MAT 550, REAL ANALYSIS
SUNY at Stony Brook, SPRING 99
Problem Set 2
Due on Friday, Febuary 5

In the next three exercises, we give two measures μ and ν on $[0,1]$ and a function $f(x,y)$ defined on the product $[0,1]^2$. In each case, show that each of iterated integrals

$$\int [\int f(x,y)d\mu(x)]d\nu(y), \quad \int [\int f(x,y)d\nu(y)]d\mu(x),$$

is well defined (e.g., f_x and f^y are measurable), but not equal to the other, and explain why this does not contradict Fubini's theorem.

1. Let $\mu = \nu$ be Lebesgue measure on $[0,1]$. Choose a sequence $0 < \delta_1 < \delta_2 < \dots$ with $\delta_n \to 1$ and let g_n be a continuous function in $(\delta_n, \delta_{n=1})$ such that $\int_0^1 g_n dt = 1$. Define

$$f(x,y) = \sum_{n=1}^{\infty} [g_n(x) - g_{n+1}(x)] g_n(y).$$

   Note that for any (x,y) there is at most one non-zero term in the series, so convergence is obvious and f is continuous except at the point (1,1).

2. Let μ be Lebesgue measure on $[0,1]$ and let ν be counting measure. Define $f(x,y) = 1$ if $x = y$ and $f(x,y) = 0$ if $x \neq y$.

3. Let $\mu = \nu$ be Lebesgue measure on $[0,1]$. Let j be a mapping of $[0,1]$ into a well ordered set so that each $j(x)$ has only countably many predecessors (the existence oF such a map is the claim of the continuum hypothesis; conjectured by Cantor and proven independent by Cohen. For the sake of this problem we assume it is true. See the article by Penelope Maddy in J. Symbolic Logic, 53(1988) 481-511 for interesting arguments for and against the "truth" of the hypothesis). Taking this map as given, define $f(x,y) = 1$ if $j(x)$ preceeds $j(y)$ and $f(x,y) = 0$ otherwise.

   In class we defined a measure μ_p on the sequence space $X = \{0,1\}^{\mathbb{N}}$ by taking the product of the probability vector $(p, 1-p)$ on the two element space $\{0,1\}$. This measure pushes forward to a probability measure (which we also call μ_p) on $[0,1]$ which gives the dyadic interval I_σ of length 2^{-n} measure $p^k(1-p)^{n-k}$ where k is the number of zeros in n-long binary string σ parameterizing I. The following questions concern this measure μ_p on $[0,1]$. As in class, let $x_n(x) : [0,1] \to \{0,1\}$ denote the nth binary digit of x.

4. Apply the law of large numbers to $f_n(x) = x_n(x) - (1-p)$ to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_n(x) = (1-p),$$

   for almost every x with respect to the measure μ_p. Deduce that $\mu_p \perp \mu_q$ for $p \neq q$.

5. Let $F_p(x) = \mu_p([0,x])$. Show that F is a homeomorphism of $[0,1]$ to itself if $0 < p < 1$.

   The following problem is harder and is optional.

6. Let F_p be as above. Show that F_p has deriviative zero almost everywhere with respect to Lebesgue measure, i.e., for almost every $x \in [0,1]$,

$$\lim_{h \to 0} \frac{F_p(x+h) - F_p(x)}{h} = 0.$$

   On the other hand, there is an uncountable, dense set of points where this limit fails to exist. (The fact that $F_p' = 0$ almost everywhere will be much easier later in the course, after we do Chapter 12.)