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CHAPTER 1

Self-affine sets.

Earlier we discussed criteria which imply that the Hausdorff and Minkowski dimensions of a set agree. In this chapter we will the class of self-affine sets for which the two dimensions usually differ. These sets are invariant under affine maps of the form $(x, y) \to (nx, my)$. We will compute the Minkowski, Hausdorff and packing dimensions as well as the Hausdorff and packing measure in the critical dimensions.

1. Construction and Minkowski dimension.

Suppose $n \geq m$ are integers and divide the unit square $[0, 1]^2$ into $m \times n$ equal rectangles , each with base n^{-1} and height m^{-1}. Choose some subset of these rectangles and throw away the rest. Divide the remaining rectangles into $n \times m$ subrectangles each of base n^{-2} and height m^{-2} and keep those corresponding to the same pattern used above. In Figure 1 we have illustrated the first two steps of the construction when $n = 3$, $m = 2$ and the pattern involves three rectangles: the middle rectangle in the top row and the left and rightmost rectangles in the bottom row.

At the nth stage we have a collection of $n^{-k} \times m^{-k}$ rectangles. To obtain the next level we subdivide each into $n \times m$ rectangles as above keep the ones corresponding to our pattern. Continuing indefinitely gives a compact set which we will denote K. Figure 1.2 shows the

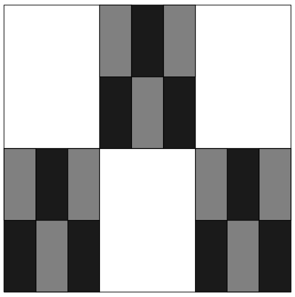

Figure 1.1. The pattern

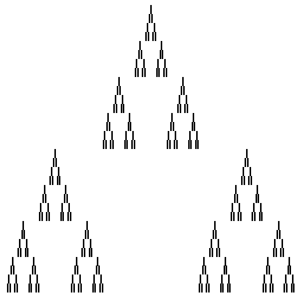

FIGURE 1.2. The McMullen set, S_0

limiting set for the pattern in Figure 1. We shall refer to this particular example as K_0, or as the McMullen set.

Here are two alternate ways of describing the same construction.

1. Given a $n \times m$ matrix A of 0's and 1's we let

$$K(A) = \{(x, y) : A_{x_n, y_n} = 1 \text{ for all } n\}.$$

2. Given a subset $D \subset \{0, \dots, n-1\} \times \{0, \dots, m-1\}$, we let

$$K(D) = \{\sum_{k=1}^{\infty} (a_k n^{-k}, b_k m^{-k}) : (a_k, b_k) \in D \text{ for all } k\}.$$

Thus the McMullen set described above corresponds either to the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

or to the pattern

$$D = \{(0, 0), (1, 1), (2, 0)\}.$$

The matrix representation is nice because it is easy to "see" the pattern in the matrix, but the subset notation is often more convenient (in what follows, sums are often indexed by the elements of D and $\#(D)$, the number of elements in D, will be an important parameter).

A self-affine set associated to the pattern D is obviously the attractor for the maps

$$g_i(x, y) = (x/n, y/m) + (a_i/n, b_i/m),$$

where $\{(a_i, b_i)\}$ are an enumeration of the points in D. If each of these maps were a similarity (which occurs if and only if $n = m$) then S would be a self-similar set and the Hausdorff and Minkowski dimensions would both be equal to $\log r / \log n$ where r is the number of rectangles in our fixed pattern. In general, however, $\dim(S) < \dim_M(S)$.

THEOREM 1.1. *Suppose every row contains a chosen square. Then*

$$dim_M(K(D)) = 1 + \log_n \frac{\#(D)}{m}.$$

The assumption that every row contains a chosen square is necessary. In general if π denote the projection onto the second coordinate, so $\#(\pi(D))$ is the number of occupied rows, we get

$$\dim_M(K(D)) = \log_m \#(\pi(D)) + \log_n \frac{\#(D)}{\#(\pi(D))}.$$

The general case is left as an exercise (see Exercise 1).

THEOREM 1.2. *Suppose every row contains a chosen square. Then*

$$dim(K(D)) = \log_m(\sum_{j=1}^{m} r(j)^{\log_n m}), \tag{1.1}$$

where $r(j)$ is the number of rectangles of the pattern lying in the jth row.

For example, the McMullen set K_0 is formed with $m = 2$, $n = 3$, $r(1) = 1$ and $r(2) = 2$. Thus

$$\dim_M(K_0) = 1 + \log_3 \frac{3}{2} = 1.36907\ldots,$$

$$\dim(K_0) = \log_2(1 + 2^{\log_3 2}) = 1.34968\ldots.$$

The difference arises because the set K_0 contains long, narrow rectangles, some of which lie next to each other. Such "groups" of thin rectangles can be efficiently covered simultaneously, rather than each covered individually. Thus allowing squares of different sizes allows much more efficient coverings. The proof of the first formula is a simple box counting argument. The proof of second is Billingsley's lemma, applied to an appropriately constructed measure. We will give this in the next section.

LEMMA 1.1. Let $r = \#(D)$ be the number of rectangles in the pattern. At stage j we have r^j rectangles of base n^{-j} and height m^{-j} (recall that $n^{-j} \ll m^{-j}$). Let $k = \lfloor \frac{\log n}{\log m} j \rfloor$ be the integer part of $\frac{\log n}{\log m} j$. Then we can cover each rectangle by m^{j-k} squares of size m^{-k} ($\sim n^{-j}$) and this many squares are needed (this is where we use the assumption that every row has a box in it; thus for any generational rectangle R the horizontal projection of $K \cap R$ onto a vertical side of R is the whole side).

Thus the total number of squares of size m^{-k} needed to cover K is $r^j m^{j-k}$ and so

$$\begin{aligned}\dim_M(S) &= \lim_{j\to\infty} \frac{\log r^j m^{k-j}}{\log n^j} \\ &= \lim_{j\to\infty} \frac{j\log r + (k-j)\log m}{j\log n} \\ &= \frac{\log r}{\log n} + \frac{\log n}{\log n} - \frac{\log m}{\log n} \\ &= 1 + \log_n \frac{r}{m}.\end{aligned}$$

□

2. The Hausdorff dimension of self-affine sets

Fix integers $m < n$ and let $\alpha = \frac{\log m}{\log n} < 1$. Following [**McM84**] we use approximate squares to calculate dimension.

DEFINITION 2.1. Suppose $(x, y) \in [0, 1]^2$ have base n and base m expansions $\{x_k\}, \{y_k\}$, the *approximate square* of generation k at (x, y), $Q_k(x, y)$, is defined to be the points $(x', y') \in [0, 1)^2$ such that the first $\lfloor \alpha k \rfloor$ digits in the base n expansions of x and x' coincide, and the first k digits in the base m expansions of y and y' coincide. We refer to the rectangle $Q_k(x, y)$ as an *approximate square of generation* r since its width $n^{-\alpha k}$ and height m^{-k} satisfy:

$$n^{-\alpha k} \le m^{-r} < m \cdot n^{-\alpha k},$$

and hence

$$n^{-\alpha k} \le \operatorname{diam}[Q(\omega)] \le n^{1-\alpha k}.$$

In the definition of Hausdorff measure, we can restrict attention to covers by such approximate squares since any set of diameter less than m^{-k} can be covered by a bounded number of approximate squares Q_k.

THEOREM 1.2). Any probability vector $\{p(d) : d \in D\} = p$ defines a probability measure μ_p on $K(D)$ which is the image of the product measure $p^{\mathbb{N}}$ under the representation map $R : D^{\mathbb{N}} \to K(D)$ given by

$$\{(a_k, b_k)\}_{k=1}^{\infty} \to \sum_{k=1}^{\infty} (a_k n^{-k}, b_k m^{-k}).$$

Any such measure is supported on $K(D)$, so the dimensions of these measures all give lower bounds for the dimension of $K(D)$. We shall show that the supremum of these dimensions is exactly $\dim(K(D))$.

In fact, we will restrict attention to measures coming from probability vectors p such that

$$p(d) \quad \text{depends only on the second coordinate of } d, \tag{2.1}$$

i.e., all rectangles in the same row get the same mass.

Suppose $\{x_\nu\}$, $\{y_\nu\}$ are the n-ary and m-ary expansions of x and y, We claim that,

$$\mu_p[Q_k(x,y)] = \prod_{\nu=1}^{k} p(x_\nu, y_\nu) \prod_{\nu=\alpha k+1}^{\ell} a(y_\nu). \tag{2.2}$$

To see this, note that the $n^{-k} \times m^{-k}$ rectangle defined by specifying the first k digits in the base n expansion of x and the first k digits in the base m expansion of y has μ_p-measure $\prod_{\nu=1}^{k} p(x_\nu, y_\nu)$. The approximate square $Q_k(x,y)$ contains

$$a(y_{\alpha k+1}) \cdot a(y_{\alpha k+2}) \cdot \cdots \cdot a(y_k),$$

such rectangles, all with the same μ_p-measure by our assumption (2.1), and so (2.2) follows. Now take logarithms in (2.2),

$$\log(\mu_p[Q_k(x,y)]) = \sum_{\nu=1}^{k} \log p(x_\nu, y_\nu) + \sum_{\nu=\alpha k+1}^{k} \log a(y_\nu) \tag{2.3}$$

Since $\{(x_\nu, y_\nu)\}_{\nu \geq 1}$ are i.i.d random variables with respect to μ_p, the strong law of large numbers yields for μ_p almost every (x,y):

$$\lim_{k\to\infty} \frac{1}{k} \log(\mu_p[Q_k(x,y)]) = \sum_{d\in D} p(d) \log p(d) + (1-\alpha) \sum_{d \in D} p(d) \log a(d)$$

where for $d = (i,j) \in D$ we denote $a(d) = a(j)$, the number of elements in row j. Now Billingsley's lemma implies that

$$\dim[\mu_p] = \frac{1}{\log m} \sum_{d\in D} p(d)[\log \frac{1}{p(d)} + \log(a(d)^{\alpha-1})]. \tag{2.4}$$

An easy and well known calculation says that if $\{a_k\}_1^n$ are positive then

$$F(p) = \sum_{k=1}^{n} p_k \log \frac{1}{p_k} + \sum_{k=1}^{n} p_k a_k,$$

attains it maximum at $p_k = a_k / \sum_k e^{a_k}$, $k = 1, \ldots, n$. This is known as Boltzman's principle. In the case at hand it says that $\dim(\mu_p)$ will be maximized if

$$p(d) = \frac{1}{Z} a(d)^{\alpha-1} \tag{2.5}$$

where

$$Z = \sum_{d \in D} a(d)^{\alpha-1} = \sum_{j=0}^{m-1} a(j)^{\alpha}.$$

For the rest of the proof we fix this choice of p and write μ for μ_p. Note that

$$\dim[\mu] = \log_m(Z),$$

so this is a lower bound for $\dim[K(D)]$. To obtain an upper bound, denote $S_k(x,y) = \sum_{\nu=1}^{k} \log a(x_\nu, y_\nu)$. Note that $\frac{1}{k}S_k(x,y)$ is uniformly bounded. Using (2.5), rewrite (2.3) in the form

$$\begin{aligned}
\log \mu[Q_k(x,y)] &= \sum_{\nu=1}^{k} \log \frac{1}{Z} a(y_k)^{\alpha-1} + (\sum_{\nu=1}^{k} \log a(y_\nu) - \sum_{\nu=1}^{\alpha k} \log a(y_\nu)) \\
&= -\sum_{\nu=1}^{k} \log Z + +(\alpha-1)S_k(x,y) + S_k(x,y) - S_{\alpha k}(x,y).
\end{aligned}$$

Thus

$$\begin{aligned}
\log \mu[Q_k(x,y)] + k \log Z &= (\alpha-1)S_k(x,y) + S_k(x,y) - S_{\alpha k}(x,y) \\
&= \alpha S_k(x,y) - S_{\alpha k}(x,y).
\end{aligned} \tag{2.6}$$

Therefore,

$$\frac{1}{\alpha k} \log \mu[Q_k(x,y)] + \frac{1}{\alpha} \log Z = \frac{S_k(x,y)}{k} - \frac{S_{\alpha k}(x,y)}{\alpha k} \tag{2.7}$$

Summing the right hand side of (2.7) along $k = \alpha^{-1}, \alpha^{-2}, \ldots,$, gives a telescoping series. Since $S_k(x,y)/k$ remains bounded for all k, it is easy to see

$$\limsup_{k \to \infty} \left[\frac{S_k(x,y)}{k} - \frac{S_{\alpha k}(x,y)}{\alpha k} \right] \geq 0$$

since otherwise the sum would tend to $-\infty$. Thus by (2.7), for every $(x,y) \in K(D)$ we have

$$\limsup_{k \to \infty} \log \mu[Q_k(x,y)] + k \log Z \geq 0.$$

Exponentiating,

$$\limsup_{k \to \infty} \mu[Q_k(x,y)] Z^k \geq 1,$$

or equivalently,

$$\liminf_{k \to \infty} \frac{m^{-k \log_m Z}}{\mu[Q_k(x,y)]} \leq 1,$$

Hence for any $\eta > \log_m Z$ we have

$$\liminf_{k\to\infty} \frac{m^{-k\eta}}{\mu[Q_k(x,y)]} = 0$$

and hence by the Frostman-Rogers-Taylor density theorem $\dim[K(D)] \leq \eta$. Combining this with the lower bound above, we get

$$\dim[K(D)] = \frac{\log Z}{\log m}$$

which is (1.1). □

3. A dichotomy for Hausdorff measure

Definition 3.1. The digit set $D \subset \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$ has *uniform horizontal fibers* if all nonempty rows in D have the same cardinality i.e., for all $j \in \pi(D)$ the preimages $\pi^{-1}(j)$ have identical cardinalities (otherwise, D has *nonuniform* horizontal fibers).

In his paper, McMullen points out that if D has uniform horizontal fibers then $K(D)$ has positive finite Hausdorff measure in its dimension, and asks what is the Hausdorff measure in other cases. A short and elegant argument found independently by Bedford and Mandelbrot (unpublished) and by Gatzouras and Lalley in [**GL92**] shows that in these cases the Hausdorff measure of $K(D)$ in any dimension is either zero or infinity. Here we prove

Theorem 3.2. *For any gauge function φ, the Hausdorff measure $\mathcal{H}^\varphi[K(D)]$ is either zero or infinity, provided D has nonuniform horizontal fibers.*

Proof. Assume that φ is a gauge function satisfying

$$0 < \mathcal{H}^\varphi[K(D)] < \infty \tag{3.1}$$

Since the intersections of $K(D)$ with the $|D|$ rectangles of the first generation are translates of each other and the Hausdorff measure $\mathcal{H}^\varphi$ is translation invariant, its restriction to $K(D)$ must assign all these rectangles the same measure. Continuing in the same fashion inside each rectangle, it follows that the restriction of $\mathcal{H}^\varphi$ to $K(D)$ is a constant multiple of μ_u where $u(d) = \frac{1}{|D|}$ for $d \in D$, i.e., u is the uniform probability vector (μ_u is one of the measures μ_p considered in the previous proof). The expression (2.4) for $\dim(\mu_p)$ shows that the function $p \to \dim[\mu_p]$, defined on probability vectors indexed by D, is *strictly* concave and hence attains a unique maximum at the vector p given in

(2.5). Since D has nonuniform horizontal fibers, this is *not* the uniform vector u and therefore

$$\dim(\mu_u) < \dim(\mu_p) = \gamma = \dim[K(D)].$$

For $\varphi(t) = t^\gamma$ (the case considered in [**GL92**]) this yields the desired contradiction immediately; for a general gauge function φ one further remark is needed. Choose β such that $\dim[\mu_u] < \beta < \gamma = \dim[K(D)]$. By Egorov's theorem there is a set $E \subset K(D)$ of positive μ_u-measure such that the convergence

$$\frac{-1}{k \log m} \log \mu_u[Q_k(x,y)] \to \dim[\mu_u]$$

as $k \to \infty$, is uniform on E. Therefore, for large k, E may be covered by $m^{k\beta}$ approximate squares Q_k.

Since the restriction of $\mathcal{H}^\varphi$ to $K(D)$ is a multiple of μ_u, necessarily $\mathcal{H}^\varphi(E) > 0$ so the coverings mentioned above force

$$\liminf_{t \downarrow 0} \frac{\varphi(t)}{t^\beta} > 0.$$

This implies that $\mathcal{H}^\varphi$ assigns infinite measure to any set of dimension strictly greater than β and in particular to $K(D)$. □

The proof of Theorem 3.2 above gives no hint whether $\mathcal{H}^\gamma[K(D)]$ is zero or infinity, but we shall prove in the next section that it is infinity.

REMARK . The expression (2.7) for the μ-measure of an approximate square implies that if D has nonuniform horizontal fibers then μ is carried by a subset of $K(D)$ of zero γ-dimensional measure. Indeed, by the law of the iterated logarithm, the set

$$K_* = \left\{ (x,y) \in K(D) : \limsup_{k \to \infty} \frac{\alpha S_k(x,y) - S_{\alpha k}(x,y)}{(k \log\log k)^{\frac{1}{2}}} > 0 \right\}$$

has full μ-measure. By (2.7), $\limsup_{k\to\infty} \mu[Q_k(x,y)] m^{k\gamma} = \infty$ for $(x,y) \in K_*$ and the Rogers-Taylor density theorem yields $\mathcal{H}^\gamma(K_*) = 0$. In [**Urb90**], M. Urbanski proves refinements of this involving gauge functions.

4. The Hausdorff measure is infinite

We saw in the last section that the Hausdorff measure of a self-affine set (in the nonuniform case) must be either 0 or ∞. Now we will prove that it must be the latter.

THEOREM 4.1. *Assume the digit set D has nonuniform horizontal fibers and let $\gamma = dim(K(D))$. Then*

$$\mathcal{H}^{\gamma}[K(D)] = \infty.$$

and furthermore $K(D)$ is not σ-finite for $\mathcal{H}^{\gamma}$.

Note the contrast with the well known self-similar constructions which have positive and finite Hausdorff measure in their dimension (see Section **??**). More precise information can be expressed using gauge functions:

THEOREM 4.2. *If D has nonuniform horizontal fibers and $\gamma = dim(K(D))$, then $\mathcal{H}^{\varphi}[K(D)] = \infty$ for*

$$\varphi(t) = t^{\gamma} \exp\left[-c\frac{|\log t|}{(\log|\log t|)^2}\right]$$

provided $c > 0$ is sufficiently small, and furthermore $K(D)$ is not σ-finite with respect to $\mathcal{H}^{\varphi}$.

Let us start with the motivation for the proof of Theorem 4.1. If the horizontal and vertical expansion factors n and m were equal, the factors $a(y_{\nu})$ would disappear from the formula (2.2) for the μ_p-measure of an approximate square and to maximize $\dim(\mu_p)$ we would take p to be uniform. In our case $m < n$ and the factors $a(y_{\nu})$ in (2.2) occur only for $\nu > \alpha k$. Thus it is reasonable to perturb McMullen's measure μ by making it more uniform in the initial generations. Thus we obtain a measure on $K(D)$ which is slightly "smoother" then the one previously constructed, which implies $K(D)$ is "large" in some sense.

Fine-tuning this idea leads to the following. Assume D has nonuniform horizontal fibers, and let p be the probability vector given by (2.5). Let $p^{(1)} = p$. Fix a small $\delta > 0$. For each $k \geq 2$ define a probability vector $p^{(k)}$ on D by

$$p^{(k)} = \left(\frac{\delta}{\log k}\right) u + \left(1 - \frac{\delta}{\log k}\right) p \tag{4.1}$$

where u is the uniform probability vector on D. Denote by λ_{δ} the image under the representation map R_T of the product measure $\prod_{k=1}^{\infty} p^{(k)}$ on $D^{\mathbf{N}}$.

THEOREM 4.3. *With the notation above, there exist $\delta > 0$ and $c > 0$ such that the measure λ_{δ} satisfies*

$$\lim_{k\to\infty} \log \lambda_{\delta}[Q_k(x,y)] - \log \varphi(m^{-k}) = -\infty \tag{4.2}$$

for $\lambda_\delta - a.e.(x,y) \in K(D)$, *where*

$$\varphi(t) = t^\gamma exp\left(-c\frac{|\log t|}{[\log|\log t|]^2}\right) \tag{4.3}$$

and

$$\gamma = dim[K(D)].$$

Proof. In complete analogy with (2.3) above, we obtain

$$\log \lambda_\delta[Q_k(x,y)] = \sum_{\nu=1}^{k} \log p^{(\nu)}(x_\nu, y_\nu) + \sum_{\nu=\alpha k+1}^{k} \log a(y_\nu)$$

On the right hand side we have sums of independent random variables, which are typically close to their expectations

$$\begin{aligned} f_\delta(k) &\stackrel{\text{def}}{=} \int_{K(D)} \log \lambda_\delta[Q_k(x,y)]d\lambda_\delta \\ &= \sum_{\nu=1}^{k}\sum_{d\in D} p^{(\nu)}(d)\log p^{(\nu)}(d) + \sum_{\nu=\alpha k+1}^{k}\sum_{d\in D} p^{(\nu)}(d)\log a(d). \end{aligned} \tag{4.4}$$

More precisely, by Kolmogorov's law of the iterated logarithm ([**Hoe63**] §18.3),

$$|\log \lambda_\delta[Q_k(x,y)] - f_\delta(k)| = O((k\log\log k)^{1/2}) \tag{4.5}$$

for λ_δ almost all $(x,y) \in K(D)$. In fact, weaker estimates $(O(k^{1-\varepsilon}))$ would suffice for our purposes. For any probability vector $\{q(d)\}_{d\in D}$ denote

$$J(q) = \sum_{d\in D} q(d)\log q(d).$$

Observe that for $\varepsilon > 0$:

$$J((1-\varepsilon)p + \varepsilon u) = J(p) - \varepsilon\Delta + O(\varepsilon^2)$$

where

$$\Delta = \sum_{d\in D}[p(d) - u(d)]\log p(d)$$

is strictly positive because

$$\Delta = \frac{1}{2|D|}\sum_{d\in D}\sum_{d'\in D}[p(d) - p(d')][\log p(d) - \log p(d')]$$

and $p \neq u$.

Returning to (4.5), we find that

$$\begin{aligned} f_\delta(k) &= \sum_{\nu=1}^{k} J(p^{(\nu)}) + \sum_{\nu=\alpha k+1}^{k} \sum_{d\in D} p^{(\nu)}(d)\log a(d) \\ &= kJ(p) - \sum_{\nu=2}^{k} \frac{\delta}{\log\nu}\Delta + O\left(\sum_{\nu=1}^{k}\left(\frac{\delta}{\log\nu}\right)^2\right) \\ &\quad +(1-\alpha)k\sum_{d\in D} p(d)\log a(d) \\ &\quad + \sum_{\nu=\alpha k+1}^{k} \frac{\delta}{\log\nu}\sum_{d\in D}[u(d)-p(d)]\log a(d) + O(1). \end{aligned}$$

Recalling that $\log p(d) = (\alpha-1)\log a(d) - \log Z$ we get

$$\begin{aligned} f_\delta(k)k\log Z &= -\Delta\delta\sum_{\nu=2}^{k}\frac{1}{\log\nu} + \frac{k}{\log^2 k}O(\delta^2) \\ &\quad +\frac{\Delta\delta}{1-\alpha}\sum_{\nu=\alpha k+1}^{k}\frac{1}{\log\nu} + O(1). \end{aligned}$$

Therefore,

$$\begin{aligned} (4.6)\quad f_\delta(k) + k\log Z &= \frac{\Delta\delta}{1-\alpha}\left[(\alpha-1)\int_2^k \frac{dt}{\log t} + \int_{\alpha k}^{k}\frac{dt}{\log t}\right] \\ &\quad +\frac{k}{\log^2 k}O(\delta^2) + O(1). \end{aligned}$$

A change of variables shows that

$$\begin{aligned} (\alpha-1)\int_2^k\frac{dt}{\log t} + \int_{\alpha k}^{k}\frac{dt}{\log t} &= \alpha\int_2^k\frac{dt}{\log t} - \int_{2\alpha^{-1}}^{k}\frac{\alpha ds}{\log(\alpha s)} \\ &= \alpha\log\alpha\int_{2\alpha^{-1}}^{k}\frac{ds}{\log s\log(\alpha s)} + O(1) \\ &\le \alpha\log\alpha\frac{k}{\log^2(k)} + O(1) \end{aligned}$$

since $\log\alpha < 0$. Thus if we choose $\delta > 0$ and $c_1 > 0$ small enough, (4.6) implies

$$f_\delta(k) + k\log Z \le -2c_1\frac{k}{\log^2 k} + O(1).$$

Utilizing (4.5) we infer that for λ_δ a.e. (x,y),

$$\log \lambda_\delta[Q_k(x,y)] \le -k \log Z - c_1 \frac{k}{\log^2 k} + O(1). \tag{4.7}$$

Finally, if c in (4.3) is small (e.g., $2c \log m \le c_1$) then (4.2) follows from (4.7). $\square$

THEOREM 4.1 AND THEOREM 4.2. Choose $\delta > 0$ and $c > 0$ such that the measure λ_δ defined in the previous proposition satisfies (4.2). The set

$$K_c = \left\{ (x,y) \in K(D) \mid \lim_{k\to\infty} \frac{\varphi(m^{-k})}{\lambda_\delta[Q_k(x,y)]} = \infty \right\}$$

has full λ_δ-measure by (4.2). The Rogers-Taylor density theorem implies that $\mathcal{H}^\varphi(K(D)) \ge \mathcal{H}^\varphi(K_c) = \infty$, proving Theorem 4.2. Since $\varphi(t) \le t^\gamma$ for small $t > 0$, it follows that $\mathcal{H}^\gamma[K(D)] = \infty$ proving Theorem 4.1. (Non σ-finiteness follows easily by changing the value of c). $\square$

5. Packing dimension and measure of self-affine sets

In Chapter 3 we defined packing dimension Pdim, the packing measure $\mathcal{P}_\varphi$ and the packing pre-measure $\tilde{\mathcal{P}}_\varphi$. We also proved (Theorem **??**): if K is a compact set which is the attractor corresponding to strictly contracting, bi-Lipschitz mappings $\{f_1, \dots f_n\}$, then the packing dimension of K equals its upper Minkowski dimension. We have previously noted that a self-affine set is the attractor for a set of affine contractions so this result applies, i.e.,

LEMMA 5.1. *Let $D \subset \{0,\dots,n-1\} \times \{0,\dots,m\}$ be pattern. Then*

$$Pdim(K(D)) = dim_M(K(D)) = log_m \#(\pi(D)) + \log_n \frac{\#(D)}{\#(\pi(D))}.$$

Next we want to consider what the packing measure is at the critical dimension.

THEOREM 5.2. *Let $\theta = Pdim(K(D))$.*

1. *If D has nonuniform horizontal fibers then $\mathcal{P}_\theta[K(D)] = \infty$ and furthermore $K(D)$ is not σ-finite for $\mathcal{P}_\theta$.*
2. *If D has uniform horizontal fibers then*

$$0 < \mathcal{P}_\theta[K(D)] < \infty.$$

Part (2) is standard and is included only for completeness. Denote by π projection to the second coordinate and let $\beta = \frac{\log n}{\log m} > 1$. Next we recall the idea of an approximate square from Section 2. Recall that $Q_k(x,y)$ consisted of the points whose base n and m expansions agreed with those of x and y in the first $\lfloor k \log_n m \rfloor$ and k positions respectively. In fact, to define $Q_k(x,y)$ all we need to know are the initial segments of the expansions.

DEFINITION 5.3. Let Ω_r be the collection of strings

$$\omega = (x_1, x_2, \ldots, x_r; y_1, y_2, \ldots, y_{[\beta r]}) \tag{5.1}$$

such that $(x_i, y_i) \in D$ for $1 \le i \le r$ and $y_i \in \pi(D)$ for $r < i \le [\beta r]$. For ω given by (5.1) we denote by $Q(\omega)$ the set of points

$$\left(\sum_{i=1}^{\infty} x_i n^{-i}, \sum_{i=1}^{\infty} y_i m^{-i} \right)$$

such that $(x_1, \ldots, x_r; y_1, \ldots, y_{[\beta r]}) = \omega$ and $0 \le x_i < n, 0 \le y_i < m$ for all $i \ge 1$.

The strings in Ω_r are exactly the initial segments of the n-ary and m-ary expansions of points in $K(D)$ and the squares $Q(\omega)$ are all of the form $Q_k(x,y)$ for some $(x,y) \in K(D)$. In principle, a natural r'th generation packing of $K(D)$ is obtained by taking a maximal ball centered at a point of $K(D)$ in each approximate square of generation r.

For some digit sets D, this is problematic since in approximate squares of a high generation, all points of $K(D)$ may lie near the boundary of the approximate square. Discarding some of the approximate squares removes the difficulty. For ω given by (5.1) and $1 \le i \le r$, write $X_i(\omega) = x_i$. For $1 \le i \le [\beta r]$ let $Y_i(\omega) = y_i$.

LEMMA 5.4. *Assume that the digit set D is not contained in a single row or a single column, so there exist digits $(\xi, \eta), (\xi', \eta') \in D$ such that $\xi \ne \xi'$ and $\eta \ne \eta'$. Denote*

$$\hat{\Omega}_r = \{\omega \in \Omega_r : Y_{r+1}(\omega) = \eta, Y_{r+2}(\omega) = \eta'\}$$

so the cardinality of $\hat{\Omega}_r$ is

$$|\hat{\Omega}_r| = |D|^r |\pi(D)|^{[\beta r] - r - 2}.$$

Then for each $\omega \in \hat{\Omega}_r$ the approximate square $Q(\omega)$ contains an open disk $B(\omega)$ centered at a point of $K(D)$ such that diam$[B(\omega)] > n^{-r-2}$.

PROOF. Let $\omega = (x_1, \ldots, x_r; y_1, \ldots, y_{[\beta r]}) \in \hat{\Omega}_r$. Choose a point $(x,y) \in K(D)$ whose expansions begin with ω and continue with $x_{r+1} = \xi, x_{r+2} = \xi'$, $y_{[\beta r]+1} = \eta \text{and} y_{[\beta r]+2} = \eta'$ (these choices of x_{r+1}, x_{r+2} are possible because $\omega \in \hat{\Omega}_r$). Clearly the distance from (x,y) to the boundary of $Q(\omega)$ is at least n^{-r-2} so we can center $B(\omega)$ at (x,y). $\square$

REMARK . In the cases which were excluded in the statement of the lemma, D has uniform horizontal fibers and moreover $K(D)$ is a self-similar set on a line. Thus in these cases $\mathcal{P}_\theta(K(D))$ is positive and finite for $\theta = \dim_P[K(D)]$.

REMARK . For the McMullen set corresponding to the pattern

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

the assertion of the lemma holds for all $\omega \in \Omega_r$ (and not just for $\omega \in \hat{\Omega}_r$). However, this is no longer true if we take the pattern

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

For this choice, taking $\omega \in \Omega_r$ such that $Y_{r+1}(\omega) = Y_{r+2}(\omega) = \cdots = Y_{r+\ell}(\omega) = 1$ yields an approximate square $Q(\omega)$ such that any disk contained in it which is centered at a point of $K(D)$ must have radius at most $n^{-r-\ell}$.

REMARK . Let $\theta = \log_n(|D|^r|\pi(D)|^{\beta-1})$. By Lemma 5.1 and Theorem 1.1 we know that $\dim_P[K(D)] = \theta$. The packings $\{B(\omega) : \omega \in \hat{\Omega}_r\}$ which arise from the previous lemma show that

$$\begin{aligned} \tilde{\mathcal{P}}_\theta[K(D)] &\geq \limsup_{r\to\infty} \sum_{\omega\in\hat{\Omega}_r} |B(\omega)|^\theta \\ &\geq \limsup_{r\to\infty} |\hat{\Omega}_r| \cdot n^{-(r+2)\theta} \\ &\geq \limsup_{r\to\infty} c\#(D)^r\#(\pi(D))^{(\beta-1)r}n^{-r\theta} \\ &= c \end{aligned}$$

by the definition of θ, where $c > 0$ does not depend on r.

REMARK . The packings of $K(D)$ above are essentially optimal packings of a *fixed* size. Indeed if $B_1, \ldots, B_N$ are disjoint discs of radius ε centered on $K(D)$, choose r such that $n^{1-r} \leq \varepsilon < n^{2-r}$. For every $\omega \in \Omega_r$, at most one disc B_j can be centered in the approximate square $Q(\omega)$ so that

$$N \leq |\Omega_r| \leq |D|^r|\pi(D)|^{(\beta-1)r} = n^{r\theta} \leq n^{-2} \cdot \varepsilon^{-\theta}.$$

Thus to obtain better packings we must use discs of many different sizes.

THEOREM 5.2. Proof of (1). We will only give the proof for the McMullen set corresponding to the pattern

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The general case involves extra difficulties due to boundary effects and can be found in [**Per94b**]. In this case we have $\beta = \frac{\log 3}{\log 2}$ and

$$\theta = \dim_P(K) = 1 + \log_3 \frac{3}{2}.$$

For each $r \geq 1$ the collection of strings $\hat{\Omega}_r = \{\omega \in \Omega_r : Y_{r+1}(\omega) = 0, Y_{r+2}(\omega) = 1\}$ has cardinality

$$3^r 2^{\lfloor \beta r \rfloor] - r - 2}.$$

Let ν_r be the normalized counting measure on $\hat{\Omega}_r$. The coordinate functions Y_i (defined immediately before Lemma 5.4) are easily seen to be independent random variables on $\langle \hat{\Omega}_r, \nu_r \rangle$. The key observation of this proof is that their distribution changes at $i = r$.

$$\int Y_i d\nu_r = \nu_r\{\omega \in \hat{\Omega}_r : Y_i(\omega) = 1\} = \begin{cases} 1/3 & i \leq r \\ 1/2 & r+2 < i \leq [\beta r] \end{cases}.$$

This, together with the law of large numbers enables us to combine packings from different scales. Let $\{r(j)\}_{j=1}^{J}$ be an increasing sequence of integers satisfying $\beta r(j) < r(j+1)$ for all j.

Define inductively sets $\Omega^*_{r(j)}$ as follows: $\Omega^*_{r(j)}$ consists of all strings $\omega \in \hat{\Omega}_{r(j)}$ satisfying

$$\sum_{i=r(j)+3}^{[\beta r(j)]} \left(Y_i(\omega) - \frac{5}{12} \right) > 0 \tag{5.2}$$

and

$$\sum_{i=r(k)+3}^{[\beta r(k)]} \left(Y_i(\omega) - \frac{5}{12} \right) < 0 \tag{5.3}$$

for all $1 \leq k < j$. (For $j = 1$, condition (5.3) is vacuous).

For each $j \leq J$, there are only j requirements in (5.2) and (5.3); Employing the weak law of large numbers, by choosing $r(1)$ large enough we can ensure that

$$\nu_{r(j)}[\Omega^*_{r(j)}] \geq \frac{1}{2} \qquad \text{for all} \qquad j \leq J. \tag{5.4}$$

As discussed in Remark 3 following Lemma 5.4,

$$\sum_{\omega \in \hat{\Omega}_r} [\text{diam } B(\omega)]^\theta \geq c$$

for some constant $c > 0$ and any r. Taking $r = r(j)$ and using (5.4) we get

$$\sum_{j=1}^{J} \sum_{\omega \in \Omega^*_{r(j)}} [\text{diam } B(\omega)]^\theta \geq \frac{Jc}{2} \tag{5.5}$$

Now the restrictions (5.2) and (5.3) imply that for any two strings $\omega \in \Omega^*_{r(j)}$ and $\omega' \in \Omega^*_{r(k)}$ with $k < j$, the corresponding approximate squares $Q(\omega)$ and $Q(\omega')$ have disjoint interiors. Thus the collection $\bigcup_{j=1}^{J} \{B(\omega) : \omega \in \Omega^*_{r(j)}\}$ furnishes a packing of K. Since we may take $r(1)$ as large as we please, (5.5) implies

$$\tilde{\mathcal{P}}_\theta(K) \geq \frac{Jc}{2}.$$

Since J is arbitrary, $\tilde{\mathcal{P}}_\theta(K) = \infty$ and Lemma 5.1 yields $\mathcal{P}_\theta(K) = \infty$.

Proof of (2). Take $(\tilde{X}_i, \tilde{Y}_i)$ for $i \geq 1$ to be i.i.d random variables uniformly distributed on the digit set D and denote by μ the distribution of

$$\left(\sum_{i=1}^{\infty} \tilde{X}_i n^{-i}, \sum_{i=1}^{\infty} \tilde{Y}_i m^{-i} \right).$$

Then μ is a probability measure on $K(D)$ which assigns each approximate square $Q(\omega)$ for $\omega \in \Omega_r$ the same mass:

$$\mu[Q(\omega)] = \left(\frac{|D|}{\pi(D)} \right)^{[\beta r] - r} |D|^{-[\beta r]}$$

Here we used the assumption that D has uniform horizontal fibers.

Let $z \in K(D)$ and let $B(z, \varepsilon)$ be a disk of radius ε centered at z. Choose an integer r such that

$$n^{1-r} < \varepsilon \leq n^{2-r}.$$

Then $z \in Q(\omega) \subset B(z, \varepsilon)$ for some $\omega \in \Omega_r$ and therefore

$$\mu[B(z, \varepsilon)] \geq |D|^{-r} |\pi(D)|^{r - \beta r} = n^{-r\theta} \geq n^{-2\theta} \varepsilon^\theta. \tag{5.6}$$

On the other hand, by projecting $B(z, \varepsilon)$ to the axis, it is easy to see that $B(z, \varepsilon) \cap K(D)$ may be covered by a bounded number of approximate squares $Q(\omega)$ with $\omega \in \Omega_r$. Consequently

$$\mu[B(z, \varepsilon)] \leq C_1 \varepsilon^\theta \tag{5.7}$$

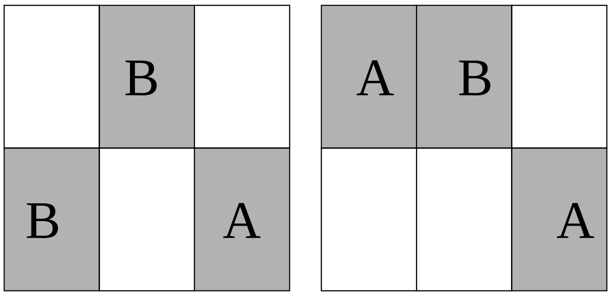

FIGURE 6.1. The "A" and "B" boxes in the finite type definition

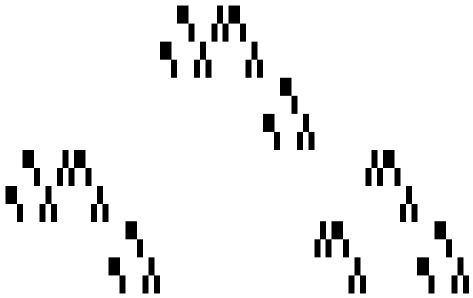

FIGURE 6.2. A Finite type, self-affine Cantor set (4 generations)

for some constant $C_1 = C_1(n, m)$. By Theorem **??** of Chapter 3 the inequalities (5.6) and (5.7) imply that $0 < \mathcal{P}_\theta[K(D)] < \infty$, as desired. $\square$

6. Self-Affine sets of finite type

There is a more complicated construction which generalizes the self-affine sets, and which is analogous to the shifts of finite type we considered earlier. In the earlier construction we had a single collection of squares which we called R. Here we have a set of patterns $\{R_1, \ldots, R_N\}$. Each pattern consists of a collection of level 1 squares and each square is labeled with one of the numbers $\{1, \ldots, N\}$ (repeats allowed). We start by assigning the unit square a label in $\{1, \ldots, N\}$, say J. We then divide the unit square into $n \times m$ rectangles and keep those corresponding to the pattern R_J. Each of the subsquares that we keep has a label assigned to it by the pattern R_J and we subdivide the squares using the appropriate patterns.

In Figure 6.1 we have two patterns labeled "A" and "B" respectively. Figure 6.2 shows the set generated after four generations by these patterns if we assume the unit square is type A.

The Minkowski dimension of these sets can be computed in terms of the spectral radius of certain matrices and the Hausdorff dimension in terms of the behavior of certain random products of matrices. For details the reader is referred to [].

7. Notes

Exercises

EXERCISE 1. Show that if only $R \le m$ of the rows are occupied then the Minkowski dimension of the self-affine set in Lemma 1.1 is

$$\log_m \#(\pi(D)) + \log_n \frac{\#(D)}{\#(\pi(D))}.$$

EXERCISE 2. Prove that the Hausdorff and Minkowski dimensions of $K(D)$ agree if and only if D has uniform horizontal fibers.

EXERCISE 3. Show that Lemma 1.2 is still correct even if every row does not contain a chosen square (we interpret $0^{\log_n m}$ as 0).

EXERCISE 4. Construct a finite-type self-affine set which is the graph of a continuous real valued function. Can this be done with a regular (not finite type) set?

EXERCISE 5. Let S_0 be the McMullen set. What is $\sup \dim(\tilde{S})$ as $\tilde{S}$ ranges over all microsets of S_0?

EXERCISE 6. Compute the dimension of a self-affine set of finite type if we assume the number of rectangles in corresponding rows is the same for each pattern.

EXERCISE 7. Prove Boltzman's principle: if $\{a_k\}_1^n$ are positive then

$$F(p) = \sum_{k=1}^n p_k \log \frac{1}{p_k} + \sum_{k=1}^n p_k a_k,$$

attains it maximum at $p_k = a_k / \sum_k e^{a_k}$, $k = 1, \dots, n$.

EXERCISE 8. Suppose K is compact and is $T_{a,b}$ invariant. How big can the difference $\dim_M(K) - dim(K)$ be?

EXERCISE 9. Fix $0 < \alpha < 1$. Use self-affine sets to construct a set K such that almost every vertical cross section has dimension 0 and almost every horizontal cross section has dimension α.

EXERCISE 10. Fix $0 < \alpha, \beta < 1$. Use self-affine sets to construct a set K such that almost every vertical cross section has dimension β and almost every horizontal cross section has dimension α.

EXERCISE 11. How small can the dimension of K be in the previous exercise? (By the slicing theorem it must be at least $1 + \max(\alpha, \beta)$. Can it be this small?)

EXERCISE 12. Suppose K is a finite type self-affine set and let $n(i,j)$ be the number rows in pattern i that have j elements. If $n(i,j)$ is independent of i show that

$$\dim(K) = \log_m \sum_j n(i,j) j^{\log_n m}.$$

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