Proceedings of 20^{th} Gökova
Geometry-Topology Conference
pp. 120 – 146

Asymptotic Plateau problem: a survey

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ABSTRACT. This is a survey of old and recent results about the asymptotic Plateau problem. Our aim is to give a fairly complete picture of the field, and present the current situation.

CONTENTS

Key words and phrases. Asymptotic Plateau problem, minimal surfaces, hyperbolic space, area minimizing surfaces, constant mean curvature surfaces.

The author is partially supported by EU-FP7 Grant IRG-226062, TUBITAK Grant 109T685 and TUBA-GEBIP Award.

1. Introduction

The asymptotic Plateau problem in hyperbolic space basically asks the existence of an area minimizing submanifold $\Sigma \subset \mathbf{H}^{n+1}$ asymptotic to given submanifold $\Gamma \subset S^n_\infty(\mathbf{H}^{n+1})$. In this survey article, we will cover old and recent results on the problem. Most of the time, we will give the essential ideas of the proofs. Our aim is to give a nice expository introduction for the interested researchers, and to present a picture of this growing field.

2. Preliminaries

In this section, we will overview the basic results which we will use in the following sections. First, we will give the definitions of area minimizing surfaces. The first set of the definitions are about compact submanifolds. The second set of the definitions are their generalizations to the noncompact submanifolds.

Definition 2.1. (Compact Case) Let D be a compact disk in a manifold X. Then, D is an *area minimizing disk* in X if D has the smallest area among the disks in X with the same boundary. Let S be a compact submanifold with boundary in a manifold X. Then, S is an *absolutely area minimizing submanifold* in X if S has the smallest volume among all submanifolds (no topological restriction) with the same boundary in X. The absolutely area minimizing surfaces and hypersurfaces can be defined likewise.

Definition 2.2. (Noncompact Case) An *area minimizing plane* (least area plane) is a complete plane in a manifold X such that any compact subdisk in the plane is an area minimizing disk in X. Let Δ be a complete submanifold in a manifold X. Then, Δ is an *absolutely area minimizing submanifold* in X if any compact part (codimension-0 submanifold with boundary) of Δ is an absolutely area minimizing hypersurface in X. The absolutely area minimizing surfaces, hypersurfaces and hyperplanes can be defined likewise.

Definition 2.3. A *minimal surface (submanifold or hypersurface)* in a manifold X is a surface (submanifold or hypersurface) whose mean curvature vanishes everywhere.

Note that the mean curvature being 0 is equivalent to being locally area minimizing [CM99]. Hence, all area minimizing surfaces and hypersurfaces are also minimal.

Definition 2.4. (Convex Hull) Let A be a subset of $S^n_\infty(\mathbf{H}^{n+1})$. Then the *convex hull* of A, $CH(A)$, is the smallest closed convex subset of $\mathbf{H}^{n+1}$ which is asymptotic to A. Equivalently, $CH(A)$ can be defined as the intersection of all supporting closed half-spaces of $\mathbf{H}^{n+1}$ [EM87].

Note that $\partial_\infty(CH(A)) = A$ for any $A \subset S^n_\infty(\mathbf{H}^{n+1})$ (Note that this is a special property of $\mathbf{H}^{n+1}$, see [HLS00]). In general, we say a subset Σ of X has the convex hull property if it is in the convex hull of its boundary in X, i.e. $\Sigma \subset CH(\partial\Sigma)$. In special case, if Σ is a complete and noncompact hypersurface in $\mathbf{H}^{n+1}$, then we say Σ has convex hull property if it is in the convex hull of its asymptotic boundary, i.e. $\Sigma \subset CH(\partial_\infty\Sigma)$. Minimal hypersurfaces in $\mathbf{H}^{n+1}$ have the convex hull property.

Lemma 2.1 ([An82]). *Let Σ be a minimal submanifold in $\mathbf{H}^{n+1}$ with $\partial_\infty\Sigma = \Gamma$. Then we have $\Sigma \subset CH(\Gamma)$.*

The idea is quite simple. Let Σ be a minimal submanifold in $\mathbf{H}^{n+1}$ with $\partial_\infty\Sigma = \Gamma$. Let K be a nonsupporting halfspace in $\mathbf{H}^{n+1}$, i.e., $\partial_\infty K \cap \Gamma = \emptyset$. Since K is a halfspace in $\mathbf{H}^{n+1}$, we can foliate K with geodesic planes whose asymptotic boundaries are in $\partial_\infty K$. Then, by maximum principle [CM99], $K \cap \Sigma = \emptyset$, and hence $\Sigma \subset CH(\Gamma)$. We should also note that instead of smooth submanifolds, if one deals with area minimizing rectifiable currents, or stationary varifolds, which might have some singularities, for this type of results, one needs *strong maximum principle* results which applies to these settings due to Simon [Si87], Solomon-White [SW89], Ilmanen [Il96] and Wickramasekera [Wi09].

Throughout the paper, $\mathbf{H}^{n+1}$ will represent the hyperbolic $n+1$-space. $\mathbf{H}^{n+1}$ has a natural compactification $\overline{\mathbf{H}^{n+1}} = \mathbf{H}^{n+1} \cup S^n_\infty(\mathbf{H}^{n+1})$ where $S^n_\infty(\mathbf{H}^{n+1})$ is the sphere at infinity of $\mathbf{H}^{n+1}$. If Σ is a subset of $\mathbf{H}^{n+1}$, the asymptotic boundary of Σ, say $\partial_\infty\Sigma$, can be defined as $\partial_\infty\Sigma = \overline{\Sigma} - \Sigma$ where $\overline{\Sigma}$ is closure of Σ in $\overline{\mathbf{H}^{n+1}}$ in the Euclidean metric. In the rest of the paper, it is mostly a good idea to imagine $\mathbf{H}^{n+1}$ in the Poincaré ball model.

3. Existence

There are basically 2 types of existence results for the asymptotic Plateau problem. The first type is the existence of absolutely area minimizing submanifolds in X for a given asymptotic boundary in $\partial_\infty X$. In this type, there are no topological restrictions on the submanifolds. The other type is the fixed topological type. The area minimizing submanifold with the given asymptotic data should also be in the given topological type.

3.1. Absolutely area minimizing submanifolds

By using geometric measure theory methods, Michael Anderson solved the asymptotic Plateau problem for absolutely area minimizing varieties for any dimension and codimension in [An82].

Theorem 3.1 ([An82]). *Let $\Gamma^p \to S^n_\infty(\mathbf{H}^{n+1})$ be an embedded closed submanifold in the sphere at infinity of $\mathbf{H}^{n+1}$. Then there exists a complete, absolutely area minimizing locally integral p+1-current Σ in $\mathbf{H}^{n+1}$ asymptotic to Γ^p at infinity.*

Proof: (Sketch) Let Γ^p be an embedded closed submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. First, Anderson proves a monotonicity formula for stationary p+1-currents such that the ratio between the volume of a stationary p+1-current restricted to an r-ball in $\mathbf{H}^{n+1}$ and the volume of p+1-dimensional r-ball is nondecreasing in r ([An82], Theorem 1). Then, he defines a sequence of closed submanifolds Γ^p_t in $\mathbf{H}^{n+1}$ such that $\Gamma^p_t \subset \partial B_t(0)$ and $\Gamma^p_t \to \Gamma^p$.

Let Σ_t be an area minimizing integral p-current with $\partial\Sigma_t = \Gamma_t$ [Fe69]. Then by using the monotonicity formula, he gives a lower bound for the volume of Σ_t restricted to r-ball, i.e. $c_r < ||\Sigma_t|_{B_r}||$. Also, by using the area minimizing property of Σ_t, he easily gives an upper bound C_r for the volume of Σ_t restricted to r-ball. Then, $c_r < ||\Sigma_t|_{B_r}|| < C_r$.

Hence, by using compactness theorem for integral currents (See [Fe69], [Mo88]), he gets a convergent subsequence for $\{\Sigma_i\}$ *for each* r-ball. Then, by using diagonal subsequence argument, he extracts a convergent subsequence $\Sigma_{i_j} \to \Sigma$ where Σ is an area minimizing integral p+1-current with $\partial_\infty \Sigma = \Gamma^p$. $\square$

Remark 3.1. This result is one of the most important results in the field. This seminal paper can be considered as the beginning of the study of the asymptotic Plateau problem. Later, we will see various generalizations of this result to different settings. Note that the embeddedness assumption on the given asymptotic boundary is very essential. In [La95], Lang constructed immersed examples in $S^n_\infty(\mathbf{H}^{n+1})$ with no solutions to the asymptotic Plateau problem.

Remark 3.2. (Interior Regularity) By interior regularity results of geometric measure theory [Fe69], [Mo88], when $p = n-1$, the currents in Theorem 3.1 are smoothly embedded hypersurfaces except for a singular set of Hausdorff dimension $n-7$. In particular when $p = n-1 < 6$, Σ is a smoothly embedded hypersurface in $\mathbf{H}^{n+1}$. In the higher codimension case ($p < n-1$), the interior regularity results say that the absolutely area minimizing currents are smoothly embedded p+1-submanifolds in $\mathbf{H}^{n+1}$ except for a singular set of Hausdorff dimension $p-1$.

Note that the varieties constructed in theorems above are absolutely area minimizing, and have no topological restrictions on them. Another interesting case is the fixed topological type.

3.2. Fixed topological type

In the result above, Anderson got absolutely area minimizing varieties asymptotic to given submanifold in the asymptotic sphere. As there are no topological restrictions on the objects, we have no idea about their topological properties.

In the case of fixed topological type for the Plateau problem, the question is to find the smallest area surface in the given topological type with the given boundary. Its generalization to the asymptotic Plateau problem is natural.

On the other hand, hyperbolic 3-manifolds, and essential 2-dimensional submanifolds in them is a very active research area. By essential, we mean π_1-injective surfaces, and they are very important tools to understand the structure of the hyperbolic manifold by using geometric topology tools. At this point, when we pass to the universal cover of the hyperbolic manifold and essential surfaces in them, the asymptotic Plateau problem in disk type becomes an important technique for construction of area minimizing representative of these essential surfaces in 3-manifolds.

In [An83], Anderson focused on the asymptotic Plateau problem in disk type, and gave an existence result in dimension 3.

Theorem 3.2 ([An83])**.** *Let* Γ *be a simple closed curve in* $S^2_\infty(\mathbf{H}^3)$*. Then, there exists a complete, area minimizing plane* Σ *in* $\mathbf{H}^3$ *with* $\partial_\infty \Sigma = \Gamma$*.*

The proof is very similar to the proof of the previous theorem. The basic difference is instead of using area minimizing surfaces $\{\Sigma_t\}$ with $\partial\Sigma_t = \Gamma_t$, he used the area minimizing *embedded* disks $\{D_t\}$ with $\partial D_t = \Gamma_t$. The existence of the disks comes from the solution of Plateau problem in disk type. However, the essential point is that the disks are embedded and they are given by [MY80]. Hence, by using similar ideas, Anderson extracted a limit $D_i \to \Sigma$ where Σ is an area minimizing plane in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma = \Gamma$.

Remark 3.3. Note that this result is for just dimension 3, it is not known if its generalization to higher dimensions is true or not. It might be possible to construct area minimizing hyperplanes in $\mathbf{H}^{n+1}$ for any dimension, by generalizing these ideas and White's results in [Wh84] to replace the sequence of disks $\{D_i\}$ in Anderson's proof with compact area minimizing hyperplanes in $\mathbf{H}^{n+1}$.

On the other hand, Gabai gave another construction for Theorem 3.2. Indeed, he needed this result for more general metrics, and he gave a topological construction for such area minimizing planes in more general settings.

Theorem 3.3 ([Ga97])**.** *Let X be $\mathbf{H}^3$ with a different Riemannian metric induced from a metric on a closed 3-manifold. Let Γ be a simple closed curve in $S^2_\infty(X)$. Then, there exists a D^2-limit lamination σ whose leaves are area minimizing planes in X with $\partial_\infty \sigma = \Gamma$.*

Proof: (Sketch) Let X be $\widetilde{M}$, the universal cover of a hyperbolic 3-manifold M with any Riemannian metric. In a similar fashion to the Anderson's proof, Gabai starts with a sequence of area minimizing disks $\{D_i\}$ in X with $\partial D_i = \Gamma_i \to \Gamma$. To get a limiting plane here, instead of using the compactness theorem of geometric measure theory, he extracts some kind of Gromov-Hausdorff limit σ of the sequence $\{D_i\}$ by using minimal surface tools and techniques of [HS88]. In particular, the sequence $\{D_i\}$ of embedded disks in a Riemannian manifold X *converges* to the lamination σ if

i) For any convergent sequence $\{x_{n_i}\}$ in X with $x_{n_i} \in D_{n_i}$ where n_i is a strictly increasing sequence, $\lim x_{n_i} \in \sigma$.

ii) For any $x \in \sigma$, there exists a sequence $\{x_i\}$ with $x_i \in D_i$ and $\lim x_i = x$ such that there exist embeddings $f_i : D^2 \to D_i$ which converge in the C^∞-topology to a smooth embedding $f : D^2 \to L_x$, where $x_i \in f_i(Int(D^2))$, and L_x is the leaf of σ through x, and $x \in f(Int(D^2))$.

We call such a lamination σ a *D^2-limit lamination.* Here, the *topological* limit σ is essentially all the limit points of a very special subsequence. Then, since locally these are limits of area minimizing disks, by using the techniques of [HS88] he shows first that the leaves of the lamination σ are minimal planes. Then by using topological arguments, Gabai proves that these planes are not only minimal, but also area minimizing. Then, he shows that this lamination must stay in a neighborhood of the convex hull of Γ, i.e., $\sigma \subset N_C(CH(\Gamma))$ where $CH(\Gamma)$ is the convex hull of Γ and C is a constant independent of Γ. Then, he shows that $\partial_\infty \sigma = \Gamma$ and finishes the proof. $\square$

Also, in [An83], Anderson constructed special Jordan curves in $S^2_\infty(\mathbf{H}^3)$ such that the absolutely area minimizing surface given by Theorem 3.1 cannot be a plane ([An83], Theorem 4.5). Indeed, he constructed examples with genus $g > g_0$ for any given genus g_0. He also used these surfaces for some nonuniqueness results which we mention later.

In the same context, de Oliveira and Soret showed the existence of a complete stable minimal surface in $\mathbf{H}^3$ for any surface of finite topology (finite genus and finitely many ends). Also, they studied the isotopy type of these surfaces in some special cases. The main difference with Anderson's existence result is that Anderson starts with the asymptotic boundary data, and gives an area minimizing hypersurface where there is no control on the topological type, while de Oliveira and Soret start with a surface with boundary and constructs a stable minimal embedded surface of this type whose asymptotic boundary is essentially determined by the surface.

Theorem 3.4 ([OS98])**.** *Let M be a compact orientable surface with boundary. Then $int(M)$ can be minimally, completely, properly and stably embedded in $\mathbf{H}^3$. Furthermore, the embeddings extend smoothly to an embedding from M to $\overline{\mathbf{H}^3}$, the compactification of $\mathbf{H}^3$.*

Recently, Brian White and Francisco Martin improved this result by showing the existence of complete, properly embedded minimal surfaces in $\mathbf{H}^3$ with *arbitrary topological type*, i.e., including infinite genus and infinite number of ends case. To get this result, they proved a bridge principle at infinity [MW13].

3.3. The bridge principle at infinity

In order to show the existence of complete, properly embedded minimal surfaces in $\mathbf{H}^3$ with arbitrary topological type, Martin and White proved the following bridge principle at infinity.

Theorem 3.5 ([MW13])**.** *Let Γ be a collection of smooth disjoint simple closed curves in $S^2_\infty(\mathbf{H}^3)$ such that Γ bounds a unique area minimizing surface Σ in $\mathbf{H}^3$. Let α be a smooth arc in $S^2_\infty(\mathbf{H}^3)$ with $\alpha \cap \Gamma = \partial\alpha$ and $\Gamma \perp \alpha$. Then, there exists an area minimizing surface Σ_α where Σ_α is* close *to $\Sigma \cup S_\alpha$. Here, S_α is a thin strip along α in $S^2_\infty(\mathbf{H}^3)$. Moreover, $\partial_\infty \Sigma_\alpha = \Gamma_\alpha$ (where $\Gamma_\alpha = \Gamma \sharp \partial S_\alpha$) bounds a unique area minimizing surface Σ_α in $\mathbf{H}^3$.*

By using this bridge principle, Martin and White showed the existence of complete, properly embedded minimal surfaces in $\mathbf{H}^3$ with *arbitrary topological type.* Notice that they greatly improved Theorem 3.4. In Theorem 3.4, de Oliveira and Soret proved the existence theorem for finite type (finite genus and finite number of ends) surfaces, while Martin and White proved the existence of any open orientable surfaces of arbitrary topology.

Theorem 3.6 ([MW13])**.** *Any open orientable surface can be properly embedded in $\mathbf{H}^3$ as an area minimizing surface.*

Proof: (Sketch) Let S be an open orientable surface S and let $S_1 \subset S_2 \subset ... S_n \subset ...$ be a simple exhaustion of S, i.e., $S = \bigcup_{n=1}^{\infty} S_n$ where S_1 is a disk , and S_{n+1} is either a pair of pants attached to S_n or a cylinder with a handle attached to S_n. By using the bridge principle at infinity (Theorem 3.5), Martin and White construct a minimal surface Σ homeomorphic to S as follows. Let Σ_1 be a geodesic plane. Then, if S_{n+1} is a pair of pants attached to S_n, they attach a bridge in $S^2_\infty(\mathbf{H}^3)$ to the corresponding component of $\partial_\infty \Sigma_n$. Similarly, if S_{n+1} is a cylinder with a handle attached to S_n, they attach two bridges successively to $\partial_\infty S_n$. Since all Σ_n's are uniquely minimizing, they can iterate this process dictated by the simple exhaustion of S, and construct a properly embedded area minimizing surface Σ in $\mathbf{H}^{n+1}$ with the same topological type of S. $\square$

Remark 3.4. Note that by proving a generalization of the bridge principle, the author showed that any open orientable surface can also be **nonproperly** embedded as a minimal surface [Co13].

4. Boundary regularity at infinity

After the above existence theorems, the next natural question was the regularity of the hypersurfaces Σ obtained as a solution of the asymptotic Plateau problem. By the interior regularity theorems of geometric measure theory, Σ is a real analytic hypersurface of $\mathbf{H}^{n+1}$ away from a singular subset of Hausdorff dimension $n-7$. The question is the behavior of the hypersurfaces near infinity, i.e., boundary regularity at infinity. In other words, if Σ is an area minimizing hypersurface in $\mathbf{H}^{n+1}$, then what can be said about the boundary regularity of $\overline{\Sigma}$ in $\overline{\mathbf{H}^{n+1}}$?

The first main result about this problem came from Hardt and Lin in [HL87]. By using geometric measure theory methods, they showed that near infinity, $\overline{\Sigma}$ is as regular as the asymptotic boundary for $C^{1,\alpha}$ asymptotic boundary data.

Theorem 4.1 ([HL87])**.** *Let Γ be a $C^{1,\alpha}$ codimension-1 submanifold of $S^n_\infty(\mathbf{H}^{n+1})$ where $0 < \alpha \leq 1$. If Σ is a complete, absolutely area minimizing locally integral n-current in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma = \Gamma$. Then, near Γ, $\Sigma \cup \Gamma$ is the union of $C^{1,\alpha}$ submanifolds with boundary with respect to the Euclidean metric on $\overline{\mathbf{H}^{n+1}}$. These submanifolds have disjoint analytic interiors, and they meet $S^n_\infty(\mathbf{H}^{n+1})$ orthogonally at Γ.*

Also, if we take the upper half space model for $\mathbf{H}^{n+1}$, then $\mathbf{R}^n \times \{0\} \cup \{\infty\}$ would represent the asymptotic sphere. Then, for a given C^1 hypersurface Γ in $\mathbf{R}^n \times \{0\}$, there is ρ_Γ with $(\Sigma \cup \Gamma) \cap \{y < \rho_\Gamma\}$ is a finite union of C^1 submanifolds with boundary which can be viewed as a graph over $\Gamma \times [0, \rho_\Gamma)$.

This result is very interesting as an area minimizing hypersurface in $\mathbf{H}^{n+1}$ has better regularity near the asymptotic boundary than in the interior. In other words, if Σ is an area minimizing hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma = \Gamma$ as above, Σ might have a singular set of Hausdorff dimension $n-7$, but this set must stay in the bounded part of Σ as $(\Sigma \cup \Gamma) \cap \{y < \rho_\Gamma\}$ is a finite union of C^1 submanifolds with boundary. In order to get this result, Hardt and Lin first get an interior regularity result "near infinity" by showing that Σ can be expressed as a union of graphs of finitely many analytic functions on vertical planes tangent to Γ. Then by using this interior regularity "near infinity" result, and hyperbolic isometries, they prove regularity at the boundary. In particular, if there was a sequence of singular points escaping to infinity (or converging to a point in the asymptotic boundary), by rescaling Σ with hyperbolic isometries, they get new area minimizing hypersurfaces, and the images of the singular points in these new area minimizing hypersurfaces would contradict the earlier interior regularity result.

Later, by studying the following quasilinear, non-uniformly elliptic equation whose solutions are minimal hypersurfaces in hyperbolic space, Lin and Tonegawa got higher regularity near the asymptotic boundary. In the upper half space model of $\mathbf{H}^{n+1}$, let $\Omega \subset \mathbf{R}^n \times \{0\}$ be a domain and $f : \Omega \to \mathbf{R}^+$ be a function. Consider $graph(f) = \Sigma_f$ which defines a hypersurface in $\mathbf{H}^{n+1}$. The volume of $\Sigma_f^K = \Sigma_f \cap \{K \times \mathbf{R}^+\}$ where K is a compact subset of Ω can be described as follows:

$$vol(\Sigma_f^K) = \int_K f^{-n}\sqrt{1+|\nabla f|^2}dx$$

Then, the corresponding Euler-Lagrange equation of this variational integral would give the following Dirichlet problem:

$$\begin{array}{ll} \nabla f - \dfrac{f_i.f_j}{1+|df|^2}f_{ij} + \dfrac{n}{f} = 0 & \text{in } \Omega \\ f > 0 & \text{in } \Omega \\ f = 0 & \text{in } \partial\Omega \end{array}$$

where $|df|^2 = \sum_{i=1}^n f_i^2$. In [An83], Anderson showed the existence and the uniqueness of the solution to this Dirichlet problem provided that $\partial\Omega$ has nonnegative mean curvature with respect to inward normal in $\mathbf{R}^n \times \{0\}$.

If one wants to focus on the boundary regularity of the solution of this Dirichlet problem, an equivalent local description of the problem can be given by considering $graph(f)$ near a point of the asymptotic boundary as a graph over a vertical plane which is tangent to the asymptotic boundary at the given point. In other words, let $\Gamma = \partial\Omega$ be at least C^1. Let P be the vertical tangent plane to Γ at p. By using hyperbolic isometries, we can assume $p = 0$ in $\mathbf{R}^n \times \{0\}$ and P is the plane $\{(x,0,y) \in \overline{\mathbf{H}^{n+1}} \mid (x,0) \in \mathbf{R}^n \text{ and } y \geq 0\}$. Then after scaling with hyperbolic isometries, we can formulate the problem as follows:

$$\begin{array}{ll} y(\nabla u - \dfrac{u_i.u_j}{1+|du|^2}u_{ij}) - n.u_y = 0 & \text{in } D \\ u(x,0,0) = \varphi(x) & \end{array}$$

where $D = \{(x,0,y) \in P \mid |x| \leq 1 \text{ and } 0 \leq y \leq 1\}$, $u : D \to \mathbf{R}$, $u(x,0,0) = \varphi(x)$ is given by Γ near p. Hence the question becomes whether u is as smooth as φ.

Lin studied first this quasi-linear degenerate elliptic partial differential equation in [Li89a] and got the following result.

Theorem 4.2 ([Li89a]). *Let Γ be a $C^{k,\alpha}$ codimension-1 submanifold of $S^n_\infty(\mathbf{H}^{n+1})$ where $1 \leq k \leq n-1$ and $0 \leq \alpha \leq 1$ or $k = n$ and $0 \leq \alpha < 1$. If Σ is a complete area minimizing hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma = \Gamma$. Then, near Γ, $\Sigma \cup \Gamma$ is union of $C^{k,\alpha}$ submanifolds with boundary in Euclidean metric on $\overline{\mathbf{H}^{n+1}}$.*

Later, Tonegawa generalized these results, and finished off the asymptotic boundary regularity problem by giving the following complete picture.

Theorem 4.3 ([To96]). *Let Γ be a $C^{k,\alpha}$ codimension-1 submanifold of $S^n_\infty(\mathbf{H}^{n+1})$ and Σ be a complete area minimizing hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma = \Gamma$. Let $k \geq n+1$ and $0 < \alpha < 1$. Then,*

1. If n is even, then $\Sigma \cup \Gamma$ is a $C^{k,\alpha}$ submanifold with boundary near Γ.

2. If n is odd, then $\Sigma \cup \Gamma$ may not be a C^{n+1} submanifold with boundary near Γ in general.

This is a very interesting result as it gives a very subtle relation between the dimension and the asymptotic regularity of area minimizing hypersurfaces. In particular, for n odd, Tonegawa gives a necessary and sufficient condition that Γ has to satisfy in the form of a PDE in order to recover $C^{k,\alpha}$ regularity. Hence, when n is odd, if Γ does not satisfy this PDE, $\Sigma \cup \Gamma$ cannot be smoother than C^{n+1} even though Γ is very smooth. Note also that in [To96], Tonegawa studied a more general form of the PDE above and generalized these results to Constant Mean Curvature (CMC) hypersurfaces in $\mathbf{H}^{n+1}$ (See Section 6.2).

For the higher codimension case ($k < n$), by the interior regularity results of geometric measure theory, absolutely area minimizing k-currents are smoothly embedded k-submanifolds in $\mathbf{H}^{n+1}$ except for a singular set of Hausdorff dimension $k-2$. For the boundary regularity at infinity in this case, Lin also showed the *existence* of an area minimizing k-current Σ in $\mathbf{H}^{n+1}$ which is as regular as the boundary at infinity, where $\Gamma = \partial_\infty \Sigma$ is a $C^{1,\alpha}$ smooth closed $k-1$-submanifold in $S^n_\infty(\mathbf{H}^{n+1})$.

Theorem 4.4 ([Li89b]). *Let Γ be a $C^{1,\alpha}$ smooth closed $k-1$-submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. Then there exists a complete area minimizing k-current in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma = \Gamma$ such that near Γ, $\Sigma \cup \Gamma$ is a $C^{1,\alpha}$ submanifold with boundary with respect to the Euclidean metric on $\overline{\mathbf{H}^{n+1}}$.*

Note that unlike the codimension-1 case, this higher codimension case does not say *any* area minimizing k-current with smooth asymptotic boundary is boundary regular at infinity. This result only tells the existence of such an area minimizing current for any given smooth asymptotic data.

5. Number of solutions

There are basically 3 types of results on the number of solutions to the asymptotic Plateau problem. The first type is the uniqueness results which classifies the asymptotic data with the unique solution to the asymptotic Plateau problem. The next type is the generic uniqueness and generic finiteness results which came out recently. The last type can be called as the nonuniqueness results which constructs the asymptotic data with more than one solution to the problem.

5.1. Uniqueness and finiteness results

Next to the existence theorems, Anderson gave very interesting uniqueness and non-uniqueness results on minimal surfaces in $\mathbf{H}^3$ and area minimizing hypersurfaces in $\mathbf{H}^{n+1}$ in [An82] and [An83]. Before visiting nonuniqueness results, we will list the uniqueness results about the the asymptotic Plateau problem.

First, in [An82], Anderson showed that if the given asymptotic boundary Γ_0 is a hypersurface bounding a convex domain in $S^n_\infty(\mathbf{H}^{n+1})$, then there exists a unique absolutely area minimizing hypersurface Σ_0 in $\mathbf{H}^{n+1}$.

Theorem 5.1 ([An82])**.** *Let Γ_0 be a hypersurface bounding a convex domain in $S^n_\infty(\mathbf{H}^{n+1})$. Then, there exists a unique absolutely area minimizing hypersurface Σ_0 in $\mathbf{H}^{n+1}$ with $\partial_\infty\Sigma_0 = \Gamma_0$.*

Proof: (Sketch) Let Γ_0 be codimension-1 submanifold bounding a convex domain in $S^n_\infty(\mathbf{H}^{n+1})$, and Σ_0 be an area minimizing hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty\Sigma_0 = \Gamma_0$ (existence of Σ_0 is guaranteed by Theorem 3.1). As Γ_0 bounds a convex domain in $S^n_\infty(\mathbf{H}^{n+1})$, we can find a continuous family of isometries $\{\varphi_t\}$ of $\mathbf{H}^{n+1}$ such that $\varphi_t(\Gamma_0) = \Gamma_t$ where $\{\Gamma_t\}$ foliates whole $S^n_\infty(\mathbf{H}^{n+1})$. Similarly, if $\varphi_t(\Sigma_0) = \Sigma_t$, then $\partial_\infty\Sigma_t = \Gamma_t$, and as $\{\Sigma_t\}$ images of continuous family of isometries, it foliates whole $\mathbf{H}^{n+1}$.

Hence, if there are two minimal hypersurfaces M_1, M_2 with $\partial_\infty M_i = \Gamma_0$, then one of them (say M_2) is not a leaf of the foliation, and M_2 must intersect a leaf Σ_{t_0} of the foliation tangentially and by lying in one side.This contradicts to the maximum principle for minimal hypersurfaces. $\square$

Later, by using similar ideas, Hardt and Lin generalized this result to the codimension-1 submanifolds bounding star shaped domains in $S^n_\infty(\mathbf{H}^{n+1})$ in [HL87].

Theorem 5.2 ([HL87])**.** *Let Γ_0 be a hypersurface bounding a star shaped domain in $S^n_\infty(\mathbf{H}^{n+1})$. Then, there exists a unique absolutely area minimizing hypersurface Σ_0 in $\mathbf{H}^{n+1}$ with $\partial_\infty\Sigma_0 = \Gamma_0$.*

While these are the only known results on the number of solutions of the asymptotic Plateau problem for a long time, many generic uniqueness results have come out recently in both general case and fixed topological type case.

For the general case, the author showed that the space of closed codimension-1 submanifolds Γ in $S^n_\infty(\mathbf{H}^{n+1})$ bounding a unique absolutely area minimizing hypersurface Σ in $\mathbf{H}^n$ is dense in the space of all closed codimension-1 submanifolds in $S^n_\infty(\mathbf{H}^{n+1})$ by using a simple topological argument.

Theorem 5.3 ([Co11a])**.** *Let B be the space of connected closed codimension-1 submanifolds of $S^n_\infty(\mathbf{H}^{n+1})$, and let $B' \subset B$ be the subspace containing the closed submanifolds of*

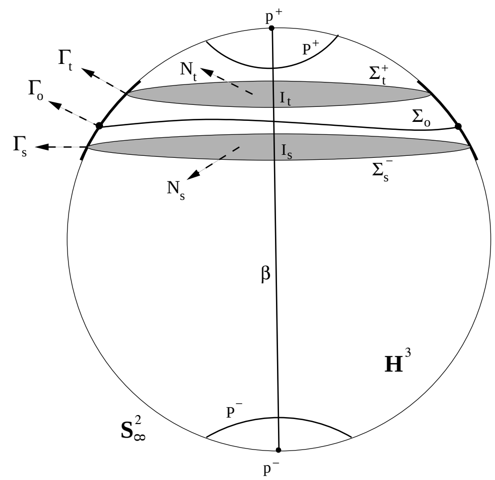

FIGURE 1. A finite segment of geodesic γ intersects the area minimizing planes Σ_t in $\mathbf{H}^n$ with $\partial_\infty \Sigma_t = \Gamma_t$ in $S^2_\infty(\mathbf{H}^3)$.

$S^n_\infty(\mathbf{H}^{n+1})$ bounding a unique absolutely area minimizing hypersurface in $\mathbf{H}^n$. Then B' is dense in B.

Proof: (Sketch) For simplicity, we will focus on the area minimizing planes in $\mathbf{H}^3$. The general case is similar. Let Γ_0 be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. First, by using Meeks-Yau exchange roundoff trick, the author establishes that if Γ_1 and Γ_2 are two disjoint simple closed curves in $S^2_\infty(\mathbf{H}^3)$, and Σ_1 and Σ_2 are area minimizing planes in $\mathbf{H}^3$ with $\partial_\infty \Sigma_i = \Gamma_i$, then Σ_1 and Σ_2 are disjoint, too. Then, by using this result, he shows that for any simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ either there exists a unique area minimizing plane Σ in $\mathbf{H}^3$ with $\partial_\infty \Sigma = \Gamma$, or there exist two *disjoint* area minimizing planes Σ^+, Σ^- in $\mathbf{H}^3$ with $\partial_\infty \Sigma^\pm = \Gamma_0$.

Then, take a small neighborhood $N(\Gamma_0) \subset S^2_\infty(\mathbf{H}^3)$ which is an annulus, and foliate $N(\Gamma_0)$ by simple closed curves $\{\Gamma_t\}$ where $t \in (-\epsilon, \epsilon)$, i.e., $N(\Gamma_0) \simeq \Gamma \times (-\epsilon, \epsilon)$. By the above fact, for any Γ_t either there exists a unique area minimizing plane Σ_t, or there are two area minimizing planes $\Sigma_t^\pm$ disjoint from each other. As disjoint asymptotic boundary implies disjoint area minimizing planes, if $t_1 < t_2$, then Σ_{t_1} is disjoint and *below* Σ_{t_2} in $\mathbf{H}^3$. Consider this collection of area minimizing planes. Note that for curves Γ_t bounding more than one area minimizing plane, we have a canonical region N_t in $\mathbf{H}^3$ between the disjoint area minimizing planes $\Sigma_t^\pm$.

Now, the idea is to consider the *thickness* of the neighborhoods N_t assigned to the asymptotic curves $\{\Gamma_t\}$. Let s_t be the length of the segment I_t of β (a fixed finite length transversal curve to the collection) between Σ_t^+ and Σ_t^-, which is the *width* of N_t assigned to Γ_t. Then, the curves Γ_t bounding more than one area minimizing planes have positive width, and contributes to the total thickness of the collection, and the curves bounding unique area minimizing plane has 0 width and do not contribute to the total thickness. Since $\sum_{t\in(-\epsilon,\epsilon)} s_t < C$, the total thickness is finite. This implies that for only countably many $t \in (-\epsilon, \epsilon)$, $s_t > 0$, i.e., Γ_t bounds more than one area minimizing plane. For the remaining uncountably many $t \in (-\epsilon, \epsilon)$, $s_t = 0$, and there exists a unique area minimizing plane for those t. This proves the space of Jordan curves of uniqueness is dense in the space of Jordan curves in $S^2_\infty(\mathbf{H}^3)$. Then, the author shows that this space is not only dense, but also generic. Also, this technique is quite general, and it can be generalized to different settings [Co06b]. □

On the other hand, there has been important progress on the number of solutions to the asymptotic Plateau problem in fixed topological type case. Recently in [Co04], the author showed a generic finiteness result for C^3 smooth Jordan curves in $S^2_\infty(\mathbf{H}^3)$ for area minimizing planes in $\mathbf{H}^3$ by using geometric analysis and global analysis methods. Later in [Co06a], he improved this result to a generic uniqueness result.

Theorem 5.4 ([Co06a])**.** *Let A be the space of C^3 simple closed curves in $S^2_\infty(\mathbf{H}^3)$. Then there exists an open dense subset $A' \subset A$ such that for any $\Gamma \in A'$, there exists a unique area minimizing plane Σ with $\partial_\infty \Sigma = \Gamma$.*

Proof: (Sketch) In [Co04], by generalizing Tomi and Tromba's global analytic techniques in [TT78] to hyperbolic setting, and by using Li and Tam's powerful results [LT93a] and [LT93b], the author showed that the boundary restriction map π from the space of minimal maps from D^2 to $\mathbf{H}^3$ with C^3 asymptotic data to the space of the C^3 immersions of S^1 into $S^2_\infty(\mathbf{H}^3)$ is Fredholm of index 0. Hence, the derivative of π is an isomorphism for the generic curves.

Fix a generic curve Γ in $S^2_\infty(\mathbf{H}^3)$. By using the inverse function theorem, there is a neighborhood U_Σ of a area minimizing plane Σ in $\pi^{-1}(\Gamma)$, mapping homeomorphically into a neighborhood V_Γ of Γ. By taking a path α in V_Γ, and by considering the corresponding path $\pi^{-1}(\alpha)$ in U_Σ, one can get a continuous family of minimal planes with disjoint asymptotic boundaries around Σ. Then, the author shows that this continuous family of minimal planes is indeed a foliation by area minimizing planes of a neighborhood of Σ. This implies the uniqueness of the area minimizing plane in $\mathbf{H}^3$ spanning Γ. Then the author proves that the same is true for any curve in a neighborhood of a generic curve, and gets an open dense subset of the C^3 Jordan curves in $S^2_\infty(\mathbf{H}^3)$ with the uniqueness result. □

Recently, in [AM10], Alexakis and Mazzeo generalized this result to any surface of genus k by using different methods. In [Co04], the author works with the space of parametrizations of minimal planes in $\mathbf{H}^3$, and hence, in order to get a generic finiteness result, he needs to deal with different parametrizations of the same minimal plane. In [AM10], Alexakis and Mazzeo showed that if $\mathcal{M}_k$ is the moduli space of all complete minimal surfaces of genus k in $\mathbf{H}^3$ with asymptotic boundary curve a $C^{3,\alpha}$ simple closed curve in

$S^2_\infty(\mathbf{H}^3)$, and ξ is the space of $C^{3,\alpha}$ simple closed curve in $S^2_\infty(\mathbf{H}^3)$, then the boundary restriction map $\pi_k : \mathcal{M}_k \to \xi$ is Fredholm of index 0 (see also Section 7.2). Moreover, they also showed that π_k is not only Fredholm of index 0, but also proper (Theorem 4.3 in [AM10]). Hence by the Sard-Smale theorem [Sm65], this implies a generic finiteness result for minimal surfaces of genus k. In other words, for a generic $C^{3,\alpha}$ simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$, there exist finitely many complete minimal surfaces Σ of genus k in $\mathbf{H}^3$ with $\partial_\infty\Sigma = \Gamma$. Indeed, their result also applies to convex cocompact hyperbolic 3-manifolds, too.

Note that the above generic uniqueness result for area minimizing planes requires some smoothness condition on the curves. Later, the author improved his result by removing the smoothness condition. This time, the author uses topological methods instead of techniques of global analysis. The technique is essentially same with the area minimizing hypersurfaces case mentioned above.

Theorem 5.5 ([Co11a])**.** *Let A be the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ and let $A' \subset A$ be the subspace containing the simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique area minimizing plane in $\mathbf{H}^3$. Then, A' is generic in A, i.e., $A - A'$ is a set of first category.*

Remark 5.1. Note that the same result is true for area minimizing surfaces in $\mathbf{H}^3$, too [Co11a]. By these results, the asymptotic Plateau problem generically has a unique solution in both area minimizing surfaces in $\mathbf{H}^3$ case and area minimizing planes in $\mathbf{H}^3$ case. In higher dimensions, the closed codimension-1 submanifolds in $S^n_\infty(\mathbf{H}^{n+1})$ bounding a unique absolutely area minimizing hypersurface in $\mathbf{H}^{n+1}$ are only dense in the closed codimension-1 submanifolds in $S^n_\infty(\mathbf{H}^{n+1})$. However, by using the similar ideas, by fixing the topological type of the closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$, it might be possible to get some generic uniqueness result, too.

Remark 5.2. Notice that except the convex and star-shaped asymptotic boundary cases, all the uniqueness results on the asymptotic Plateau problem are about area minimizing surfaces or area minimizing planes. Unfortunately, the techniques used for these results cannot be extended to the minimal surfaces or minimal planes cases. The main obstacle here is that while two area minimizing surfaces with disjoint asymptotic boundaries must be disjoint, the same statement may not be true for minimal surfaces. In any case, it would be an interesting problem to study this case in order to understand whether the simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique minimal surface (or plane) is dense in the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ or not. The author believes that the similar statements are not true in minimal surfaces (or planes) case.

Recently in [AM10], Alexakis and Mazzeo obtained a substantial result on the problem by using degree theory [TT78], [Wh87] . In a sense, this is the strongest result on the number of complete minimal surfaces spanning given asymptotic data in $\mathbf{H}^3$. In particular, let $\Lambda(S^2_\infty(\mathbf{H}^3))$ be the space of $C^{3,\alpha}$ simple closed curves in $S^2_\infty(\mathbf{H}^3)$, and let $\mathcal{M}_k(\mathbf{H}^3)$ be the space of all complete, properly embedded minimal surfaces Σ of genus g in $\mathbf{H}^3$ with $\partial_\infty\Sigma \in \Lambda(S^2_\infty(\mathbf{H}^3))$.

Theorem 5.6 ([AM10]). *Let $\Pi : \mathcal{M}_k(\mathbf{H}^3) \to \Lambda(S^2_\infty(\mathbf{H}^3))$ be the natural map defined by $\Pi(\Sigma) = \partial_\infty \Sigma$. Then,*

i. Π *is Fredholm with index* 0.

ii. Π *is proper.*

Remark 5.3. Notice that this result immediately implies generic finiteness for each k. On the other hand, by using the fact that for a convex curve Γ in $S^2_\infty(\mathbf{H}^3)$, there exists a unique minimal surface Σ in $\mathbf{H}^3$, and Σ is a plane, it is easy to see that $deg(\Pi_0) = 1$ and $deg(\Pi_k) = 0$ for any $k > 0$. Hence, this automatically implies nonuniqueness for a generic regular curve $\Gamma \in \Lambda(S^2_\infty(\mathbf{H}^3))$ bounding a complete minimal surface of genus k, i.e., if $\Pi_k^{-1}(\Gamma) \neq \emptyset$, then there are even number of genus k minimal surfaces with asymptotic boundary Γ. Note also that their result is indeed more general, and it is true for any convex cocompact hyperbolic manifolds rather than just $\mathbf{H}^3$ (see Section 7.2).

5.2. Nonuniqueness results

Beside his existence results, Anderson also gave many different nonuniqueness results for the asymptotic Plateau problem in the fixed topological type in [An83].

Theorem 5.7 ([An83]). *There exists a simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ such that there are infinitely many complete minimal surfaces $\{\Sigma_i\}$ in $\mathbf{H}^3$ with $\partial_\infty \Sigma_i = \Gamma$.*

In the proof of this theorem, Anderson first constructs a simple closed curve such that the absolutely area minimizing surface given by his existence theorem is not a plane (positive genus) (a similar construction can be found in [Ha92]). Then, by modifying this curve, he constructs a curve Γ with the same property such that it is also a limit set for a quasi-Fuchsian group Λ. Since the absolutely area minimizing surface Σ is Λ invariant and has positive genus, this implies the area minimizing surface Σ/Λ in the compact hyperbolic manifold $\mathbf{H}^3$ is not π_1-injective. This implies that the absolutely area minimizing surface Σ in $\mathbf{H}^3$ with $\partial_\infty \Sigma = \Gamma$ must have infinite genus. Then, by using this property, he shows that there exist infinitely many complete *minimal* surfaces asymptotic to Γ.

Note that this result shows nonuniqueness for *minimal surfaces* for fixed topological type. Notice also that [AM10] gives a stronger nonuniqueness result by Remark 5.3. In particular, by Theorem 3.4, for every $n \in \mathbf{N}$, there exists a simple closed curve $\Gamma_n \subset S^2_\infty(\mathbf{H}^3)$ where the area minimizing surface Σ_n with $\partial_\infty \Sigma_n = \Gamma_n$ has genus n. Hence, by using the ideas in Remark 5.3, it can be shown that for every $n \in \mathbf{N}$, there exists a simple closed curve $\Gamma_n \subset S^2_\infty(\mathbf{H}^3)$ where Γ_n bounds more than one minimal surface of genus n.

So far, we described the existence of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding more than one *minimal* surface in $\mathbf{H}^3$. On the other hand, the author showed that a similar existence result is true for least area planes, and area minimizing surfaces.

Theorem 5.8 ([Co11a]). *There exists a simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ such that there are more than one area minimizing surfaces $\{\Sigma_1, \Sigma_2, ..., \Sigma_n\}$ in $\mathbf{H}^3$ with $\partial_\infty \Sigma_i = \Gamma$. The same is true for least area planes, too.*

Remark 5.4. In the nonuniqueness results above, only Hass's result gives an explicit example of a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ bounding more than one minimal surface in $\mathbf{H}^3$. All other results on nonuniqueness so far shows the existence of such a curve, but it does not give one. So, it would be interesting to construct an explicit simple closed curve in $S^2_\infty(\mathbf{H}^3)$ bounding more than one area minimizing surface (or plane).

Remark 5.5. Although there are many examples of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding more than one minimal surface or more than one area minimizing surface (or plane) in $\mathbf{H}^3$, there is no example in higher dimensions so far. It would be interesting to generalize the nonuniqueness results to higher dimensions by showing whether there exists a closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$ bounding more than one absolutely area minimizing hypersurface in $\mathbf{H}^{n+1}$.

On the other hand, recently, B. Wang and Z. Huang obtained a very interesting result on the nonunique solutions for the asymptotic Plateau problem [WH12]. A quasi-Fuchsian 3-manifold is a hyperbolic 3-manifold M which is homeomorphic to $\Sigma_g \times \mathbf{R}$ where Σ_g is the closed genus g surface. In [WH12], for any given $N > 0$, they construct a quasi-Fuchsian manifold M_N which contains at least 2^N incompressible minimal surfaces. This would automatically imply that the limit set $\Gamma_N \subset S^2_\infty(\mathbf{H}^3)$ of M_N bounds at least 2^N minimal planes in $\mathbf{H}^3$. In other words;

Theorem 5.9 ([WH12])**.** *There are simple closed curves in $S^2_\infty(\mathbf{H}^3)$ which bound arbitrarily many minimal planes in $\mathbf{H}^3$.*

Note that the simple closed curves in this theorem are far from being smooth as they are limit sets of quasi-Fuchsian 3-manifolds. Indeed, they are completely nonrectifiable (see the end of section 6.4).

6. CMC hypersurfaces

After many important results on minimal hypersurfaces in hyperbolic space, like existence, regularity, etc., the question of generalization of these results to constant mean curvature (CMC) hypersurfaces was naturally raised: For a given codimension-1 submanifold Γ in $S^n_\infty(\mathbf{H}^{n+1})$, does there exist a complete CMC hypersurface Σ with specified mean curvature H in $\mathbf{H}^{n+1}$ and $\partial_\infty\Sigma = \Gamma$?

For simplicity, from now on, we will call CMC hypersurfaces with mean curvature H as *H-hypersurfaces*.

Note that for this generalization of the asymptotic Plateau problem, we need to assume that $|H| < 1$ (after fixing an orientation on $\mathbf{H}^{n+1}$). This is because it is impossible to have a complete H-hypersurface Σ in $\mathbf{H}^{n+1}$ with $|H| \geq 1$ and $\partial_\infty\Sigma = \Gamma$ as we can always find a horosphere ($H = 1$) in $\mathbf{H}^{n+1}$ with tangential intersection with such a Σ which contradicts to the maximum principle.

We should also note that H-hypersurfaces in $\mathbf{H}^3$ with $H = 1$ and $H > 1$ are an area of active research. A basic reference for CMC hypersurfaces in hyperbolic space with $H > 1$ would be [KKMS92]. For the case $H = 1$, we refer to Rosenberg's survey [Ro99], and Bryant's seminal paper [Br87] where he showed that any minimal surface in $\mathbf{R}^3$ is isometric to a CMC surface in $\mathbf{H}^3$ with $H = 1$.

We should point out that the generalization of area minimizing (or minimal) hypersurfaces to CMC hypersurfaces is quite natural. As we see the minimal hypersurfaces ($H = 0$) as the critical points of the area functional, CMC hypersurfaces occurs as the critical points of some modification of the area functional with a volume constraint. In particular, let Σ^n be a compact hypersurface, bounding a domain Ω^{n+1} in some ambient Riemannian manifold. Let A be the area of Σ, and V be the volume of Ω. Let us vary Σ through a one parameter family Σ_t, with corresponding area $A(t)$ and volume $V(t)$. If f is the normal component of the variation, and H is the mean curvature of Σ, then we get $A'(0) = -\int_\Sigma nHf$, and $V'(0) = \int_\Sigma f$ where n is the dimension of Σ, and H is the mean curvature.

Now, let Σ be a hypersurface with boundary Γ. We fix a hypersurface M with $\partial M = \Gamma$, and define $V(t)$ to be the volume of the domain bounded by M and Σ_t. Now, we define a new functional as a combination of A and V. Let $I_H(t) = A(t) + nHV(t)$. Note that $I_0(t) = A(t)$. If Σ is a critical point of the functional I_H for any variation f, then this will imply Σ has constant mean curvature H [Gu73]. Note that critical points of the functional I_H are independent of the choice of the hypersurface M since if $\widehat{I}_H$ is the functional which is defined with a different hypersurface $\widehat{M}$, then $I_H - \widehat{I}_H = C$ for some constant C. On the other hand, we will call Σ a *minimizing CMC hypersurface* if Σ is the absolute minimum of the functional I_H among hypersurfaces with the same boundary. From this point of view, CMC hypersurfaces are natural generalization of minimal hypersurfaces and area minimizing hypersurfaces as the area functional is just the $H = 0$ case for the functional I_H. This point of view is very useful and essential to generalize the geometric measure theory methods developed for area minimizing case to CMC case as in [To96] and [AR97].

Now, we continue with the basic notions on H-hypersurfaces in $\mathbf{H}^{n+1}$. Fix a codimension-1 closed submanifold Γ in $S^n_\infty(\mathbf{H}^{n+1})$. Γ separates $S^n_\infty(\mathbf{H}^{n+1})$ into two parts, say Ω^+ and Ω^-. By using these domains, we will give orientation to hypersurfaces in $\mathbf{H}^{n+1}$ asymptotic to Γ. With this orientation, the mean curvature H is positive if the mean curvature vector points towards the positive side of the hypersurface, negative otherwise. The following fact is known as the maximum principle.

Lemma 6.1 (Maximum Principle)**.** *Let Σ_1 and Σ_2 be two hypersurfaces in a Riemannian manifold which intersect at a common point tangentially. If Σ_2 lies in the positive side (mean curvature vector direction) of Σ_1 around the common point, then H_1 is less than or equal to H_2 ($H_1 \leq H_2$) where H_i is the mean curvature of Σ_i at the common point. If they do not coincide in a neighborhood of the common point, then H_1 is strictly less than H_2 ($H_1 < H_2$).*

The other important notion about CMC hypersurfaces in $\mathbf{H}^{n+1}$ is the generalization of the convex hull property to this context. Now, let Γ be a codimension-1 submanifold of $S^n_\infty(\mathbf{H}^{n+1})$ and orient all spheres accordingly. If T is a round $n-1$-sphere in $S^n_\infty(\mathbf{H}^{n+1})$, then there is a unique H-hypersurface P_H in $\mathbf{H}^{n+1}$ asymptotic to T for $-1 < H < 1$

[NS96]. T separates $S^n_\infty(\mathbf{H}^{n+1})$ into two parts Δ^+ and Δ^-. Similarly, P_H divides $\mathbf{H}^{n+1}$ into two domains D^+_H and D^-_H with $\partial_\infty D^\pm_H = \Delta^\pm$. We will call these regions as *H-shifted halfspaces*. If the asymptotic boundary of a H-shifted halfspace contains Γ, then we will call this H-shifted halfspace as *supporting H-shifted halfspace*, i.e., if $A \subset \Delta^+$, then D^+_H is a supporting H-shifted halfspace. Then the *H-shifted convex hull* of Γ, $CH_H(\Gamma)$ is defined as the intersection of all supporting closed H-shifted halfspaces of $\mathbf{H}^{n+1}$.

Now, the generalization of convex hull property of minimal hypersurfaces in $\mathbf{H}^{n+1}$ to H-hypersurfaces in $\mathbf{H}^{n+1}$ is as follows [Co06b]. Similar versions of this result have been proved by Alencar-Rosenberg in [AR97], and by Tonegawa in [To96].

Lemma 6.2 ([To96], [AR97], [Co06b]). *Let Σ be a H-hypersurface in $\mathbf{H}^{n+1}$ where $\partial_\infty\Sigma$ is Γ and $|H| < 1$. Then Σ is in the H-shifted convex hull of Γ, i.e., $\Sigma \subset CH_H(\Gamma)$.*

6.1. Existence

In the decade after Anderson's existence ([An82], [An83]) and Hardt-Lin's regularity results ([HL87],[Li89a]), there have been many important generalizations of these results to CMC hypersurfaces in the hyperbolic space. In [To96], Tonegawa generalized Anderson's techniques to this case, and proved existence for CMC hypersurfaces by using geometric measure theory methods. In the same year, by using similar techniques, Alencar and Rosenberg got a similar existence result in [AR97].

Theorem 6.3 ([To96], [AR97]). *Let $\Gamma \subset S^n_\infty(\mathbf{H}^{n+1})$ be a codimension-1 closed submanifold, and let $|H| < 1$. Then there exists a CMC hypersurface Σ with mean curvature H in $\mathbf{H}^{n+1}$ where $\partial_\infty\Sigma = \Gamma$. Moreover, any such CMC hypersurface is smooth except a closed singularity set of dimension at most $n - 7$.*

We should also note that Nelli and Spruck showed the existence of a CMC hypersurface asymptotic to a $C^{2,\alpha}$ codimension-1 submanifold Γ which is the boundary of a mean convex domain in $S^n_\infty(\mathbf{H}^{n+1})$ by using analytic techniques in [NS96]. Later, Guan and Spruck generalized this result to $C^{1,1}$ codimension-1 submanifolds bounding star shaped domains in $S^n_\infty(\mathbf{H}^{n+1})$.

Theorem 6.4 ([GS00]). *Let Ω be a star shaped (mean convex for [NS96]) domain in $S^n_\infty(\mathbf{H}^{n+1})$ where $\Gamma = \partial\Omega$ is a $C^{1,1}$ ($C^{2,\alpha}$ for [NS96]) codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. Then, for any $0 < H < 1$, there exists a complete smoothly embedded CMC hypersurface Σ with mean curvature H and $\partial_\infty\Sigma = \Gamma$. Moreover, Σ can be represented as the graph of a function $u \in C^{1,1}(\overline{\Omega})$ ($u \in C^{2,\alpha}(\overline{\Omega})$ in [NS96]).*

Even though this second existence result is for a fairly restricted class of asymptotic boundary data (star shaped condition), the CMC hypersurfaces obtained are smoothly embedded with no singularity in any dimension (unlike the first one), and they can be represented as a graph like $x_{n+1} = u$ for a function $u \in C^{1,1}(\overline{\Omega})$ in the half space model for $\mathbf{H}^{n+1}$. We should also note that, in [AA02], Aiyama and Akutagawa gave a completely different construction for CMC surfaces of disk type in $\mathbf{H}^3$ with asymptotic boundary a $C^{1,\alpha}$ smooth simple closed curve in $S^2_\infty(\mathbf{H}^3)$ by studying a Dirichlet problem at infinity.

These results gave a positive answer to existence question of asymptotic Plateau problem for H-hypersurfaces in $\mathbf{H}^{n+1}$. On the other hand, like in the minimal case, which topologies can occur as a solution to this problem is an interesting question. In dimension 3, the author generalized Martin and White's result [MW13] to this case.

Theorem 6.5 ([Co13]). *Any open orientable surface can be properly embedded in $\mathbf{H}^3$ as a minimizing H-surface.*

In other words, for any given open orientable surface S, there exists a collection of simple closed curves Γ in $S^2_\infty(\mathbf{H}^3)$ such that the minimizing H-surface Σ with $\partial_\infty\Sigma = \Gamma$ is homeomorphic to S, i.e., $\Sigma \simeq S$. The techniques are quite similar with [MW13]. The author first generalized the bridge principle for uniquely minimizing surfaces in [MW13] to uniquely minimizing H-surfaces. Then, following a compact exhaustion of the given surface S, he constructs the surface by using the bridges inductively (See the proof of the Theorem 3.6.)

6.2. Boundary regularity at infinity

Beside the existence results, in [To96], Tonegawa studied the following quasi-linear degenerate elliptic PDE which is a more general form of the PDE in Section 4 for H-hypersurfaces with $|H| < 1$, and got important regularity results for these hypersurfaces near asymptotic boundary.

$$\begin{aligned} &y(\nabla u - \frac{u_i.u_j}{1+|du|^2}u_{ij}) - n(u_y - H\sqrt{1+|Du|^2}) = 0 \quad \text{in } D \\ &u(x,0,0) = \varphi(x) \end{aligned}$$

For $k \leq n$, Tonegawa generalized the Lin's result for minimal hypersurfaces ($H = 0$) in [Li89a].

Theorem 6.6 ([To96]). *Let Γ be a $C^{k,\alpha}$ codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$ where $1 \leq k \leq n-1$ and $0 \leq \alpha \leq 1$ or $k = n$ and $0 \leq \alpha < 1$. If Σ is a complete CMC hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty\Sigma = \Gamma$, then $\Sigma \cup \Gamma$ is a $C^{k,\alpha}$ submanifold with boundary in $\overline{\mathbf{H}^{n+1}}$ near Γ.*

On the other hand, Tonegawa showed that for higher regularity case, $H = 0$ case is fairly different form the $H \neq 0$ case. As we mentioned in Section 4, in $H = 0$ case, Tonegawa showed that when n is even, higher regularity is always true, but when n is odd, higher regularity depends on the asymptotic boundary Γ (Theorem 4.3). In the $H \neq 0$ case, Tonegawa got a very surprising result that while the similar result is true for $n = 2$, it is not true for $n = 4$.

Theorem 6.7 ([To96]). **a.** **($n = 2$ case)** *Let Γ be a $C^{k,\alpha}$ smooth simple closed curve in $S^2_\infty(\mathbf{H}^3)$ with $k \geq n+1 = 3$, $0 < \alpha < 1$. Let Σ be a H-hypersurface in $\mathbf{H}^3$ with $\partial_\infty\Sigma = \Gamma$. Then, $\Sigma \cup \Gamma$ is a $C^{k,\alpha}$ submanifold with boundary near Γ.*

b. **($n = 4$ case)** *For $n = 4$, $H \neq 0$ and $|H| < 1$, there exists a smooth Γ such that $\Sigma \cup \Gamma$ is not a $C^{n+1} = C^5$ submanifold with boundary where Σ is a H-hypersurface with $\partial_\infty \Sigma = \Gamma$.*

We should also note that by studying the PDE above, or by using some barrier arguments, it is not hard to show that the intersection angle θ_H between an H-hypersurface and the asymptotic boundary $S^n_\infty(\mathbf{H}^{n+1})$ is $\arctan(\frac{\sqrt{1-H^2}}{H})$ [To96]. In other words, let Γ be a codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$, and Σ be an H-hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma = \Gamma$. Then for any $p \in \Gamma$, the angle θ_H between $\Sigma \cup \Gamma$ and $S^n_\infty(\mathbf{H}^{n+1})$ at p would be $\arccos H$.

6.3. Number of solutions

By using analytic techniques, Nelli and Spruck generalized Anderson's uniqueness result for mean convex domains in area minimizing hypersurfaces case to CMC context in [NS96]. Then, Guan and Spruck extended Hardt and Lin's uniqueness results for star-shaped domains in area minimizing hypersurfaces case to CMC hypersurfaces in hyperbolic space in [GS00].

Theorem 6.8 ([GS00])**.** *Let Ω be a star shaped (mean convex for* [NS96]*) domain in $S^n_\infty(\mathbf{H}^{n+1})$ where $\Gamma = \partial\Omega$ is a $C^{1,1}$ ($C^{2,\alpha}$ for* [NS96]*) codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. Then, for any $0 \leq H < 1$, there exists a unique complete CMC hypersurface Σ with mean curvature H and $\partial_\infty \Sigma = \Gamma$.*

On the other hand, the author got a generic uniqueness result for CMC hypersurfaces by generalizing his methods in [Co11a]. In particular, he defined the notion of *minimizing CMC hypersurfaces* as generalizations of area minimizing hypersurfaces. In other words, as minimal hypersurfaces are critical points of the area functional, and area minimizing hypersurfaces are not just critical but minimum points of the functional, the same generalization is defined for CMC hypersurfaces in [Co06b]. The CMC hypersurfaces are the hypersurfaces with constant mean curvature and they correspond to critical points of the functional $I_H(t) = A(t) + nHV(t)$, and *minimizing CMC hypersurfaces* correspond to minimizers of the functional I_H. Note that the existence result Theorem 6.3 by Tonegawa and Alencar-Rosenberg indeed gives minimizing CMC hypersurfaces.

Theorem 6.9 ([Co06b])**.** *Let A be the space of codimension-1 closed submanifolds of $S^n_\infty(\mathbf{H}^{n+1})$, and let $A' \subset A$ be the subspace containing the closed submanifolds of $S^n_\infty(\mathbf{H}^{n+1})$ bounding a unique minimizing CMC hypersurface with mean curvature H in $\mathbf{H}^{n+1}$. Then A' is generic in A, i.e., $A - A'$ is a set of first category.*

On the other hand, there is no known result for nonuniqueness of CMC hypersurfaces. In particular, there is no known example of a codimension-1 submanifold Γ in $S^n_\infty(\mathbf{H}^{n+1})$ such that Γ is the asymptotic boundary of more than one CMC hypersurface with mean curvature H for any $0 < H < 1$. For $H = 0$, Anderson [An83], Hass [Ha92], and the

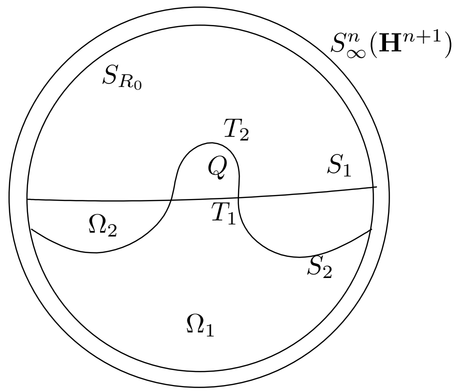

FIGURE 2. For $0 < H_1 < H_2 < 1$, S_1 is above S_2 near the boundary of the ball $B_{R_0}(p)$ by [To96].

author [Co11a] gave such examples. It might be possible to generalize these techniques to prove nonuniqueness in CMC case for any $H \in (-1, 1)$.

6.4. Foliations of hyperbolic space

While discussing the uniqueness of CMC hypersurfaces for a given asymptotic data in asymptotic boundary, there is a related problem in the subject: For a given codimension-1 closed submanifold Γ in $S^n_\infty(\mathbf{H}^{n+1})$, does the family of CMC hypersurfaces $\{\Sigma_H\}$ with mean curvature H foliate $\mathbf{H}^{n+1}$ or not, where $-1 < H < 1$ and $\partial_\infty \Sigma_H = \Gamma$. This question is related to the uniqueness question as the existence of such a foliation automatically implies the uniqueness of the CMC hypersurface Σ_H with mean curvature H where $\partial_\infty \Sigma_H = \Gamma$ by maximum principle. In the reverse direction, the author showed the following result.

Theorem 6.10 ([Co10]). *Let Γ be a $C^{2,\alpha}$ closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. Also assume that for any $H \in (-1, 1)$, there exists a unique CMC hypersurface Σ_H with $\partial_\infty \Sigma_H = \Gamma$. Then, the collection of CMC hypersurfaces $\{\Sigma_H\}$ with $H \in (-1, 1)$ foliates $\mathbf{H}^{n+1}$.*

Proof: (Sketch) First, by using the boundary regularity results in [To96] and some cut-and-paste arguments similar to the exchange roundoff trick, the author shows that two different minimizing H-hypersurfaces with the same asymptotic boundary must be disjoint (see Figure 2). In particular, he proves that if Γ is a $C^{2,\alpha}$ closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$, and Σ_{H_1} and Σ_{H_2} are minimizing CMC hypersurfaces in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma_{H_i} = \Gamma_i$ and $-1 < H_1 < H_2 < 1$, then Σ_{H_1} and Σ_{H_2} are disjoint. Hence, $\{\Sigma_H\}$ for $-1 < H < 1$ is a disjoint family of hypersurfaces in $\mathbf{H}^{n+1}$. Now, there are two points to check to show that $\{\Sigma_H\}$ foliates $\mathbf{H}^{n+1}$. The first point is that there is no gap between the leaves of $\{\Sigma_H\}$, and the second point is that $\{\Sigma_H\}$ fills $\mathbf{H}^{n+1}$.

For the first point the idea is to use the assumption that Γ bounds a unique H-hypersurface for any $H \in (-1, 1)$. If there was a gap between the family $\{\Sigma_H\}_{H\in(-1,H_0]}$ and $\{\Sigma_H\}_{H\in(H_0,1)}$, then one can construct a sequence of hypersurfaces $\{S_i\}$ such that $S_i \subset \Sigma_{H_i}$ where $H_i \searrow H_0$ and $\partial S_i \to \Gamma$. Then, by showing that $S_i \to \Sigma'_{H_0}$ where Σ'_{H_0} is another minimizing H_0-hypersurface with $\partial_\infty \Sigma'_{H_0} = \Gamma$, he gets a contradiction as Γ must bound a unique H_0-hypersurface in $\mathbf{H}^{n+1}$.

For the second point, if the family $\{\Sigma_H\}$ of hypersurfaces does not fill $\mathbf{H}^{n+1}$, then by constructing a suitable horosphere in the unfilled region, and by using the maximum principle, the author gets a contradiction. $\square$

Hence, by the uniqueness results in [GS00] and [NS96], for the star shaped asymptotic data and mean convex asymptotic data, the above result gives positive answer for the question. Note that in [CV03], Chopp and Velling studied this problem by using computational methods, and had an interesting result that for many different types of curves in $S^2_\infty(\mathbf{H}^3)$, CMC surfaces give a foliation of $\mathbf{H}^3$.

On the other hand, recently in [Wa08], Wang showed that if a quasi-Fuchsian 3-manifold M contains a minimal surface whose principle curvature is less than 1, than M admits a foliation by CMC surfaces by using volume preserving mean curvature flow. If we lift this foliation to the universal cover, we get a foliation of $\mathbf{H}^3$ by CMC surfaces with same asymptotic boundary Γ where Γ is a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ and it is the limit set of the quasi-Fuchsian 3-manifold M. However, the limit set of quasi-Fuchsian manifolds are far from being smooth, even they contain no rectifiable arcs ([Be72]). Existence of one smooth point in the limit set implies the group being Fuchsian, which means the limit set is a round circle in $S^2_\infty(\mathbf{H}^3)$. Hence, in addition to smooth examples in [Co10], [Wa08] gives completely nonrectifiable simple closed curve examples where CMC hypersurfaces with the given asymptotic data foliate the hyperbolic space. Also in [Wa08], Wang constructs a simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ (as limit set of a quasi-Fuchsian 3-manifold) which is similar to the one in [Ha92], where there cannot be a foliation of $\mathbf{H}^3$ by CMC surfaces with asymptotic boundary Γ.

7. Further results

Other than existence, regularity and number of solutions to the asymptotic Plateau problem, there have been other important features which are studied.

7.1. Properly embeddedness

The properly embeddedness of the solution of the asymptotic Plateau problem is one of the interesting problems which is under investigation. Namely, the question is whether a solution to the asymptotic Plateau problem Σ with $\partial_\infty \Sigma = \Gamma$ where Γ is a codimension-1 closed submanifold in $S^n_\infty(\mathbf{H}^{n+1})$ is properly embedded, or not? In other words, if $\varphi : S \to \mathbf{H}^{n+1}$ is an embedding with $\varphi(S) = \Sigma$, then is φ proper, i.e., is the preimage of a compact subset K of $\mathbf{H}^{n+1}$, $\varphi^{-1}(K)$, is compact in S for any K?

In recent years, the properly embeddedness of complete minimal surfaces is under serious investigation in $\mathbf{R}^3$ case (see the survey [Al06]). This is called the Calabi-Yau Conjecture for minimal surfaces, and has been shown by Colding and Minicozzi in [CM08]. After this result, Meeks and Rosenberg generalized this result by showing that any embedded minimal surface with positive injectivity radius in an ambient space with nonpositive curvature must be properly embedded [MR06]. This would automatically suggest the question of whether the Calabi-Yau Conjecture is true in $\mathbf{H}^3$.

Recently, the author obtained the first positive answer in this direction. In particular, he showed that for any area minimizing plane Σ in $\mathbf{H}^3$ with asymptotic boundary Γ which is a simple closed curve with at least one smooth point, then Σ is properly embedded in $\mathbf{H}^3$. The technique is very different from Colding and Minicozzi's techniques. While Colding-Minicozzi relates intrinsic distances and extrinsic distances for embedded minimal surface in $\mathbf{R}^3$ by using very powerful analytical techniques, the author's techniques are purely topological.

Theorem 7.1 ([Co09]). *Let Σ be a complete embedded area minimizing plane in $\mathbf{H}^3$ with $\partial_\infty \Sigma = \Gamma$ where Γ is a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ with at least one smooth (C^1) point. Then, Σ must be proper.*

Proof: (Sketch) Assume that Σ is a non-properly embedded area minimizing plane in $\mathbf{H}^3$ with $\partial_\infty \Sigma = \Gamma$ where Γ is a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ with at least one smooth point. The author gets a contradiction by analyzing the disks in the intersection of Σ with the balls $B_R(0)$ which exhaust $\mathbf{H}^3$. First, he shows that for sufficiently large generic $R > 0$, $\Sigma \cap B_R(0)$ contains infinitely many disjoint disks. Then, he categorizes these disks as separating and nonseparating depending on their boundary in the annulus $A_R = CH(\Gamma) \cap \partial B_R(0)$ being essential or not.

Then, he establishes the Key Lemma which shows that the nonseparating disks in $B_R(0)$ must stay close to the boundary $\partial B_R(0)$. In particular, he proves that if D_r is a nonseparating disk in $B_r(0) \cap \Sigma$, then there is a function F which is a monotone increasing function with $F(r) \to \infty$ as $r \to \infty$, such that $d(0, D_r) > F(r)$ where d is the distance. He proves the Key Lemma by using a barrier argument (see Figure 3). In other words, by using the smooth point assumption, he proves the existence of a least area annulus $\mathcal{A}_r$ in $\mathbf{H}^3$ with $\partial_\infty \mathcal{A}_r = \Gamma^+_r \cup \Gamma^-_r$, where $\Gamma^\pm_r$ are simple closed curves sufficiently close to Γ in opposite sides. Since they are area minimizing, any nonseparating disk D_r must stay in one side of the least area annulus $\mathcal{A}_r$. As $r \to \infty$ the distance from 0 to $\mathcal{A}_r$ will give the desired function. Hence, this shows that nonseparating disks do not come close to the point 0, and stay close to the boundary $\partial B_r(0)$.

Finally, the author proves the main result by using the Key Lemma as follows. A separating disk D_{R_1} in $\Sigma \cap B_{R_1}(0)$ will be a subdisk in a nonseparating disk E_{R_2} in $\Sigma \cap B_{R_2}(0)$ where $R_2 > R_1$. By choosing R_2 appropriately and by using the fact that the separating disk D_{R_1} is a subset of the disk E_{R_2}, he shows that the nonseparating disk E_{R_2} comes very close to the point 0, which is a contradiction. $\square$

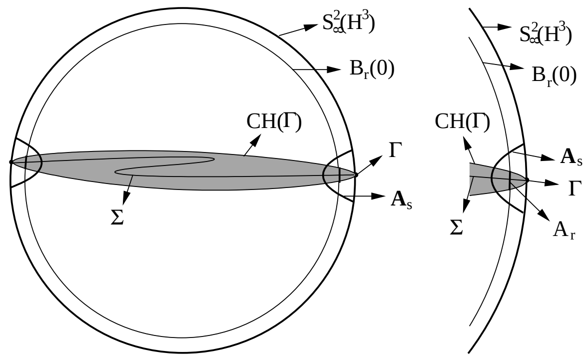

FIGURE 3. The least area annulus $\mathcal{A}_s$ is used as barrier in the proof of Key Lemma.

However, in the following years, the author showed that the Calabi-Yau Conjecture is not true in $\mathbf{H}^3$ by constructing a complete, nonproperly embedded minimal plane in $\mathbf{H}^3$ [Co11b]. Furthermore, by combining Martin and White's construction [MW13] with this nonproperly embedded minimal plane, he showed the following:

Theorem 7.2 ([Co13]). *Any open orientable surface S can be embedded in $\mathbf{H}^3$ as a complete nonproper minimal surface.*

7.2. The moduli space

On the other hand, the space of all solutions to the asymptotic Plateau problem (the Moduli Space) is another interesting subject, and its structure gives powerful global analysis tools to get important results on the number of solutions to the asymptotic Plateau problem . In particular, the author showed that the space of minimal planes in $\mathbf{H}^3$ with asymptotic boundary a $C^{3,\alpha}$ smooth simple closed curve is a manifold and its projection to the asymptotic boundary is a Fredholm map in [Co04]. By using these results, the author showed a generic uniqueness result (Theorem 5.4) for C^3 smooth simple closed curves in $S^2_\infty(\mathbf{H}^3)$, [Co06a].

Very recently, by using different techniques, Alexakis and Mazzeo generalized the author's results to complete properly embedded minimal surfaces of any fixed genus in convex cocompact hyperbolic 3-manifolds ($\mathbf{H}^3$ is a special case).

Theorem 7.3 ([AM10]). *Let X be a convex cocompact hyperbolic 3-manifold, and $M_k(X)$ is the space of properly embedded minimal surfaces in X of genus k with asymptotic boundary a $C^{3,\alpha}$ simple closed curve in $\partial_\infty X$. Let ξ be the space of all $C^{3,\alpha}$ curves in $\partial_\infty X$. Then, both $M_k(X)$ ($M_0(\mathbf{H}^3)$ case in [Co04]) and ξ are Banach manifolds, and the projection map $\pi_k : M_k(X) \to \xi$ is a smooth proper Fredholm map of index 0.*

Note that being Fredholm map of index 0 is a very strong property, and it can be considered as the map is locally one-to-one for generic points. Indeed, they showed that

π_k is not only Fredholm of index 0, but also proper. Hence, by using this result, they developed a powerful $\mathbf{Z}$-valued degree theory for π_k as follows:

$$deg(\pi_k) = \sum_{\Sigma \in \pi_k^{-1}(\Gamma)} (-1)^{n(\Sigma)}$$

where Γ is a regular value of π_k and $n(\Sigma)$ is the number of negative eigenvalues of the Jabobi operator $-L_\Sigma$. By combining this degree theory with the techniques in [TT78] and [Wh87], one can get very interesting results on complete minimal surfaces in $\mathbf{H}^3$ (see Section 4 in [AM10]).

7.3. Renormalized area

In [AM10], in addition to the study of the global structure of moduli spaces of complete minimal surfaces in $\mathbf{H}^3$ and a $\mathbf{Z}$-valued degree theory on them (see Section 7.2), Alexakis and Mazzeo defined a notion called *renormalized area* $\mathcal{A}(Y)$ for properly embedded minimal surfaces Y in $\mathbf{H}^3$ (or more generally convex cocompact hyperbolic 3-manifolds) where $\partial_\infty Y = \Gamma$ is a $C^{3,\alpha}$ simple closed curve in $S^2_\infty(\mathbf{H}^3)$. They showed that if a minimal surface minimizes renormalized area, it must be an area minimizing surface.

Theorem 7.4 ([AM10])**.** *Let Γ be a $C^{3,\alpha}$ embedded curve in $S^2_\infty(\mathbf{H}^3)$. Suppose that Y_1 and Y_2 are two properly embedded minimal surfaces in $\mathbf{H}^3$ with $\partial_\infty Y_1 = \partial_\infty Y_2 = \Gamma$. If Y_1 is area minimizing in $\mathbf{H}^3$, then $\mathcal{A}(Y_1) \leq \mathcal{A}(Y_2)$, and equality holds if and only if Y_2 is also an area minimizer.*

Moreover, they also showed that the renormalized area functional $\mathcal{A}$ is connected with the Willmore functional $\mathcal{W}$, which is the total integral of the square of the mean curvature, in the following way. The renormalized area functional is defined for any convex cocompact hyperbolic 3-manifold X. After modifying the metric on X in a suitable way such that it induces a Z_2-invariant smooth metric on the double of X, denoted by $2X$, consider the double of any surface Σ in $M_k(X)$ (see Section 7.2), denoted by 2Σ, in $2X$. Then, Alexakis and Mazzeo showed that $\mathcal{A}(\Sigma) = -\frac{1}{2}\mathcal{W}(2\Sigma)$ for any $\Sigma \in M_k(X)$.

On the other hand, they also define an extended renormalized area $\mathcal{R}$ which is defined for all properly embedded surfaces Y which intersect $S^2_\infty(\mathbf{H}^3)$ orthogonally and $\partial_\infty Y = \Gamma$ is a $C^{3,\alpha}$ simple closed curve in $S^2_\infty(\mathbf{H}^3)$. Then the extended renormalized area behaves just like the area for these surfaces.

Theorem 7.5 ([AM10])**.** *Let Γ be a $\mathcal{C}^{3,\alpha}$ closed curve in $S^2_\infty(\mathbf{H}^3)$. Then the infimum of $\mathcal{R}(Y)$ where Y ranges over the set of all $\mathcal{C}^{3,\alpha}$ surfaces with $\partial_\infty Y = \Gamma$ which intersect $S^2_\infty(\mathbf{H}^3)$ orthogonally is attained only by absolutely area-minimizing surfaces. Also, if Y is a critical point for $\mathcal{R}$, then Y must be a minimal surface.*

Notice that renormalized area behaves just like the area for these infinite surfaces in many ways. Hence, many techniques from the compact area minimizing surfaces can be generalized to these surfaces with this new tool.

Acknowledgements. I would like to thank Urs Lang for very valuable remarks.

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