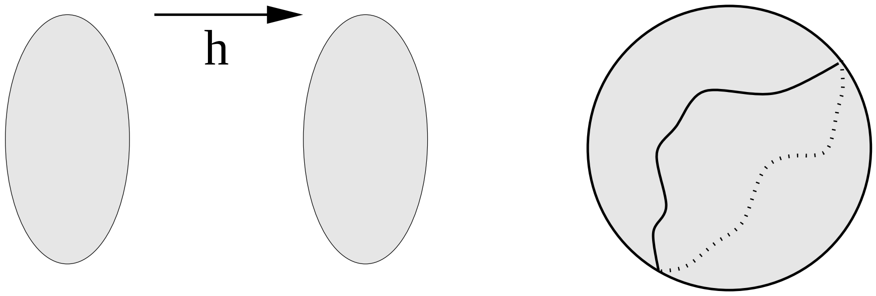
Suppose $h : \mathbb{T} \to \mathbb{T}$ is a orientation preserving homeomorphism. Attach two copies of the disk along their boundaries via h. We get a sphere S^2.



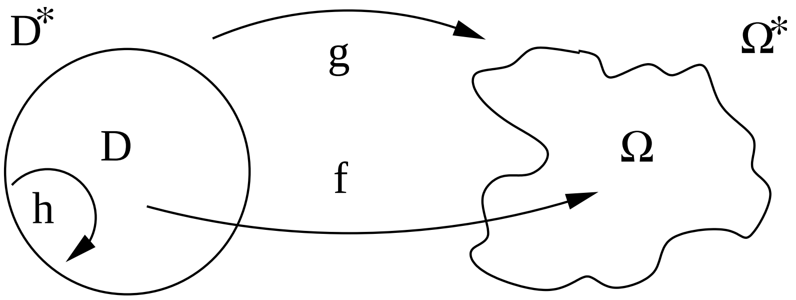
The disks and the sphere all have conformal structures. Can we make the identification respect these structures?

Suppose Γ is a closed Jordan curve and

$$S^2 \setminus \Gamma = \Omega \cup \Omega^*.$$

Let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be conformal. Then $h = g^{-1} \circ f$ is homeomorphism of the circle.

This gives a map from closed curves in the plane to circle homeomorphisms (all modulo Möbius transformations).

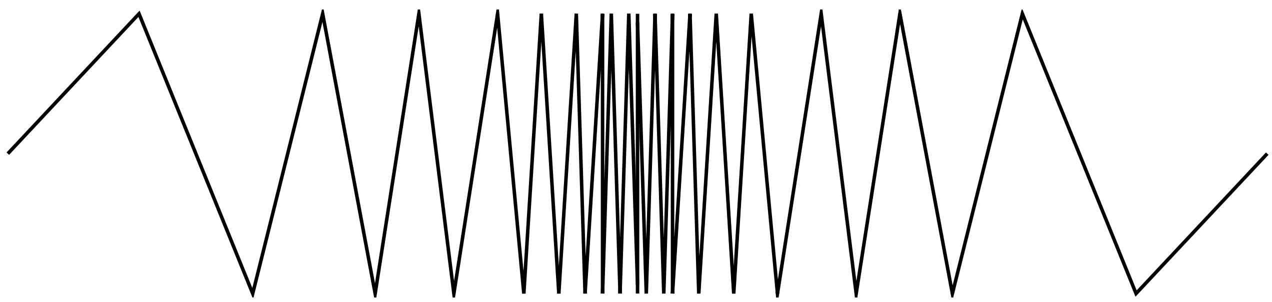


Is the map $\Gamma \to h$ onto?

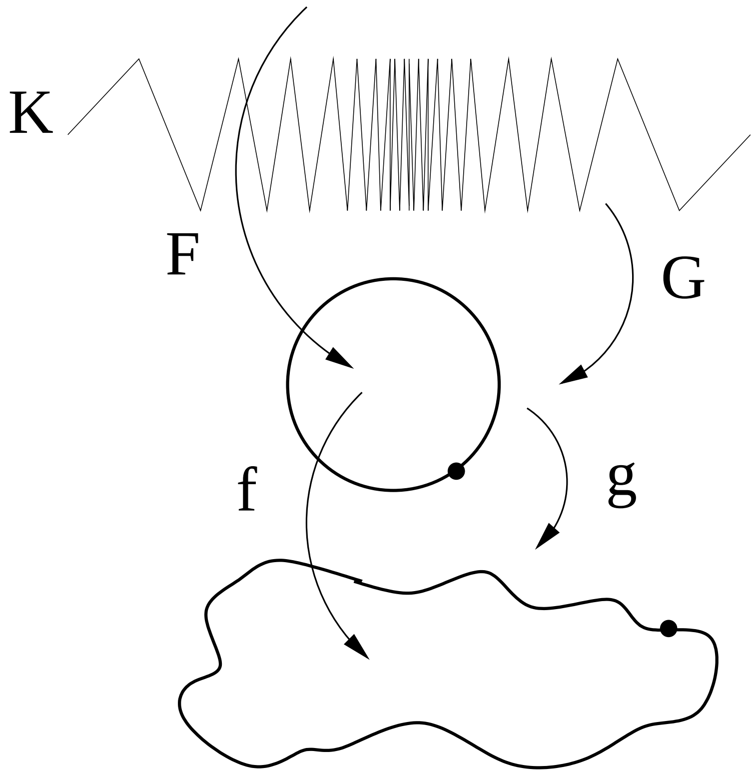
Is the map $\Gamma \to h$ 1-to-1?

The answer to both questions is no.

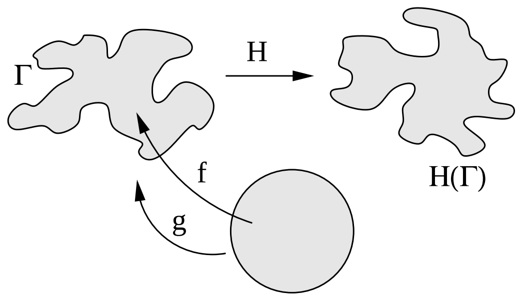
Let K be the closure of the graph of $\sin(1/x)$. This divides the plane into two domains. Let F and G be maps corresponding maps to $\mathbb{D}, \mathbb{D}^*$. Then $h = G \circ F^{-1}$ is well defined, continuous and 1-1 except at one point. Thus it extends to a homeomorphism of circle.



This h is not a conformal welding. Suppose $h = g^{-1} \circ f$ for some curve Γ. Then $f \circ F$ and $g \circ G$ would be conformal off K and continuous except on the segment $I = [i, -i]$. By Morera's theorem they extend to conformal map from complement of I to complement of a point. This contradicts Liouville's theorem.



To see that the map is not 1-to-1 is more difficult. However, one can prove there are closed curves Γ and homeomorphisms of the sphere H which are conformal off Γ but which are not Möbius. Then Γ and $\Gamma' = H(\Gamma)$ are not equivalent, but they do give the same homeomorphism h.



For example, every curve of positive area is non-removable (define QC map with non-zero dilatation on Γ).

Non-uniqueness of Γ can fail more extremely. There are h's so that the set of corresponding Γ's is dense in the set of all closed curves (in Hausdorff metric). We call these flexible curves.

Fundamental theorem of conformal welding: h is a conformal welding if it is quasisymmetric, i.e., there is an $M < \infty$.

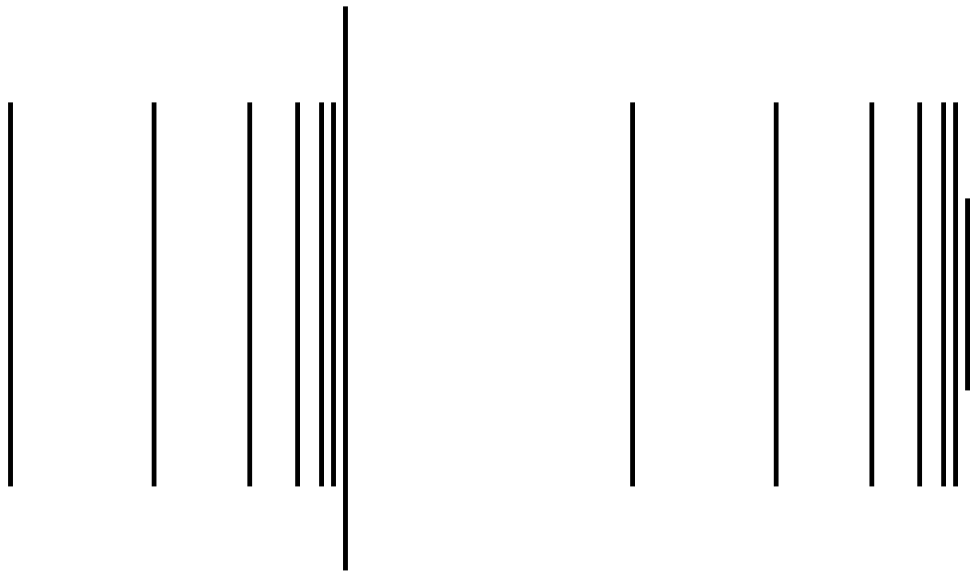
$$\frac{1}{M} \leq \frac{|h(I)|}{|h(J)|} \leq M,$$

for any adjacent intervals I, J of equal length.

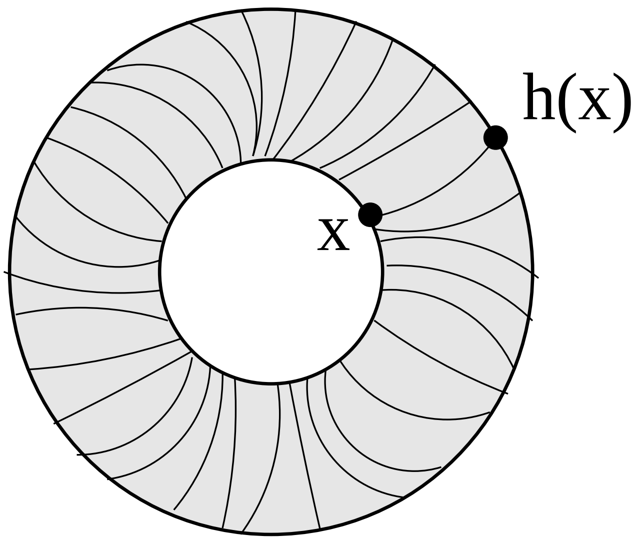
By a famous theorem of Ahlfors and Beurling, these are same as boundary values of quasiconformal selfmappings of the disk. First proof of fundamental theorem was by Pfluger (1960) using measurable Riemann mapping theorem.

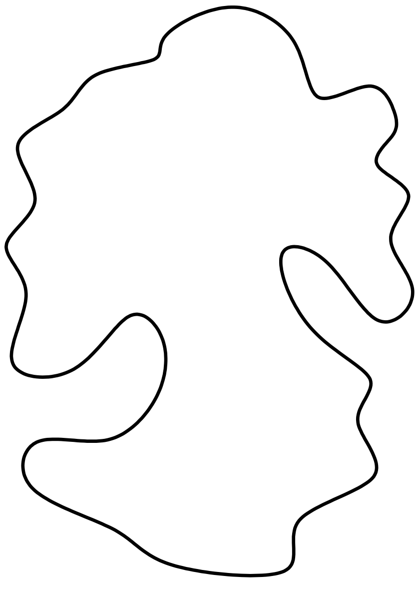
QS map include all diffeomorphisms and include many non-smooth maps ($h' = 0$ a.e. is possible), but are still far from all homeomorphisms. For example, every QS map must be Hölder continuous.

A decomposition of compact set K is a collection of pairwise disjoint closed sets whose union is all of K.
A collection $\mathcal{C}$ of closed sets in the plane is called upper semi-continuous if a collection of elements which converges in the Hausdorff metric must converge to a subset of another element.

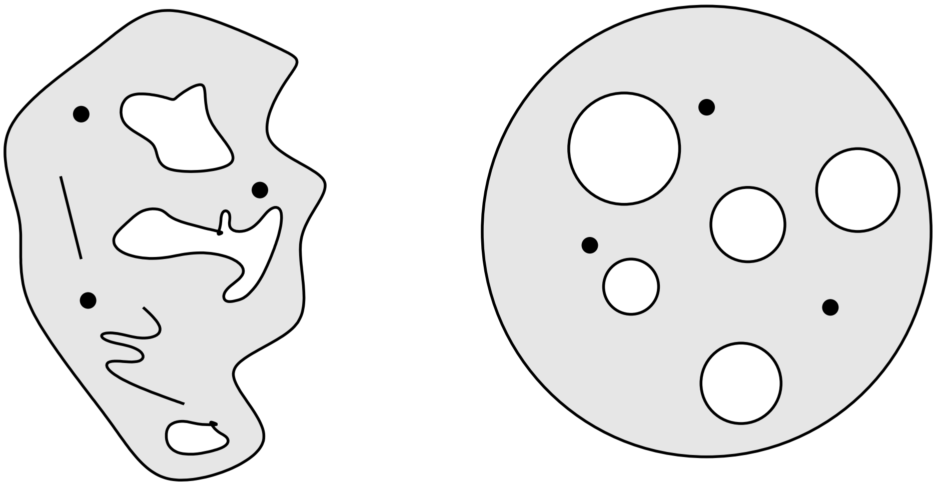


Theorem (R.L. Moore, 1925): Suppose $\mathcal{C}$ is an upper semi-continuous collection of disjoint continua (compact, connected sets) in $\mathbb{R}^2$ each of which does not separate $\mathbb{R}^2$. Then the quotient space formed by identifying each set to a point is homeomorphic to $\mathbb{R}^2$.





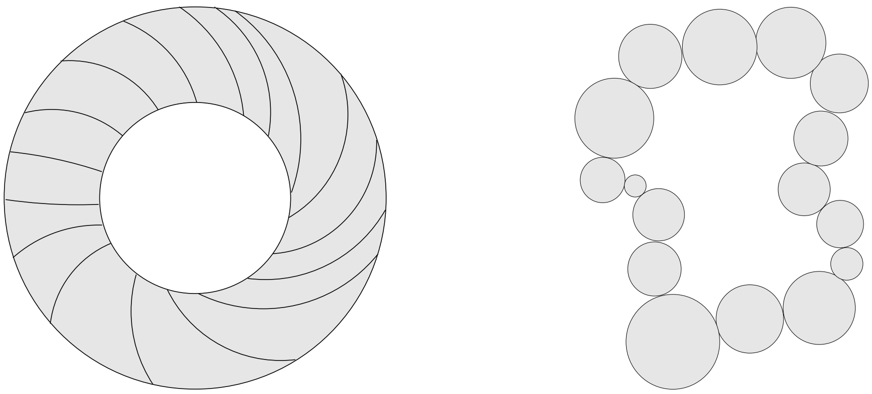
Koebe's circle domain theorem (1908): Every finitely connected domain can be conformally mapped to a domain bounded by circles and points.



Koebe's Conjecture: Every planar domain can be mapped to a domain whose boundary components are all points or circles.

Koebe's conjecture has been proven in various special cases. He and Schramm proved it for countably many boundary components (1993).

Connect points $\{x_k\}_1^n \subset \mathbb{T}$ by disjoint curves $\{\gamma_n\}$ to the points $2h(x_k) \in \{|z| = 2\}$. Let Ω be the union of $\mathbb{D}$, $2\mathbb{D}^*$ and an ϵ-neighborhood of each γ_n. By Koebe's circle domain theorem, this can be mapped to a circle domain. Taking $\epsilon \to 0$ we obtain a closed chain of tangent circles.

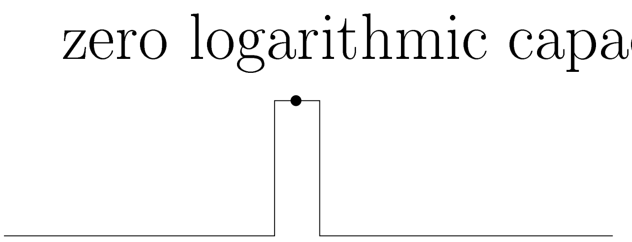


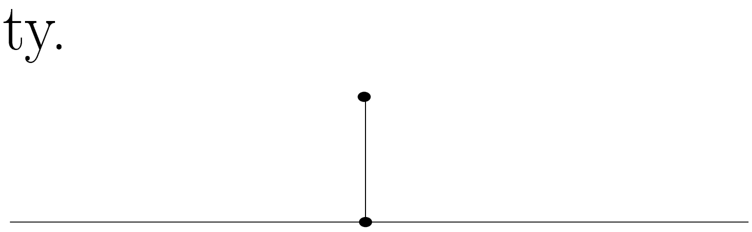
Assume chains stay inside $\{|z| < R\}$ for all n. Then there are at most $(R/\epsilon)^2$ disks of size $> \epsilon$, independent of n.

Theorem 1: For any orientation preserving homeomorphism h there are non-degenerate sequences $f_n : \mathbb{D} \to \Omega_n$ and $g_n : \mathbb{D}^* \to \Omega_n^*$ of conformal maps such that $|f_n(x) - g_n(h(x))| \to 0$, for all $x \in \mathbb{T} \setminus E$, where E is countable.

Conjecture: Suppose $h : \mathbb{T} \to \mathbb{T}$ is any orientation preserving homeomorphism. Then there are conformal maps f, g onto disjoint domains such that $h = g^{-1} \circ f$ on $\mathbb{T} \setminus E$ where E is a countable set.

The problem is passing to the limit. We can have $\lim_n f_n = f$, $\lim_n g_n = g$, and $f_n(x) = g_n(h(x)) \forall n$, but still $f(x) \neq g(x)$. Moreover, we only know f and g have radial limits off a set of zero logarithmic capacity.





Generalized Koebe conjecture: Suppose $\mathcal{C}$ is a upper semi-continuous decomposition of S^2 and suppose none of the elements of $\mathcal{C}$ separate the plane. Let Ω be the interior of the set of singletons. Then there is a closed decomposition $\mathcal{D}$ of S^2 such that every element is either a point or a disk and a bijection $f : \mathcal{C} \to \mathcal{D}$ so that f is continuous in the Hausdorff metric and f is conformal on Ω.

This contains Koebe's conjecture as a special case, because if Ω is connected then the decomposition of $E = S^2 \setminus \Omega$ into its connected components is a upper semi-continuous decomposition (R.L. Moore 1925).

It would also be nice to have conditions on $\mathcal{C}$ which imply all elements of $\mathcal{D}$ are points (the analog of welding).

Suppose $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ onto disjoint simply connected domains. If $E \subset \mathbb{T}$ and $h = g^{-1} \circ f$ on E, we say h is a generalized conformal welding on E. David Hamilton (1991) introduced this idea.

Theorem 2: Every orientation preserving homeomorphism h is a generalized conformal welding on $\mathbb{T} \setminus (F_1 \cup F_2)$, where F_1 and $h(F_2)$ have zero logarithmic capacity.

Corollary: If h preserves sets of zero capacity then it is a generalized welding except on a set of zero capacity.

Corollary: If $\mathrm{cap}(E) = 0$ implies $|h(E)| = 0$ then h is a generalized welding except on a set of Lebesgue measure zero.

Theorem 2 gives no information if $\mathbb{T} = F_1 \cup F_2$, but amazingly in this case we have

Theorem 3: Suppose h is a orientation preserving homeomorphism. Then it is the welding of a flexible curve iff there is a Borel set E such that both E and $h(E)$ have zero logarithmic capacity.

Thus we know h is a conformal welding if it is either good enough (quasisymmetric) or wild enough (log-singular)!

Theorem 4: Every orientation preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ agrees with a conformal welding homeomorphism H, except on a set of measure ϵ (for any $\epsilon > 0$).

This requires another idea: interpolating sets for conformal maps.

Suppose $\mu > 0$ is a Borel measure and define

$$I(\mu) = \iint \log \frac{2}{|z-w|} d\mu(z) d\mu(w).$$

Let Prob(E) be mass 1 measures on E and define

$$\operatorname{cap}(E) \sup\{I(\mu)^{-1} : \mu \in \operatorname{Prob}(E)\}.$$

For $E \subset \mathbb{T}$, cap > 0, monotone and countably subadditive.

For compact subsets of the circle E has zero logarithmic capacity iff

(i) E is the zero set of some conformal map f.

(ii) Planar Brownian motion never hits E a.s.

(iii) The extremal length from $D(0, 1/2)$ to E is infinite.

(iv) Radial limits for a conformal f don't exist on E.

(v) E is an interpolation set for conformal maps.

To illustrate the Koebe's theorem approach to conformal welding, we will give a new proof that boundary values of QC maps are weldings.

IDEA OF PROOF:

Step 1: Choose large n and take n-circle chain associated to h. Let Γ be limit set of corresponding reflection group.

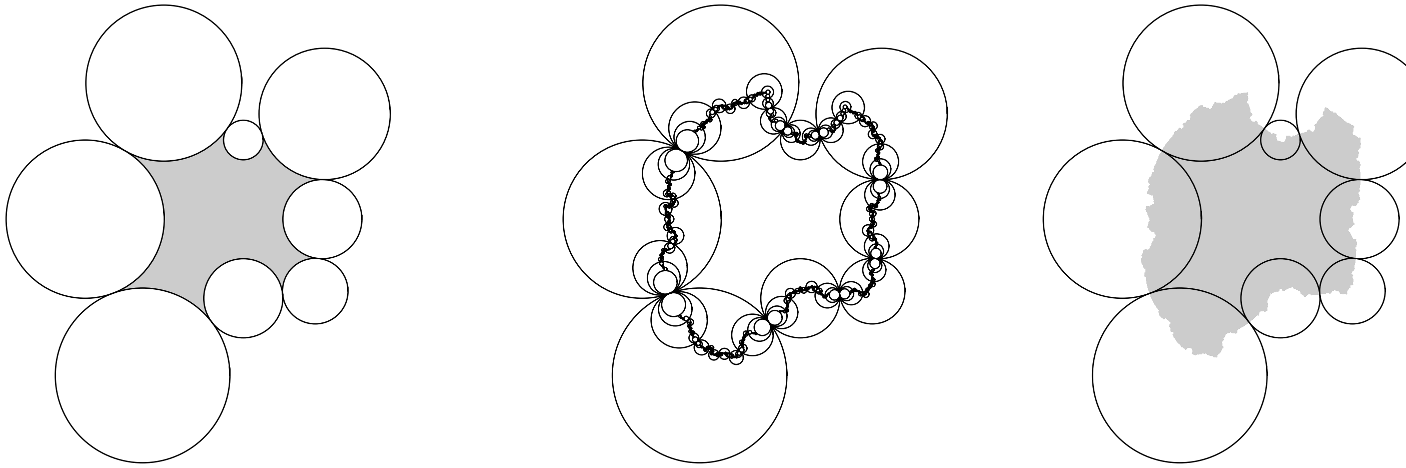
Step 2: Let f and g be conformal maps from $\mathbb{D}$ and $\mathbb{D}^*$ to the two sides of circle chain.

Step 3: Extend f and g by Schwarz reflection to maps from universal cover of a n-punctured plane to either side of Γ.

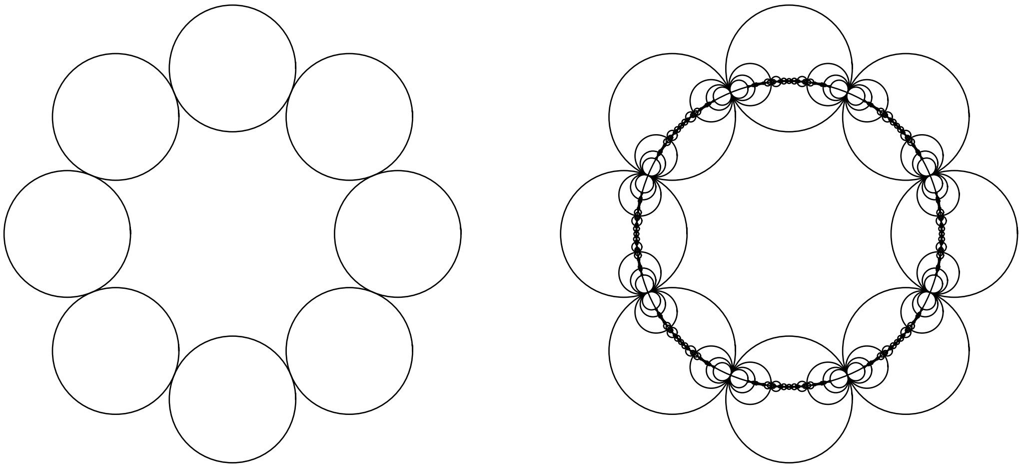
Step 4: By uniformization theorem this gives maps from $\mathbb{D}$ and $\mathbb{D}^*$ to either side of Γ.

Step 5: Lift H to $\mathbb{D}^*$ and show two maps agree. Thus Γ is K-quasiconformal image of circle.

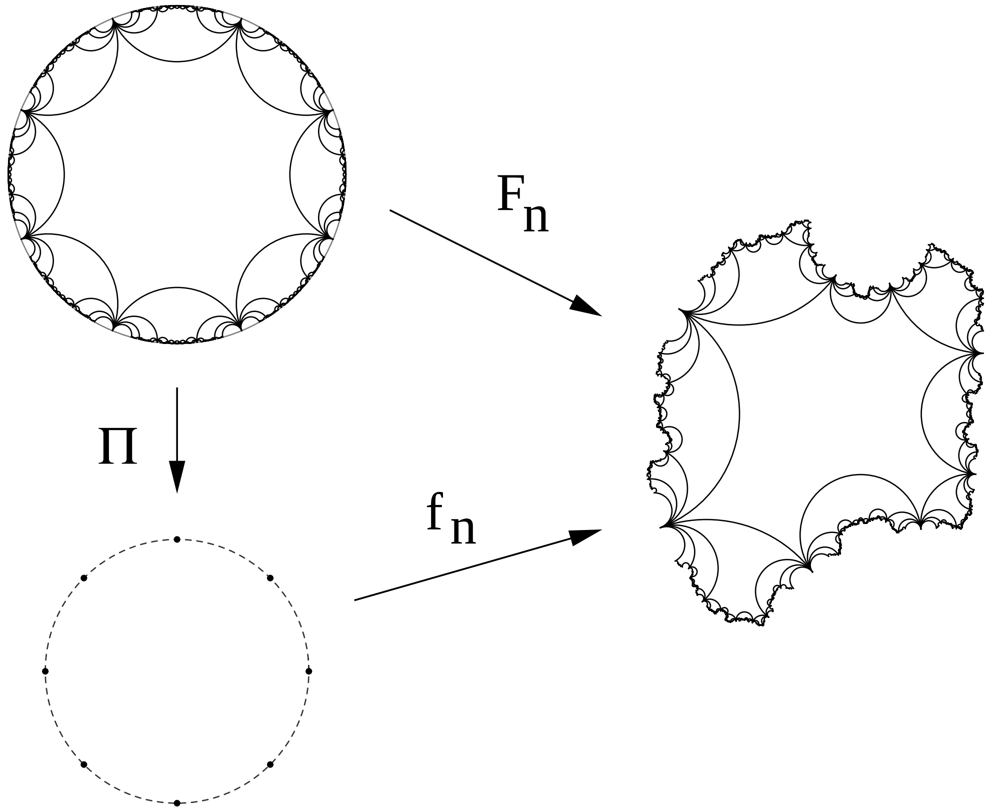
Step 6: Let $n \to \infty$ and pass to limit.



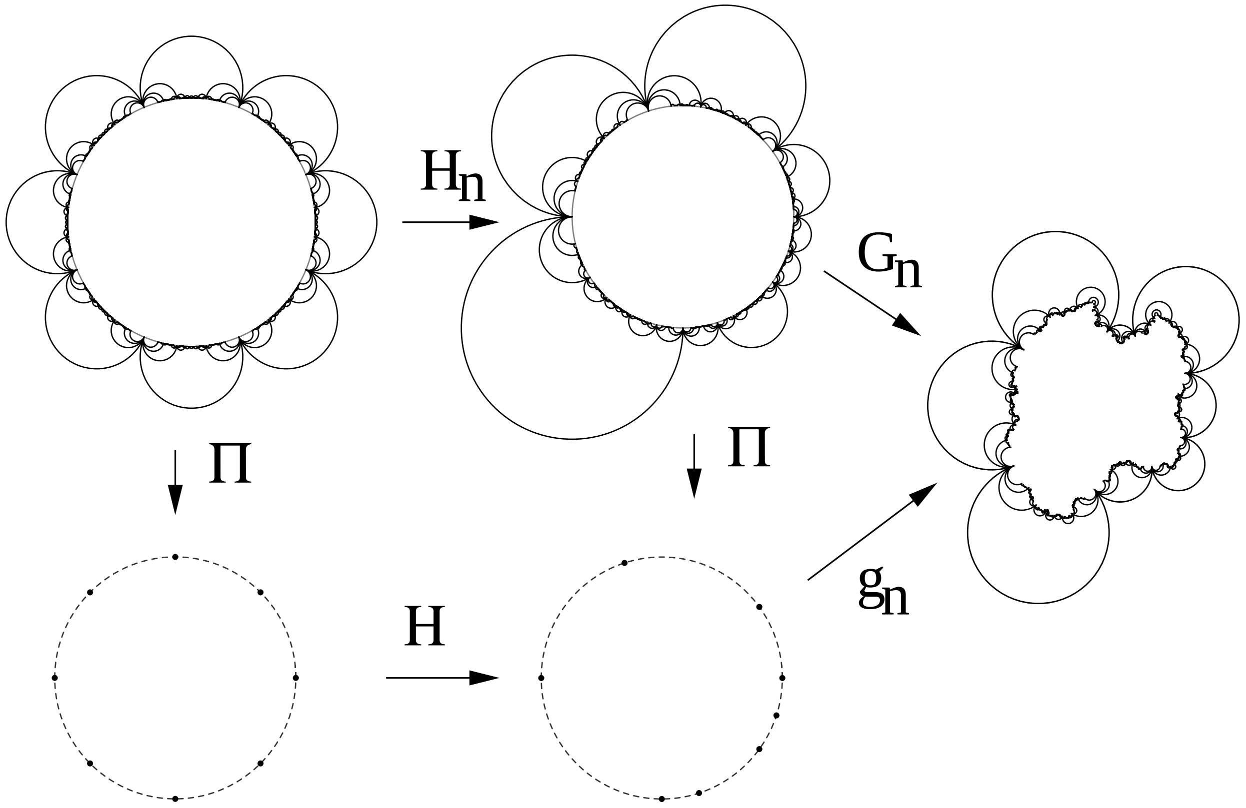
Here is a circle chain and several self-reflections. The limit of these reflections is a closed Jordan curve Γ. The two sides of the curve will be denoted D and D^*.



Same figure as before, but with a chain corresponding to the identity homeomorphism.



The map $f : \mathbb{D} \to \Omega_n$ can be extended by reflection to a map of the universal cover of $W_n = S^2 \setminus \{x_1, \ldots, x_n\}$ to D_n (the inside of Γ). Since the universal cover can be identified with $\mathbb{D}$, we get a map of $\mathbb{D}$ to D_n.



Same construction gives a conformal map from $\mathbb{D}^*$ to D^* (the outside of Γ). If we first extend H to the plane by reflection and then lift it to the universal cover of $S^2 \setminus \{y_1, \dots, y_n\}$, we get a K-quasiconformal map of $\mathbb{D}^*$ which conjugates the group action to the symmetric one (i.e., the reflection of the inside).

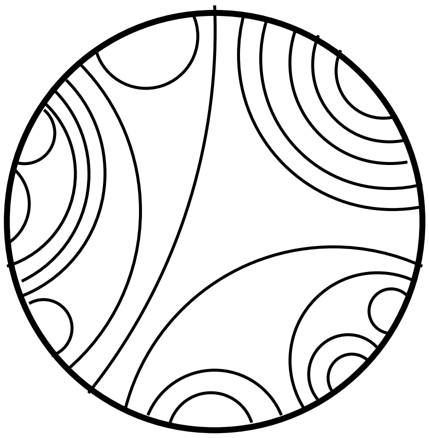
Idea to prove Theorem 4: If h is log-regular (maps sets of zero capacity to zero length), then results above are enough. If h maps zero capacity to positive length, then need new idea.

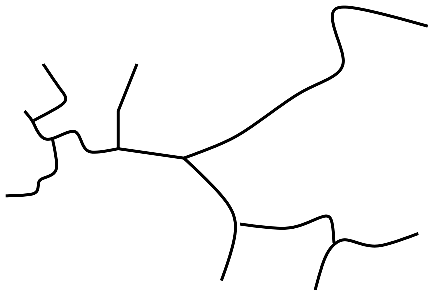
Theorem 5: If E has zero log capacity and h is any o.p. homeomorphism of the circle then there is a conformal map of $\mathbb{D} \to \Omega \subset \mathbb{D}$ such that $f|_E = h|_E$.

Given such an h we can take f inside the disk and $g(z) = z$ outside. Then $g^{-1} \circ f = h$ (at least on the set of zero capacity). Combining this with ideas from log-regular case gives all cases.

We say the decomposition is realized by a function $f : K \to S^2$ if it consists of the level sets $\{f^{-1}(z) : z \in S^2\}$ of f. If $K \subset \mathbb{T}$ and f is the boundary values of a conformal map on $\mathbb{D}$, we will call the decomposition conformal.

We will say a decomposition of the unit circle is separated and if any two distinct sets are contained in disjoint intervals.





Theorem 6: Suppose $E \subset \mathbb{T}$ is compact. Then the following are equivalent.

1. E has logarithmic capacity zero.
2. Given any homeomorphism $g : \mathbb{D} \to \Omega \subset \mathbb{R}^2$ which extends continuously to $\mathbb{T}$, there is a conformal map $f : \mathbb{D} \to \Omega$ which extends continuously to $\mathbb{T}$ and $f|_E = g|_E$.
3. Given any continuous map $g : E \to \mathbb{R}^2$ such that $z \neq w$ implies $g^{-1}(z)$ and $g^{-1}(w)$ lie in disjoint arcs of $\mathbb{T}$, there is a conformal map $f : \mathbb{D} \to \Omega$ which extends continuously to $\mathbb{T}$ and $f|_E = g|_E$.
4. A decomposition of E is conformal iff it is upper semi-continuous and separated.