MEASURES, MARTINGALES, MANIFOLDS AND MAPPINGS

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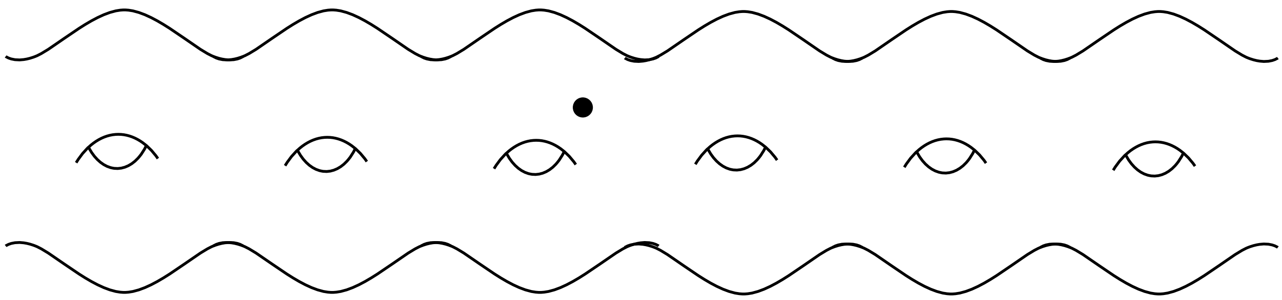
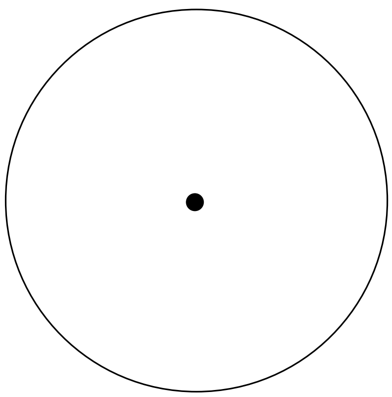
1998 Barrett lectures
Knoxville, TN, June 1998

Suppose M is a hyperbolic $d+1$ manifold, $z_0 \in M$ and identify the geodesic rays emanating from z_0 with points of the S^d. We say the geodesic is

1. **recurrent** if it hits some compact set infinitely often
2. **escaping** if it leaves every compact set eventually



Problem: How many geodesics of each type are there?



Hausdorff content:

$$\mathcal{H}^{\varphi}_{\delta}(E) = \inf\{\sum \varphi(r_j) : E \subset \cup U_j, \mathrm{diam U_j = r_j} \leq \delta\}.$$

If $\varphi(t) = t^{\alpha}$, denote by $\mathcal{H}^{\alpha}_{\delta}$. Taking $\delta = \infty$ gives Hausdorff content.

Hausdorff dimension:

$$\mathrm{dim(E)} = \inf\{\alpha : \mathcal{H}^{\alpha}_{\infty}(\mathrm{E}) = 0\}$$

Hausdorff measure:

$$\mathcal{H}^{\varphi}(E) = \lim_{\delta \to 0} \mathcal{H}^{\varphi}_{\delta}(E).$$

Fact: $\mathcal{H}^{\varphi}_{\infty}(E) = 0$ iff $\mathcal{H}^{\varphi}(E) = 0$.

An nth generation **dyadic interval** in R is of the form $[j2^{-n}, (j+1)2^{-n})$ and a **dyadic cube** in R^d is a product of such intervals.

The **parent** Q^* of Q is the unique dyadic square of twice the size which contains it.

We let $\ell(Q) = 2^{-n}$ denote the side length of Q and $|Q| = |Q|_d = 2^{-dn}$ be its volume.

Given a point $x \in \mathrm{R}^d$ we let $Q_n(x)$ be the dyadic cube of generation n containing x.

Fact: Any cube in R^d can be covered by $C(d)$ dyadic intervals of smaller size. Thus in the definition of Hausdorff content and measure we may consider coverings only by dyadic cubes.

Dyadic cubes form a tree (each cube is adjacent to its parent and children). A compact set $E \subset Q_0$ corresponds to the subtree Γ_E of dyadic cubes which hit E.

A dyadic covering of E is a **cutset** $\mathcal{C}$ of this tree, i.e., a subset of edges which separates the root from the leaves. Let $\varphi(\mathcal{C}) = \Sigma_{Q \in \mathcal{C}} \varphi(Q)$.

A measure is the same as a **flow** on the tree, i.e., a function f so that

$$f(Q') = \sum_{Q^* = Q'} f(Q).$$

Note $\|f\| = f(Q_0) = \mu(Q_0)$.

We say φ dominates the flow if $f(Q) \leq \varphi(Q)$ for all Q. Think of f as a flow of fluid through a network of pipes, each with a given maximum capacity.

Frostman's Lemma: $\mathcal{H}^{\varphi}_{\infty}(E) > 0$ iff E supports a measure μ such that $\mu(Q) \leq \varphi(Q) \, \forall \, \mathrm{dyadic\, Q}$.

This is same as:

Max flow = Min Cut Theorem: Given φ

$$\max_{\text{flows } f \leq \varphi} \|f\| = \min_{\text{cutsets}} \varphi(\mathcal{C}).$$

Proof: The direction $\leq$ is easy. To prove $\geq$, choose f attaining max and consider the set of maximal saturated edges, i.e., Q so that $f(Q) = \varphi(Q)$. This is a cutset (or else can increase flow along some path to a leaf; contradicts maximality of f) which gives equality. $\square$

Frostman-Rogers-Taylor Theorem: Let μ be a finite measure on $[0,1]^2$ and let $A \subset [0,1]^2$ be a closed set. Given a point x let $Q_n(x)$ denote the nth generation dyadic square containing x. Then if for all $x \in A$,

1. $\limsup_{n\to\infty} \frac{\mu(Q_n(x))}{\varphi(2^{-n})} \geq C$, then $\mathcal{H}^\varphi(A) \leq MC^{-1}\mu(A)$.
2. $\limsup_{n\to\infty} \frac{\mu(Q_n(x))}{\varphi(2^{-n})} \leq C$, then $\mathcal{H}^\varphi(A) \geq M^{-1}C^{-1}\mu(A)$.

Example: Singular measure on middle third Cantor set satifies this for $\varphi(t) = t^{\log 2/\log 3}$.

Defn: A sequence of functions $\{f_n\}$, $n = 0, 1, \ldots$ on Q_0 is a dyadic martingale if each f_n is constant on each nth generation dyadic cube and satisfies

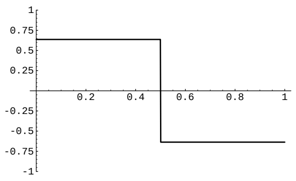
$$\int_Q f_{n+1} = \int_Q f_n,$$

if Q is an nth generation cube.

Examples of martingales:

1. If f is a L^1 function let $f_n(x) = |Q_n(x)|^{-1} \int_{Q_n(x)} f dx$. Then $f_n \to f$ a.e.

2. If μ is a singular measure, let $f_n(x) = \mu(Q_n(x))/|Q_n(x|$. Then $f_n(x) \to 0$ Lebesgue a.e. and $f_n(x) \to \infty$ μ a.e.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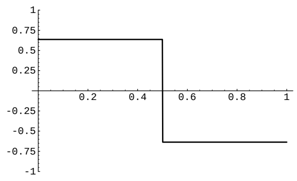
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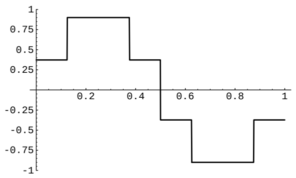
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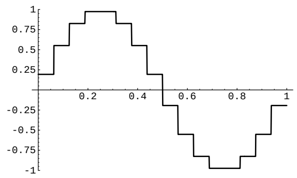
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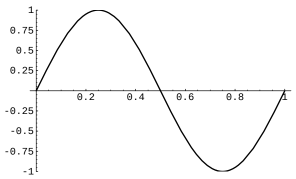
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3. The simple random walk on the integers: Suppose $d = 1$. Let $f_0 = 0$. Given a dyadic interval I, let

$$f_{n+1}(x) = f_n(x) + 1$$

if x is in the left child and

$$f_{n+1}(x) = f_n(x) - 1$$

if it is in the right child.

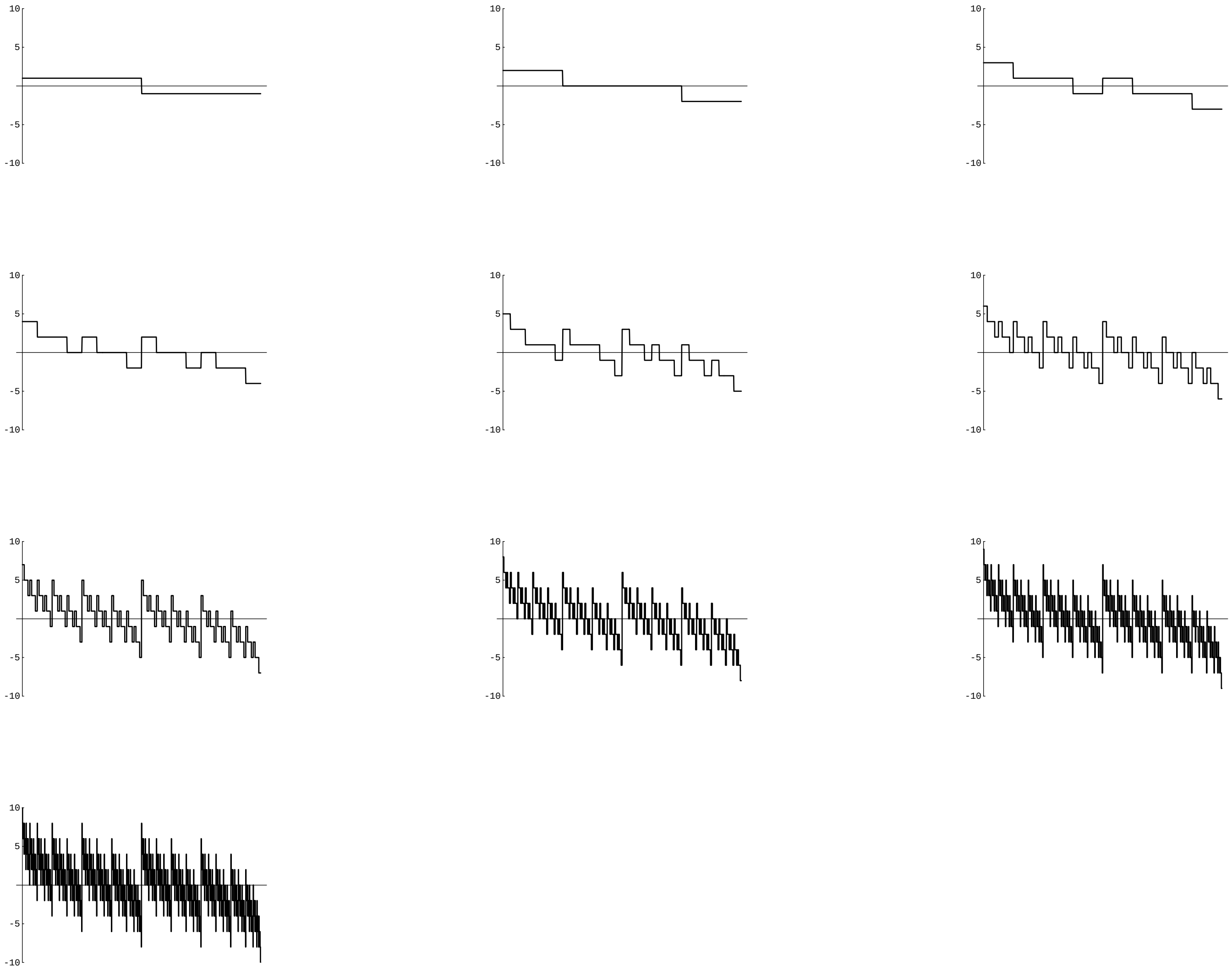
This martingale diverges for Lebesgue a.e. point.

4. A random walk on the positive integers: Let $g_0 = 1$. If $g_n(x) = 0$ then $g_{n+1}(x) = 0$. Otherwise, $g_{n+1}(x) = g_n(x) \pm 1$, as above. We say $\{g_n\}$ is "stopped" at zero.

Theorem: If $\{f_n\}$ is a positive martingale then there is a positive, finite measure μ on Q_0 such that

$$f_n(x) = \frac{1}{|Q_n(x)|} \int_{Q_n(x)} d\mu(y),$$

for $x \in Q$.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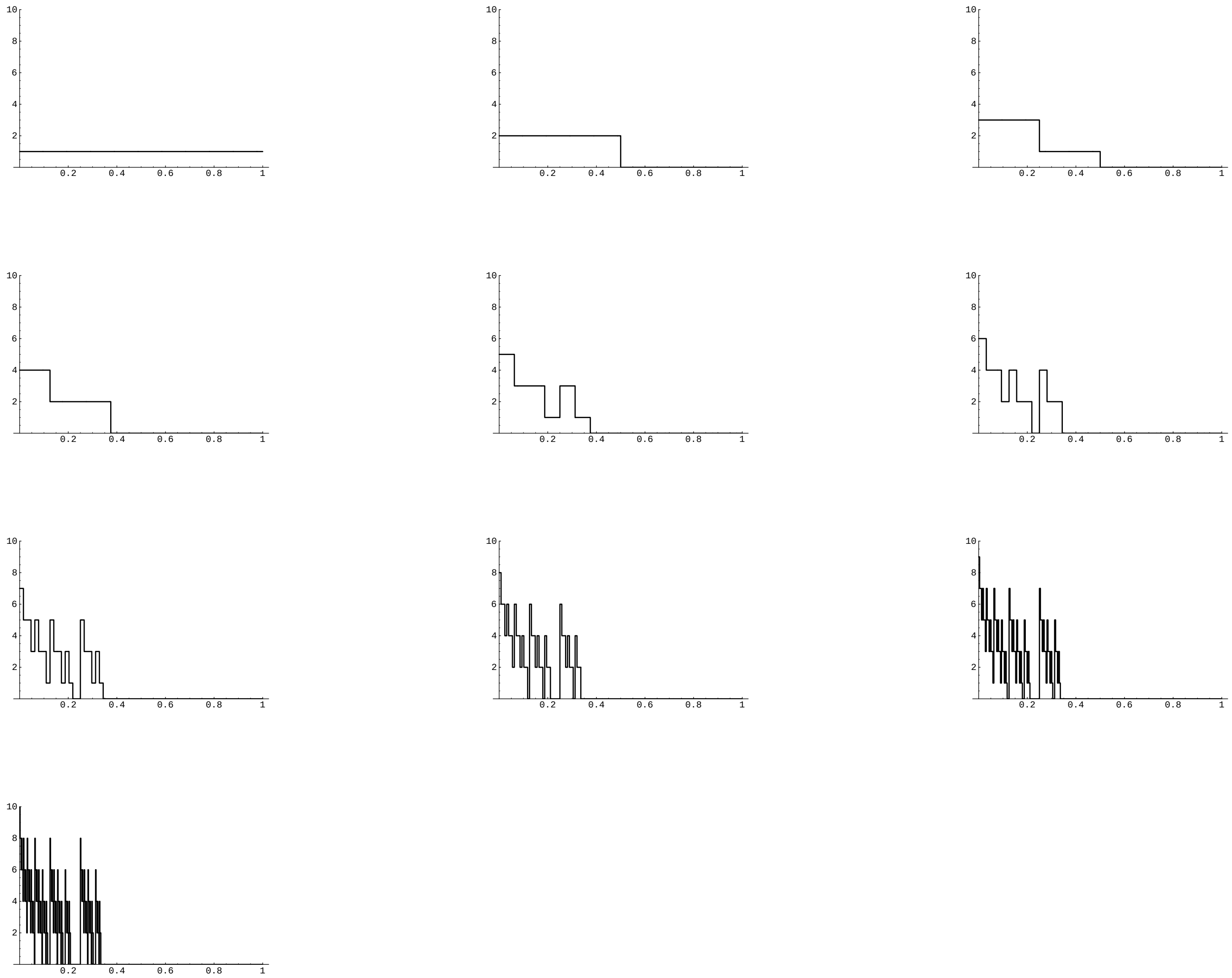
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We let $\Delta f_n = f_n - f_{n-1}$ be the **martingale differences**. Note that $\Delta f_n \perp \Delta f_m$, $n \neq m$.

We say f is **Bloch** is Δf_n is uniformly bounded. This is analogous to a Bloch harmonic function U on the upper half-space H^{d+1}:

$$|\nabla U(z)| \leq \frac{C}{\mathrm{dist(z, R^d)}}.$$

This is the same as having bounded hyperbolic gradient.

Lemma: If U is a Bloch harmonic function on H^{d+1}, then there is a Bloch dyadic martingale $\{f_n\}$ on R^d so that

$$|U(x, 2^{-n}) - f_n(x)| \leq A,$$

for any $x \in \mathrm{R}^d$. If U is the Poisson integral of a positive measure μ then we can take $f(Q) = \mu(Q)/|Q|$.

Lemma: Suppose $\{f_n\} \geq 0$ is a Bloch martingale on R^d with $f_0 > 0$. Let

$$E = \{x : f_n(x) > 0 \ \forall \ n\}.$$

Then dim(E) = d.

Proof: Let μ be associated measure. Since $\{f_n\}$ is Bloch then trivially,

$$\frac{\mu(Q_n(x))}{|Q_n(x)|} = |f_n(x)| \leq An.$$

Thus

$$\mu(Q_n(x)) \leq An|Q_n(x)|_d \leq A\psi(\ell(Q_n(x))),$$

where

$$\psi(t) = t^d \log \frac{1}{t},$$

and hence $\mathcal{H}^\psi(E) > 0$ by the F-R-T theorem. Since

$$t^d \log \frac{1}{t} < t^{d-\epsilon}, \qquad t \to 0,$$

for any $\epsilon > 0$, we have dim(E) = d. $\square$

Corollary: Suppose $\{f_n\}$ is a dyadic martingale so that $f_0 > 0$ and $\liminf_n f_n(x) \le 0$ for Lebesgue almost every x. Then $f_n(x) \nearrow +\infty$ on a set of dimension d.

Proof: Form a new martingale $\{g_n\}$ by stopping $\{f_n\}$ the first time it is ≤ 0.

Then

$$-A \le \lim_n g_n(x) \le 0,$$

exists a.e. Thus g_n is the integral of $g + \mu$ where $g \le 0$ is a bounded function and μ is a positive, singular measure.

Its easy to see by a covering lemma that $\mu \ll dx$ on the set where $\liminf_n g_n(x) \le M < \infty$, and hence this set has zero μ measure. Thus $g_n(x) \to \infty$ at μ almost every x and hence on a set of dimension d. $\square$

Theorem: Suppose $M = \mathrm{H}^{d+1}/G$ is a hyperbolic manifold with no Greens function. If M supports a non-constant u Bloch harmonic function, then the escaping geodesics have dimension d (in fact, positive $\mathcal{H}^\varphi$ measure).

Proof: Let U be the lift of u to H^{d+1} and let $\{f_n\}$ be the corresponding Bloch martingale so that $|U - f_n| \leq A$.

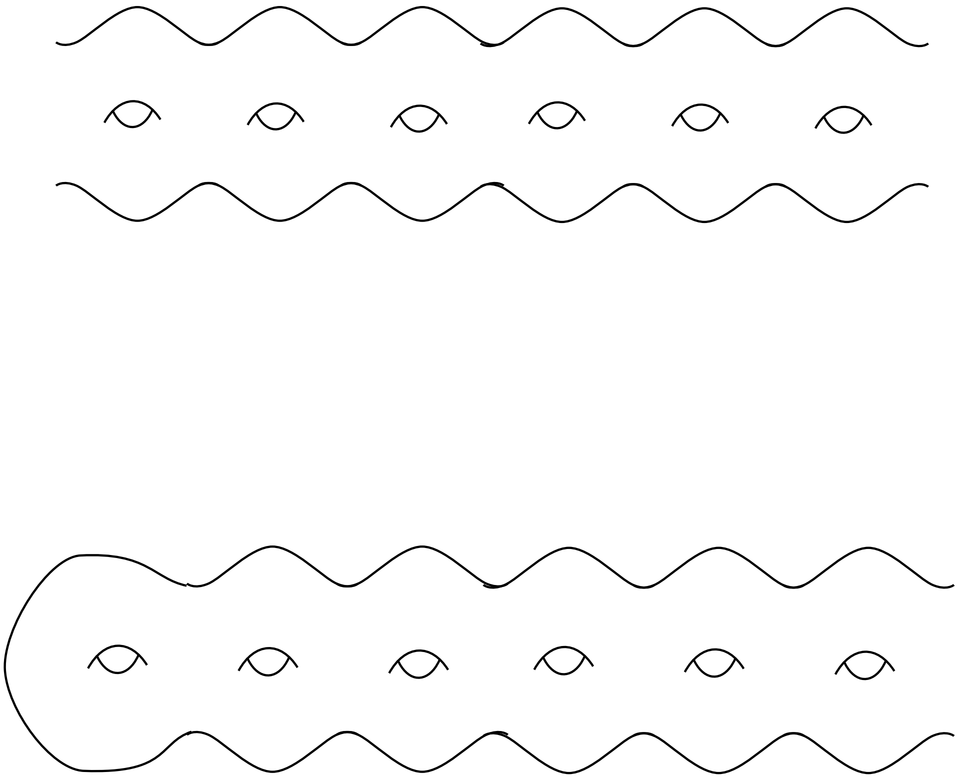
Fact: The G-orbit of any point in H^{d+1} is non-tangentially dense Lebesgue almost everywhere on the boundary.

In particular, U has non-tangential limits almost nowhere (just consider the orbits of two points where U has different values). Standard results (e.g., Stein's book) imply

$$\liminf_{y\to 0} U(x,y) = -\infty, \quad \limsup_{y\to 0} U(x,y) = \infty \quad \text{a.e. x.}$$

$$\Rightarrow \liminf_n f_n(x) = -\infty, \quad \text{a.e. x.}$$

By the corollary, $f_n(x) \to \infty$ on a set of dimension d, so same is true of U. These rays project to geodesics on M along which u tends to infinity, so they must be escaping (since u is bounded on any compact set). $\square$



Top surface supports a non-constant harmonic Bloch function, but bottom does not. (It does if we add a boundary.)

When $d = 1$, M is a Riemann surface and behavior of escaping geodesics is completely described by a result of Fernández and Melián:

Theorem: On a Riemann surface the escaping geodesics are

1. countable if R has finite area,
2. dimension 1 and zero length if R has no Greens function but is infinite area,
3. positive length if R has a Greens function.

Corollary: If M is a non-compact, hyperbolic $d+1$ manifold with no Greens function and a positive lower bound on the injectivity radius, then $M \setminus B(x, r)$ supports a non-constant harmonic Bloch function. Thus the escaping geodesics have dimension d.

Sketch of proof of corollary:

Step 1: Remove a small ball B from M to get a manifold N which does have a Greens function. Because of the injectivity radius we can prove the gradient of Greens function is uniformly bounded away from the pole (independently of the position of the pole).

Step 2: Move the pole to infinity. We obtain a non-constant, positive Bloch harmonic function U on N which vanishes on ∂B.

Step 3: List to a function U on H^{d+1}. We can construct a region Ω and a positive, Bloch harmonic V on all of H^{d+1} so that $|U - V| \leq A$ on Ω and $V \leq A$ off Ω. (i.e. V is U "stopped" when we hit a lift of B).

Step 4: By previous arguments, V tends to ∞ on a set of dimension d, hence so does U, and the corresponding geodesics in M are escaping. $\square$

Theorem: If the martingale differences are bounded away from zero whenever $f_n > 0$ then $E = \{f_n(x) > 0 : \forall n\}$ has finite ψ measure for $\psi(t) = t^d\sqrt{\log \frac{1}{t}}$.

Proof: Define the martingale square function as

$$Sf_n(x) = (\sum_{k=1}^{n} \Delta f_n(x)^2)^{1/2}.$$

By orthogonality, $\|f_n\|_2 = \|Sf_n\|_2$.

The L^1 norm of f_{2^n} is C and its BMO norm is $\leq 2^{n/2}$. By interpolation, the L^2 norm is $\leq 2^{n/4}$. On the other hand, for $x \in E$, the square function of f_{2^n} is bounded below by $C2^{n/2}$, so

$$2^{n/4} \sim \|f_{2^n}\|_2 = \|Sf_{2^n}\|_2 \geq (\int_{E_n} 2^n dx)^{1/2},$$

which implies

$$|E_n|_d \leq C2^{-n/2},$$

where E_n be the union of dyadic cubes of generation 2^n which hit $E = \text{supp}(\mu)$.

Hence the number of cubes in E_n is $\leq 2^{d \cdot 2^n} 2^{-n/2}$. Since E_n is a cover of E, this implies the result.

Theorem: Suppose $\{f_n\}$ is a positive Bloch martingale such that the martingale differences are bounded uniformly away from zero. Then for μ a.e. x,

$$\liminf_n \frac{f_n(x)}{\sqrt{n}(\log n)^{-2}} = +\infty.$$

Proof: Let E_n be the union of dyadic cubes of generation 2^n which hit $E = \text{supp}(\mu)$. We showed that

$$|E_n| \leq C2^{-n/2}.$$

Let B_n be the collection of dyadic cubes of generation between 2^n and 2^{n+1} so that

$$f_n(x) \leq n^{-2}2^{n/2}, \qquad x \in B_n.$$

Then

$$\mu(Q) = f_n(x)|Q|_d \leq n^{-2}2^{n/2}|Q|_d$$

This and the observation $B_n \subset E_n$ imply

$$\sum_n \mu(B_n) \leq n^{-2}2^{n/2}|E_n|_d \leq n^{-2},$$

so Borel-Cantelli implies the lemma. $\square$

We have shown that E has positive measure for

$$t^d \log \frac{1}{t},$$

and finite measure for

$$t^d \sqrt{\log \frac{1}{t}}.$$

The sharp result is that E has positive and finite measure for

$$t^d \sqrt{\log \frac{1}{t} \log \log \log \frac{1}{t}}.$$

Proof of sharp result requires Hardy-Littlewood maximal theorem and a Gaussian type estimate for martingales with bounded square function.

Now we sketch how to prove sharper version of estimates.

Lemma: Suppose $\{f_n\} \geq 0$ is a Bloch martingale on R^d with $f_0 > 0$. Let

$$E = \{x : f_n(x) > 0 \, \forall \, n\}.$$

Then $\mathcal{H}^\psi(E) > 0$ where

$$\psi(t) = t^d \sqrt{\log \frac{1}{t}} (\log \log \frac{1}{t} \log \log \log \frac{1}{t})^2.$$

Proof: Define the maximal function

$$M f_n(x) = \sup_{k \leq n} |f_k(x)|.$$

Hardy-Littlewood theorem implies$\|M f_n\|_2 \leq C \|f_n\|_2$. Since the martingale differences are orthogonal, and $\{f_n\}$ is Bloch,

$$\|M f_n\|_2 \leq C \|f_n\|_2 = C(\|\Delta f_1\|^2 + \ldots \|\Delta f_n\|_2^2)^{1/2} \leq A\sqrt{n},$$

so by Tchebyshev's inequality,

$$|\{M f_n > \lambda\}| \leq \frac{Cn}{\lambda^2}.$$

Choose a sequence so that

$$a_{n+1} \leq C a_n, \qquad \sum_n \frac{n}{a_n} = \sum_n \sum_{k>n} \frac{1}{a_k} < \infty \tag{1}$$

say $a_n = n^2 \log^2 n$,

$$\begin{aligned} \mu(Mf_{2^n} > \sqrt{2^n} a_n) &\leq \sum_{k>n} \mu(\sqrt{2^k} a_{n+k}) \\ &\leq Mf_n < \sqrt{2^{k+1}} a_{k+1} \\ &\leq \sum_{k>n} a_k^2 \cdot a_{k+1} \end{aligned}$$

Because of (1), the Borel-Cantelli lemma implies that for μ a.e. x

$$Mf_{2^n} \leq \sqrt{2^n} a_n \leq \sqrt{2^n} (\log 2^n \log\log 2^n)^2,$$

for all sufficiently large n. Thus for $2^{n-1} < k \leq 2^n$,

$$f_k \leq C\sqrt{k} (\log k \log\log k)^2,$$

Hence, as before $\mathcal{H}^\psi(E) > 0$ by the F-R-T theorem.

Instead of just having an L^2 estimate for Bloch martingales an exponential estimate is true

$$\int_Q \exp(C_1 \frac{1}{n} |f_n|^2) dx \leq C_2.$$

(See e.g., paper of Chang-Wilson-Wolff, Comm. Math. Helv., 60(1985), 217–246).

Using this stronger starting result we can easily show

$$|Mf_n > \lambda| \leq C \exp(-C \frac{\lambda^2}{n}) \ll \frac{Cn}{\lambda^2}.$$

Using this estimate in the previous proof gives $\mathcal{H}^\varphi(E) > 0$, for the function

$$\varphi(t) = t^d \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}.$$

Theorem: If the martingale differences are bounded away from zero then E has finite φ measure.

Sketch of Proof: Cover E by dyadic squares of size 2^{-2^n}. If $U(z_Q) \geq \epsilon\sqrt{2^n \log n}$ put the square in a collection B_n. Otherwise, subdivide it into squares of size $2^{-2^{n+1}}$ and repeat. Stop when $n = N$ and put the remaining squares into a collection called G_N.
If $Q \in B_1 \cup \ldots B_N$ then

$$\varphi(\ell(Q)) \leq \frac{C}{\epsilon}\mu(Q).$$

Since μ is a finite measure, the φ-sum over such squares is bounded. A very technical argument shows that the φ-sum over the G_N squares tends to 0 with N: roughly one must show that a martingale $\{f_n\}$ with differences bounded away from zero can only satisfy

$$1 \leq f_k \leq \epsilon\sqrt{k \log\log k},$$

for $n_0 < k < n$ on a set of measure $\leq \exp(-cn^{-\alpha})$. See Bishop-Jones, "The LIL for Klienian groups", Contem. Math. 211, for details.

Techniques discussed earlier can also prove:

Theorem If $\{f_n\}$ is a Bloch martingale then

$$\limsup_n \frac{f_n(x)}{\sqrt{n \log \log n}} < \infty,$$

for Lebesgue a.e. x. If the martingale differences are bounded away from zero then we also have > 0 for a.e. x.

This idea was used by N.G. Makarov to prove his famous theorem on harmonic measure. If $F : \mathrm{D} \to \Omega$ is a Riemann mapping then $\log |F'|$ is a Bloch harmonic function (this is a consequence of the Koebe distortion theorem). Thus

$$|F'(x+iy)| \le C \exp(C\sqrt{\log \frac{1}{y} \log \log \log \frac{1}{y}}),$$

for a.e. x. From this one can deduce (with some work!) that the image of Lebesgue measure under F is a measure ω such that $w \ll \mathcal{H}^{\psi}$, where

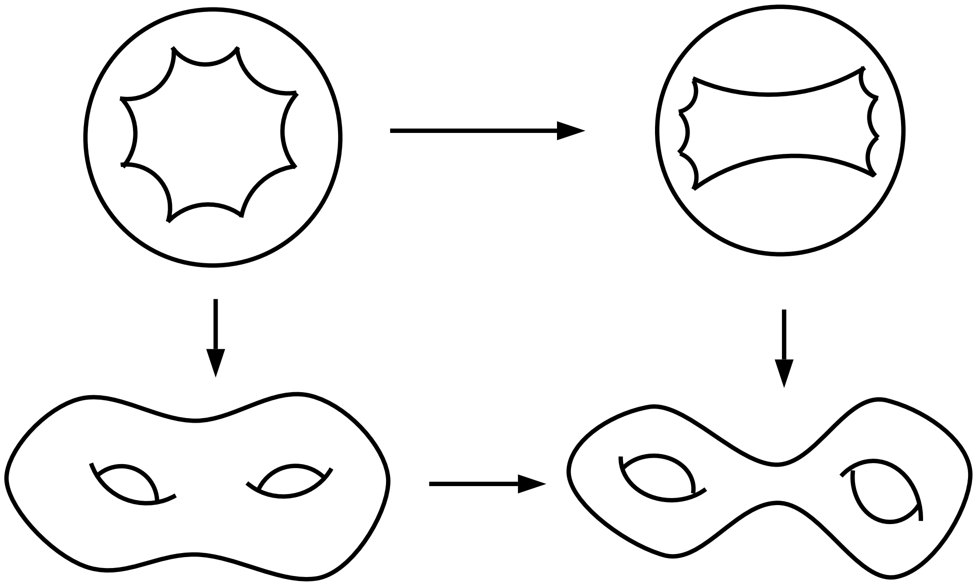
$$\psi(t) = t \exp(C\sqrt{\log \frac{1}{t} \log \log \log \frac{1}{t}}).$$

In particular, $\dim(\mathrm{E}) < 1 \Rightarrow \omega(\mathrm{E}) = 0$.

A mapping h is K-quasiconformal if

$$K_h(x) = \limsup_{r\to 0} = \frac{\max_{|x-y|=r} |h(x) - h(y)|}{\min_{|x-y|=r} |h(x) - h(y)|} \leq K.$$

Suppose G is a Fuchsian group covering a Riemann surface R and let h be a quasiconformal mapping conjugating G to a group G'. If R is compact, h is nowhere differentiable on the unit circle (Mostow rigidity). More generally, Tukia showed h is not differentiable at endpoints of recurrent geodesics (conical limit set).



Theorem: If R is the "jungle gym" surface and $K_h - 1$ has compact support (mod R) then h is differentiable on a set positive $\mathcal{H}^\varphi$ measure for $\varphi(t) = t\sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}$.

Proof: A theorem of Lehto (1965, Ann. Acad. Sci. Fenn) says that

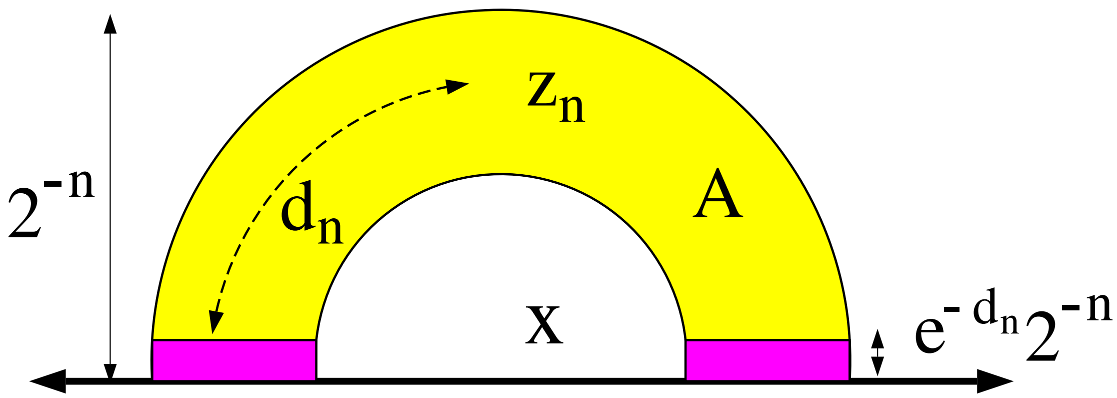
$$\int_{|x-y|<1} \frac{K_h(x) - 1}{|x-y|^2} dx < \infty$$

implies h is differentiable (and conformal) at y.

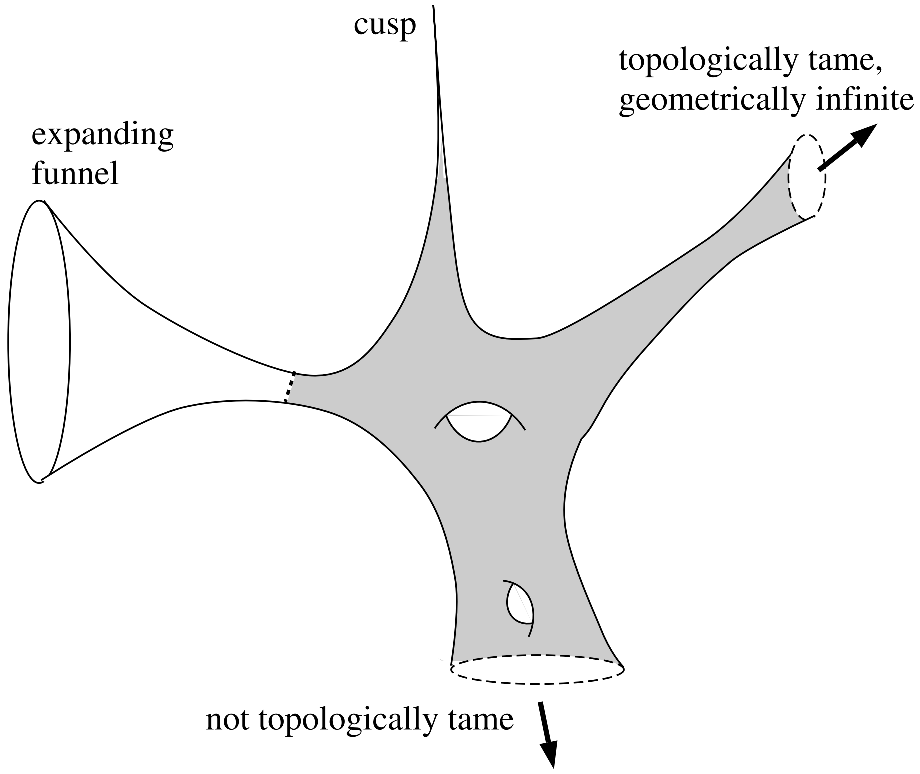
Claim: If $y \in \mathrm{T}$ is endpoint of a geodesic γ which escapes to ∞ at speed $\geq M \log t$, then Lehto's condition is satisfied, hence h is differentiable at y. By our earlier lemma, this is true on a set of positive $\mathcal{H}^\varphi$ measure.

Proof of Claim: If $A = \{x : 2^{-n-1} \leq |x-y| < 2^{-n}\}$, $z_n = y + i2^{-n}$, $d_n = \mathrm{dist(z_n, supp(K_h - 1))}$, then

$$\int_A \frac{K_h(x) - 1}{|x-y|^2} dx \leq C \exp(-C d_n).$$



Suppose G is a discrete group of isometries of the hyperbolic space H^3. This is the same as a discrete group of Möbius transformations acting on the plane R^2. Let $M = \mathrm{H}^3/G$ be the quotient manifold.



Definitions:

Limit set, Λ: accumulation set of any orbit. Corresponds endpoints on R^2 of rays except those escaping in funnels.

Conical limit set, Λ_c: endpoints of recurrent geodesics.

Geometrically finite : only ends are funnels and cusps.

Topologically tame: only ends are funnels, cusps and tame ends.

Convex core: M minus the funnels.

The **Poincaré exponent** of G is

$$\delta = \inf\{s : \sum_{g\in G} \exp(-s\rho(0, g(0))) < \infty\}.$$

Also $\lambda = \delta(d-\delta)$ where λ is base eigenvalue M. By inequalities of Cheeger and Buser $\lambda > 0$ iff

$$h = \inf_{N\subset M} \frac{|\partial N|_d}{|N|_{d+1}} > 0.$$

Theorem: For any hyperbolic manifold with more than one limit point, $\delta = \dim(\Lambda_c) = \dim(\Lambda_b)$.

Theorem: If $d = 2$, G is finitely generated and $\text{area}(\Lambda) = 0$ then $\delta = \dim(\Lambda)$. This always occurs if G is topologically tame.

Theorem: If $d = 2$, G is finitely generated then $\dim(\Lambda) < 2$ iff G is geometrically finite and $\Lambda \neq S^2$.

Ahlfors Conjecture: If $d = 2$, G is finitely generated and $\Lambda \neq S^2$ then $\text{area}(\Lambda) = 0$.

This is known to be true if G is topologically tame.

Equivalent version of Ahlfors Conj.: If G is finitely generated and there is a funnel in M, then Brownian motion on M almost surely enters a funnel.

Theorem: If G is geometrically infinite and has a lower bound on its injectivity radius then Λ has positive $\mathcal{H}^\varphi$ measure for

$$\varphi(t) = t^2 \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}.$$

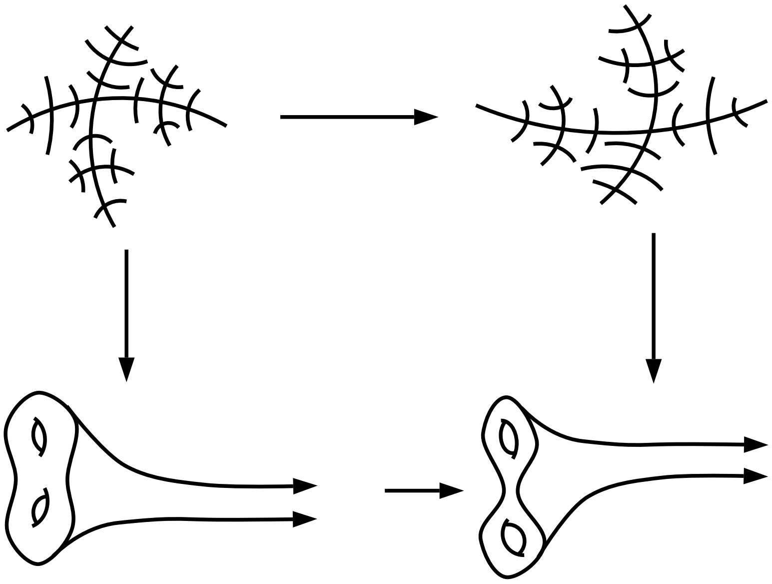
If in addition, G is topologically tame and $\Lambda \neq S^2$, then it has finite $\mathcal{H}^\varphi$ measure.

Sketch of proof: The hypotheses allow us to construct a positive Bloch harmonic function on U by moving the pole of Greens function to infinity. U tends to ∞ in a geometrically infinite end and tends to zero in all geometrically finite ends. Thus the lower bound follows from the analogous result for Bloch martingales.

The topologically tame assumption implies that the gradient of U is bounded away from 0 on average whenever $U > 1$. This implies the approximating martingale has differences bounded away from 0. Thus the upper bound follows from the analogous result for martingales. $\square$

Theorem: Suppose G_1, G_2 are geometrically infinite, topologically tame and have a lower bound on their injectivity radii. If h is quasiconformal conjugacy between them then it is differentiable everywhere on Λ_1 except a set of zero $\mathcal{H}^\varphi$ measure ($\varphi(t) = t^2\sqrt{\log \frac{1}{t} \log \log \log \frac{1}{t}}$).

This is surprising because we can have $K_h \geq 1+\epsilon$ (Lebesgue) almost everywhere. Thus Lehto's integral diverges everywhere! The idea is to extend h from 2 to 3 dimensions and apply 3-dimensional analog of Lehto's result.



Sketch of Proof:

Step 1: By a result of Curt McMullen h extends to a quasi-isometry of H^3 with local distortion

$$K_h(y) \leq 1 + \exp(-C\mathrm{dist}(\mathrm{y}, \mathrm{H}^3 \setminus \mathrm{C}(\Lambda))),$$

i.e., the mapping is exponentially an isometry.

Step 2: We know that at μ almost every point, the distance to the convex core boundary increases like at least $\sqrt{t}/\log^2 t$.

Step 3: Just as before we show that if y is limit point of a geodesic which tends to ∞ in convex core at speed $\geq M \log t$, then

$$\int \frac{K_h(x) - 1}{|x - y|^3} dx < \infty.$$

Step 4: A recent result of Gutlyanskii, Martio and Vuorinen extends Lehto's result and implies differentiability.