MAT 550, REAL ANALYSIS
SUNY at Stony Brook, SPRING 99
Problem Set 4, REVISED VERSION
Due on Friday, Febuary 26

Two Banach spaces X and Y are said to be isomorphic if there is a linear map $L : X \to Y$ which is 1-1, onto and preserves norms, i.e., $\|Lx\|_Y = \|x\|_X$ for every $x \in X$. As usual, let ℓ^p denote the L^p space on the positive integers with respect to counting measure. The following exercises combine to show that none of the spaces ℓ^p, $1 \leq p < \infty$ is isomorphic to any of the others (in fact, none is isomorphic to a closed subspace of any of the others). If you can't do one of the exercises, you may assume it for the subsequent ones.

A sequence $\{x_n\}$ in a Banach space X is called a Schauder basis if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}$ such that $x = \sum_n a_n x_n$. For example, let $e_n \in \ell^p$ be the sequence which is all 0's except for a 1 in the nth position. Then $\{e_n\}$ is a Schauder basis for each of the spaces ℓ^p, $1 \leq p < \infty$. A sequence $\{x_n\}$ is called a basic sequence if it is a Schauder basis for its closed linear span. A sequence is called normalized if $\|x_n\|_X = 1$ for all n. Two basic sequences, $\{x_n\}$ of X and $\{y_n\}$ of Y are called equivalent if the series $\sum_{n=1}^{\infty} a_n x_n$ converges iff $\sum_{n=1}^{\infty} a_n y_n$ converges. Suppose $\{x_n\}$ is a Schauder basis of X. A block basis of $\{x_n\}$ is a sequence of non-zero vectors of the form $u_k = \sum_{n=p_k+1}^{p_{k+1}} a_n x_n$ with $\{a_n\}$ scalars and $\{p_k\}$ an increasing sequence of integers. A sequence $\{y_n\} \subset L^p(\mu)$ is said to converge weakly to zero if $\langle y_n, x\rangle = \int y_n x d\mu \to 0$ for every x in the dual space L^q, $\frac{1}{p} + \frac{1}{q} = 1$ (equivalently, $L(y_n) \to 0$ for every bounded linear functional L on L^p).

1. Suppose $\{x_n\}$ is a normalized Schauder basis in X. Suppose $\{y_n\}$ is a basic sequence in X with $\sum_{n=1}^{\infty} \|x_n - y_n\|_X < \infty$. Show that $\{x_n\}$ and $\{y_n\}$ are equivalent. (You may assume $X = \ell^p$ and the sum is strictly less than 1 if you wish.)

2. Show that the usual basis $\{e_n\}$ in ℓ^p converges weakly to zero. Show that if $T : \ell^{p_1} \to Y \subset \ell^{p_2}$ is an isomorphism then the image sequence $\{y_n\} = \{T(e_n)\}$ converges weakly to zero in ℓ^{p_2}. Deduce the following: suppose that with respect to the standard basis $\{e_n\}$ we have the expansions $y_n = \sum_{k=1}^{\infty} a_{k,n} e_k$. Show that for any $N > 0$, $\lim_{n\to\infty} \| \sum_{k=1}^{N} a_{k,n} e_k \|_{p_2} = 0$.

3. Suppose $\{y_n\} \subset \ell^p$ is as in the previous problem. Show there is a **normalized** block basis $\{u_k\}$ of $\{e_n\}$ which is equivalent to some subsequence of the $\{y_n\}$.

4. Suppose $1 \leq p < \infty$. Suppose $\{u_n\}$ is a **normalized** block basis of the standard basis $\{e_n\}$. Then $\{e_n\}$ is equivalent to $\{u_n\}$.

5. If ℓ^p is isomorphic to any closed subspace Y of ℓ^q, prove the standard bases $\{e_n\}$ in ℓ^p and ℓ^q are equivalent. Show this is impossible unless $p = q$ (and thus ℓ^p cannot be isomorphic to a closed subsapce of ℓ^q unless $p = q$).