis find the unique adjacent vertex closer to the root (the "parent"). Include vertices of P as leaves of medial axis tree. Also choose 3 reference points on boundary of each non-trivial disk.



• Define a certain explicit Möbius transformation between each disk and its parent. Rrecord cross ratios of images of reference points with respect to reference points for parent.



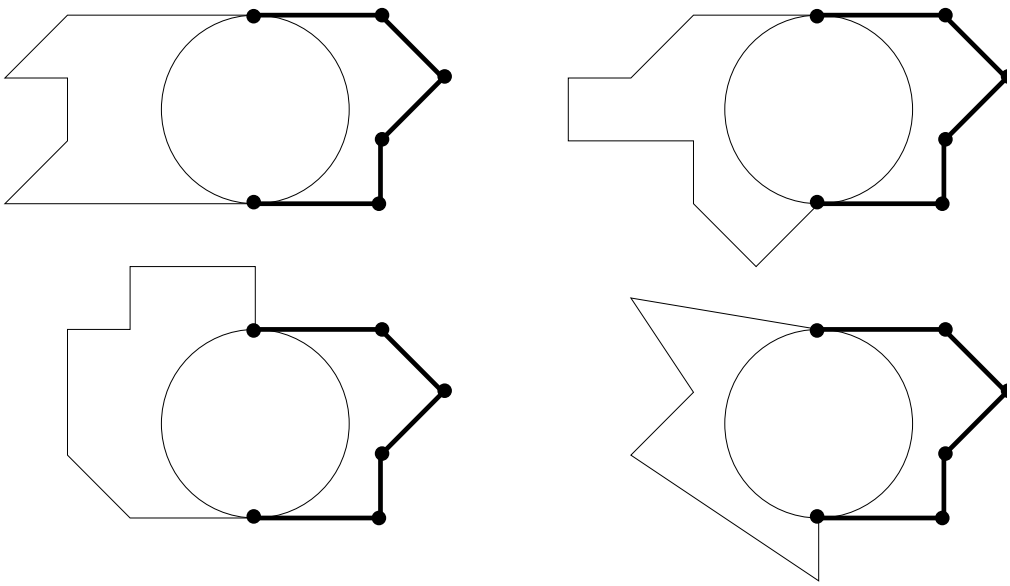
• ι is defined by composing the maps along path from leaf to root.

• Working from the root outwards, we compute images of reference points using recorded cross ratio information. When we reach a vertex of P we ave ι. Time is $O(n)$.

Other properties of ι

• The map ι shrinks arclength along P. (Collapsing a crescent decreases arclength on 'outer' edge; this gives finitely bent case and pass to limit for general case.)

• If $\gamma \subset \partial\Omega$ is an arc whose endpoints lie on $\partial D \cap \partial\Omega$ for some open disk $D \subset \Omega$, then the images of vertices on γ can be chosen to depend only γ and not on the rest of $\partial\Omega$.

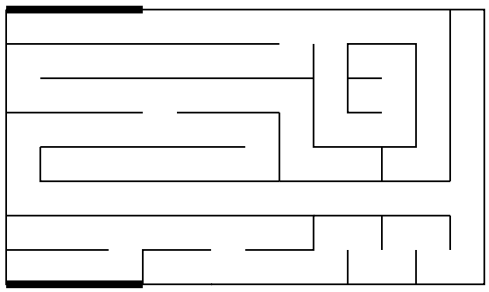


• If $\mathbf{v} = \{v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}\} \subset \partial D \cap \partial\Omega$ for some open disk $D \subset \Omega$ then the cross ratio of $\iota(\mathbf{v})$ equals the cross ratio of $\mathbf{v}$.

• For any four vertices $v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}$ of P the corresponding true and approximate prevertices on $\mathbb{T}$ satisfy

$$\frac{1}{8} \leq \frac{\mathrm{Mod}_{\mathbb{D}}(z_{j_1}, z_{j_2}, z_{j_3}, z_{j_4})}{\mathrm{Mod}_{\mathbb{D}}(w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})} \leq 8,$$

where Mod denotes conformal modulus on the disk.



There is an $O(n)$ algorithm to compute the medial axis of an n-gon. There are much simplier $O(n \log n)$ methods which may be faster in practice.

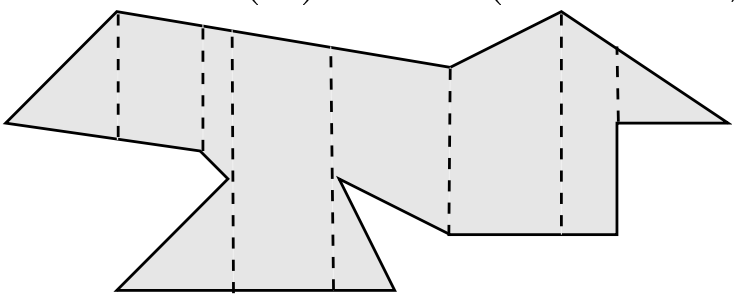
It is equivalent to compute the internal Voronoi diagram for P where the sites are the vertices and open edges. A key step is the

Merge Lemma: Suppose n sites S are divided into S_1 and S_2 by a line and that the Voronoi diagrams of S_1 and S_2 are given. Then the Voronoi diagram of S can be found in at most $O(n)$ additional time.

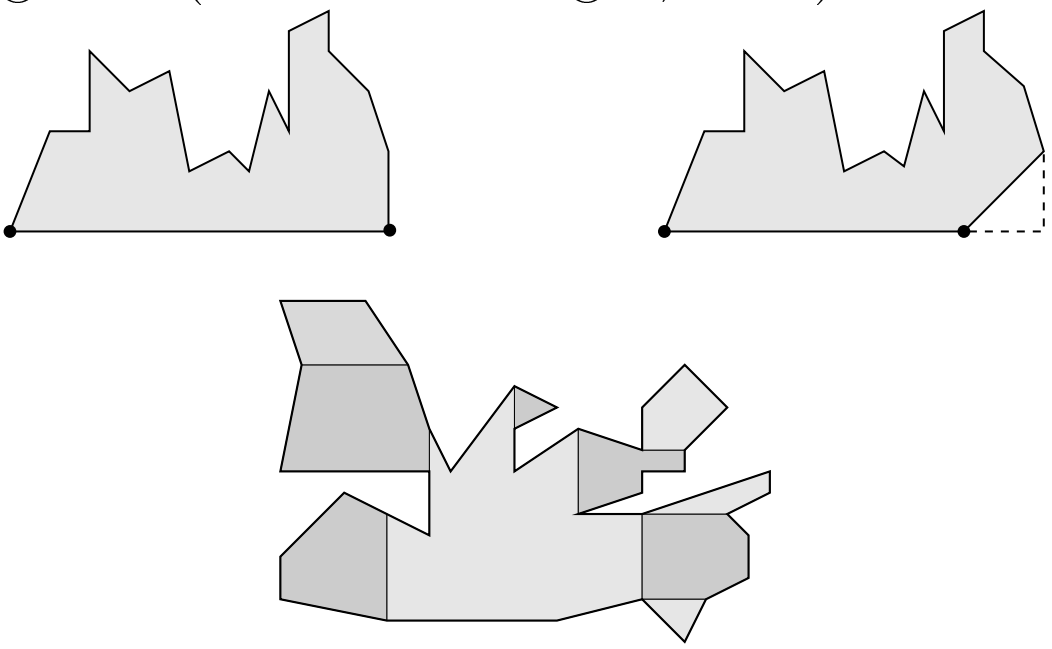
Sort x-coordinates of vertices and group into vertical slabs. Recursively dividing P into two almost equal sized pieces gives a $O(n \log n)$ algorithim (Yap 1993).

The $O(n)$ estimate is more involved. The basic steps are (Chin-Snoeyink-Wang, 1998)

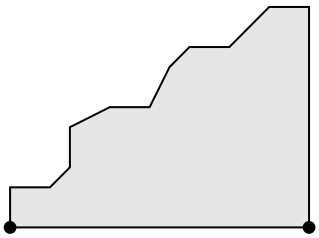
• Cut interior of P into trapedzoids with vertical sides. Possible in $O(n)$ time (Chazelle, 1991).



• Use trapezoids to divide P into pseudo-normal histograms (Klein and Lingas, 1993).



• Cut each pseudo-normal histogram into monotone histograms.



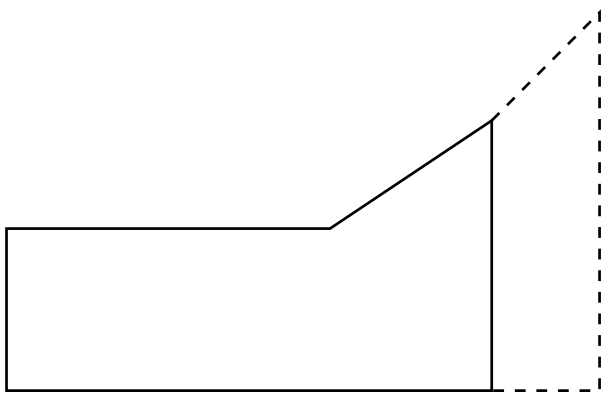
• Compute Voronoi diagrams of monotone histograms (Aggarwal, Guibas, Saxe, Shor, 1989 and Djidev, Lingas 1991)

• Merge monotone diagrams into diagrams for pseudo-normal histograms and merge the results into diagram for P.

Davis's algorithm: Suppose P is a polygon with vertices $\mathbf{v} = \{v_1, \ldots, v_n\}$ and $\mathbf{w} = \{w_1, \ldots, w_n\} \in \mathbb{T}^n$ is the current guess for the prevertices. Apply Schwarz-Christoffel to $\mathbf{w}$ and let $\mathbf{v}' = \{v'_1, \ldots, v'_n\}$ be resulting vertices. Define $\mathbf{w}'$ by

$$|w'_k - w'_{k+1}| = k|w_k - w_{k+1}| \frac{|v_k - v_{k+1}|}{|v'_k - v'_{k+1}|}$$

where k is a normalizing factor. Often works in practice but must fail in some cases.



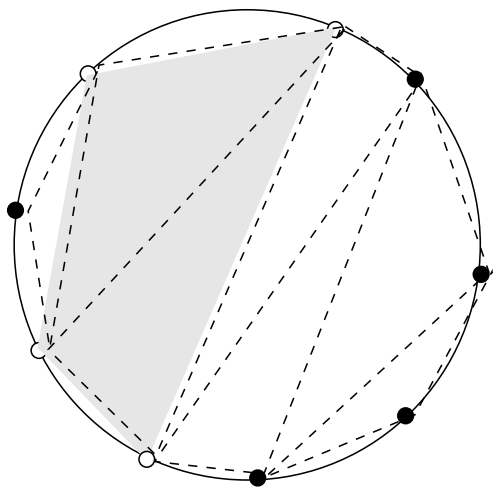
Can we do better using medial axis?

Coordinates on n-tuples: Let X_n denote the space of n-tuples of distinct points on the unit circle and let $\tilde{X}_n$ be X_n with any two Möbius equivalent n-tuples identified.

Give $\tilde{X}_n$ coordinates by taking a triangulation of the disk with the given vertices and using the $n-3$ numbers

$$\log \mathrm{Mod}_{\mathbb{D}}(z_{j_1}, z_{j_2}, z_{j_3}, z_{j_4})$$

for each 4-tuple corresponding to a pair of adjacent triangles.



Fix n angles and let S denote the Schwarz-Christoffel map from n-tuples on $\mathbb{T}$ to polygons determined by these angles. Consider $\Psi = \iota \circ S$ mapping n-tuples in the circle to n-tuples. Easy to check Ψ is well-defined on $\tilde{X}_n$.

Given a simple polygon with vertices $\mathbf{v}$ the conformal prevertices are a solution of $\Psi(\mathbf{z}) = \mathbf{w}$. Are they the only solution?

Is Ψ one-to-one? A diffeomorphism? Not hard to see it is onto since $\|\Psi(\mathbf{w}) - \mathbf{w}\| \leq C$.

Does $\|\Psi(\mathbf{w}) - \iota(v)\|^2$ have additional local minima?

The QC extension of ι has an explicit Beltrami coefficient. Can we improve our guess by quickly solving (or approximately solving) the Beltrami equation on the disk?

On the disk the Beltrami coefficient can be written as a sum $\mu = \sum \mu_n$ where each μ_n corresponds to an explicitly given circle homeomorphism. The composition of these maps may give a good approximation to the circle homeomorphism corresponding to μ, especially if all the norms are small.

If Ω is finitely bent we can multiply each crescent angle by a factor of $t \in [0,1]$ to get a surface Ω_t. Thus $\Omega_1 = \Omega$ and $\Omega_0 = \mathbb{D}$.

Then $(\iota_t)^{-1} \circ \iota_s$ is a map of $\partial\Omega_s$ to $\partial\Omega_t$ with QC extension of size $1 + C|s-t|$. This means the conformal prevertices are close in the d_{QC} metric.

Suppose we have a map F so that $d_{QC}(z,w) < \epsilon$ implies $d_{QC}(z, F(w)) < \lambda\epsilon$ for some $\lambda < 1$, i.e., we can solve for conformal pervertices $\mathbf{z}$ by iteration if we start close enough.

Choose $t_0 = 0 < t_1 < \ldots t_N = 1$ and let $\Omega_k = \Omega_{t_k}$. Choose points so prevertices differ by $\epsilon/2$ in d_{QC} metric. Iterate F to get within $\epsilon/2$ of kth set of prevertices then use this as initial guess for next set. Leads us to correct prevertices in $C/(\epsilon \log(1/\lambda))$ steps.

• What is best K in Sullivan's theorem. I do not know a way to compute K directly, but given a polygon we could compute the conformal and ι images for various 4-tuples of vertices and comparing these would give a lower bound for K.

• Can we get a "simple" map that approximates the conformal prevertices within d_{QC} distance ϵ in time $O(n \log \epsilon)$?

• Driscoll and Vavasis report that the iteration

$$w_{n+1} = w_n + (\iota(v) - \Psi(w_n)),$$

converges to the true prevertices in practice. This would be true if the derivative of Ψ was close to the identity. Is this often true? Is there an alternate explaination for why the iteration converges?

What works in $\mathbb{R}^3$?

If Ω is obtained by adding non-overlapping spherical caps repeated onto a ball then the medial axis will still be a tree and the proof should be unchanged: such a domain can be mapped to a ball by a uniformly QC map with a bounded derivative.

In general, medial axis of 3-D domain is a 2-D object. If each boundary point is associated to a medial axis ball containing it and each point of medial axis is connected by a path to some base point, this induces a map from $\partial\Omega$ to a sphere, but is not clear when this is "nice", or even continuous.