Phys. 540 Statistical Mechanics

Problem Set 2 Due Tuesday. Feb. 26, 2008

1. The gamma function is defined as

$$\Gamma(z+1) = \int_0^\infty dt e^{-t} t^z \tag{1}$$

Demonstrate as $z \to \infty$ that

$$\Gamma(z+1) \sim z^{z+1/2} e^{-z} \sqrt{2}\pi \{1 + c_1/z + O(z^{-2})\} \tag{2}$$

and compute c_1 by use of the method of steepest descents.

2. The Bessel function $J_\nu(z)$ of order ν and argument z is defined as

$$J_\nu(z) = \int_C dw e^{z \sinh w - \nu w} \tag{3}$$

on a contour C which begins at $+\infty - i\pi$ and ends at $+\infty + i\pi$

For $z \to \infty$ and $\nu \to \infty$ with z/ν fixed

A. Find the points of steepest desents.

B. Compute the leading term in the asymptotic expansion in the three separate cases $z/\nu > 1,\ z/\nu = 1$ and $z/\nu < 1$.

C. Discuss why the expansions in the three cases have different forms. What needs to be done to obtain a uniform expansion which will interpolate between the three cases?

3. Demonstrate that the contour integral

$$I_1(t) = \frac{1}{2\pi i} \oint_C \frac{dz}{z} \frac{1+z}{1-z} [(1-tz)(1-tz^{-1}]^{-1/2} \tag{4}$$

where C it the contour $|z| = 1 - \epsilon$ is evaluated as

$$I_1(t) = \frac{1}{1-t} \tag{5}$$

4.EXTRA CREDIT

Evaluate by the same method as used in 3

$$I_2(t) = \frac{1}{2\pi i)^2} \oint_{c_1} dz_1 \oint_{c_2} dz_2 \frac{1 + z_1 z_2}{(1 - z_1 z_2)^3} \left[\frac{(1 - tz_2)(1 - tz_2^{-1})}{(1 - tz_1)(1 - tz_1^{-1})} \right]^{1/2} \tag{6}$$

where the contours c_1 and $_2$ are $|z_{1,2}| = 1 - \epsilon$

5.EXTRA EXTRA CREDIT

It has been found by means of MAPLE the the integral

$$f(t) = \frac{1}{(2\pi)^2} \oint_{c_1} dz_1 \oint_{c_2} dz_2 \frac{1}{(1 - z_1 z_2)^2} \left[\frac{(1 - tz_2)(1 - tz_2^{-1})}{(1 - tz_1)(1 - tz_1^{-1})} \right]^{1/2} \tag{7}$$

with c_1 and c_2 as given in problem 4 is evaluated as

$$f(t) = K(K - E)/2 \tag{8}$$

where

$$K = F(1/2, 1/2; 1; t^2) \qquad E = F(1/2, -1/2; 1; t^2) \tag{9}$$

with the hypergeometric function $F(a, b; c; z)$ defined as

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt t^{b-1} (1 - t)^{c-b-1} (1 - tz)^{-a} \tag{10}$$

Provide an analytic derivation of this result. If you find such a derivation it is publishable material because no such derivation is in the literature