Contemporary Mathematics

The law of the iterated logarithm for Kleinian groups

Christopher J. Bishop and Peter W. Jones

Abstract. Suppose G is an analytically finite, geometrically infinite Kleinian group and that there is a lower bound on the injectivity radius for $M = \mathbb{B}/G$. Then the limit set Λ has positive Hausdorff measure with respect to the gauge function $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$. If, in addition, the group is topological tame and $\Lambda \neq S^2$, then the limit set has finite measure with respect to this gauge. We also obtain corollaries on the size of the conical limit set, the existence and ergodicity of conformal densities, and the differentiability of quasiconformal conjugacies.

1. Statement of results

Suppose G is a Kleinian group acting on the hyperbolic three ball $\mathbb{B}$ and let Λ and Λ_c denote the limit set and conical limit set respectively (all definitions will be reviewed in the next section). The purpose of this paper is to prove the following result.

Theorem 1.1. *Suppose G is an analytically finite but geometrically infinite Kleinian group such that the injectivity radius of $M = \mathbb{B}/G$ is bounded away from zero. Then $\Lambda \setminus \Lambda_c$ has positive Hausdorff measure with respect to the function $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$. If, in addition, G is topologically tame and $\Lambda \neq S^2$, then Λ has finite Hausdorff φ-measure and Λ_c has zero φ-measure.*

This had been conjectured for certain examples by Sullivan in [**26**] and is extremely close to results of Makarov for conformal mappings [**16**]. The topological tameness is needed for the second conclusion since there are analytically finite groups with injectivity radius bounded away from zero such that the limit sets are nowhere dense and have positive area [**20**]. The first statement of the theorem

---

1991 *Mathematics Subject Classification.* 30F40 ; Secondary 28A78, 30C35, 31A15 .

Key words and phrases. Hausdorff dimension, Kleinian groups, limit set, degenerate group, geometrically infinite, conical limit set, Poincaré exponent, critical exponent, Martin boundary, conformal density.

The first author is partially supported by NSF Grant DMS 95-00577 and an Alfred P. Sloan research fellowship. The second author is partially supported by NSF grant DMS-94-23746.

strengthens the result from [**5**] that the limit set has dimension 2. The final statement strengthens the results of Bonahon, Canary and Thurston that limit sets of such groups have zero area (see [**6**]).

The lower bound in Theorem 1.1 can be easily deduced from known results of Makarov on laws of the iterated logarithm for Bloch functions and martingales [**17**]. The upper bound is in some sense new, but we point out that Makarov (Proposition 4.7 of [**17**]) proved a similar result in an analogous setting (he considered the support of Kahane's measure). His results use classical estimates for random walks which do not seem to apply directly to our case.

If the injectivity radius of M is bounded away from zero, then G has no parabolic elements and in this case G is geometrically finite iff $\Lambda = \Lambda_c$ (see [**2**], [**4**]). Thus

COROLLARY 1.2. *Suppose G is an analytically finite Kleinian group such that the injectivity radius of $M = \mathbb{B}/G$ is bounded away from zero. Then $\Lambda \setminus \Lambda_c$ is either empty or has dimension 2, depending on whether G is geometrically finite or infinite.*

A *conformal density* of dimensional α is a positive measure μ on Λ with the property that

$$\mu(g(E)) = \int_E |g'|^\alpha d\mu, \quad \text{for all } g \in G.$$

From this it follows that the Poisson extension of μ is a G-invariant, positive harmonic function on $\mathbb{B}$. There is always a conformal density of dimension δ, where δ is the critical exponent of G (e.g., [**23**], Chapter 4). If G is topologically tame and $\Lambda = S^2$ then $\delta = 2$ and Lebesgue measure is the unique conformal density.

COROLLARY 1.3. *Suppose G is topologically tame, geometrically infinite, has a positive lower bound for the injectivity radius, and $\Lambda \neq S^2$. Then $\mu = \mathcal{H}^\varphi$ is a conformal density of dimension 2 where $\mathcal{H}^\varphi$ is the Hausdorff measure associated to the guage function $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$.*

If M has only one geometrically infinite end and $\Lambda \neq S^2$, then, up to constant multiples, there is only one conformal density and G will be ergodic with respect to this measure. If M has more than one geometrically infinite end then there will be a different conformal density associated to each geometrically infinite end of M and G will be ergodic with respect to each of them (Corollary 9.6). Examples of groups with one geometrically infinite end are well known (e.g., [**22**]) and examples with more than one such end can be obtained by combination theorems (e.g., [**20**]). In the latter case, $\mathcal{H}^\varphi$ on Λ is the sum of distinct conformal densities for each geometrically infinite end and thus is a conformal density for G which is not ergodic. Moreover, the measures corresponding to different ends are mutually singular, so that Theorem 4.2.1 of [**23**] is false (it says that a conformal density is ergodic iff it is unique up to positive multiples).

Since conformal densities for G correspond to positive harmonic functions on M, these remarks can also be interpreted in terms of the Martin boundary of M (e.g., [**19**]) Suppose G is topologically tame, has positive lower bound for the injectivity radius and $\Lambda \neq S^2$. Then the Martin boundary for M is $S \cup E$, where S is the compact surface $(S^2 \setminus \Lambda)/G$ and E is a finite set, one point corresponding to each geometrically infinite end of M.

It is a classical result of Hopf [**12**] that the conical limit set of a Fuchsian group has (normalized) Lebesgue measure either zero or one, and Sullivan proved the same in higher dimensions [**26**]. Λ_c always has either zero or full measure for a conformal density of dimension α (e.g., [**23**], Theorem 4.4.4), and the measure must be zero if G is convergence type for exponent α (e.g., [**23**], Theorem 4.4.1). If $\Lambda \neq S^2$ then G must be convergence type for the exponent 2.

COROLLARY 1.4. *Suppose G is a topologically tame Kleinian group such that the injectivity radius of $M = \mathbb{B}/G$ is bounded away from zero. Then Λ_c has either full 2-dimensional Lebesgue measure or zero Hausdorff measure with respect to $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$, depending on whether $\Lambda = S^2$ or not. In the latter case, Λ_c even has zero Hausdorff measure with respect to the functions $\psi(t) = t^2\log\frac{1}{t}(\log\log\frac{1}{t})^{-1-\epsilon}$, for any $\epsilon > 0$.*

The geodesic flow is not ergodic for the groups we are considering since the conical limit set has zero μ measure (Theorem 8.3.5, [**23**]). One can also see this directly as follows. Given $r > 0$ one can consider the set of doubly infinite geodesics in $C(M)$ which are never closer than distance r to $\partial C(M)$. It is easy to check that such a set is G invariant and that we can choose r so that it has $\mu \times \mu$ measure strictly between 0 and 1.

Suppose G_1 and G_2 are two groups as above which are quasiconformally conjugate via a map Φ. McMullen asked if the image of a conformal density for G_1 is singular or absolutely continuous with respect to a density for G_2. We shall show the latter is true.

COROLLARY 1.5. *Suppose G_1 and G_2 are two groups which are topologically tame, geometrically infinite and have a positive lower bound for the injectivity radius. Let μ_1 and μ_2 be $\mathcal{H}^\varphi$ restricted to Λ_1 and Λ_2 respectively. Suppose also that the groups are quasiconformally conjugate, i.e., there is a quasiconformal map Φ on S^2 so that $\Phi^{-1}\circ G_1 \circ \Phi = G_2$. Then $\Phi^*\mu_1$ is absolutely continuous with respect to μ_2. In fact, Φ is differentiable and conformal at μ_1 almost every point.*

Here $\Phi^*\mu(E) = \mu(\Phi^{-1}(E))$. The proof of the corollary uses McMullen's result that the quasiconformal conjugacy on S^2 extends to be a quasi-isometry on M with constant approaching 1 exponentially fast in the geometrically infinite ends, and the fact that μ almost every geodesic escapes into a geometrically infinite end sufficiently quickly. McMullen calls $x \in \Lambda$ a *deep point* if the geodesic γ ending at x satisfies

$$\liminf_{t\to\infty} \frac{\mathrm{dist}(\gamma(t), M \setminus C(M))}{t} > 0, \tag{1.1}$$

and proves the quasiconformal conjugacy Φ is differentiable at such points. In fact, one only needs

$$\liminf_{t\to\infty} \frac{\mathrm{dist}(\gamma(t), M \setminus C(M))}{\log t} = \infty, \tag{1.2}$$

to deduce Φ is differentiable at x. We shall show that for μ almost every geodesic,

$$\liminf_{t\to\infty} \frac{\mathrm{dist}(\gamma(t), M \setminus C(M))}{\sqrt{t}\log^{-2} t} = \infty, \tag{1.3}$$

which shows Φ is differentiable μ almost everywhere. The deep points in Λ are a set of dimension zero, and Corollary 1.5 improves this by saying Φ is differentiable on a subset of Λ having dimension 2.

We will now outline the proof of Theorem 1.1. The basic idea is that the lower bound for the dimension is deduced from the existence of a positive harmonic function U on M with bounded gradient, i.e., a harmonic function which can grow at most linearly in the geometrically infinite end of M. The upper bound for dimension can be deduced if we know that $|\nabla U|$ is bounded away from zero (at least on average) in the geometrically infinite end, i.e., the growth is at least linear.

The function U is constructed by taking the Greens function on M and moving the pole to infinity through a geometrically infinite end, while normalizing the function to have value 1 at some fixed base point. To show the limit U has bounded gradient we need to show the Greens function does (at least away from the pole). A version of the following result is proved in Sullivan's paper [**26**]; we shall give a different proof in this paper which applies in greater generality.

LEMMA 1.6. *Suppose $M = \mathbb{B}/G$ and that Greens function $G(w,z)$ exists on M. Then*
$$\sup_{z\in M} \sup_{\rho(w,z)\geq 1} |\nabla_w G(w,z)| \leq \frac{A}{vol(B(w,1))}$$
i.e., the hyperbolic gradient of Green's function is bounded outside a neighborhood of the pole by the reciprocal of the local volume. In particular, if the injectivity radius is bounded away from zero, then the gradient is uniformly bounded away from the pole.

By moving the pole of Greens function off to infinity and passing to a limit we can deduce the following.

LEMMA 1.7. *Suppose G is analytically finite, $M = \mathbb{B}/G$ has injectivity radius bounded below by $\epsilon > 0$ and that Greens function $G(w,z)$ exists on M. Then there exists a positive harmonic function U on M such that*

1. $\sup_{z\in M} |\nabla U(z)| \leq 1$,
2. *U tends to zero in the geometrically finite ends of M.*

If, in addition, G is topologically tame then for any $a_0 > 0$ there are constants a_1 and a_2 so that
$$\int_{B(z,a_1)} |\nabla U|^2 dV \geq a_2,$$
for every z such that $dist(z, C(M)) \leq a_0$. Moreover, $U(z)$ tends to $+\infty$ in the geometrically infinite ends as $dist(z, \partial C(M)) \to \infty$.

U lifts to a hyperbolically harmonic function (which we shall also call U) on $\mathbb{B}$, and this function must be the Poisson integral of some positive measure μ which is supported on the limit set. Since U has bounded hyperbolic gradient, it satisfies the estimate
$$|U(z)| \leq U(0) + A \log \frac{1}{1-|z|}.$$
The explicit formula for the Poisson integral implies
$$\mu(D(x,r))(1-r)^{-2} \leq CU(z),$$
where x is the radial projection of z onto S^2 and $r = |z|$. Thus
$$\mu((D(x,r)) \leq C(1-r)^2 \log \frac{1}{1-r}, \qquad \forall x \in S^2, 0 < r < 1. \tag{1.4}$$
The mass distribution principle then implies that Λ has positive Hausdorff measure with respect to the function $\varphi_1(t) = t^2 \log \frac{1}{t}$. This is due to Sullivan [**26**].

This estimate is easily improved using a law of the iterated logarithm for harmonic functions as found, e.g., in the paper of Chang, Wilson and Wolff [**8**]. They show that a harmonic function with bounded square function satisfies an exponential integrability estimate. In our case, their estimate says there are absolute constants C_1, C_2 depending only on the supremum of $|\nabla_H U|$, so that

$$\int_{S^2} \exp(\frac{C_1}{n} U_n^2(x)) dx \leq C_2,$$

where $U_n(x) = U((1-e^{-n})x)$. Tchebyshev's inequality then implies

$$|E_n| = |\{x \in S^2 : |U_n(x) - U(0)| > \lambda\sqrt{n}\}| \leq C_2 \exp(-\lambda^2/C_1).$$

Taking $\lambda = C_1 C_2 M \log n$ and doing some arithmetic gives $|E_n| \leq n^{-M}$. Since $|\nabla U_n| \leq Ce^n$ on S^2, the set E_n can be covered by $Ce^{2n} n^{-M}$ disks of diameter e^{-n}. Thus the μ measure of E_n is bounded by

$$\mu(E_n) \leq Ce^{2n} n^{-M} e^{-2n} n = Cn^{M-1}.$$

Hence the μ measure of $F_n = \cup_{k>n} E_n$ is bounded by Cn^{M-2} and hence $F = \cap_n F_n$ has μ measure zero if $M > 2$. Thus

$$\limsup_{n\to\infty} \frac{|U_n(x) - U(0)|}{\sqrt{n \log n}} < \infty$$

on a set of full μ measure. Since U has bounded hyperbolic gradient, this implies

$$\limsup_{r\to 1} \frac{|U(rx) - U(0)|}{\sqrt{\log \frac{1}{1-r} \log\log \frac{1}{1-r}}} < \infty, \tag{1.5}$$

for μ-almost every $x \in S^2$. Thus

$$\mu(D(x,t)) \leq C_x t^2 \sqrt{\log \frac{1}{t} \log\log \frac{1}{t}},$$

for μ-almost every x. By the Frostman-Rogers-Taylor theorem, this means that Λ has positive measure for the gauge $\varphi(t) = t^2 \sqrt{\log \frac{1}{t} \log\log \frac{1}{t}}$.

To get Theorem 1.1 we need to improve (1.5) by proving the following

THEOREM 1.8. *Suppose U is the hyperbolic Poisson integral of a positive measure μ and has bounded hyperbolic gradient in $\mathbb{B}$. Then for μ-almost every $x \in S^2$,*

$$\limsup_{r\to 1} \frac{|U(rx) - U(0)|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} < \infty. \tag{1.6}$$

It follows from the Chang-Wilson-Wolff theorem that this is true for Lebesgue almost every x. However, in our case we already know U has radial limit 0 Lebesgue almost everywhere and we really need the sharper version. This completes the sketch of how we prove the lower bound in Theorem 1.1. Next we describe how to prove the upper bound.

To show that Λ has finite $\mathcal{H}^\varphi$ measure, we need to find a cover $\{Q_j\}$ of Λ by arbitrarily small squares so that $\sum_j \varphi(\mathrm{diam}(Q_j))$ is uniformly bounded. We do this by running a stopping time argument. Move to the upper half-space model $\mathbb{R}^3_+$ and assume Λ is contained in the unit square Q_0. First we show there is an $c_1 > 0$ so that if $U(z) < c_1$ somewhere in the top half of a Carleson cube then the base of the cube does not intersect Λ. Next, we define a cover of Λ inductively as follows. For the first step, choose some integer n_0 and take all the dyadic squares of side length $2^{-2^{n_0}}$ which intersect Λ. Call this collection $\mathcal{G}_{n_0}$.

Fix an $\epsilon > 0$ and for a general $n > n_0$ and a collection of squares $\mathcal{G}_n$, subdivide each square in the collection into dyadic subsquares of side length $2^{-2^{n+1}}$. If $U(z) < c_1$ at any point of the top half of the corresponding Carleson cube throw the square away. If $U(z) > \epsilon\sqrt{2^n \log n}$ in the top half of the Carleson cube, put the square into a collection $\mathcal{B}_n$ for Λ. If neither condition holds, put the square into the new collection $\mathcal{G}_{n+1}$. We stop the construction at some large value $n = N-1$ and cover Λ by $\mathcal{B} \cup \mathcal{G}_N$, where $\mathcal{B} = \cup_{n=n_0}^{N-1} \mathcal{B}_n$.

For squares in $\mathcal{B}$ we have $\mu(Q) \geq \frac{C}{\epsilon}\varphi(\mathrm{diam}(Q))$, so the φ-sum of such squares is uniformly bounded since μ is a finite measure. Thus we need to show that if ϵ is small enough, then the φ sum of squares in $\mathcal{G}_N$ is uniformly bounded (in fact, we will show it tends to zero). This argument is based on analogous estimates for random walks and Brownian motion (it roughly corresponds to computing the probability that a random walk started at 1 stays positive for n steps and is never larger than $\max(1, \epsilon\sqrt{k \log\log k})$ for $n_0 \leq k \leq n$). This is the most technical part of the paper.

The remainder of the paper is organized as follows.

Section 2: We review definitions and related results.
Section 3: We prove some technical estimates used in the following sections.
Section 4: We prove the lower bound in Theorem 1.1
Section 5: We prove the upper bound in Theorem 1.1.
Section 6: We give an upper bound for the size of Λ_c.
Section 7: We prove Corollary 1.5.
Section 8: We prove Lemma 1.6.
Section 9: We prove Lemma 1.7.

2. Definitions

If A, B are quantities that depend on some parameter we write $A \simeq B$ if the ratios B/A and A/B are bounded uniformly independent of the parameter. If $E \subset \mathbb{R}^d$ we let $|E|_d$ denote the d-dimensional Lebesgue measure of E and use $|E|$ if the dimension is clear from context. Suppose φ is a increasing continuous function from $[0,\infty)$ to itself such that $\varphi(0) = 0$. We define the *Hausdorff content* of a set $E \subset \mathbb{R}^2$ as

$$\mathcal{H}^\varphi_\infty = \inf\{\sum \varphi(r_j) : E \subset \cup_j D(x_j, r_j)\}.$$

If $\varphi(t) = t^\alpha$ we denote $\mathcal{H}^\varphi$ by $\mathcal{H}^\alpha$. The *Hausdorff dimension* of E is

$$\dim(E) = \inf\{\alpha : \mathcal{H}^\alpha_\infty(E) = 0\},$$

and the *Hausdorff measure* of E is

$$\mathcal{H}^\varphi(E) = \lim_{\delta\to 0}[\inf\{\sum \varphi(r_j) : E \subset \cup_j D(x_j, r_j), r_j \leq \delta\}].$$

A nth generation *dyadic cube* in $\mathbb{R}^d$ is a product of intervals of the form $(j2^{-n}, (j+1)2^{-n}]$. The collection of such cubes is denoted $\mathcal{D}_n$ (or $\mathcal{D}^d_n$ is the dimension is not clear from context). Given a point $x \in \mathbb{R}^d$, we let $Q_n(x)$ denote the unique cube in $\mathcal{D}_n$ which contains x. If $Q \in \mathcal{D}_n$, we let $\ell(Q) = 2^{-n}$ denote the side length of Q. The *parent* of a dyadic cube $Q \in \mathcal{D}_n$ is the unique dyadic cube in $\mathcal{D}_{n-1}$ which contains it; its *children* are the 2^d cubes of $\mathcal{D}_{n+1}$ which it contains. It is sometimes convenient to consider the Hausdorff measure defined using only coverings of a set only by dyadic cubes. The resulting content and measure are easily seen to be comparable to the usual definitions, with constants depending only on the dimension d.

If E supports a measure μ such that $\mu(D(x,r)) \leq Cr^{\alpha}$ for every disk $D(x,r)$, then it is trivial to see that $\mathcal{H}^{\alpha}(E) \geq C^{-1}\mu(E) > 0$ and hence $\dim(E) \geq \alpha$. This is called the *mass distribution principle.* A more sophisticated version of this is the Frostman-Rogers-Taylor density theorem, [**24**],

THEOREM 2.1. *Let μ be a finite measure on $[0,1]^2$ and let $A \subset [0,1]^2$ be a closed set. Then*

1. *If for all $x \in A$,*
$$\limsup_{n\to\infty} \frac{\mu(Q_n(x))}{\varphi(2^{-n})} \geq C,$$
then $\mathcal{H}^{\varphi}(A) \leq MC^{-1}\mu(A)$.
2. *If for all $x \in A$,*
$$\limsup_{n\to\infty} \frac{\mu(Q_n(x))}{\varphi(2^{-n})} \leq C,$$
then $\mathcal{H}^{\varphi}(A) \geq M^{-1}C^{-1}\mu(A)$.

We let $\mathbb{B}^n$ denote the unit ball of $\mathbb{R}^n$ equipped with the hyperbolic metric
$$d\rho = \frac{2|dz|}{1-|z|^2}.$$
In most of this paper we take $n=3$ and simply write $\mathbb{B}$ instead of $\mathbb{B}^3$. We will also use the upper half space model $\mathbb{R}^{n+1}_+ = \{(x_1, x_2, \dots, x_n, y) : y > 0\}$ for hyperbolic space with the metric
$$d\rho = \frac{|dz|}{y}.$$
The hyperbolic volume in $\mathbb{R}^{n+1}_+$ will be denoted
$$dV = \frac{dx_1 dx_2 \dots dx_n dy}{y^n}.$$
The hyperbolic Laplacian on $\mathbb{B}^n$ is given by
$$\Delta_H = \frac{(1-r^2)^2}{4}[\Delta + \frac{2(n-2)r}{1-r^2}\frac{\partial}{\partial r}],$$
and on the upper half space $\mathbb{R}^n_+$ it is
$$\Delta_H = y^2[\Delta - \frac{n-2}{y}\frac{\partial}{\partial y}].$$
The hyperbolic Poisson kernel on $\mathbb{B}^n$ is
$$P(z,\zeta) = w_n[\frac{1-|z|^2}{|z-\zeta|^2}]^{n-1},$$
where w_n is chosen to make the integral 1. On the upper half space $\mathbb{R}^3_+$ the hyperbolic Poisson kernel is
$$P(z,\zeta) = v_n[\frac{y}{|z-\zeta|^2}]^{n-1},$$
again with v_n chosen to make the integral 1. See Chapter 5 of [**23**].

A discrete group G of isometries of the hyperbolic metric on $\mathbb{B}$ is called *Kleinian group*. Such a group can also be considered as a group of linear fractional transformations on S^2. G is called *elementary* if it contains a finite index Abelian subgroup. In this paper we are only concerned with non-elementary groups. For a non-elementary G, the accumulation of any orbit in $\mathbb{B} \cup S^2$ is a closed set $\Lambda \subset S^2$ which is independent of the particular orbit. This is the *limit set.* The *conical limit set* Λ_c is the set of points $x \in \Lambda$ for which there is a sequence of orbit points converging to x within a non-tangential cone in $\mathbb{B}$. It is easy to see that x is a conical limit point iff a geodesic ray in $\mathbb{B}$ ending at x projects to a geodesic in $M = \mathbb{B}/G$ which returns to some compact set of M infinitely often.

We call $\Omega = S^2 \setminus \Lambda$ the *ordinary set* of G. G acts discontinuously on Ω, and Ω/G is a (branched) Riemann surface. If it is a Riemann surface of finite type (i.e., is a finite union of compact surfaces each with at most a finite number of punctures) we call G *analytically finite.* The famous Ahlfors finiteness theorem [**1**], [**3**] says that any finitely generated Kleinian group is analytically finite.

G is called *geometrically finite* if there is a finite sided fundamental polygon for G's action on $\mathbb{B}$. One can show ([**2**], [**4**]) that G is geometrically finite iff every limit point of G is either a conical limit point or a fixed point of some parabolic element of G.

If G is a Kleinian group with limit set Λ we let $C(\Lambda) \subset \mathbb{B}$ denote the convex hull of Λ and let $C(M) = C(\Lambda)/G \subset M = \mathbb{B}/G$ be the *convex core* of M. $C(M)$ can be defined more intrinsically as the closed convex hull of the closed geodesics in M. If G is analytically finite then G is geometrically finite iff $C(M)$ has finite hyperbolic volume. In general, G is geometrically finite iff the unit neighborhood of $C(M)$ has finite volume.

If $x \in M$ then the *injectivity radius*, $\mathrm{inj}(x)$, is half the length of the shortest loop through x; alternatively it is half the distance between the two closest distinct lifts of x to $\mathbb{B}$. In this paper we will assume that the injectivity radius is bounded away from zero uniformly on M. In particular, this implies that G has no parabolic elements. With this extra assumption, G is geometrically finite iff $C(M)$ is compact iff $\Lambda = \Lambda_c$. Theorem 1.1 gives a very strong converse of this; if G is not geometrically finite then "most" limit points are non-conical.

G is called *topologically tame* if M is homeomorphic to the interior of a compact 3-manifold with boundary. By a result of Canary [**6**], this implies that G is *analytically tame*, i.e., there are finite area surfaces which sweep out the geometrically infinite ends. In this case $C(M)$ consists of a compact piece and a finite number of ends E_j each of which is topologically equivalent to $\mathbb{R}_+ \times S$ for some compact surface S. The only property we will use is that for each $x \in E_j$ there is a finite diameter smooth surface $S_j(x)$ which is a bounded hyperbolic distance from x and which separates E_j.

Next we recall some facts about dyadic martingales (these can be found in [**10**]). Suppose Q_0 is a unit cube in $\mathbb{R}^d$. We say that a sequence of functions $\{f_n\}$, $n = 0, 1, \dots$ on Q_0 is a *dyadic martingale* if each function is constant on each nth generation dyadic cube and satisfies

$$\int_Q f_{n+1} = \int_Q f_n,$$

if $Q \in \mathcal{D}_n$. A martingale is said to be in the space H^p if

$$\|\{f_n\}\| = (\sup_n \int_{Q_0} |f_n|^p)^{1/p} < \infty.$$

The martingale *maximal function* is defined for $x \in Q_0$ as

$$Mf(x) = \sup_n |f_n(x)|,$$

and is a bounded operator from H^p to $L^p(Q_0)$, $1 < p \leq \infty$, and from H^1 to weak L^1. Moreover, if $\{f_n\} \in H^p$, $1 < p \leq \infty$, then there is an $f \in L^p$ such that

$$f_n(x) = \frac{1}{|Q_n(x)|} \int_{Q_n(x)} f(y)dy.$$

If $\{f_n\}$ is a positive martingale then there is a positive, finite measure μ on Q_0 such that

$$f_n(x) = \frac{1}{|Q_n(x)|} \int_{Q_n(x)} d\mu(y).$$

We let $\Delta f_n = f_n - f_{n-1}$. A martingale is called *Bloch* if $\sup_n \|\Delta f_n\|_\infty < \infty$, i.e., the steps are uniformly bounded. We define the martingale *square function* as

$$S(f)(x) = (\sum_{n=1}^{\infty} [\Delta f_n(x)]^2)^{1/2}.$$

This is a bounded operator from H^p, $1 < p < \infty$ to L^p. Actually, the square function considered in the paper of Chang, Wilson and Wolff [**8**] which we will use is slightly different and defined by

$$S(f)(x) = (\sum_{n=1}^{\infty} \|\chi_{Q_n(x)} \Delta f_n\|_\infty^2)^{1/2}.$$

The two definitions agree when $d = 1$ and in general they are comparable up to a dimension dependent constant. The following can be found, e.g. in [**8**],

THEOREM 2.2. *Let $\{f_n\}$ be a dyadic martingale on $Q_0 \subset \mathbb{R}^d$ with limit function f. Suppose $\|Sf\|_\infty < \infty$. Then for $\lambda \geq 0$,*

$$|\{x \in Q_0 : f(x) - f_0(x) \geq \lambda\}| \leq \exp(-\frac{1}{2}\lambda^2/\|Sf\|_\infty^2).$$

In particular, there are constants C_1, C_2 such that

$$\int_Q \exp(\frac{C_1|f - f_0|^2}{\|Sf\|_\infty^2})dx \leq C_2.$$

This theorem implies the following estimate for the maximal function.

COROLLARY 2.3. *Suppose $\{f_n\}$ is a martingale with limit function f and suppose $Sf \in L^\infty(\mathbb{R}^2)$. Then there are constants $C_1 > 0$ and $C_2 < \infty$ (depending only on α) so that*

$$|\{x \in Q : |Mf(x) - \frac{1}{|Q|}\int_Q f| > \lambda\}| \leq C_2|Q| \exp(-\frac{C_2}{4}\frac{\lambda^2}{\|Sf\|_\infty^2}).$$

Proof: Without loss of generality assume $\frac{1}{|Q|}\int_Q f = 0$. The set $\{Mf(z) > \lambda\}$ can be covered by dyadic squares $\{Q_j\}$ such that

$$\frac{1}{|Q_j|}\int_{Q_j} f dx > \lambda/2.$$

For such a square Q_j we get

$$\exp(\frac{C_1\lambda^2}{4\|Sf\|_\infty^2}) \quad \leq \quad \exp(\frac{C_1}{\|Sf\|_\infty^2}(\frac{1}{|Q_j|}\int_{Q_j} f(y)dy)^2).$$

Now apply Cauchy-Schwarz to get

$$\leq \exp(\frac{C_1}{\|Sf\|_\infty^2}\frac{1}{|Q_j|}\int_{Q_j} f^2(y)dy),$$

and Jensen's inequality $(\exp(\frac{1}{|Q|}\int_Q |f|^2) \leq \frac{1}{|Q|}\int_Q \exp(|f|^2))$,

$$\leq \frac{1}{|Q_j|}\int_{Q_j} \exp(\frac{C_1}{\|Sf\|_\infty^2} f^2(y))dy.$$

Thus by Theorem 2.2,

$$\begin{aligned}
\sum_j |Q_j|\exp(\frac{C_1\lambda^2}{\|Sf\|_\infty^2}) &\leq \sum_j |Q_j|\frac{1}{|Q_j|}\int_{Q_j}\exp(\frac{C_1}{\|Sf\|_\infty^2} f^2(y))dy \\
&\leq \int_{Q_0}\exp(\frac{C_1}{\|Sf\|_\infty^2} f^2(y))dy \\
&\leq C_2.
\end{aligned}$$

Hence

$$|\{Mf > \lambda\}| \leq \sum_j |Q_j| \leq C_2 \exp(\frac{-C_1\lambda^2}{\|Sf\|_\infty^2}).$$

□

For $x \in \mathbb{R}^2$ let $\Gamma_\alpha(x)$ be the non-tangential cone in $\mathbb{R}^3_+$ with vertex x and angle α,

$$\Gamma_\alpha(x) = \{(w, y) \in \mathbb{R}^2 \times \mathbb{R}^1_+ : y > \alpha|x - w|\}.$$

Define the square function

$$S_\alpha f(x) = (\int_{\Gamma_\alpha(x)} |\nabla_H f(y,t)|^2 y dy dt)^{1/2}.$$

An important property of the square function is that for $1 < p < \infty$, $\|S_\alpha(f)\|_p \simeq \|f\|_p$ if f has mean value zero, (and the constants depend only on p and α). See Chapter IV of [**25**].

Next we recall some facts about the space BMO ("bounded mean oscillation"). The space BMO is defined to be the class of locally integrable functions f on $\mathbb{R}^d$ so that

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|_d}\int_Q |f(x) - \frac{1}{|Q|_d}\int_Q f(y)dy|dx < \infty.$$

This space contains L^∞ and is contained in every L^p, $1 < p < \infty$. One of the most important estimates concerning BMO functions is the John-Nirenberg theorem ([**14**], or Theorem VI.2.1 of [**9**]). It says that there are constants C_3, C_4

(depending only on d) so that if $f \in \mathrm{BMO}$ and $Q \subset \mathbb{R}^d$ is a cube then

$$\{x \in Q : |f(x) - \frac{1}{|Q|}\int_Q f| > \lambda\}| \leq C_3 |Q| \exp(-C_4 \frac{\lambda}{\|f\|_{\mathrm{BMO}}}). \tag{2.1}$$

If Q is a cube in $\mathbb{R}^d$, then $\hat{Q} = Q \times [0, \ell(Q)]$ is called the Carleson cube in $\mathbb{R}^{d+1}_+$ with base Q. We let $T(Q) = Q \times [\frac{1}{2}\ell(Q), \ell(Q)]$ denote the top half of Q and let z_Q be the center of $\hat{Q}$. A measure μ on $\mathbb{R}^{d+1}_+$ is called *Carleson measure* if is a $M < \infty$ so that

$$\|\mu\|_C = \sup_Q \frac{\mu(\hat{Q})}{|Q|_d} \leq M,$$

where the sup is over all cubes $Q \in \mathbb{R}^d$. Suppose u is the hyperbolic harmonic extension of f from $\mathbb{R}^d$ to the upper half-space $\mathbb{R}^{d+1}_+$. Then $f \in \mathrm{BMO}$ iff

$$\mu = |\nabla_E u|^2 y dx_1 \ldots dx_n dy = |\nabla_H u|^2 y^{n-1} dV$$

is a Carleson measure. Moreover, $\|f\|_{\mathrm{BMO}} \simeq \sqrt{\|\mu\|_C}$.

3. Some estimates for Bloch and BMO functions

In this section we record a few technical estimates that we will use in the following sections. The first is an estimate which says that various ways of averaging a harmonic function with bounded gradient all give essentially the same answer. The second is the so called "$L^2 - L^4$" trick and the last allows us to control the L^2 norm of a function in terms of its L^1 and BMO norms.

The following was proven by Carleson (Lemma 3, [**7**]) in dimension two.

LEMMA 3.1. *Suppose U on $\mathbb{R}^3_+$ is the (hyperbolic) Poisson integral of the positive measure μ and satisfies $|\nabla_H U(z)| \leq 1$. For a square $Q \in \mathbb{R}^2$, let $Q_t = \{(x_1, x_2, t) : (x_1, x_2) \in Q\}$. Then there is an $A_1 < \infty$ so that*

$$|U(z_Q) - \frac{1}{|Q|_2}\int_Q U(x_1, x_2, t) dx_1 dx_2| \leq A_1,$$

for any $0 < t \leq \ell(Q)$ and

$$|U(z_Q) - \frac{1}{|Q|_2}\int_Q d\mu| \leq A_1.$$

Proof: By rescaling we may assume $z_Q = (0, 0, 1)$ and by replacing U by $U - U(z_Q)$ we may assume $U(z_Q) = 0$. Let $\psi \in C^\infty(\mathbb{R}^3_+)$ be chosen with $\mathrm{supp}(\psi) \subset 3Q$. Our first goal is to prove

$$\iint_Q \psi(x_1, x_2, t) U(x_1, x_2, t) dx_1 dx_2 = O(1),$$

with bounds depending only on the C^2 norm of ψ (in particular, independent of t).

We begin by assuming U is smooth up to the boundary (later we will apply these estimates to functions of the form $U(x_1, x_2, y+t)$). Let $W(z)$ be the hyperbolic Poisson extension of the characteristic function of Q to $\mathbb{R}^3_+$ and let $F(z) = y^2 \psi(z) W(z)$. We now apply Greens theorem (in the hyperbolic setting) to $U(z)$ and $F(z)$ to get

$$\iint_{\mathbb{R}^3_+} U \Delta_H F - F \Delta_H U dV = \int_{\mathbb{R}^2} U \frac{\partial F}{\partial n} - F \frac{\partial U}{\partial n} dA.$$

Since $\Delta_H U = 0$ this becomes

$$\iint_{\mathbb{R}^3_+} U \Delta_H F dV = \int_{\mathbb{R}^2} U \frac{\partial F}{\partial n} - F \frac{\partial U}{\partial n} dA. \tag{3.1}$$

First we estimate the right hand side. Since $|\nabla_H U| = O(1)$ and $F = O(y^2)$ we get,

$$\int_{\mathbb{R}^2} F \frac{\partial U}{\partial n} dA = O(1).$$

To estimate the first term on the right hand side of (3.1) note that

$$\begin{aligned} \frac{\partial F}{\partial n}(z) &= \frac{\partial \psi(z)}{\partial n} \cdot y^2 \cdot W(z) + \psi(z) \cdot \frac{\partial y^2}{\partial n} \cdot W(z) + \psi(z) \cdot y^2 \cdot \frac{\partial W(z)}{\partial n} \\ &= O(y^3) + \psi(z) y^2 W(z) + O(y^2 \min(W(z), 1 - W(z))). \end{aligned} \tag{3.2}$$

The last term uses the fact that the hyperbolic gradient of a bounded harmonic function is bounded by a multiple of the function itself, applied to both W and $1 - W$.

Now break the integral $\int U \frac{\partial F}{\partial n}$ into three terms according to (3.2). The integral of the first term over the boundary tends to zero as $y \to 0$. The integral of the second term gives

$$\int_{Q_t} U(x)\psi(x) dx.$$

Since $\min(W(z), 1 - W(z)) \to 0$ almost everywhere as $y \to 0$, the Lebesgue dominated convergence theorem implies the integral of the third term also tends to zero. Thus we deduce

$$\int_{\mathbb{R}^2} U \frac{\partial F}{\partial n} - F \frac{\partial U}{\partial n} dx_1 dx_2 = \int_Q U(x)\psi(x) dx_1 dx_2 + O(1). \tag{3.3}$$

Now we estimate the left hand side of (3.1). Since ψ is C^2 up to the boundary, we have $\nabla_H \psi = O(y)$ and $\Delta_H \psi = O(y^2)$. Thus

$$\begin{aligned} \Delta_H F &= \Delta_H(\psi \cdot y^2 \cdot W) \\ &= \Delta_H \psi \cdot y^2 \cdot W + \psi \cdot \Delta_H y^2 \cdot W + \psi \cdot y^2 \cdot \Delta_H W \\ &\qquad + \nabla_H \psi \cdot \nabla_H y^2 \cdot W + \nabla_H \psi \cdot y^2 \cdot \nabla_H W + \psi \cdot \nabla_H y^2 \cdot \nabla_H W \\ &= O(y^4) + 0 + 0 + O(y^3) + O(y^3) + O(y^2 \nabla_H W). \end{aligned}$$

The integral $\iint_{\mathbb{R}^3_+} U \Delta_H F dV$ breaks into six terms, the first five of which are either zero or are obviously bounded (since $U(z) = O(\log \frac{1}{y})$). To bound the last term, we use the fact that $|\nabla_H W| = O(y|\nabla W|)$ and the simple estimate

$$\int_{3Q} |\nabla W(x_1, x_2, y)| dx_1 dx_2 = O(1),$$

independent of y. This gives of order

$$\begin{aligned} \iint_{|z|<3} y^2 |\nabla_H W| \log \frac{1}{y} dV &= \int_{0<y<3} \int_{|x_1|,|x_2|<3} y |\nabla W| \log \frac{1}{y} dx_1 dx_2 dy \\ &= \int_{0<y<3} y \log \frac{1}{y} y \\ &= O(1). \end{aligned}$$

Thus by (3.3), we have now proven that if U is hyperbolically harmonic, has bounded hyperbolic gradient and $U(z_Q)=0$ then

$$\int_Q U\psi dx_1 dx_2 = O(1), \tag{3.4}$$

with bounds depending only on the C^2 norm of ψ. To deduce the first part of the lemma, set $F(z)=U(z+(0,0,t))-U(0,0,1-t)$ and apply the preceding case to F. Choose ψ so that $\psi\equiv 1$ on Q. Then (3.4) implies

$$\int_Q U(x_1,x_2,t)dx_1dx_2 = \int_Q F dx_1 dx_2 + U(0,0,1-\epsilon) = U(z_Q)+O(1),$$

as desired. To prove the second part of the lemma we simply wish to note that

$$\lim_{t\to 0}\frac{1}{|Q|_2}\int_Q U(x_1,x_2,t)dx_1dx_2 = \frac{1}{|Q|_2}\int_Q d\mu.$$

To prove this, we first observe that $\mu(\partial Q)=0$. This is because we can cover ∂Q by $4\cdot 2^n$ squares of size 2^{-n}, and U is bounded by Cn in the tops of the corresponding Carleson boxes. Hence the μ measure of each square is at most $C2^{-2n}n$ and the total μ measure of ∂Q is at most $C2^{-n}n\to 0$. Next note that

$$\int_Q P_H(x_1,x_2,t)dx_1dx_2,$$

converges to 1 inside Q and to 0 outside Q, and hence to χ_Q μ-almost everywhere. Now use the fact that U is the Poisson integral of μ, Fubini's theorem and the Lebesgue dominated convergence theorem to get

$$\begin{aligned}\frac{1}{|Q|_2}\int_Q U(x_1,x_2,t)dx_1dx_2 &= \frac{1}{|Q|_2}\int_Q[\int_{\mathbb{R}^2}P_H(x_1,x_2,t)d\mu]dx_1dx_2\\ &= \int_{\mathbb{R}^2}[\frac{1}{|Q|_2}\int_Q P_H(x_1,x_2,t)dx_1dx_2]d\mu\\ &\to \frac{1}{|Q|_2}\int_{\mathbb{R}^2}\chi_Q d\mu.\end{aligned}$$

This completes the proof of the lemma. □

The previous lemma says that if we form the dyadic martingale corresponding to the measure μ then the value of the martingale on a dyadic square Q is (up to an additive constant) the same as the value of the harmonic function U in the top half of the corresponding Carleson cube. Thus the boundary behavior of U exactly mimics the behavior of the martingale.

The second estimate which we wish to record is the standard "L^2-L^4 trick". This says that a function which has mean value zero, big L^2 norm and small L^4 norm necessarily satisfies certain pointwise estimates.

Lemma 3.2. *Suppose μ is a probability measure on X. Suppose F is a real valued function on X so that $\int_X F d\mu = 0$ and $\|F\|_4\le B\|F\|_2$. Then*

$$\mu(\{x : F(x)\le -\frac{1}{\sqrt{8}B^2}\|F\|_2\})\ge\frac{1}{64B^{12}}.$$

Proof: Without loss of generality we may assume $\|F\|_2=1$. Hölder's inequality implies that

$$1=\|F\|_2\le\|F\|_1^{1/3}\|F\|_4^{2/3}\le\|F\|_1^{1/3}B^{2/3},$$

which implies $\|F\|_1 \geq B^{-2}$. Write $F = F_+ - F_-$ as the difference of its positive and negative parts. Since F has mean value zero we must have $\|F_-\|_1 \geq \frac{1}{2}B^{-2}$, and hence $F(x) \leq -\frac{1}{2}B^{-2}$ at some point.

To show F is negative on a set of large measure, let

$$E_1 = \{x \in Q : F_- \leq \frac{1}{\sqrt{8}B^2}\}, \quad E_2 = \{x \in Q : F_- \geq \frac{1}{\sqrt{8}B^2}\}.$$

Since $\|F_-\|_2^2 \geq \|F_-\|_1^2 \geq B^{-4}/4$ and $\int_{E_1}(F_-)^2 d\mu \leq \frac{1}{8}B^{-4}$, we deduce $\int_{E_2}(F_-)^2 d\mu \geq \frac{1}{8}B^{-4}$. Thus if $\mu(E_2) < (64B^{12})^{-1}$, we would get

$$\frac{1}{8}B^{-4} \leq \int_{E_2} F_-^2 d\mu \leq (\int F_-^4 d\mu)^{1/2}\mu(E_2)^{1/2} < B^2\frac{1}{8}B^{-6} = \frac{1}{8}B^{-4},$$

which is a contradiction. This proves the lemma. □

LEMMA 3.3. *There is a $C < \infty$ so that if $f \in L^1 \cap BMO$ and f has mean value zero then*

$$\|f\|_2 \leq C(\|f\|_1\|f\|_{BMO})^{1/2}.$$

PROOF. There are several ways to do this, e.g., use the fact that interpolating BMO and L^1 gives L^2 ([**13**]). A more direct proof uses the John-Nirenberg theorem as follows. Without loss of generality we may assume $\|f\|_1 = 1$ and set $\lambda = \|f\|_{\mathrm{BMO}}$. Write (using the formula for L^p norms on page 7 of [**25**])

$$\begin{aligned}
\|f\|_2^2 &= 2\int_0^\infty t|\{x : |f(x)| > t\}|dt \\
&\leq 2\int_0^\lambda \lambda|\{x : |f(x)| > t\}|dt + 2\int_\lambda^\infty t|\{x : |f(x)| > t\}|dt.
\end{aligned}$$

The first term is clearly bounded by $2\lambda\|f\|_1 = 2\lambda$. The second term can be estimated by the John-Nirenberg theorem as follows.

$$\begin{aligned}
\int_\lambda^\infty t|\{x : |f(x)| > t\}|dt &\leq \sum_{n=1}^\infty \lambda n|\{x : |f(x)| > \lambda n\}| \\
&\leq \sum_{n=1}^\infty \lambda n \exp(-C\lambda n/\|f\|_{\mathrm{BMO}}) \\
&\leq \sum_{n=1}^\infty \lambda n \exp(-Cn) \\
&\leq C\lambda.
\end{aligned}$$

Thus $\|f\|_2^2 \leq C\|f\|_1\|f\|_{\mathrm{BMO}}$, as desired. □

To see why we need mean value zero, consider the constant function 1; it has L^1 and L^2 norm 1 but BMO norm 0.

4. The lower bound in the law of the iterated logarithm

Suppose U is the hyperbolic Poisson integral of a positive measure μ on $\mathbb{R}^2$ and has bounded hyperbolic gradient. We want to show the support of μ has positive Hausdorff measure for the function $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$. This could be deduced from the methods of Makarov [**17**], but we include a proof for completeness. Define a dyadic martingale by

$$S_n(x) = \frac{\mu(Q_n(x))}{|Q_n(x)|_2}.$$

Then by Lemma 3.1, S_n is a Bloch martingale. Therefore it is enough to prove the following (Lemma 4.4 of of [**17**]; Theorem 1 of [**18**]).

THEOREM 4.1. *Suppose Y_n is a positive Bloch martingale. Let $E = \{x \in [0,1] : Y_n(x) > 0$ for all $n\}$. Then $\mathcal{H}^\varphi(E) > 0$ for $\varphi(t) = t\sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}$.*

Proof: $\{Y_n\}$ is a positive martingale and hence there is a positive measure μ on $[0,1]$ so that $Y_n(x) = \mu(Q_n(x))/|Q_n(x)|_2$ where $Q_n(x)$ is the nth generation dyadic interval containing x. Without loss of generality, we may assume μ has mass 1. For μ-almost every x, $Y_n > 0$ for all n, so E has full μ measure. We claim that there is a $A < \infty$ so that for μ-almost every x, the estimate

$$\mu(Q_k(x)) \le 2A2^{-2k}\sqrt{k \log\log k},$$

holds for all large enough k. If we can show this, then by the Frostman-Rogers-Taylor theorem, E has positive Hausdorff measure with respect to φ.

If $2^{n-1} \le k \le 2^n$, and Q is a kth generation square such that

$$\mu(Q) \ge 2A2^{-2k}\sqrt{k \log\log k},$$

then $Y_k(x) \ge 2A\sqrt{k\log\log k}$. Therefore the maximal function

$$\begin{aligned} MY_{2^n}(x) &\ge 2A\sqrt{k \log\log k} \\ &\ge 2A\sqrt{2^{n-1}\log\log 2^{n-1}} \\ &\ge A\sqrt{2^n \log\log 2^n}. \end{aligned}$$

Thus, by the Borel-Cantelli lemma it will be enough to show

$$\mu(F_n^A) \equiv \mu(\{x : MY_{2^n}(x) \ge A\sqrt{2^n \log\log 2^n}) \le n^{-2},$$

for all large enough n.

Fix an integer $m \ge n$ and let $\{Q_j\}$ be the collection of dyadic squares $Q_k(x)$ with $k \le 2^n$, which are maximal with respect to the property

$$\mu(Q_j) \ge \frac{A}{8} 2^{m/2}|Q_j|_2.$$

Thus the total Lebesgue measure of such squares is less than

$$\sum_j |Q_j|_2 \le 2^{-m/2}\frac{8}{A}\sum_j \mu(Q_j) \le 2^{-m/2}\frac{8}{A}.$$

Moreover, by maximality, $\mu(Q_j) \le \frac{A}{2}2^{m/2}|Q_j|_2$, for such a square. Now fix such a square and define

$$F_n^A(Q_j) = \{x \in Q_j : MY_{2^n}(x) > A\sqrt{2^m \log\log 2^m}\}.$$

Since Y has average $\le 2^{m/2}A/2$ on Q_j, this is contained in the set

$$\{x \in Q_j : MY_{2^n}(x) - \frac{1}{|Q_j|_2}\int_{Q_j} Y_{2^n}\,dx > \frac{A}{2}\sqrt{2^m \log\log 2^m}\}.$$

Since the square function of Y is bounded above by $C2^{n/2}$, Corollary 2.3 gives

$$|F_n^A(Q_j)|_2 \leq C_1|Q_j|_2 \exp(-C_2\frac{1}{2}A2^{m-n}\log\log 2^m).$$

Since $m \geq n$, then summing over j shows (if A is large enough),

$$\begin{aligned} |\{x : MY_{2^n}(x) \geq \sqrt{m\log\log m}\}|_2 &\leq C_2\frac{2}{A}2^{-m/2}\exp(-C_2\frac{1}{2}A\log\log m) \\ &\leq C_2\frac{2}{A}2^{-m/2}m^{-4}. \end{aligned}$$

Now we want to convert this estimate of Lebesgue measure into an estimate of μ measure.

$$\begin{aligned} &\mu(\{x : MY_{2^n}(x) \geq A\sqrt{2^n\log\log 2^n}\}) \\ &\quad\leq \sum_{m=n}^{\infty}\mu(\{x : A\sqrt{2^m\log\log 2^m} \leq MY_{2^n}(x) \leq A\sqrt{2^{m+1}\log\log 2^{m+1}}\}) \\ &\quad\leq \sum_{m=n}^{\infty}|\{x : A\sqrt{2^m\log\log 2^m} \leq MY_{2^n}(x)\}|_2(A\sqrt{2^{m+1}\log\log 2^{m+1}}) \\ &\quad\leq \sum_{m=n}^{\infty}(C2^{-m/2}m^{-4}\frac{1}{A})(\sqrt{2^{m+1}\log\log 2^{m+1}}) \\ &\quad\leq \sum_{m=n}^{\infty}C\sqrt{2}\log(m+1)m^{-4} \\ &\quad\leq Cn^{-2}. \end{aligned}$$

This is what we wanted to prove, so we are done. □

5. The upper bound in the law of the iterated logarithm

The proof in this section is stated in terms of harmonic functions in the upper half space, but would work equally well for positive Bloch martingales with an appropriate lower bound on the square function.

Suppose U is the positive harmonic function given by Lemma 1.7. By moving to the upper half space and conjugating the group, we may assume U is a hyperbolically harmonic positive function on $\mathbb{R}^3_+$, and that it is the Poisson integral of a positive measure μ supported on the limit set $\Lambda \subset [0,1]^2$. We may also assume $\text{diam}(\Lambda) \geq 1/2$. By multiplying U by a constant we may also assume $|\nabla_H U(z)| \leq 1$. For a square $Q \in \mathbb{R}^2$, recall that $T(Q)$ is the top half of the corresponding Carleson square in $\mathbb{R}^3_+$.

LEMMA 5.1. *With U as above, there is a $0 < c_1 < \infty$ such that $U(z) \leq c_1$ for some $z \in T(Q)$ implies $Q \cap \Lambda = \emptyset$.*

Proof: Since $\partial C(M)$ is compact and U and tends to ∞ in the geometrically infinite ends, U is bounded below on $C(M)$, say by m_0. Thus if $U(z) \leq c_0 \ll m_0$ for some $z \in T(Q)$, then this point must be a large distance from $\partial C(\Lambda)$ and outside $C(\Lambda)$ (since $U \geq m_0$ on all of $C(\Lambda)$). This easily implies the base of Q misses Λ. □

The next lemma quantifies the fact that U oscillates everywhere in $C(M)$ and hence has large square function.

LEMMA 5.2. *With U as above, let $F_n(x_1, x_2, y) = U(x_1, x_2, 2^{-n} + y)$. There are constants $\alpha, b_1, b_2 > 0$ so that for any $x = (x_1, x_2, 0)$ such that $F_n(x) > c_1$, we have*

$$b_1\sqrt{n} \leq S_\alpha(F_n)(x) \leq b_2\sqrt{n}. \tag{5.1}$$

The right hand side actually holds for every $x \in \mathbb{R}^2$.

Proof: Since $|\nabla_H U| \leq 1$, we have $|\nabla_E U(x_1, x_2, y)| \leq Cy^{-1}$ and hence the right hand side of (5.1) holds for every x for some b_2 depending only on α.

To prove the left hand side, note that if $U(z) \geq c_1$, then $\mathrm{dist}(z, C(\Lambda))$ is uniformly bounded and hence by the convexity of Λ, the axis of the Stolz cone with vertex z remains within a bounded distance a_0 of $C(\Lambda)$. Let a_1, a_2 be the constants from Lemma 1.7 and choose α so small that the Stolz cone Γ_α contains an a_1-neighborhood of its axis. Then the Stolz cone will contain approximately $n/(2a_1)$ disjoint balls of radius a_1 and on each of them

$$\int |\nabla_H U|^2 dx_1 dx_2 \frac{dy}{y^3} \geq a_2,$$

by Lemma 1.7. Thus there is a b_1, such that $S_\alpha(F_n)(x) \geq b_1\sqrt{n}$, as desired. □

Fix a large integer n_0. Let $\epsilon > 0$. A particular value for ϵ will be chosen following Lemma 5.8. We use induction to define two classes of dyadic cubes. First, we say $Q \in \mathcal{B}_{n_0}$ if $Q \in \mathcal{D}_{2^{n_0}}$ and $U(z_Q) \geq \epsilon\sqrt{2^{n_0}\log n_0}$. Secondly, we say $Q \in \mathcal{G}_{n_0}$ if $Q \in \mathcal{D}_{2^{n_0}}$, $u(z_Q) \geq c_0$ and $Q \notin \mathcal{B}_{n_0}$. In general, for $n \geq n_0$, we let $\mathcal{C}_n \subset \mathcal{D}_{2^n}$ be the dyadic squares of side length 2^{-2^n} which are contained in some square of $\mathcal{G}_{n-1}$. Then set

$$\mathcal{B}_n = \{Q \in \mathcal{C}_n : U(z_Q) \geq \epsilon\sqrt{2^n \log n}\},$$

$$\mathcal{G}_n = \{Q \in \mathcal{C}_n : c_1 \leq U(z_Q) \leq \epsilon\sqrt{2^n \log n}\}.$$

If we fix a large integer $N > n_0$, then the collection $\mathcal{F}_N = \mathcal{B}_{n_0} \cup \cdots \cup \mathcal{B}_N \cup \mathcal{G}_N$ is a cover of Λ. To see this, note that any square in $\mathcal{D}_{2^N}$ which is not covered by this collection has $U(z_Q) < c_1$, which implies it does not hit Λ.

We claim the φ-sum for this covering becomes uniformly bounded for large enough N (recall that $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$). For the squares in $\mathcal{B}_n$ this is trivial, because for each such square Q,

$$\varphi(\ell(Q)) \quad \leq \quad \frac{C}{\epsilon}U(z_Q)|Q|_2 \leq \frac{C}{\epsilon}\mu(Q). \tag{5.2}$$

(Here we have used the fact $U(z_Q) \simeq \frac{1}{|Q|_2}\int_Q d\mu$ when U is large, i.e., that the two terms are bounded by multiples of each other. In fact, more is true; they are within an additive constant of each other by Lemma 3.1.) Since μ is a finite measure, the sum over all such squares is bounded by $C\|\mu\|/\epsilon$.

Thus to finish the proof we need only prove the φ-sum over $\mathcal{G}_N$ is bounded for large N. In fact, we shall show this sum tends to zero as $N \to \infty$ with the explicit estimate

$$\sum_{Q \in \mathcal{G}_N} \varphi(Q) \leq A_{n_0} e^{-N^{1/4}}, \tag{5.3}$$

if n_0 is large enough (depending on ϵ). To prove (5.3) we start by splitting $\mathcal{G}_n$ into three smaller classes. Let c_1 and A_1 be the constants from Lemmas 5.1 and 3.1

and suppose $\delta > 0$ is a small number (which will be fixed following Lemma 5.5). Suppose $Q \in \mathcal{G}_n$. Then we let

$$Q \in \begin{cases} \mathcal{G}_n^0, & \text{if } c_1 \le U(z_Q) \le 4(A_1 + c_1) \\ \mathcal{G}_n^1, & \text{if } 4(A_1+c_1) < U(z_Q) \le \delta 2^{-n/2} \\ \mathcal{G}_n^2, & \text{if } U(z_Q) \ge \delta 2^{-n/2} \end{cases}.$$

It will be easy to bound the sum over $\mathcal{G}_n^0$ by the corresponding sum over $\mathcal{G}_n^1$, so we really only have to control the size of the latter two collections.

LEMMA 5.3. *Let $\eta > 0$. If δ is small enough and $Q \in \mathcal{G}_n^0 \cup \mathcal{G}_n^1$, then*

$$|\cup_{Q' \in \mathcal{G}_{n+1} \cup \mathcal{B}_{n+1}} Q' \cap Q|_2 \le \eta |Q|_2.$$

Proof: Suppose δ is very small, but that the conclusion fails for Q. We will apply the "$L^2 - L^4$ trick" (i.e., Lemma 3.2) to $F = U_n - U_n(z_Q)$, where $U_n((x,y) = U(x, 2^{-2^{n+1}} + y)$. By Lemma 5.2, the square function of U_n is bounded by $b_2 2^{n/2}$ on Q and hence $\frac{1}{|Q|_2}\int_Q F^4 \le B_2 2^{2n}$. Similarly

$$\begin{aligned} \|F\|^2_{L^2(Q)} &\simeq \|S_\alpha(F)\|^2_{L^2(Q)} \\ &\ge C\int_Q S_\alpha(U_n)^2 dx \\ &\ge C\int_{\cup_{Q'\in\mathcal{G}_{n+1}\cup\mathcal{B}_{n+1}} Q'\cap Q} S_\alpha(U_n)^2 dx \\ &\ge C\int_{\cup_{Q'\in\mathcal{G}_{n+1}\cup\mathcal{B}_{n+1}} Q'\cap Q} 2^n dx \\ &\ge C\eta |Q|_2 2^n. \end{aligned}$$

The penultimate inequality above follows from the lower bound in Lemma 5.2. Thus $\|F\|_4 \le C\eta^{-1/2}\|F\|_2$. Hence by Lemma 3.2, $F = U - U(z_Q) \le -C\eta\eta^{1/2}2^{n/2}$ at some point of Q. But this means

$$U(z) \le U(z_Q) - C\eta^{3/2}\sqrt{2^n} \le \delta\sqrt{2^n} - C\eta^{3/2}\sqrt{2^n}.$$

Since $U > 0$, taking $\delta < C\eta^{3/2}$ gives a contradiction and proves the lemma. □

COROLLARY 5.4. *If $Q^* \in \mathcal{G}_n^0 \cup \mathcal{G}_n^1$ then*

$$\sum_{Q\in\mathcal{G}_{n+1}\cup\mathcal{B}_{n+1}, Q\subset Q^*} |Q|_2 \le \eta |Q^*|_2,$$

and if $\eta < 1/8$ then

$$\sum_{Q\in\mathcal{G}_{n+1}Q^*, Q\subset Q^*} \varphi(Q) \le \frac{1}{2}\varphi(Q^*).$$

Proof: The first inequality is an obvious consequence of the previous lemma. To prove the second, note that Q^* has side length 2^{-2^n} and Q has side length $2^{-2^{n+1}}$, so

$$\varphi(Q^*) = 2^{-2^{n+1}}\sqrt{2^n \log\log 2^n} = |Q^*|_2\sqrt{2^n\log\log 2^n},$$
$$\varphi(Q) = 2^{-2^{n+2}}\sqrt{2^{n+1}\log\log 2^{n+1}} = |Q|_2\sqrt{2^{n+1}\log\log 2^{n+1}},$$

and hence

$$\begin{aligned}\sum_{Q\in G_{n+1}\cap Q^*}\varphi(Q) &\le \eta|Q^*|_2\sqrt{2^{n+1}\log\log 2^{n+1}}\\ &\le \eta\varphi(Q^*)\sqrt{2}\sqrt{\frac{\log n+1}{\log n}}\\ &\le \frac{1}{2}\varphi(Q^*),\end{aligned}$$

if $\eta < 1/8$ and $n \ge 2$. □

LEMMA 5.5. *Suppose $Q^* \in \mathcal{G}_n^1$. If δ is small enough, then*

$$\sum_{Q\in\mathcal{G}^2_{n+1}\cup\mathcal{B}_{n+1},Q\subset Q^*} U(z_Q)|Q|_2 \ge CU(z_{Q^*})|Q^*|_2.$$

Proof: Let $a = U(z_{Q^*})$ and, as above, let $U_n(x_1, x_2, y) = U(x_1, x_2, 2^{-2^{n+1}} + y)$. Then since $U > 0$, $\|U_n\|_{L^1(Q^*)} \simeq a$. Moreover, using the Carleson measure characterization on BMO, and the fact that U has bounded hyperbolic gradient we get $\|U_n\|_{\mathrm{BMO}(Q^*)} \le C\sqrt{2^n}$. Now by Lemma 3.3 we get

$$\|U_n\|_{L^2(Q^*)} \le C(\|U\|_1\|U\|_{\mathrm{BMO}})^{1/2} \le Ca^{1/2}2^{n/4}. \tag{5.4}$$

(Actually we interpolate with the function $U(z)-U(z_{Q^*})$ which has L^1 norm $\le 2a$, because the lemma requires functions which vanish at the z_{Q^*}.)

Let $E = \{x \in Q^* : U_n(x) > c_1\}$. We claim that $\int_E U_n dx \ge \frac{3}{4}a|Q^*|_2$. To prove this, first use Lemma 3.1 to deduce that since $Q^* \in \mathcal{G}_n^1$,

$$\int_{Q^*} U_n dx \ge (U_n(z_{Q^*}) - A_1)|Q^*|_2.$$

Furthermore,

$$\int_{Q\setminus E} U_n dx \le c_1|Q^*|_2,$$

since U is uniformly bounded by c_1 off E. Thus

$$\int_E U_n dx \ge (U(z_Q) - A_1 - c_1)|Q^*|_2 \ge \frac{3}{4}a|Q^*|_2,$$

which is the claim.

The square function $S(U_n) > C\sqrt{2^n}$ on E by Lemma 5.2. Thus

$$|E|_2 2^n \le C\int_E S(U_n)^2dx \le C\int_{Q^*} S(U_n)^2 dx \simeq \int_{Q^*} U_n^2 dx \le Ca2^{n/2}|Q^*|_2,$$

where the last inequality follows by (5.4). Therefore $|E|_2 \le Ca2^{-n/2}|Q^*|_2$. Now define

$$E_\delta = \{x \in E : U_n(x) \le \delta\sqrt{2^n}\}.$$

Then

$$\int_{E_\delta} U_n \le \delta\sqrt{2^n}|E|_2 \le C\delta a|Q^*|_2 \le \frac{a}{2}|Q^*|_2,$$

if δ is small. Since $\int_E U_n \ge \frac{3}{4}a|Q^*|_2$, we have $\int_{E\setminus E_\delta} U_n \ge \frac{1}{4}a|Q^*|_2 = \frac{1}{4}U_n(z_{Q^*})|Q^*|_2$, as desired. □

We now fix a value of δ in the definition of $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ so that the previous lemmas hold with $\eta = 1/8$.

LEMMA 5.6. *Suppose $Q \in \mathcal{G}_n^2$. Let $y_Q = 2^{-2^n}$. If γ_0 is small enough (depending only on δ), then for any y_0 such that $2^{-\gamma_0 2^n} y_Q \leq y_0 \leq y_Q$ we have*

$$|\{z = (x_1, x_2, y_0) \in Q : U(z) > c_1\}|_2 \geq \frac{1}{2}|Q|_2.$$

Proof: Let $A = -\log_2 y$ and define $\mathcal{L} = \{z = (x_1, x_2, y_0) : (x_1, x_2) \in Q\}$. Then by the usual square function estimates,

$$\int_{\mathcal{L}} |U - U(z_Q)|^2 \leq CA|Q|_2 \leq C\gamma_0 2^n |Q|. \tag{5.5}$$

Tchebyshev's inequality implies

$$\begin{aligned} |\{x : U > c_1\}|_2 &\leq |\{x \in Q : |U - U(z_Q)| > \delta 2^{n/2} - c_1\}|_2 \\ &\leq C\gamma_0 2^n |Q|_2/(\delta 2^{n/2} - c_1)^2 \\ &\leq \frac{1}{2}|Q|_2, \end{aligned}$$

if γ_0 is small enough and n large enough. □

LEMMA 5.7. *There are constants $\alpha, \beta > 0$ such that if $\gamma < \gamma_0$, $Q \in \mathcal{D}_k$ with $2^n \leq k \leq 2^{n+1}$ and $U(z_Q) \geq \delta 2^{n/2}$, then*

$$|\{z = (x_1, x_2, 2^{-\gamma 2^n} 2^{-k}) : (x_1, x_2) \in Q, U(z) - U(z_Q) \geq \alpha\sqrt{\gamma 2^n}\}|_2 \geq \beta |Q|_2.$$

Proof: By Lemma 5.6,

$$|\{z = (x_1, x_2, 2^{-\gamma 2^n} y_Q) : U(z) > c_1\}|_2 \geq \frac{1}{2}|Q|_2.$$

The cones above these points lie near $C(\Lambda)$, so the standard square function estimate for the L^2 norm gives

$$C_0\sqrt{\gamma 2^n} \leq \left(\frac{1}{|Q|}\int_Q |U(z) - U(z_Q)|^2\right)^{1/2} \simeq \left(\frac{1}{|\mathcal{L}|}\int_{\mathcal{L}} |U(z) - U(z_Q)|^2\right)^{1/2},$$

and gives an L^4 norm bounded above by $C\sqrt{\gamma}2^{n/2}$. Thus the L^2 and L^4 norms are comparable with constants independent of γ, so by the "$L^2 - L^4$ trick" (Lemma 3.2), the lemma follows. □

We now iterate Lemma 5.7.

LEMMA 5.8. *Suppose $Q^* \in \mathcal{D}_{2^n}$ and $U(z_{Q^*}) \geq \delta 2^{n/2}$. Then if ϵ is small enough,*

$$\sum |Q|_2 \geq n^{-1/3}|Q^*|_2, \tag{5.6}$$

where the sum is over $\{Q \in \mathcal{D}_{2^{n+1}} : Q \subset Q^* \text{ and } U(z_Q) \geq 2\epsilon\sqrt{2^n \log n}\}$.

Proof: Let $y_0 = y_{Q^*}$ (the y coordinate of z_{Q^*}), $y_1 = 2^{-\gamma 2^n} y_0$, and in general $y_k = 2^{-\gamma 2^n} y_{k-1}$ for $k = 1, \ldots, M \geq \gamma^{-1}$. If we choose $\gamma = 1/(c_3 \log n)$, then iterating Lemma 5.7 M times gives

$$\begin{aligned} |Q^*|_2 \beta^M &\leq |\{z = (x_1, x_2, y_M) : (x_1, x_2) \in Q^*, U(z) - U(z_{Q^*}) > M\alpha\sqrt{\gamma 2^n}\}|_2 \\ &\leq |\{z = (x_1, x_2, y_M) : (x_1, x_2) \in Q^*, U(z) > M\alpha\sqrt{\gamma 2^n}\}|_2 \\ &= |\{z = (x_1, x_2, y_M) : (x_1, x_2) \in Q^*, U(z) > c_4\alpha\sqrt{2^n \log n}\}|_2, \end{aligned}$$

where $\beta^M \geq \beta^{c_4 c_3 \log n} = n^{c_4 c_3 \log \beta} \geq n^{-1/3}$. By Harnack's inequality, if $U > a$ at some point of $T(Q)$ then it is bigger than Ca on all of $T(Q)$ (and in particular at z_Q). This gives the lemma if $\epsilon \leq Cc_4\alpha$. □

Fix ϵ in the definition of $\mathcal{B}$ and $\mathcal{G}$ so that the previous lemma holds. Let $G_n^i = \cup_{Q\in\mathcal{G}_n^i} Q$ for $i = 0, 1, 2$ and let $G_n = G_n^0 \cup G_n^1 \cup G_n^2$. Similarly, let $B_n = \cup_{Q\in\mathcal{B}_n} Q$.

LEMMA 5.9. *If $Q^* \in \mathcal{G}_n^2$ then $\mu(G_{n+1} \cap Q^*) \le (1 - n^{-1/2})\mu(Q^*)$.*

Proof: By Lemma 5.8 $|Q^* \cap B_{n+1}|_2 \ge n^{-1/3}|Q^*|_2$. Since $Q^* \in \mathcal{G}_n^2$ we have

$$\mu(Q^*) \le \epsilon 2^{n/2}\sqrt{\log n}|Q^*|_2.$$

Therefore

$$\begin{aligned}\mu(Q^* \setminus G_{n+1}) &\ge \sum_{Q\subset Q^*} \mu(Q \cap B_{n+1}) \\ &\ge Cn^{-1/3}|Q^*|_2\epsilon 2^{(n+1)/2}\sqrt{\log(n+1)} \\ &\ge Cn^{-1/3}\mu(Q^*).\end{aligned}$$

The last term is $\ge n^{-1/2}\mu(Q^*)$ if $n \ge n_0(\epsilon)$, so we are done. □

LEMMA 5.10. *If $Q^* \in \mathcal{G}_n^1$ then*

$$\mu(G_{n+1} \cap Q^*) \le (1 - n^{-1/2})\mu(Q^*).$$

Proof: Use the proof of Lemma 5.5 to deduce $\mu(\cup Q) \ge C\mu(Q^*)$, where the union is over all dyadic $Q \in Q^*$ of side length $= 2^{-\frac{9}{8}2^n}$ such that $U(z_Q) \ge c_5\sqrt{2^n}$. Now apply the argument of Lemma 5.9 to each Q to obtain

$$\mu(G_{n+1} \cap Q) \le (1 - n^{-1/3})\mu(Q),$$

$$\int_{B_{n+1}\cap Q} d\mu \ge n^{-1/3}\int_Q d\mu.$$

Thus

$$\mu(B_{n+1} \cap Q^*) \ \ge\ \sum_Q \mu(B_{n+1} \cap Q) \ge \sum_Q n^{-1/3}\mu(Q) \ge Cn^{-1/3}\mu(Q^*).$$

Since $Cn^{-1/3} \ge n^{-1/2}$ for large n, this proves the lemma. □

LEMMA 5.11. $\mu(G_n^0) \le B2^{-n/2}$.

Proof: Recall that the L^1 norm of $U_n = U(x_1, x_2, 2^{-2^n} + y)$ is $\simeq 1$ and the BMO norm is $\le C2^{n/2}$. Thus by Lemma 3.3 the L^2 norm is at most $C2^{n/4}$. On the set G_n^0, the square function of U_n is bounded below by $b_1 2^{n/2}$ which means the L^2 norm of U_n is bounded below by $b_1\sqrt{2^n|G_n^0|_2}$. Thus $|G_n^0|_2 \le C2^{-n/2}$. For $Q \in \mathcal{G}_n^0$, $U(z_Q) \simeq 1$, so $\mu(Q) \le C|Q|_2$. This proves the lemma. □

LEMMA 5.12. $\mu(G_n) \le A_{n_0}e^{-\frac{1}{4}n^{1/2}}$.

Proof: We prove this by induction. For $k = n_0$ it is true with $A_{n_0} = \|\mu\|e^{\frac{1}{4}\sqrt{n_0}}$. Assume it is true for $n \ge n_0$. From the previous three lemmas we can deduce

$$\begin{aligned}\mu(G_{n+1}) &\le \mu(G_{n+1} \cap G_n^0) + \mu(G_{n+1} \cap (G_n^1 \cup G_n^2)) \\ &\le B2^{-n/2} + (1 - n^{-1/2})\mu(G_n^1 \cup G_n^2) \\ &\le B2^{-n/2} + (1 - n^{-1/2})A_{n_0}\exp(-\frac{1}{4}n^{1/2}).\end{aligned}$$

Now if n_0 is so large that $n \geq n_0$ implies $B2^{-n/2} \leq \frac{1}{2}n^{-1/2}\exp(-\frac{1}{4}n^{1/2})$ we get

$$\begin{aligned} &\leq (1-\frac{1}{2}n^{-1/2})A_{n_0}\exp(-\frac{1}{4}n^{1/2}) \\ &\leq A_{n_0}\exp(\log(1-\frac{1}{2}n^{-1/2})-\frac{1}{4}n^{1/2}) \\ &\leq A_{n_0}\exp(-\frac{1}{2}n^{-1/2}-\frac{1}{4}n^{1/2}) \\ &\leq A_{n_0}\exp(-\frac{1}{4}(n+1)^{1/2}), \end{aligned}$$

where we have used the facts $\log(1-x) \leq -x$ and $\sqrt{n+1}-\sqrt{n} \leq \frac{1}{2}n^{-1/2}$. □

LEMMA 5.13. $\sum_{Q\in\mathcal{G}_{n+1}}\varphi(Q) \leq \frac{1}{2}\sum_{Q^*\in\mathcal{G}_n}\varphi(Q^*)+A'_{n_0}e^{-n^{1/4}}$.

Proof: Note that by Corollary 5.4

$$\sum_{Q^*\in\mathcal{G}_n^0\cup\mathcal{G}_n^1}[\sum_{Q\subset Q^*,Q\in\mathcal{G}_{n+1}}\varphi(Q)] \leq \frac{1}{2}\sum_{Q^*\in\mathcal{G}_n^0\cup\mathcal{G}_n^1}\varphi(Q^*).$$

If $Q\in\mathcal{G}_{n+1}$ and $Q^*\in\mathcal{G}_n$, then

$$\varphi(\mathrm{diam}(Q)) \leq \sqrt{2}(\frac{\log n+1}{\log n})^{1/2}\varphi(\mathrm{diam}(Q^*)).$$

Furthermore, since $U \geq \delta\sqrt{2^n}$ on the top half of $Q^*\in\mathcal{G}_n^2$,

$$\varphi(\mathrm{diam}(Q^*)) \leq \frac{C}{\delta}\sqrt{\log n}\,\mu(Q^*).$$

Thus

$$\begin{aligned} \sum_{Q^*\in\mathcal{G}_n^2}[\sum_{Q\subset Q^*,Q\in\mathcal{G}_{n+1}}\varphi(\mathrm{diam}(Q))] &\leq \sqrt{2}(\frac{\log n+1}{\log n})^{1/2}\sum_{Q^*\in\mathcal{G}_n^2}\varphi(\mathrm{diam}(Q^*)) \\ &\leq \frac{C}{\delta}\sqrt{\log n}\sum_{Q^*\in\mathcal{G}_n^2}\mu(Q^*) \\ &\leq \frac{C}{\delta}\sqrt{\log n}A_{n_0}e^{-\frac{1}{4}n^{1/2}} \\ &\leq A_{n_0}e^{-n^{1/3}}, \end{aligned}$$

if n_0 is large enough. □

COROLLARY 5.14. $\sum_{Q\in\mathcal{G}_n}\varphi(Q) \leq A''_{n_0}e^{-n^{1/4}}$.

Proof: Let $a_n = \sum_{Q\in\mathcal{G}_n}\varphi(Q)$. Then the previous lemma says

$$a_{n+1} \leq \frac{1}{2}a_n + A'_{n_0}e^{-n^{1/3}}.$$

To prove the corollary we use induction. It is true for $n = n_0$ by an appropriate choice of A''_{n_0}. In general we need only prove

$$\frac{1}{2}(A''_{n_0}\exp(-n^{1/4})) + A'_{n_0}\exp(-n^{1/3}) \leq A''_{n_0}\exp(-(n+1)^{1/4}).$$

Assuming we have chosen $A''_{n_0} \geq A'_{n_0}$ this reduces to showing

$$\exp(-n^{1/3}) \leq \exp(-(n+1)^{1/4}) - \frac{1}{2}\exp(-n^{1/4}).$$

This in turn follows from the two following estimates

$$\frac{1}{2}\exp(-n^{1/4}) \le \frac{2}{3}\exp(-(n+1)^{1/4}),$$

$$\exp(-n^{1/3}) \le \frac{1}{3}\exp(-(n+1)^{1/4}),$$

each of which is easy and left to the reader. □

6. The measure of the conical limit set

As noted in the introduction, the theory of conformal densities implies the conical limit set must have zero measure for the gauge $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$. We can see this more directly as follows. Note that the proof in the previous section actually shows that for any square Q, $\mathcal{H}^\varphi(\Lambda \cap Q) \le C\mu(Q)$. Since U has radial limit ∞ at μ almost every point of Λ (by standard differentiation theorems, e.g. Section 2.9 of [**21**]) but at no point of Λ_c, we see Λ_c must have zero μ measure and hence zero $\mathcal{H}^\varphi$ measure. In fact, it is not too hard to prove a better estimate.

LEMMA 6.1. *If G is topologically tame, $\Lambda \ne S^2$, and the injectivity radius is bounded away from zero, then $\mathcal{H}^\psi(\Lambda_c) = 0$ where $\psi(t) = t^2\log\frac{1}{t}(\log\log\frac{1}{t})^{-1-\epsilon}$ for any $\epsilon > 0$.*

PROOF. Fix a $M < \infty$ and let Λ_c^M be the limit points which are covered by infinitely many dyadic squares Q with $U(z_Q) \le M$. Then Λ_c is the countable union of the sets $\Lambda_c^1, \Lambda_c^2, \dots$, so it suffices to show each one has zero measure.

Let $\mathcal{A}_n$ be the collection of dyadic squares in $\cup_{k=2^n}^{2^{n+1}}\mathcal{D}_k$ which are maximal with respect to the property $c_1 \le U(z_Q) \le M$. Let E_n be the squares in $\mathcal{D}_{2^n}$ where $U(z_Q) \ge c_1$ and let $E_n(Q)$ be the squares $Q' \subset Q$ with $\operatorname{diam}(Q') = 2^{-2^n}\operatorname{diam}(Q)$ and $U(z_{Q'}) \ge c_1$. We claim that $|E_n|_2 \le C2^{-n/2}$. To prove this, let $U_n(x,y) = U(x, 2^{-2^{n+1}} + y)$ and recall that the L^1 norm of $U \simeq 1$ and the BMO norm is $\simeq 2^{n/2}$. Thus by Lemma 3.3 the L^2 norm is bounded by $C2^{n/4}$. On the other hand, at points of E, the square function of U_n is bounded below by $b_1 2^{n/2}$ which means that the L^2 norm of the square function is at least $(b_1^2 2^n |E|_2)^{1/2}$. Since the L^2 norms of U and $S_\alpha U$ are equivalent, we deduce

$$|E_n|_2 \le C2^{-n/2}, \tag{6.1}$$

as desired. A similar argument shows $|E_n(Q)|_2 \le C2^{-n/2}|Q|_2$, if $U(z_Q) \simeq 1$.

Cover Λ_c^M with the collection $\cup_{n=n_0}^\infty \cup_{Q\in\mathcal{A}_n} E_n(Q)$. Note that $\cup_{Q\in\mathcal{A}_n} E_n(Q) \subset E_n$. Thus

$$\sum_{Q\in A_n} |E_n(Q)|_2 \le C\sum_{Q\in A_n} 2^{-n/2}|Q|_2 \le C2^{-n/2}|E_n|_2 \le C2^{-n/2}\cdot 2^{-n/2},$$

and therefore,

$$\sum_{Q\in A_n} |E_n(Q)|_2 \le C2^{-n},$$

which implies

$$\sum_{Q'\in E_n(A_n)} |Q'|_2 \log\frac{1}{\operatorname{diam}(Q)} \le C2^{-n}2^n \le C.$$

Thus

$$\sum_{Q' \in E_n(A_n)} \frac{|Q'|_2 \log \frac{1}{\operatorname{diam}(Q)}}{(\log\log \frac{1}{\operatorname{diam}(Q)})^{1+\epsilon}} \leq C n^{-1-\epsilon}.$$

Since this is a convergent series, we have proved the lemma. □

A similar proof shows Λ_c has zero measure for any gauge of the form $\psi(t) = t^2 \log \frac{1}{t} \eta(t)$ where $\sum_n \eta(2^{-2^n}) < \infty$.

7. Absolute continuity of quasiconformal conjugacies

In this section we prove Corollary 1.5. There are two facts needed for the proof. The first is the following result of McMullen, Theorem 2.11 of [**22**]

THEOREM 7.1. *Suppose Φ is a K-quasiconformal conjugacy on S^2 between two Kleinian groups G_1 and G_2, with quotient manifolds M_1 and M_2. Suppose the injectivity radius for M_1 in the convex core lies in $[R_0, R_1]$ with $0 < R_1 \leq R_1 < \infty$. Then Φ extends to a volume preserving, quasi-isometry Φ between M_1 and M_2 and the pointwise quasi-isometry constant $L(\Phi, p)$ satisfies*

$$\log L(\Phi, p) \leq C \exp(-\alpha dist(p, M \setminus C(M)).$$

the constants C and $\alpha > 0$ depend only on K, R_0, R_1.

The second fact we need is that μ almost every geodesic tends to infinity in the convex core with a certain estimate.

LEMMA 7.2. *Suppose G is a geometrically infinite, topologically tame group and let μ be a conformal density for G. Then for μ almost every geodesic γ,*

$$\liminf_{t \to \infty} \frac{dist(\gamma(t), M \setminus C(M))}{\sqrt{t}(\log t)^{-2}} = +\infty.$$

PROOF. We use the notation of the two previous sections. Let E_n be the union of dyadic squares of generation 2^n which hit the Λ. Then Equation (6.1) says that $|E_n|_2 \leq C 2^{-n/2}$. Let $A > 0$ and set

$$\mathcal{B}_n = \{Q : 2^{-n-1} \leq \log \ell(Q)^{-1} \leq 2^{-n} \text{ and } \mu(Q) \leq A n^{-2} 2^{n/2} |Q|_2\}.$$

Then if $B_n = \cup_{Q \mathcal{B}_n} Q$, we see

$$\sum_n \mu(B_n) \leq A \sum_n n^{-2} 2^{n/2} |E_n|_2 \leq CA \sum_n n^{-2} < \infty.$$

Thus by the Borel-Cantelli lemma, μ almost every point of Λ is in only finitely many of the sets B_n, which proves the lemma. □

We can now prove Corollary 1.5. Let Φ be the conjugacy given by Theorem 7.1 lifted to $\mathbb{B}$ and assume it maps the origin to itself. Let μ_1 and μ_2 be the restrictions of $\mathcal{H}^\varphi$ to Λ_1 and Λ_2 respectively. Let $x \in \Lambda_1$ be a point where the conclusion of Lemma 7.2 holds and let $B = B(x,t)$ denote the (Euclidean) ball of radius t around x in $\mathbb{R}^3$. Let $y = x(1-t)$. Since Φ is quasiconformal in $\mathbb{R}^3$, $\Phi(B)$ contains a ball of size at least $C(1 - |\Phi(y)|)$ around $\Phi(x)$. We claim this is comparable to $1 - |y|$. To see this, note that $\rho(0, \Phi(y))$ can be estimated from above by integrating the pointwise quasi-isometric constant of Φ along the geodesic from

0 to y. By combining McMullen's estimate and the previous lemma this is bounded by

$$\int_0^{\rho(0,y)} 1 + C\exp(-\alpha A_x \sqrt{t}/\log^2 t)dt \leq \rho(0,y) + C_x.$$

Thus for μ almost every point $x \in \Lambda_1$ there is a constant C_x so that $\Phi(B(x,t))$ contains the ball $B(\Phi(x), C_x t)$.

Finally we have to check that this implies $\Phi^*\mu_1$ is absolutely continuous with respect to μ_2. If not, then there is a compact set E of positive μ_1 measure so that $\Phi(E)$ has zero μ_2 measure, i.e., zero $\mathcal{H}^\varphi$ measure. By passing to a subset we may assume the same constant $C = C_x$ works for all $x \in E$. Let $\{B_j\}$ be a covering of $\Phi(E)$ with small φ-sum. Then $\{\Phi^{-1}(B_j)\}$ is a covering of E by sets of comparable diameter, hence also has small φ-sum. Thus $\mu_2(\Phi(E)) = 0 \Rightarrow \mathcal{H}^\varphi(\Phi(E)) = 0 \Rightarrow \mathcal{H}^\varphi(E) = 0 \Rightarrow \mu_1(E) = 0$. This is a contradiction, so the first part of the corollary is proved.

Finally, we would like to show that Φ is differentiable μ_1 almost everywhere. Let $x \in \Lambda$ be a point where the corresponding geodesic γ satisfies

$$\mathrm{dist}(\gamma(t), M \setminus C(M)) \geq \eta(t),$$

for some function η which satisfies

$$\int_a^\infty \exp(-\alpha\eta(t))dt < \infty. \tag{7.1}$$

Then for points $t_1 > t_2$ on the geodesic, we have (using estimates for Φ and Φ^{-1})

$$\rho(\Phi(\gamma(t_1)), \Phi(\gamma(t_2))) = |t_2 - t_1| + O(\int_{t_1}^{t_2} \exp(-\alpha\eta(t))dt) = |t_2 - t_1| + o(1),$$

uniformly as $t_1 \to \infty$. This implies the image of γ is asymptotically close to a geodesic ending at $\Phi(x)$.

Assume $x = 0 \in \Lambda$ and consider the rays $R_0 = \{(0,0,t) : t > 0\}$, $R_1 = \{(t,0,t) : t > 0\}$ and $R_2 = \{(0,t,t) : t > 0\}$ in the upper half space which end at x. Let $z_n^i = R_i(e^{-n})$, $i = 0,1,2$ and let w_n^i be the orthogonal projection of z_n^i onto $\mathbb{R}^2$. The position of z_{n+1}^i is determined, to within $O(a_n) = O(\exp(-\alpha\eta(n)))$, by its distances to the points z_n^i, $i = 0,1,2$ (e.g., Lemma B.23 of [**22**]). In particular, the argument of w_{n+1}^i, $i = 1,2$, is the same as the argument of w_n^i, to within $O(a_n)$. Since this is a summable sequence we see $\Phi(R_1)$ and $\Phi(R_2)$ are asymptotic to rays at $\Phi(x)$ and the angles between $\Phi(R_0)$, $\Phi(R_1)$ and $\Phi(R_2)$ are the same as the corresponding angles between R_0, R_1 and R_2. If we renormalize the maps so that $z_n^0 = (0,0,1)$, we see (by symmetry and equicontinuity) that the maps converge uniformly on compact sets of $\mathbb{R}^3_+$ to a conformal map and hence converge uniformly on compact subsets of $\mathbb{R}^3$ to a conformal map. This proves Φ is differentiable at x, as desired. Since all we need to prove Φ is differentiable was (7.1), we see that Equation (1.2) suffices. Thus Lemma 7.2 implies Φ is differentiable μ_1 almost everywhere, finishing the proof.

It is conjectured (e.g., Equation (3.17) of [**11**]) that if f is a quasiconformal map in $\mathbb{R}^n$ with real dilatation K_f, then

$$\int_{|x-x_0|<t} \frac{K_f(x) - 1}{|x - x_0|^n} dx < \infty,$$

implies f is differentiable at x_0 and is conformal there (this is known for $n = 2$, [**15**]). Our mapping Φ satisfies such an estimate at μ_1 almost every x, so this conjecture would give an alternate proof that Φ is differentiable μ_1 almost everywhere.

The set where Φ is not differentiable is only slightly larger than the conical limit set. Indeed, if we repeat the proof of Lemma 6.1, but use the fact that U has mean value norm $\leq C \log n$ on a square Q where $U(z_Q) \leq \log n$ ($\ell(Q) = 2^{-2^n}$), then we can show Φ is differentiable everywhere on Λ except on a set of zero Hausdorff measure for $\psi(t) = t^2 \log \frac{1}{t} (\log\log \frac{1}{t})^{-1-\epsilon}$ for any $\epsilon > 0$. (The extra growth gives a $\log\log\log$ term, but this can be absorbed into the $\log\log^{-1-\epsilon}$ term.)

8. An estimate for the gradient of Green's function

LEMMA 8.1. *There is a constant $A < \infty$ so that the following holds. Suppose $x \in M = \mathbb{B}/G$ and $y_1, y_2 \in M$ with $\rho(x, y_1), \rho(x, y_2) \geq 1$ and $\rho(y_1, y_2) \leq 1$. Then $|G(y_1, x) - G(y_2, x)| \leq A vol(B(1, y_1))^{-1}$.*

PROOF. We will first consider the case when $\rho(x, y_1) = \rho(x, y_2) = 1$, and will deduce the general case using the maximum principle at the end of the proof.

By the symmetry of Greens function it is enough to prove

$$|G(x, y_1) - G(x, y_2)| \leq A/\mathrm{vol}(B(1, y_1)).$$

Let $B_r = \{z \in M : \rho(z, x) < r\}$ and let S_r be the boundary of B_r. Let W_y be the space of Brownian paths on M started at y at time $t = 0$. For a path $b \in W_y$, define stopping times by

$$\tau_1^y = \inf t > 0 \text{ such that } b(t) \in S_1.$$

$$\sigma_1^y = \inf t > \tau_1^y \text{ such that } b(t) \in S_2.$$

In general, we let

$$\tau_n^y = \inf t > \sigma_{n-1}^y \text{ such that } b(t) \in S_1.$$

$$\sigma_n^y = \inf t > \tau_n^y \text{ such that } b(t) \in S_2.$$

Let

$$E_n^y = \{b \in W : \tau_n < \text{ finite }\},$$

and note that for $b \in E_n^y$, σ_n^y is finite almost surely. Let μ_n^y be the measure defined on S_2 by the stopping time σ_n^y, i.e., for $E \subset S_2$,

$$\mu_n^y(E) = \mathbb{P}(\sigma_n^y < \infty \text{ and } b(\sigma_n^y) \in E).$$

Then clearly $\|\mu_n^y\|$ is decreasing in n and

$$\|\mu_n^y\| = \int_{S_2} \omega(w, S_1, M \setminus B_1) d\mu_{n-1}^y(w) = \|\mu_{n-1}^y\| \int_{S_2} \omega(w, S_1, M \setminus B_1) d\tilde{\mu}_{n-1}^y(w).$$

where $\tilde{\mu}_n^y = \mu_n^y / \|\mu_n^y\|$.

For $y \in M$ let $T(y)$ be the expected time a path started at y spends in B_1 before σ_1^y (i.e., before the first time it hits S_2 following its first intersection with B_1). This is harmonic on $M \setminus B_1$ and is uniformly bounded. Then for $y \in M \setminus B_1$, let $\mu_1^y = \omega(y, \cdot, M \setminus B_1)$ and

$$\mu_n^y = L(\tilde{\mu}_{n-1}^y) \int_{S_2} \omega(w, S_1, M \setminus B_1) d\mu_{n-1}^y(w).$$

Then if $y \in B_2$ we get (by the mean value property)

$$
\begin{aligned}
(8.1)\quad G(x,y)\mathrm{vol}(B_1) &= \int_{B_1} G(x,y)dx \\
&= \text{expected time a path started at } y \text{ spends in } B_1 \\
&= T(y) + \sum_{n=1}^{\infty} \int_{S_2} T(w) d\mu_n^y(w).
\end{aligned}
$$

Thus to estimate G it will be enough to estimate $\|\mu_n^y\|$.

We claim that there is a probability measure μ on S_2, so that $\tilde{\mu}_n^y \to \mu$ exponentially fast, i.e., $\|\mu - \tilde{\mu}_n^y\| \leq \lambda^n$, for some $\lambda < 1$. To prove this we will show there is a strict contraction L on the convex set of probability measures on S_2 such that $\tilde{\mu}_n^y = L(\tilde{\mu}_{n-1}^y)$. Then μ will simply be the fixed point of the contraction.

For $u \in B_2$ and $v \in S_2$, let $P_{B_2}(u,v)$ be the Poisson kernel for B_2 with respect to the point u, i.e., for $E \subset S_2$,

$$\omega(u, E, B_2) = \int_{S_2} \chi_E(v) P_{B_2}(u,v) dv,$$

(where dv denotes area measure on S_2). Similarly, define a kernel $P_{M\setminus B_1}(w,u)$ on ∂B_1 by

$$\omega(u, E, M \setminus B_1) = \int_{S_1} \chi_E(v) P_{M\setminus B_1}(w,u) du,$$

Define a kernel $K(v,w)$ on $S_2 \times S_2$ by

$$K(w,v) = P_{M\setminus B_1}(w,u) P_{B_2}(u,v).$$

This corresponds to the first hitting probability on S_2 of a Brownian motion started at $v \in S_2$, conditioned to first hit B_1. Let

$$\tilde{K}(w,v) = \frac{K(w,v)}{\int_{S_2} K(w,v) dw}.$$

Thus

$$\int_{S_2} \tilde{K}(w,v) dw = 1, \tag{8.2}$$

We claim that area$(S_2)K(w,v)$ is bounded away from zero by some absolute constant (independent of x). This is easiest to see by lifting to $\mathbb{B}$. Let $\Omega \subset \mathbb{B}$ be the lift of B_2 and assume x lifts to the origin. Since $B(0,2) \subset \Omega$ the Poisson kernel for Ω with respect to $0 \in \Omega$ is uniformly bounded below on $\partial B(0,2) \cap \partial\Omega$. If the volume of B_2 is large we are done. If it is small then there are $\simeq \mathrm{vol}(B_1)^{-1}$ lifts of x in $B(0,1)$ and by Harnack's inequality, the Poisson kernel for each of these is comparable on $\partial\Omega$ to the kernel for 0. Thus summing over the orbit of 0 gives a sum which bounded below by area$(S_2)K(w,v)$ since area$(S_2) \simeq \mathrm{vol}(B_1)$.

Thus there is a $C_1 > 0$ so that

$$\int_{S_2} |\tilde{K}(w,v) - C_1| dv \leq \lambda, \tag{8.3}$$

for some fixed $\lambda < 1$. Note that λ only depends of the behavior of the Dirichlet problem in the ball B_2 and hence may be chosen independent of the point x (and even the group G).

For a probability measure ν on S_2 define a measure $L(\nu)$ by

$$L(\nu)dv = [\int_{S_2} \tilde{K}(w,v)d\nu(w)]dv.$$

Then by Equation 8.2, L maps probability measures to probability measures. To see L is a strict contraction, let ν_1, ν_2 be two probability measures on S_2, Then using Equation 8.3 and the fact that $\nu_1 - \nu_2$ has mean value zero we get

$$\begin{aligned} \|L(\nu_1) - L(\nu_2)\| &= |\int_{S_2}[\int_{S_2} \tilde{K}(w,v)d(\nu_1-\nu_2)(w)]dv| \\ &\leq |\int_{S_2}[\int_{S_2} (\tilde{K}(w,v) - C_0)d(\nu_1-\nu_2)(w)]dv \\ &\qquad + \int_{S_2}[\int_{S_2} C_0 d(\nu_1-\nu_2)(w)]dv| \\ &\leq |\int_{S_2}[\int_{S_2} (\tilde{K}(w,v) - C_0)d(\nu_1-\nu_2)(w)]dv + 0| \\ &\leq \int_{S_2}[\int_{S_2} |\tilde{K}(w,v) - C_0|dvd|\nu_1-\nu_2|(w) \\ &\leq \lambda\|\nu_1-\nu_2\|. \end{aligned}$$

Thus L is a strict contraction. It is easy to check that $\tilde{\mu}_n = L(\tilde{\mu}_{n-1})$, so there is a probability measure μ on S_2 which is a fixed point for L and so that $\tilde{\mu}_n$ converges to μ exponentially fast.

Define

$$\eta \equiv 1 - \int_{S_2} \omega(w, S_1, M\setminus B_1)d\mu(w).$$

Observe that η is bounded away from 1 uniformly and can be bounded away from zero depending only on $\mathrm{dist}(x, \partial C(M))$. Fix a probability measure μ_0 on S_2 and let

$$\mu_n = L(\tilde{\mu}_{n-1})\|\mu_{n-1}\| \int_{S_2} \omega(w, S_1, M\setminus B_1)d\tilde{\mu}_{n-1}(w).$$

We want to estimate $\|\mu_n\|$. Note,

$$\begin{aligned} \|\mu_n\| &= \|\mu_{n-1}\| \int_{S_2} \omega(w, S_1, M\setminus B_1)d\tilde{\mu}_{n-1}(w) \\ &= \|\mu_{n-1}\|[\int_{S_2} \omega(w, S_1, M\setminus B_1)d\mu(w) \\ &\qquad + \int_{S_2} \omega(w, S_1, M\setminus B_1)d(\tilde{\mu}_{n-1}-\mu)(w)] \\ &= \|\mu_{n-1}\|[(1-\eta) + \int_{S_2} \omega(w, S_1, M\setminus B_1)d(\tilde{\mu}_{n-1}-\mu)(w)] \\ &= \|\mu_{n-1}\|[(1-\eta) + \int_{S_2} ((1-1) + \omega(w, S_1, M\setminus B_1))d(\tilde{\mu}_{n-1}-\mu)(w)] \\ &= \|\mu_{n-1}\|[(1-\eta) - \int_{S_2} (1 - \omega(w, S_1, M\setminus B_1))d(\tilde{\mu}_{n-1}-\mu)(w)]. \end{aligned}$$

By Harnack's inequality we have $C\eta \geq 1 - \omega(w, S_1, M \setminus B_1)$, for all $w \in S_2$, so

$$[(1-\eta) - C\eta\|\tilde{\mu}_{n-1} - \mu\|] \leq \frac{\|\mu_n\|}{\|\mu_{n-1}\|} \leq [(1-\eta) + C\eta\|\tilde{\mu}_{n-1} - \mu\|],$$

which gives

$$[(1-\eta) - C\eta\lambda^n] \leq \frac{\|\mu_n\|}{\|\mu_{n-1}\|} \leq [(1-\eta) + C\eta\lambda^n].$$

Thus if we assume $\eta < 1/2$,

$$\prod_{k=1}^{n}[1 - 2C\eta\lambda^k] \leq \frac{\|\mu_n\|}{(1-\eta)^n} \leq \prod_{k=1}^{n}[1 + 2C\eta\lambda^k].$$

$$1 - 2C\eta \leq \frac{\|\mu_n\|}{(1-\eta)^n} \leq 1 + 2C\eta.$$

Thus

$$|\|\mu_n\| - (1-\eta)^n| \leq C\eta(1-\eta)^n.$$

Next we want to show G does not oscillate much on S_2. Let

$$A = \sum_{n=1}^{\infty} \int_{S_2} T(w)(1-\eta)^n d\mu(w) \simeq \frac{1}{\eta}.$$

For $y \in S_2$, (8.1) implies

$$\begin{aligned}
|G(x,y)\mathrm{vol}(B_1) - A| &\leq T(y) + |\sum_{n=1}^{\infty}[\int_{S_2} T(w)(1-\eta)^n d\mu(w) - \int_{S_2} T(w) d\mu_n^y(w)]| \\
&\leq C + C\sum_{n=1}^{\infty} \|(1-\eta)^n\mu(w) - \mu_n^y(w)\| \\
&\leq C + C\sum_{n=1}^{\infty}[\|(1-\eta)^n\mu(w) - (1-\eta)^n\tilde{\mu}_n^y(w)\| \\
&\qquad + \|(1-\eta)^n\tilde{\mu}_n^y(w) - \mu_n^y\|] \\
&\leq C + C\sum_{n=1}^{\infty}[(1-\eta)^n\lambda^n + \eta(1-\eta)^n] \\
&\leq C + C[\frac{\lambda}{1-\lambda} + \frac{\eta(1-\eta)}{\eta}] \\
&\leq C.
\end{aligned}$$

From this it follows that $G(x,y)\mathrm{vol}(B_1)$ oscillates by at most a uniform constant for $y \in S_2$. This is what we wanted to prove. Since $A \simeq 1/\eta$, this also shows

$$G(x,y) \leq \frac{C\eta^{-1}}{\mathrm{vol}(B_1(x))}. \tag{8.4}$$

If y_1, y_2 are both distance ≥ 1 from x then the maximum principle,

$$\max_{M\setminus B(x,1)} G(y,x) = \max_{\partial B(x,1)} G(y,x),$$

implies

$$\begin{aligned} |G(y_1,x)-G(y_2,x)| &= |G(x,y_1)-G(x,y_2)| \\ &\leq \sup_{w\in S(2,y_1)} |G(w,y_1)-G(w,y_2)| \\ &\leq \sup_{w\in S(2,y_1)} |G(y_1,w)-G(y_2,w)| \\ &\leq A\mathrm{vol}(B(1,y_1))^{-1}. \end{aligned}$$

□

A standard estimate for harmonic functions says there is an absolute $C<\infty$ so that

$$|\nabla U(x)| \leq C(\sup_{B(x,1)} U(y) - \inf_{B(x,1)} U(y)),$$

so we can now deduce,

Corollary 8.2. *If dist$(y,x)\geq 1$ then $|\nabla_y G(y,x)| \leq C vol(B(1,y))^{-1}$.*

9. A positive harmonic function with linear growth on M

Using the bound for the gradient of Green's function from the previous section, we now want to construct a positive harmonic function U on M by taking the limit as we push the pole off to infinity. We start with the following observations.

Lemma 9.1. *Suppose $\partial C(M)$ is compact and $x\in C(M)$ with dist$(x,\partial C(M))\geq 1$. Then $G(y,x)$ is bounded on $\partial C(M)$ (independent of x).*

Proof. By symmetry we only have to show that $G(x,y)$ is bounded outside the ball of radius one around y. By the maximum principle it is enough to show it is bounded on the boundary of this ball. However, this follows from the (8.4). □

Lemma 9.2. *Suppose $\partial C(M)$ is compact and area$(\Lambda)=0$. Then there is an $\epsilon>0$ so that for any $x\in C(M)$ with dist$(x,\partial C(M))>10$ there is a point $y\in \partial C(M)$ where $G(y,x)\geq\epsilon$*

Proof. This is obvious by Harnack's inequality and the fact that

$$\int_{\partial C(M)} |\nabla G| ds \geq \int_{\partial C(M)} |\frac{\partial G}{\partial n}| ds \geq 1.$$

□

Lemma 9.3. *Suppose G is geometrically infinite, $\partial C(M)$ is compact and there is a lower bound for the injectivity radius in $C(M)$. Fix a point $x_0\in\partial C(M)$. Then there is a positive harmonic function U on $M=\mathbb{B}/G$ so that $U(x_0)=1$, $U(z)\to 0$ in the geometrically finite ends and $|\nabla U|\leq A$ on all of M. In particular, $U(z)\leq C+C dist(z,x_0)$ for $z\in C(M)$.*

Proof. If Λ has positive area, this is easy (take the Poisson integral of its characteristic function). Thus we may assume Λ has zero area. Since G is geometrically infinite we can choose a sequence of points $x_n\in C(M)$ so that $\mathrm{dist}(x_0,x_n)\to\infty$. Then $G(z,x_n)$ is uniformly bounded on any compact subset of $C(M)$ and hence there is a subsequence which converges uniformly on compact subsets to a harmonic function U. Clearly U satisfies the same upper bounds as $G(z,x_n)$ which gives the last two conditions. Finally, Lemma 9.2 and Harnack's inequality give a lower bound for $G(y, x_n)$ on $\partial C(M)$ which is independent of x_n. Hence U also satisfies this lower bound, so by a bounded rescaling we can get $U(x_0) = 1$. $\square$

LEMMA 9.4. *Suppose G is topologically tame but geometrically infinite group and there is a positive lower bound for the injectivity radius. Then there is a positive harmonic function U satisfying the conclusions of Lemma 9.3 and in addition satisfies the following condition. For every $a_0 < \infty$ there are positive constants a_1 and a_2 so that*

$$\int_{B(z,a_1)} |\nabla U|^2 dV \geq a_2,$$

for every $z \in M$ with $dist(z, C(M)) \leq a_0$. Moreover, U tends to ∞ in the geometrically infinite ends.

PROOF. By standard results (see [**6**]) if G is topologically tame and has a positive lower bound for the injectivity radius then $C(M)$ can be written as a compact part K and a finite number of ends E_i each of which is topologically of the form $E_j = S_j \times \mathbb{R}^+$, $j = 1, \dots, N$. For $z \in K$ the estimate is true at each point (since U is not constant) and hence is true uniformly on K by compactness.

Next consider an end E_j. Choose a sequence of points converging to ∞ in this end and let U_j denote the limiting harmonic function, normalized to be 1 at a fixed point $x_0 \in \partial C(M)$. For any $x \in E_j$ there is a surface $S_j(x)$ containing x which divides E_j and which has uniformly bounded area and diameter. If z is a point in the unbounded component of $E_j \setminus S_j(x)$ then the integral over $S_j(x)$ of the gradient of Greens function with pole at z is at least 1 (since in this case the limit set has zero area, hence zero harmonic measure with respect to z). Moreover, the same integral for a Greens function with pole in a different end E_k, $k \neq j$, is zero over $S_j(x)$. Letting $z \to \infty$ we see the same is true for the limiting functions U_j and U_k.

Now let $U = \sum_j U_j$. Then for each end E_j and each $x \in E_j$, the integral of $|\nabla U|$ over $S_j(x)$ is at least 1. Thus there is a point on $S_j(x)$ where the $|\nabla U| \geq (\text{area}(S_j(x))^{-1}$ is bounded uniformly away from zero. Because $|\nabla U|$ is also bounded above uniformly, so are all second derivatives of U and hence $|\nabla U|$ is actually bounded uniformly away from zero on some ball of fixed size centered at some point of $S_j(x)$.

Finally we note that the gradient lower bound implies U cannot be bounded in any end E_j. Together with the maximum principle, Harnack's inequality and the fact that the ends have bounded diameter cross sections, we easily see that U must tend to infinity in each end. This proves the lemma. $\square$

LEMMA 9.5. *Suppose $M = \mathbb{B}/G$ is topologically tame with $N \geq 1$ geometrically infinite ends $E_1, \dots, E_N$, that there is a lower bound for the injectivity radius on M and that $\Lambda \neq S^2$. Then for each $j = 1, \dots, N$, there is a unique (up to constant multiples) positive harmonic function U_j which tends to infinity in E_j, is bounded in the other geometrically infinite ends and tends to zero in the geometrically finite ends. Any positive harmonic function on M which tends to zero in the geometrically finite ends is of the form $\sum a_j U_j$, where $a_j \geq 0$, $j = 1, \dots, N$.*

PROOF. Suppose U is a positive harmonic function on M which tends to zero in the geometrically finite ends. Choose a base point z_0 and with loss of generality assume $U(z_0) = 1$. First we will show that the gradient of U is uniformly bounded.

Suppose R is a large positive number and define $F(z) = \min(U(z), R)$. Then F is clearly superharmonic, so ΔF is a negative measure σ_R. Let $H(z)$ be the convolution of σ_R with the Green's function for M. Then H is bounded (since σ_R locally looks like surface measure on $\{U = R\}$ and $\Delta H = \sigma_R$. Thus $F - H$ is a bounded harmonic function on M which tends to zero in the geometrically finite ends. Thus the lift of $F - H$ to the ball is a bounded harmonic function with radial limits zero almost everywhere. Hence $H = F$. Since the Greens function with pole in the convex core of M is bounded below on $\partial C(M)$, it will be uniformly bounded below at z_0 by Harnack's inequality. Thus we see that $\|\sigma_R\| \simeq H(z_0) = 1$ is uniformly bounded, independent of R. Thus $\nabla_H U$ is uniformly bounded since the gradient of Green's function is uniformly bounded.

Now suppose U_1 and U_2 are two positive harmonic functions on M, tending to zero in the geometrically finite ends, and both are bounded on $M \setminus E_1$. Without loss of generality assume $U_1(z_0) = U_2(z_0) = 1$. Then $F = U_1 - U_2$ has uniformly bounded gradient in E_1 and must be zero on every connected surface S which divides E_1 (otherwise solving the Dirichlet problem on $M \setminus S$ with boundary values $F|_S$ would show $F(z_0) \neq 0$). Since E has a family of dividing surfaces with bounded diameter we deduce that F is uniformly bounded in E_1. As above the lift of F to the ball is a bounded harmonic function with radial limits zero almost everywhere and hence $U_1 = U_2$.

Given any positive harmonic function U on M which tends to zero in the geometrically finite ends, we can can write it as a sum of harmonic functions, each of which is unbounded in at most one end. To do this, suppose U is unbounded in E_1 and choose a family of dividing surfaces S_t which tend to infinity in E_1. Solve the Dirichlet problem with boundary values $U|_{S_t}$ (and zero elsewhere) and take the limit as $t \to \infty$. This gives a positive harmonic function U_1 on M. Moreover if $C(M) = K \cup_j E_j$ where K is compact, let C be an upper bound for U on K. Then the maximum principle implies $U - C \leq U_1 \leq U$ in E_1 and hence $U - U_1$ is bounded in E_1. Now repeat this procedure in each of the other ends to obtain the desired decomposition. It is unique since two decompositions would differ by a bounded harmonic function in M which if $U = \sum_j U_j = \sum_j U'_j$ then $U_j - U'_j$ would be bounded and hence zero on M. □

The previous result can be summarized using the language of Martin boundaries. It says that there is exactly one Martin boundary point corresponding to each geometrically infinite end. The result can also be interpreted in terms of conformal densities on the ball.

Corollary 9.6. *Suppose G is a Kleinian group and $M = \mathbb{B}/G$ is as in the previous lemma. Then associated to each geometrically infinite end of M there is a unique (up to constant multiples) conformal density μ_j of dimension 2. G is ergodic with respect to each of these densities. The densities $\{\mu_j\}$ are mutually singular and any conformal density for G is of the form $\sum_j a_j \mu_j$, $a_j \geq 0$.*

Proof. Conformal densities of dimension 2 correspond to positive harmonic functions on M via the Poisson extension, so existence and uniqueness follow from the previous lemma. Similarly for the decomposition of densities. The action is ergodic on each μ_j since an invariant subset E with measure $0 < \mu_j(E) < \mu_j(\Lambda)$ would imply the corresponding harmonic function U_j could be decomposed as the sum of two other positive harmonic functions (the Poisson extensions of $\mu_j|_E$ and $\mu_j|_{E^c}$) which were not constant multiplies of each other (since they are Poisson integrals of mutually singular measures). To see that the densities corresponding to different ends are mutually singular recall that given harmonic functions U_1 and U_2 corresponding to different ends, U_2 remains bounded along geodesics where U_1 tends to infinity. Thus if we consider the corresponding conformal densities, the Poisson integral of μ_1 tends to infinity μ_1 almost everywhere and the Poisson integral of μ_2 remains bounded μ_1 almost everywhere. Hence μ_1 and μ_2 must be singular. $\square$

References

[1] L. Ahlfors. Finitely generated Kleinian groups. *Amer. J. of Math.*, 86:413–429, 1964.

[2] A.F. Beardon and B. Maskit. Limit points of Kleinian groups and finite sided fundamental polyhedra. *Acta Math.*, 132:1–12, 1974.

[3] L. Bers. Inequalities for finitely generated Kleinian groups. *J. d'Analyse Math.*, 18:23–41, 1967.

[4] C.J. Bishop. Geometric exponents and Kleinian groups. *Invent. Math.*, 127:33–50, 1997.

[5] C.J. Bishop and P.W. Jones. Hausdorff dimension and Kleinian group. To appear in Acta Math., also appeared as Stony Brook IMS preprint 1994/5.

[6] R.D. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 6:1–35, 1993.

[7] L. Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn.*, 10:113–123, 1985.

[8] S.-Y. A. Chang, J.M. Wilson, and T. Wolff. Some weighted norm inequalities concerning the Schrödinger operators. *Comment. Math. Helv.*, 60:217–246, 1985.

[9] J.B. Garnett. *Bounded analytic functions*. Academic Press, 1981.

[10] A.M. Garsia. *Martingale inequalities: seminar notes on recent progress*. W.A. Benjamin, Inc., Reading, Massachusetts, 1973.

[11] V.Ya. Gutlyanskii and M. Vuorinen. On maps almost conformal at the boundary. 1995. Helsinki preprint 98, to appear in Complex Variables.

[12] E. Hopf. Fuchsian groups and ergodic theory. *Trans. Amer. Math. Soc.*, 39:299–314, 1936.

[13] S. Janson and P.W. Jones. Interpolation between H^p spaces: the complex method. *J. Funct. Analysis*, 48:58–80, 1982.

[14] F. John and L. Nirenberg. On functions of bounded mean oscillation. *Comm. Pure Appl. Math.*, 14:415–416, 1961.

[15] O. Lehto. On the differentiability of quasiconformal mappings with prescribed complex dilatation. *Ann. Acad. Sci. Fenn, Ser. A I*, 275, 1960.

[16] N.G. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.*, 51:369–384, 1985.

[17] N.G. Makarov. Probability methods in the theory of conformal mappings. *Albegra i Analiz*, 1:3–59, 1989. English translation in *Leningrad J. Math.* 1:1-56, 1990.

[18] N.G. Makarov. Smooth measures and the law of the iterated logarithm. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53:439–446, 1989. English translation in *Math. USSR-Izv.* 34:455–463, 1990.

[19] R.S. Martin. Minimal positive harmonic functions. *Trans. Amer. Math. Soc.*, 49:137–172, 1941.

[20] B. Maskit. *Kleinian groups*. Springer-Verlag, 1988.

[21] P. Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Cambridge University Press, 1995.

[22] C. McMullen. *Renormalization and* 3*-manifolds which fiber over the circle*. Annals of Math. Studies, 142. Princeton University Press, 1996.

[23] P.J. Nicholls. *The ergodic theory of discrete groups*. LMS Lecture Notes 143. Cambridge University Press, 1989.

[24] C.A. Rogers and S.J. Taylor. Functions continuous and singular with respect to a Hausdorff measure. *Mathematika*, 8:1–31, 1961.

[25] E. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, 1971.

[26] D. Sullivan. Growth of positive harmonic functions and Kleinian group limit sets of planar measure 0 and Hausdorff dimension 2. In *Geometry Symposium (Utrecht 1980)*, Lecture Notes in Math. 894, pages 127–144. Springer-Verlag, 1981.

C.J. Bishop, Mathematics Department, State University of New York at Stony Brook, Stony Brook, NY 11794-3651
E-mail address: bishop@math.sunysb.edu

P.W. Jones, Mathematics Department, Yale University, New Haven, CT 06520