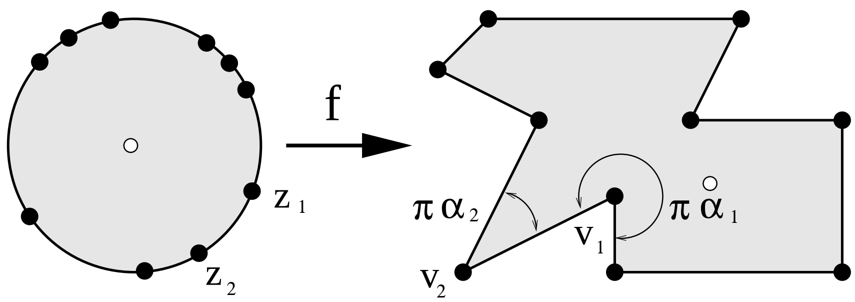
A fast approximation to the Riemann map

Christopher Bishop

www.math.sunysb.edu/~bishop

Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \mathbb{D} \to \Omega$.



If $P = \partial\Omega$ is a simple polygon, preimages of the vertices are called the **conformal prevertices**. In general, there is no simple formula for them in terms of P.

The Schwarz-Christoffel formula gives a formula for the Riemann map of the disk onto a polygonal region Ω: if the interior angles of P are $\alpha\pi = \{\alpha_1\pi, \ldots, \alpha_n\pi\}$, i.e.,

$$f(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$

This maps the disk to a polygon with the correct angles but the edge lengths depend on the parameters $\mathbf{z} = \{z_1, \ldots, z_n\}$. How to find $\mathbf{z}$ in practice?

Theorem: There is a $C < \infty$ so that if Ω is bounded by a simply polygon P with n vertices we can find $\mathbf{w} = \{w_1, \dots, w_n\} \subset \mathbb{T}$ so that

1. $\mathbf{w}$ can be computed in at most Cn steps.
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \log 8$, $\mathbf{z}$ = conformal prevertices.

$d_{QC}(\mathbf{w}, \mathbf{z})$ is the infimum of $\log K$ such that there is a K-quasiconformal $h : \mathbb{D} \to \mathbb{D}$ such that $h(\mathbf{z}) = \mathbf{w}$.

This is not quite a formula for the conformal prevertices, but it is pretty close. This result originates with a theorem about convex sets in hyperbolic 3-manifolds.

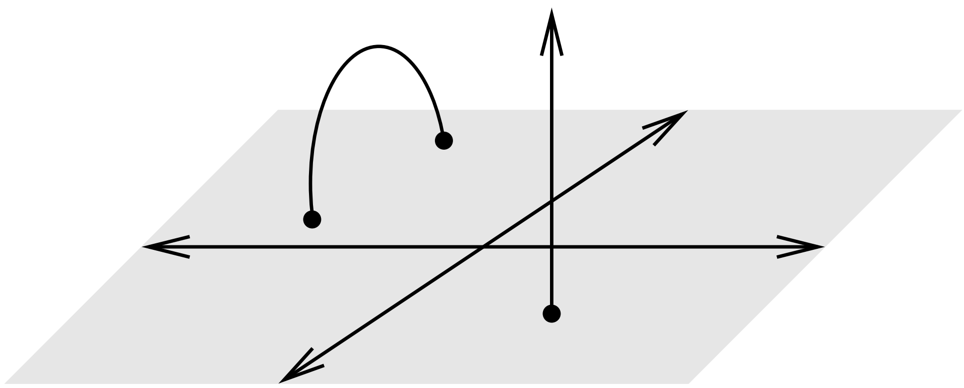
Hyperbolic space: The hyperbolic metric on the disk or ball is

$$d\rho = 2|dz|/(1-|z|^2).$$

The hyperbolic metric on the upper half space

$$d\rho = |dz|/\mathrm{dist}(z, \mathbb{R}^2).$$

Geodesics are circles orthogonal to the boundary



The hyperbolic metric on a simply connected domain plane Ω is defined by transferring the metric on the disk by the Riemann map. It satisfies

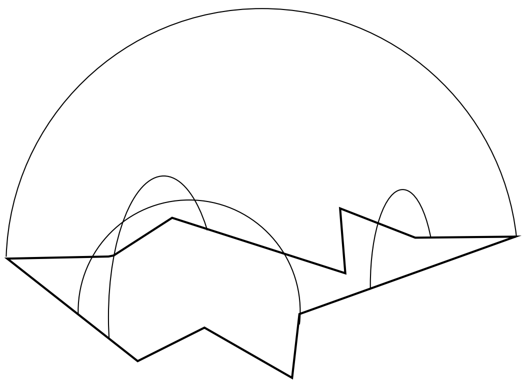
$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}.$$

A **Kleinian group** is a discrete group of isometries of hyperbolic space. A **hyperbolic manifold** is $M = \mathbb{R}^3_+/G$, G a Kleinian group.

An orbit accumulates on a set $\Lambda \subset S^2 = \partial\mathbb{B}$ called the **limit set**. The complement $\Omega = S^2 \setminus \Lambda$ is called the **ordinary set** and $\partial_\infty M = \Omega/G$ is a Riemann surface.

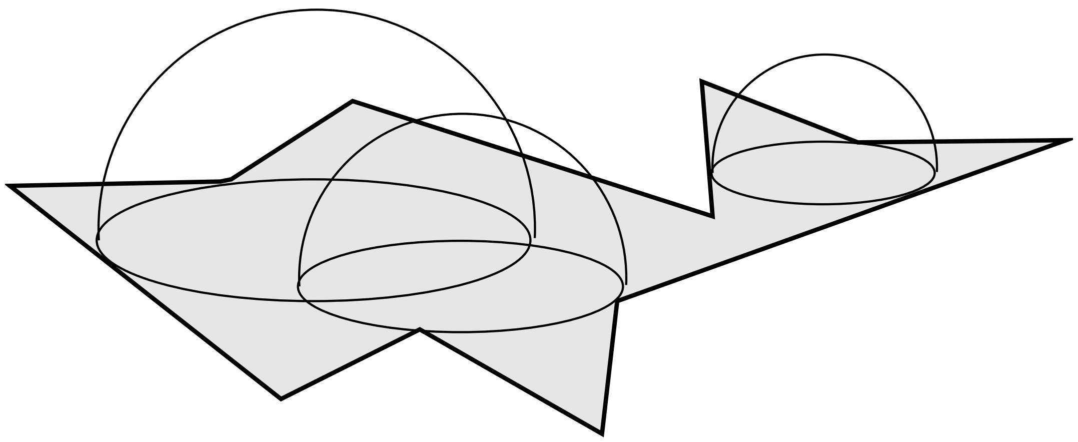
The **convex core** is smallest convex set in M containing all closed geodesics.
• $\partial C(M) = \partial_\infty M$ $(M \setminus C(M) \sim \partial C(M) \times \mathbb{R}^+)$.
• G has a finite sided fundamental domain iff $C(M)$ has finite volume.
• Λ has zero area iff Brownian motion almost surely leaves $C(M)$ (Ahlfors conjectures claims this happens unless $C(M) = M$).

The convex core and the limit set are closely related. Let $C(\Lambda) \subset \mathbb{H}^3$ be the smallest convex set in $\mathbb{R}^3_+$ which contains all the infinite hyperbolic geodesics with both endpoints in Λ. This is G-invariant and $C(M) = C(\Lambda)/G$.



There is one component of $\partial C(M)$ corresponding to each complementary component of Λ. Let Ω be one such component and let $S \subset \mathbb{R}^3_+$ be the component of $\partial C(M)$ "facing " Ω. S is called the **dome** of Ω.

The dome can also be defined as the boundary of the union of all hemispheres centered on $\mathbb{R}^2$ whose bases are disks contained in Ω.



Similar to Euclidean space where the complement of a closed convex set is a union of half-spaces.

If Λ is connected, Ω is simply connected.

Each point on S is on the boundary of at least one hyperbolic half-space H whose interior misses S. ∂H is called a **support plane**. Its intersection with S is either an infinite geodesic (**bending line**) or a **geodesic face** bounded by geodesics. Thus the base disk hits $\partial\Omega$ in at least two points. The centers of such disks define the **medial axis** in computational geometry.

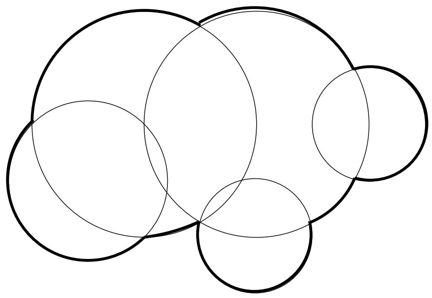
- Medial Axis =

  set of support planes =

  dual of bending lamination.

MA of a polygon is well known to be a finite tree. It is a $\mathbb{R}$-tree for any simply connected domain.

If Ω is simply connected and a finite union of disks we call it **finitely bent**. The dome is a finite union of geodesic faces which meet along infinite geodesics with a well defined **bending angle**. Every simply connected domain can be approximated by such domains from inside.

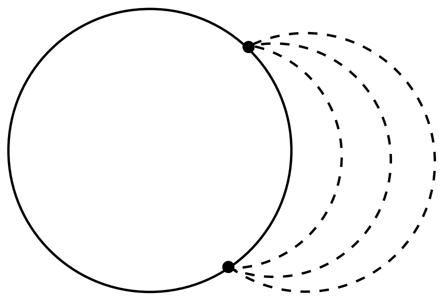


In general the dome is still a union of geodesic faces (possibly empty) and bending geodesics (possibly uncountably many). There is a **transverse measure** which says how much bending a transverse arc encounters as it crosses the bending lamination. Thus dome has a measured lamination, called the bending lamination.

Let ρ_S be the hyperbolic path metric on S.

Theorem (Thurston): There is an isometry ι from (S, ρ_S) to the hyperbolic disk.

For finitely bent domains one can see this by simply rotating around each bending geodesic by an isometry to remove the bending. Doing this for all of them maps the surface isometrically to a hemisphere, which is isometric to the hyperbolic disk.



• Recall that a map is **biLipschitz** if

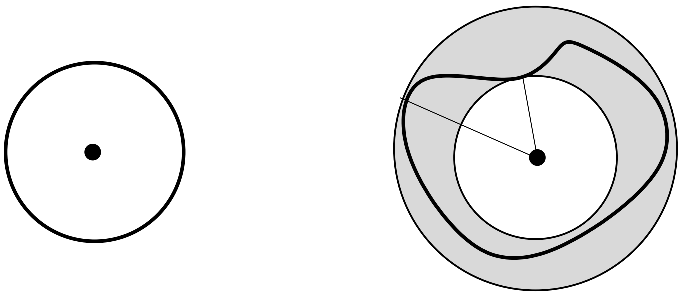
$$\frac{1}{A}d(x,y) \le d(f(x), f(y)) \le Ad(x,y).$$

• f is a **quasi-isometry** (or rough isometry) if

$$\frac{1}{A}\rho(x,y) - B \le \rho(f(x), f(y)) \le A\rho(x,y) + B.$$

• A homeomorphism is K**-quasiconformal** if for every $x \in \Omega$

$$\limsup_{r \to 0} \frac{\max_{y:|x-y|=r} |f(x) - f(y)|}{\min_{y:|x-y|=r} |f(x) - f(y)|} \le K.$$



• For self-maps of hyperbolic disk,
BiLipschitz $\Rightarrow$
Quasiconformal $\Rightarrow$
Quasi-isometry $\Rightarrow$
there is BiLipschitz map with same boundary values

Thus all three classes have the same type of boundary values which can be identified as:

• A circle homeomorphism is **quasisymmetric** if

$$\frac{1}{M} \leq \frac{|f(I)|}{|f(J)|} \leq M$$

whenever I, J are adjacent intervals of equal length.

Theorem (Sullivan, Epstein-Marden): There is a K_0 biLipschitz map $\sigma : \Omega \to S_\Omega$ so that $\sigma = \text{Id}$ on $\partial\Omega = \partial S$. Thus σ is also K-QC for some K independent of Ω.

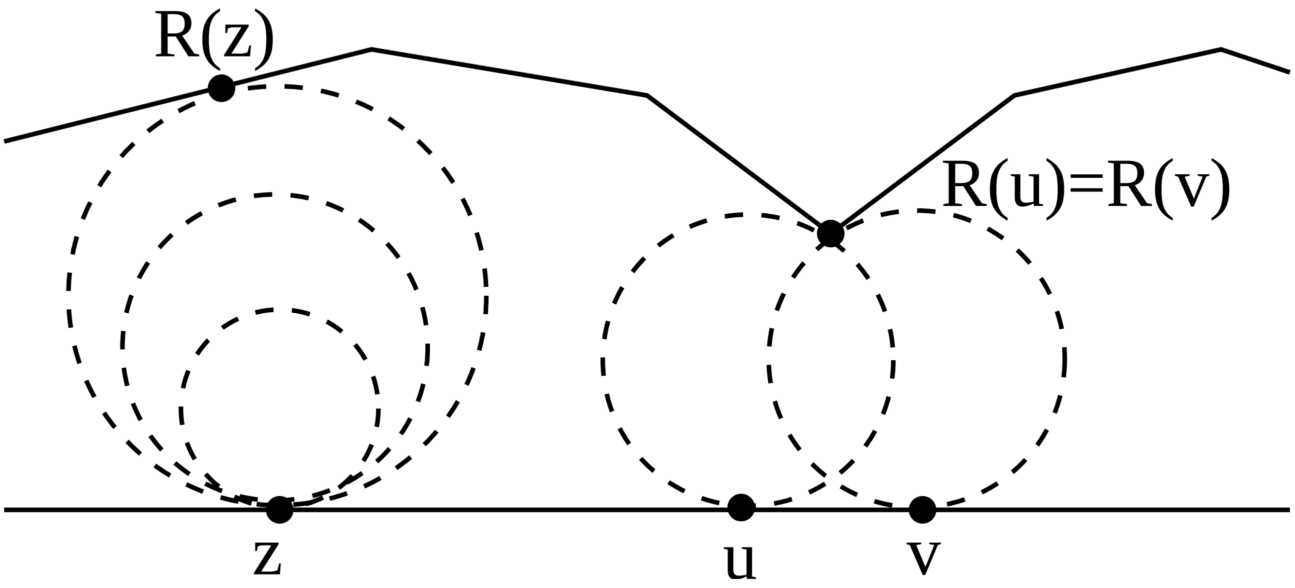
If Ω is invariant under a Kleinian group, so is σ.

Sullivan sketched proof for some invariant domains without explicit constant. Epstein-Marden proved it for all simply connected domains with $K \approx 80$. Best current bounds are $K \leq 7.82$ (B.) and $K > 2.1$ (Epstein-Markovic).

I currently know of five proofs involving explicit constructions, Teichmüller theory, quasisymmetric mappings and martingales. I will give the "easy" one.

A proof of Sullivan's Theorem:

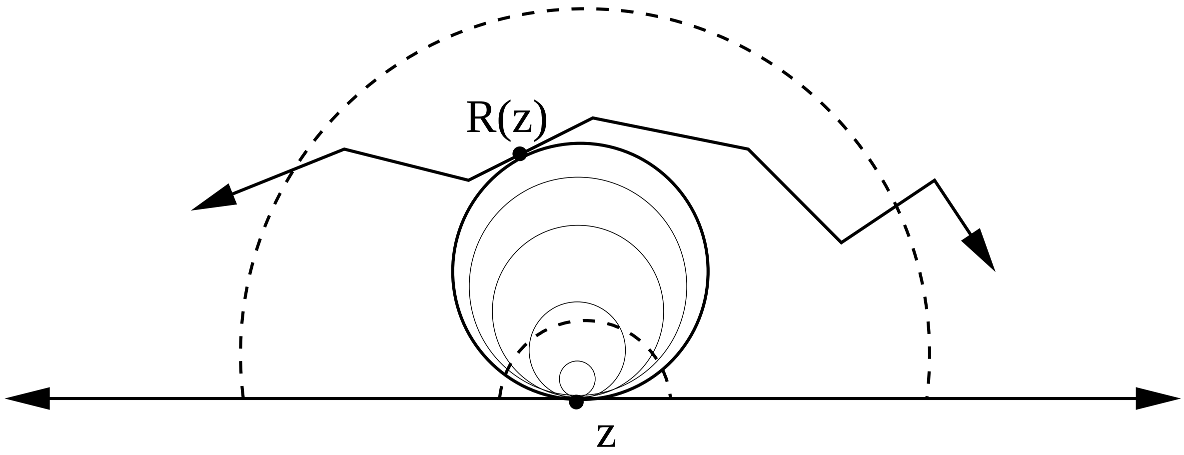
Define nearest point retraction $R : \Omega \to S$ by expanding horoball tangent at $z \in \Omega$ until it first hits S at $R(z)$. It suffices to show R is a quasi-isometry.



In general, the retraction is **not** 1-to-1 in crescents, so can't be QC.

Fact 1: If $z \in \Omega$,

$$\mathrm{dist}(z, \partial\Omega) \simeq \mathrm{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



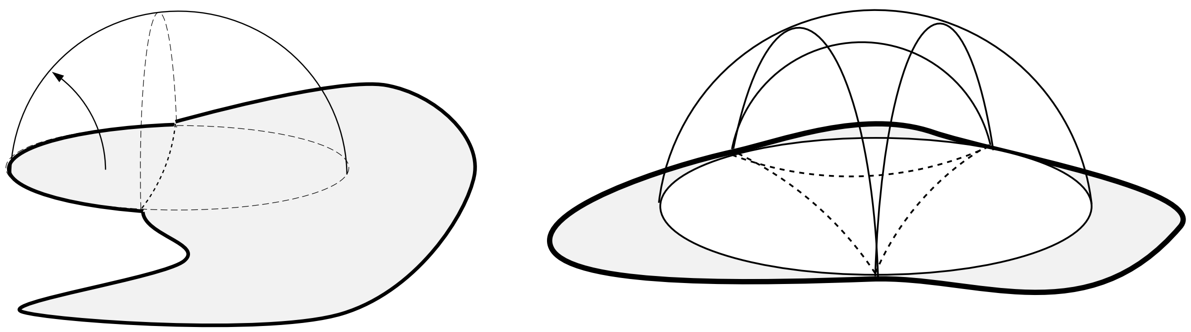
Fact 2: R is Lipschitz. For simply connected domain

$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)},$$

and for disk $D \subset \Omega$,

$$\mathrm{dist}(z, \partial\Omega) \lesssim \mathrm{dist}(z, \partial D) \leq \mathrm{dist}(z, \partial\Omega)$$

for z in convex hull (in D) of $\partial D \cap \partial\Omega$.

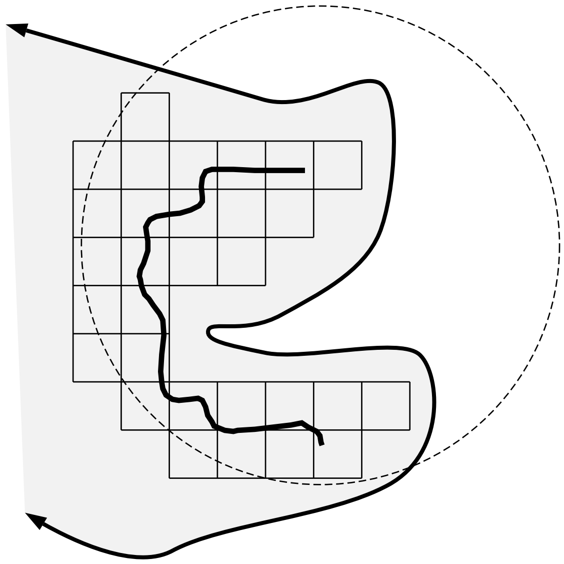


Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z, w) \leq C$.

Suppose $R(z)$ is at height r. Every point on S-geodesic from $R(z)$ to $R(w)$ has height $\simeq r$. Thus every preimage point v has

$$\mathrm{dist}(v, \partial\Omega) \simeq r \simeq |v - z|.$$

Thus preimage of path is covered by $O(1)$ squares of hyperbolic diameter $\leq C$.

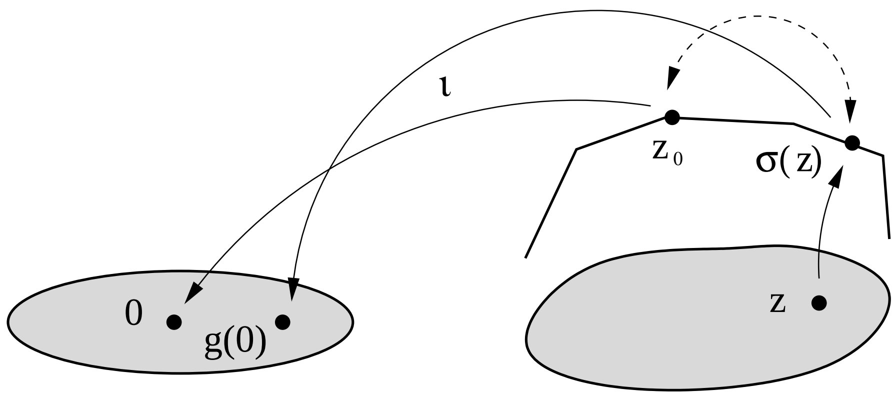


Moreover, $g = \iota \circ \sigma : \Omega \to \mathbb{D}$ is locally Lipschitz. Standard estimates show

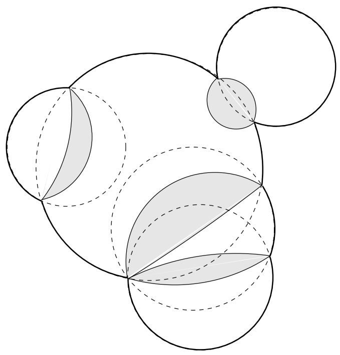
$$|g'(z)| \simeq \frac{\mathrm{dist}(g(z), \partial\mathbb{D})}{\mathrm{dist}(z, \partial\Omega)}.$$

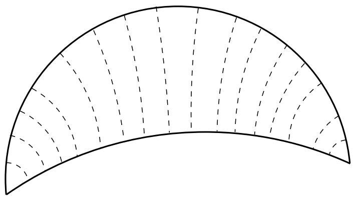
Use Fact 1

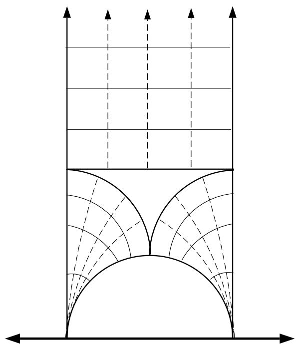
$$\begin{aligned}
\mathrm{dist}(z, \partial\Omega) &\simeq \mathrm{dist}(\sigma(z), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(\sigma(z), z_0)) \\
&\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \mathrm{dist}(g(z), \partial D)
\end{aligned}$$



Proof giving explicit constant: Divide finitely bent Ω into crescents and gaps. Foliate crescents by orthogonal circles. Triangulate gaps and extend foliation to gaps. Define explicit map of Ω into $\cup$gaps which locally QC and continuous when followed by $R : \Omega \to S$.







Requires explict upper bound of fraction of length that a foliation interval can spend in crescents.

Corollary: Every simply connected domain can be mapped to the disk by a locally Lipschitz homeomorphism.

Corollary: Any conformal map $f : D \to \Omega$ can be written as $f = g \circ h$ where h is a K-QC self-map of D and $|g'|$ is bounded away from zero. Indeed $|g'(tz)| \leq C|g'(z)|$, $0 \leq t \leq 1$

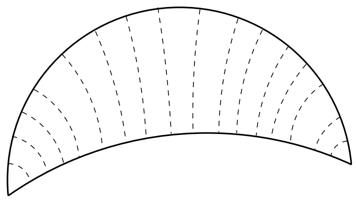
Corollary: $K = 2$ in Sullivan's theorem implies Brennan's conjecture: for any conformal $f : \Omega \to \mathbb{D}$, $f' \in L^p(\Omega, dxdy)$ for all $p < 4$.

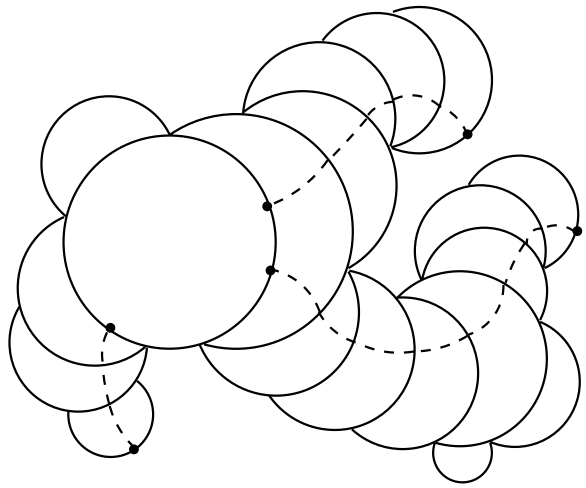
Epstein and Markovic showed $K > 2.1$.

Corollary: If Ω/G has no Greens function, then $\partial\Omega$ is either a circle or has dimension > 1.

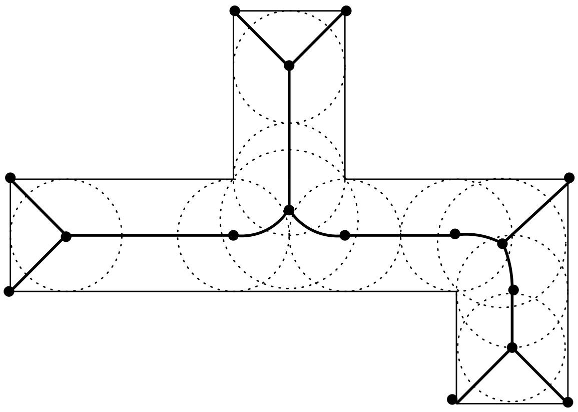
ι is easy to visualize for for finitely bent domains. Write Ω as a disk and a union of crescents. Map 'outer' edge of each crescent to 'inner' edge by a Möbius transformation. Composing gives ι.

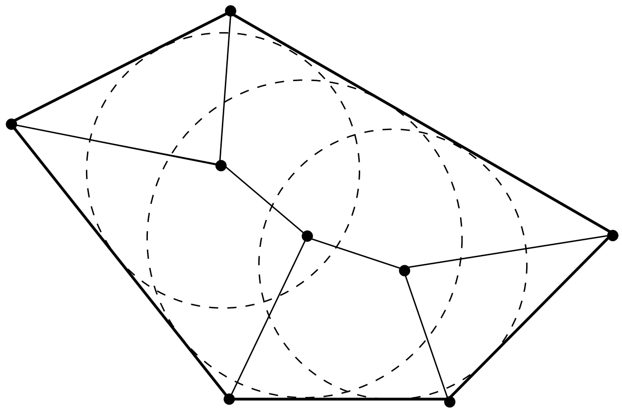
Equivalently, foliate crescents by circles passing though endpoints. The orthogonal foliation consists of circular arcs which are orthogonal to both boundary arcs of the crescent. Following leaves gives $\iota : \partial\Omega \to \partial D$.





The medial axis: The medial axis of a domain is the set of points in the interior which are equidistant from two or more boundary points.





Chin-Snoeyink-Wang proved the medial axis can be computed in time $O(n)$.

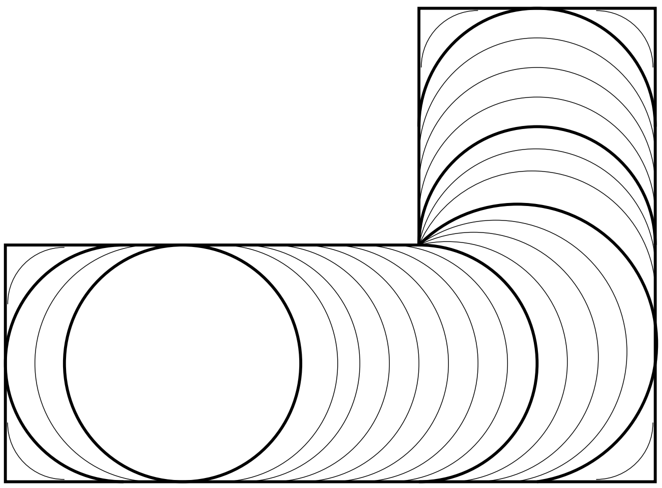
Medial axis is a type of Voronoi diagram, where sites are vertices and open edges and distance is measured along paths inside P. This constrained Voronoi diagram has extra edges at the concave vertices.

Applications of the medial axis:

- Analysis of chromosones
- Designs of type fonts
- Describe statisical features of porus materials
- Shape recognition
- Time critical collision detection
- Robotic motion
- Biological description of shape
- Mesh generation
- Computer vision
- Radiosurgery

However, medial axis is unstable under perturbations. For example, medial axis of a disk is a point, but for regular n-gon has n radial segments.

Medial axis flow: Fix a MA-disk D_0. If D is a distinct MA-disk, $\Omega \setminus D$ has at least two components and exactly one of these intersects D_0. Take the boundary arcs of D corresponding to the components that don't hit D_0. This foliates $\Omega \setminus D_0$



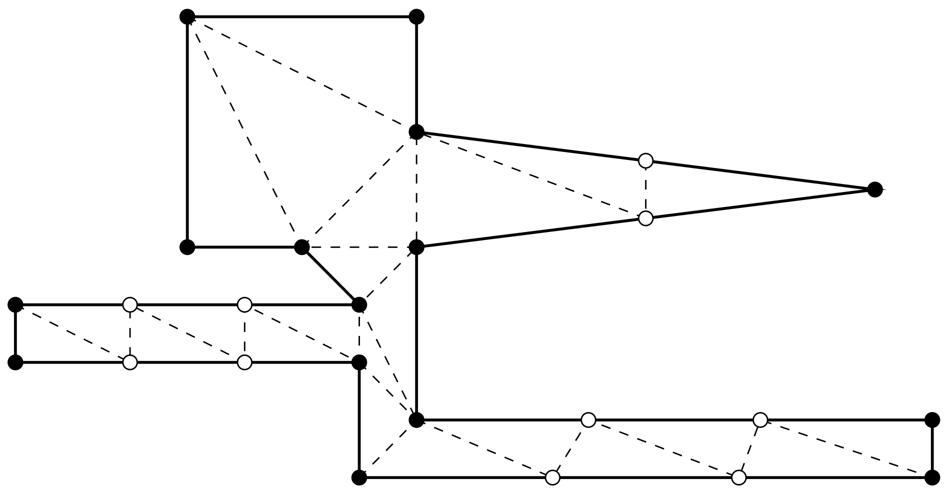
Following orthogonal flow gives $\iota : \partial\Omega \to \mathbb{T} = \partial D_0$. This map differs from Riemann map by a K-QC map (where K is independent of Ω).

Theorem: There is a $C < \infty$ so that if Ω is bounded by a simply polygon P with n vertices we can find points $\mathbf{w} = \{w_1, \ldots, w_n\} = \iota(\mathbf{v}) \subset \mathbb{T}$ so that

1. $\mathbf{w}$ can be computed in at most Cn steps.
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \log 8$, $\mathbf{z}$ = conformal prevertices.

This theorem was motivated by 1998 paper of Toby Driscoll and Steve Vavasis "Numerical conformal mappings using cross ratios and Delaunay triangulations". They define an mapping which turns out to be the ι map for a domain assoicated to P.

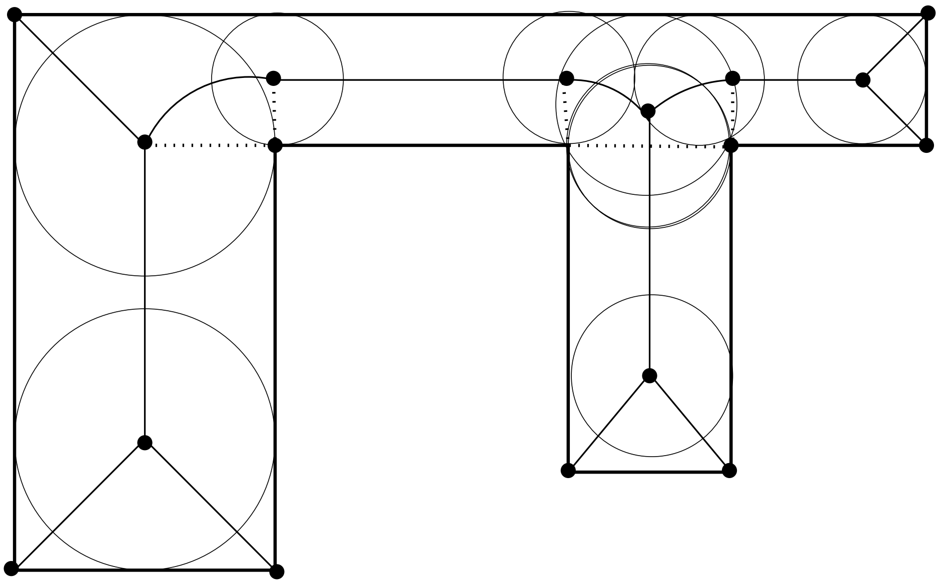
However, this approximation requires new vertices to be added to P (arbitarily many depending on the geometry), so their method is not $O(n)$.



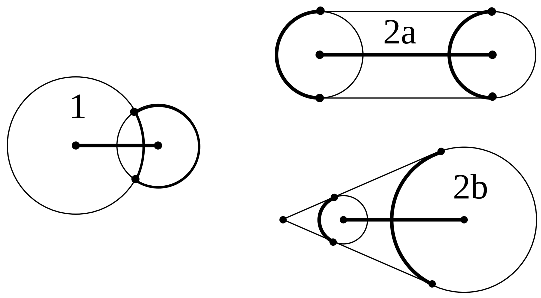
If we know the medial axis we can compute $\iota(\mathbf{v})$ in $O(n)$ steps:

• Compute the medial axis (which is a tree) and fix a root vertex.

• For each vertex of medial axis find the unique adjacent vertex closer to the root (the "parent"). Include vertices of P as leaves of medial axis tree. Also choose 3 reference points on boundary of each non-trivial disk.



• Define a certain explicit Möbius transformation between each disk and its parent. Record cross ratios of images of reference points with respect to reference points for parent.



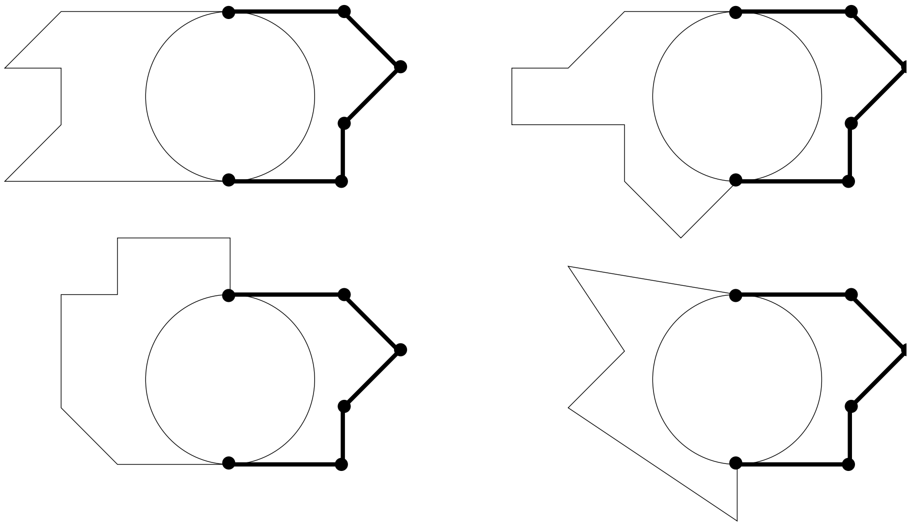
• ι is defined by composing the maps along path from leaf to root.

• Working from the root outwards, we compute images of reference points using recorded cross ratio information. When we reach a vertex of P we have ι. Time is $O(n)$.

Other properties of ι

• The map ι shrinks arclength along P. (Collapsing a crescent decreases arclength on 'outer' edge; this gives finitely bent case and pass to limit for general case.)

• If $\gamma \subset \partial\Omega$ is an arc whose endpoints lie on $\partial D \cap \partial\Omega$ for some open disk $D \subset \Omega$, then the images of vertices on γ can be chosen to depend only γ and not on the rest of $\partial\Omega$.

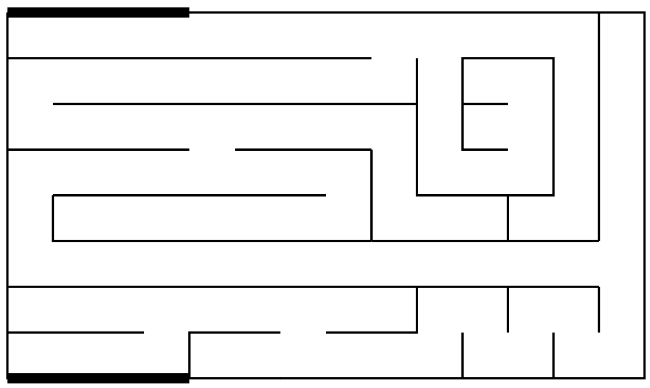


• If $\mathbf{v} = \{v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}\} \subset \partial D \cap \partial\Omega$ for some open disk $D \subset \Omega$ then the cross ratio of $\iota(\mathbf{v})$ equals the cross ratio of $\mathbf{v}$.

• For any four vertices $v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}$ of P the corresponding true and approximate prevertices on $\mathbb{T}$ satisfy

$$\frac{1}{8} \leq \frac{\mathrm{Mod}_{\mathbb{D}}(z_{j_1}, z_{j_2}, z_{j_3}, z_{j_4})}{\mathrm{Mod}_{\mathbb{D}}(w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})} \leq 8,$$

where Mod denotes conformal modulus on the disk.



There is an $O(n)$ algorithm to compute the medial axis of an n-gon. There are much simplier $O(n \log n)$ methods which may be faster in practice.

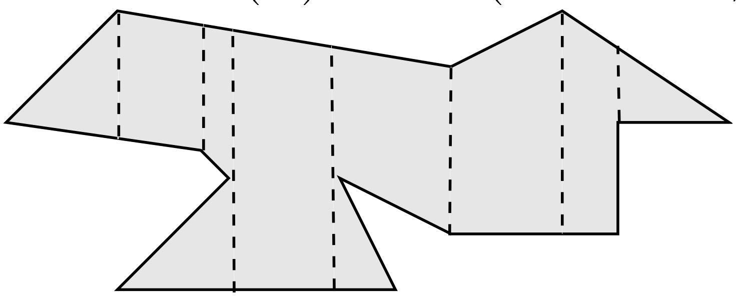
It is equivalent to compute the internal Voronoi diagram for P where the sites are the vertices and open edges. A key step is the

Merge Lemma: Suppose n sites S are divided into S_1 and S_2 by a line and that the Voronoi diagrams of S_1 and S_2 are given. Then the Voronoi diagram of S can be found in at most $O(n)$ additional time.

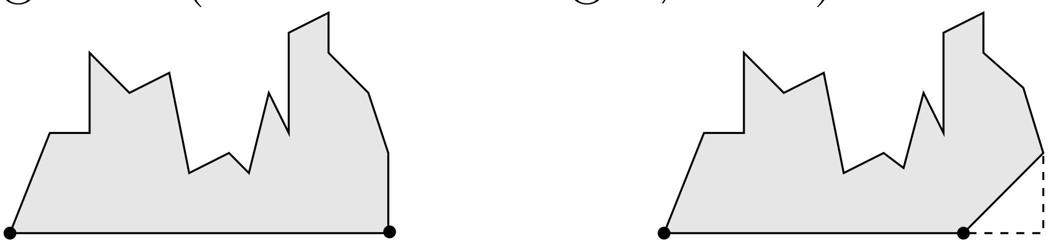
Sort x-coordinates of vertices and group into vertical slabs. Recursively dividing P into two almost equal sized pieces gives a $O(n \log n)$ algorithim (Yap 1993).

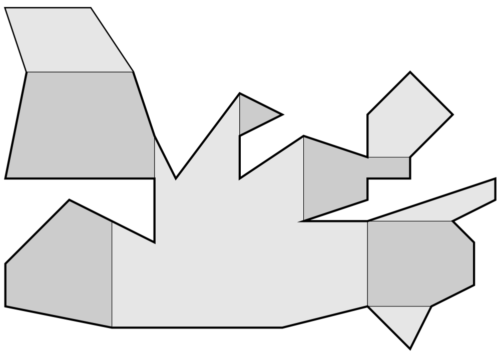
The $O(n)$ estimate is more involved. The basic steps are (Chin-Snoeyink-Wang, 1998)

• Cut interior of P into trapedzoids with vertical sides. Possible in $O(n)$ time (Chazelle, 1991).

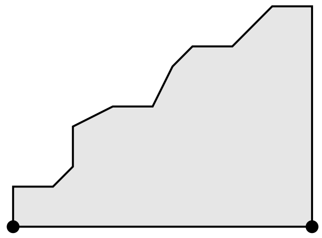


• Use trapezoids to divide P into pseudo-normal histograms (Klein and Lingas, 1993).





• Cut each pseudo-normal histogram into monotone histograms.



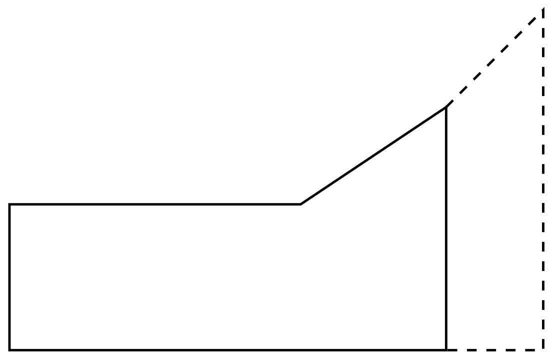
• Compute Voronoi diagrams of monotone histograms (Aggarwal, Guibas, Saxe, Shor, 1989 and Djidev, Lingas 1991)

• Merge monotone diagrams into diagrams for pseudo-normal histograms and merge the results into diagram for P.

Davis's algorithm: Suppose P is a polygon with vertices $\mathbf{v} = \{v_1, \ldots, v_n\}$ and $\mathbf{w} = \{w_1, \ldots, w_n\} \in \mathbb{T}^n$ is the current guess for the prevertices. Apply Schwarz-Christoffel to $\mathbf{w}$ and let $\mathbf{v}' = \{v_1', \ldots, v_n'\}$ be resulting vertices. Define $\mathbf{w}'$ by

$$|w_k' - w_{k+1}'| = k|w_k - w_{k+1}|\frac{|v_k - v_{k+1}|}{|v_k' - v_{k+1}'|}$$

where k is a normalizing factor. Often works in practice but must fail in some cases.



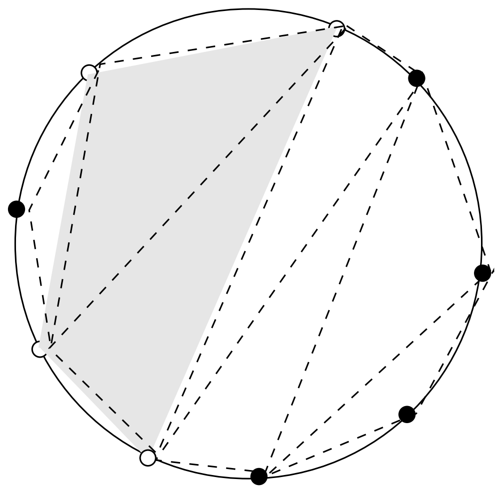
Can we do better using medial axis?

Coordinates on n-tuples: Let X_n denote the space of n-tuples of distinct points on the unit circle and let $\tilde{X}_n$ be X_n with any two Möbius equivalent n-tuples identified.

Give $\tilde{X}_n$ coordinates by taking a triangulation of the disk with the given vertices and using the $n-3$ numbers

$$\log \mathrm{Mod}_{\mathbb{D}}(z_{j_1}, z_{j_2}, z_{j_3}, z_{j_4})$$

for each 4-tuple corresponding to a pair of adjacent triangles.



Fix n angles and let S denote the Schwarz-Christoffel map from n-tuples on $\mathbb{T}$ to polygons determined by these angles. Consider $\Psi = \iota \circ S$ mapping n-tuples in the circle to n-tuples. Easy to check Ψ is well-defined on $\tilde{X}_n$.

Given a simple polygon with vertices $\mathbf{v}$ the conformal prevertices are a solution of $\Psi(\mathbf{z}) = \iota(v)$. Is this the only solution?

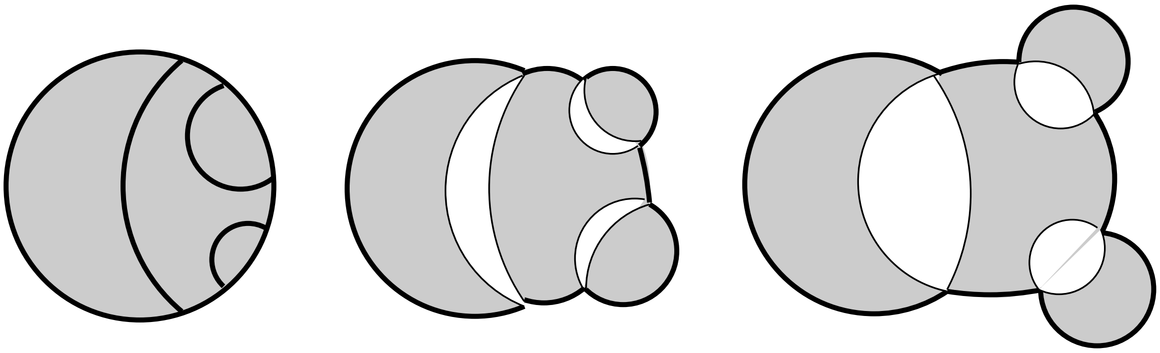
Is Ψ one-to-one? A diffeomorphism? Not hard to see it is onto since $\|\Psi(\mathbf{w}) - \mathbf{w}\| \leq C$.

Does $\|\Psi(\mathbf{w}) - \iota(v)\|^2$ have additional local minima?

The QC extension of ι has an explicit Beltrami coefficient. Can we improve our guess by quickly solving (or approximately solving) the Beltrami equation on the disk?

On the disk the Beltrami coefficient can be written as a sum $\mu = \sum \mu_n$ where each μ_n corresponds to an explicitly given circle homeomorphism. The composition of these maps may give a good approximation to the circle homeomorphism corresponding to μ, especially if all the norms are small.

If Ω is finitely bent we can multiply each crescent angle by a factor of $t \in [0, 1]$ to get a surface Ω_t. Thus $\Omega_1 = \Omega$ and $\Omega_0 = \mathbb{D}$.



Then $(\iota_t)^{-1} \circ \iota_s$ is a map of $\partial\Omega_s$ to $\partial\Omega_t$ with QC extension of size $1 + C|s - t|$. This means the conformal prevertices are close in the d_{QC} metric.

Suppose we have a map F so that $d_{QC}(z, w) < \epsilon$ implies $d_{QC}(z, F(w)) < \lambda\epsilon$ for some $\lambda < 1$, i.e., we can solve for conformal pervertices $\mathbf{z}$ by iteration if we start close enough.

Choose $t_0 = 0 < t_1 < \ldots t_N = 1$ and let $\Omega_k = \Omega_{t_k}$. Choose points so prevertices differ by $\epsilon/2$ in d_{QC} metric. Iterate F to get within $\epsilon/2$ of kth set of prevertices then use this as initial guess for next set. Leads us to correct prevertices in $C/(\epsilon \log(1/\lambda))$ steps.

• What is best K in Sullivan's theorem? Given a polygon could compute the conformal and ι images for various 4-tuples of vertices to get lower bounds.

• Can we get a "simple" map that approximates the conformal prevertices within d_{QC} distance ϵ in time $O(n \log \epsilon)$?

• Driscoll and Vavasis report that the iteration

$$w_{n+1} = w_n + (\iota(v) - \Psi(w_n)),$$

converges to the true prevertices in practice. Why does it? This would be true if the derivative of Ψ was close to the identity.

What works in $\mathbb{R}^3$?

If Ω is obtained by adding non-overlapping spherical caps repeated onto a ball then the medial axis will still be a tree and the proof should be unchanged: such a domain can be mapped to a ball by a uniformly QC map with a bounded derivative.

In general, medial axis of 3-D domain is a 2-D object. If each boundary point is associated to a medial axis ball containing it and each point of medial axis is connected by a path to some base point, this induces a map from $\partial\Omega$ to a sphere, but is not clear when this is "nice", or even continuous.