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# MEASURE AND INTEGRAL

## An Introduction to Real Analysis

**Richard L. Wheeden**
Department of Mathematics
Rutgers, the State University
of New Jersey
New Brunswick, New Jersey

**Antoni Zygmund**
Department of Mathematics
University of Chicago
Chicago, Illinois

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11. For which $p > 0$ does $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

12. Give an example of a bounded continuous $f$ on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0,\infty)$ for any $p > 0$.

13. (a) Let $\{f_k\}$ be a sequence of measurable functions on $E$. Show that $\sum f_k$ converges absolutely a.e. in $E$ if $\sum \int_E |f_k| < +\infty$. [Use theorems (5.16) and (5.22).]

(b) If $\{r_k\}$ denotes the rational numbers in [0,1] and $\{a_k\}$ satisfies $\sum |a_k| < +\infty$, show that $\sum a_k |x - r_k|^{-1/2}$ converges absolutely a.e. in [0,1].

14. Prove the following result (which is obvious if $|E| < +\infty$), describing the behavior of $a^p\omega(a)$ as $a \to 0+$. If $f \in L^p(E)$, then $\lim_{a\to 0+} a^p\omega(a) = 0$. (If $f \geq 0$, $\varepsilon > 0$, choose $\delta > 0$ so that $\int_{\{f\leq\delta\}} f^p < \varepsilon$. Thus, $a^p[\omega(a) - \omega(\delta)] \leq \int_{\{a<f\leq\delta\}} f^p < \varepsilon$ for $0 < a < \delta$. Now let $a \to 0$.)

15. Suppose that $f$ is nonnegative and measurable on $E$ and that $\omega$ is finite on $(0,\infty)$. If $\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha$ is finite, show that $\lim_{a\to 0+} a^p\omega(a) = \lim_{b\to+\infty} b^p\omega(b) = 0$. (Consider $\int_{a/2}^a$ and $\int_{b/2}^b$.)

16. Suppose that $f$ is nonnegative and measurable on $E$ and that $\omega$ is finite on $(0,\infty)$. Show that (5.51) holds without any further restrictions [that is, $f$ need not be in $L^p(E)$ and $|E|$ need not be finite] if we interpret $\int_0^\infty \alpha^p\, d\omega(\alpha) = \lim_{\substack{a\to 0+\\ b\to+\infty}} \int_a^b$. [Use $E_{ab}$ to obtain the relation $\int_E f^p = -\int_0^\infty \alpha^p\, d\omega(\alpha)$. If either $\int_0^\infty \alpha^p\, d\omega(\alpha)$ or $\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha$ is finite, use (5.50) and the results of Exercises 14 or 15 to integrate by parts.]

17. If $f \geq 0$ and $\omega(\alpha) \leq c(1+\alpha)^{-p}$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

18. If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^{+\infty} 2^{kp}\omega(2^k) < +\infty$. (Use Exercise 16.)

19. Derive analogues of (5.52) and (5.54) for integrals over intervals in $\mathbf{R}^n$, $n > 1$.

20. Let $\mathbf{y} = T\mathbf{x}$ be a nonsingular linear transformation of $\mathbf{R}^n$. If $\int_E f(\mathbf{y})\, d\mathbf{y}$ exists, show that

$$\int_E f(\mathbf{y})\, d\mathbf{y} = |\det T| \int_{T^{-1}E} f(T\mathbf{x})\, d\mathbf{x}.$$

[The case when $f = \chi_{E_1}$, $E_1 \subset E$, follows from integrating the formula $\chi_{E_1}(T\mathbf{x}) = \chi_{T^{-1}E_1}(\mathbf{x})$ over $T^{-1}E$, and then applying (3.35).]

21. If $\int_A f = 0$ for every measurable subset $A$ of a measurable set $E$, show that $f = 0$ a.e. in $E$.

# Chapter 6

# Repeated Integratio[n]

Let $f(x,y)$ be defined in a rectangle

$$I = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}.$$

If $f$ is continuous, we have the classical formula

$$\iint_I f(x,y)\, dx\, dy = \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx,$$

and there is an analogous formula for functions of $n$ variables.

The first two sections of this chapter extend this and related results o[n] repeated integration to the case of Lebesgue integrable functions. The la[st] section contains some applications.

## 1. Fubini's Theorem

We shall use the following notation. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a point of a[n] $n$-dimensional interval $I_1$,

$$I_1 = \{\mathbf{x} = (x_1, \ldots, x_n) : a_i \leq x_i \leq b_i,\ i = 1, \ldots, n\},$$

and let $\mathbf{y}$ be a point of an $m$-dimensional interval $I_2$,

$$I_2 = \{\mathbf{y} = (y_1, \ldots, y_m) : c_j \leq y_j \leq d_j,\ j = 1, \ldots, m\}.$$

Here $I_1$ and $I_2$ may be all of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. The Cartesian produ[ct] $I = I_1 \times I_2$ is an $(n+m)$-dimensional interval consisting of points $(x_1, \ldots,$ $x_n, y_1, \ldots, y_m)$. We shall denote such points by $(\mathbf{x},\mathbf{y})$. A function $f(x_1, \ldots,$ $x_n, y_1, \ldots, y_m)$ defined in $I$ will be written $f(\mathbf{x},\mathbf{y})$, and its integral $\int_I f$ wi[ll] be denoted by $\iint_I f(\mathbf{x},\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}$.

**(6.1) Theorem (*Fubini's Theorem*)** *Let $f(\mathbf{x},\mathbf{y}) \in L(I)$, $I = I_1 \times I_2$. Then*

(i) *for almost every $\mathbf{x} \in I_1$, $f(\mathbf{x},\mathbf{y})$ is measurable and integrable on $I$[$_2$] as a function of $\mathbf{y}$;*

(ii) *as a function of* $\mathbf{x}$, $\int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}$ *is measurable and integrable on* $I_1$, *and*

$$\iint_I f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int_{I_1}\left[\int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x}.$$

Setting $f = 0$ outside $I$, we see that it is enough to prove the theorem when $I_1 = \mathbf{R}^n$, $I_2 = \mathbf{R}^m$, and $I = \mathbf{R}^{n+m}$. For simplicity, we then drop $I_1$, $I_2$, and $I$ from the notation and write $\int f(\mathbf{x},\mathbf{y})\,d\mathbf{x}$ for $\int_{I_1} f(\mathbf{x},\mathbf{y})\,d\mathbf{x}$, $L(d\mathbf{x})$ for $L(I_1)$, $L(d\mathbf{x}\,d\mathbf{y})$ for $L(I)$, etc.

We will prove the theorem by considering a series of special cases. The first two lemmas below will help in passing from one case to the next. In these lemmas, we say that a function $f$ in $L(d\mathbf{x}\,d\mathbf{y})$ for which Fubini's theorem is true has *property* $\mathscr{F}$.

**(6.2) Lemma** *A finite linear combination of functions with property* $\mathscr{F}$ *has property* $\mathscr{F}$.

This follows immediately from (4.9) and (5.28).

**(6.3) Lemma** *Let* $f_1, f_2, \ldots, f_k, \ldots$, *have property* $\mathscr{F}$. *If* $f_k \nearrow f$ *or* $f_k \searrow f$ *and* $f \in L(d\mathbf{x}\,d\mathbf{y})$, *then* $f$ *has property* $\mathscr{F}$.

*Proof.* Changing signs if necessary, we may assume that $f_k \nearrow f$. For each $k$, there exists by hypothesis a set $Z_k$ in $\mathbf{R}^n$ with measure zero such that $f_k(\mathbf{x},\mathbf{y}) \in L(d\mathbf{y})$ if $\mathbf{x} \notin Z_k$. Let $Z = \bigcup_k Z_k$, so that $Z$ has $\mathbf{R}^n$-measure zero. If $\mathbf{x} \notin Z$, then $f_k(\mathbf{x},\mathbf{y}) \in L(d\mathbf{y})$ for all $k$, and therefore, by the monotone convergence theorem applied to $\{f_k(\mathbf{x},\mathbf{y})\}$ as functions of $\mathbf{y}$,

$$h_k(\mathbf{x}) = \int f_k(\mathbf{x},\mathbf{y})\,d\mathbf{y} \nearrow h(\mathbf{x}) = \int f(\mathbf{x},\mathbf{y})\,d\mathbf{y} \qquad (\mathbf{x} \notin Z).$$

By assumption, $h_k(\mathbf{x}) \in L(d\mathbf{x})$, $f_k \in L(d\mathbf{x}\,d\mathbf{y})$ and $\int\int f_k(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int h_k(\mathbf{x})\,d\mathbf{x}$. Therefore, using the monotone convergence theorem again, we obtain that $\int\int f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int h(\mathbf{x})\,d\mathbf{x}$. Since $f \in L(d\mathbf{x}\,d\mathbf{y})$, it follows that $h \in L(d\mathbf{x})$, which implies that $h$ is finite a.e. This completes the proof.

The next three lemmas prove special cases of Fubini's theorem.

**(6.4) Lemma** *If* $E$ *is a set of type* $G_\delta$, *viz.*, $E = \bigcap_{k=1}^{\infty} G_k$, *and if* $G_1$ *has finite measure, then* $\chi_E$ *has property* $\mathscr{F}$.

*Proof. Case 1.* Suppose that $E$ is a bounded open interval in $\mathbf{R}^{n+m}$: $E = J_1 \times J_2$, where $J_1$ and $J_2$ are bounded open intervals in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Then $|E| = |J_1||J_2|$, where $|J_1|$ and $|J_2|$ denote the measures of $J_1$ and $J_2$ in $\mathbf{R}^n$ and $\mathbf{R}^m$. For every $\mathbf{x}$, $\chi_E(\mathbf{x},\mathbf{y})$ is measurable as a function of $\mathbf{y}$. If $h(\mathbf{x}) = \int \chi_E(\mathbf{x},\mathbf{y})\,d\mathbf{y}$, then $h(\mathbf{x}) = |J_2|$ for $\mathbf{x} \in J_1$, and $h(\mathbf{x}) = 0$ otherwise. Therefore, $\int h(\mathbf{x})\,d\mathbf{x} = |J_1||J_2|$. But also, $\int\int \chi_E(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = |E| = |J_1||J_2|$, and the lemma is proved in this case.



*Case 2.* Suppose that $E$ is any set (of type $G_\delta$ or not) on the boundary of an interval in $\mathbf{R}^{n+m}$. Then for almost every $\mathbf{x}$, the set $\{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in E\}$ has $\mathbf{R}^m$-measure zero. Therefore, if $h(\mathbf{x}) = \int \chi_E(\mathbf{x},\mathbf{y})\,d\mathbf{y}$, it follows that $h(\mathbf{x}) = 0$ a.e. Hence, $\int h(\mathbf{x})\,d\mathbf{x} = 0$. But also, $\int\int \chi_E(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = |E| = 0$.

*Case 3.* Suppose next that $E$ is a partly open interval in $\mathbf{R}^{n+m}$. Then $E$ is the union of its interior and a subset of its boundary. It follows from cases 1 and 2 and (6.2) that $\chi_E$ has property $\mathscr{F}$.

*Case 4.* Let $E$ be an open set in $\mathbf{R}^{n+m}$ with finite measure. Write $E = \bigcup I_j$, where the $I_j$ are disjoint, partly open intervals. If $E_k = \bigcup_{j=1}^{k} I_j$, then $\chi_{E_k} = \sum_{j=1}^{k} \chi_{I_j}$, so that $\chi_{E_k}$ has property $\mathscr{F}$ by case 3 and (6.2). Since $\chi_{E_k} \nearrow \chi_E$, $\chi_E$ has property $\mathscr{F}$ by (6.3).

*Case 5.* Let $E$ satisfy the hypothesis of (6.4). We may assume that $G_k \searrow$ by considering the open sets $G_1$, $G_1 \cap G_2$, $G_1 \cap G_2 \cap G_3$, etc. Then $\chi_{G_k} \searrow \chi_E$ and the lemma follows from case 4 and (6.3).

**(6.5) Lemma** *If* $Z$ *is a subset of* $\mathbf{R}^{n+m}$ *with measure zero, then* $\chi_Z$ *has property* $\mathscr{F}$. *Hence, for almost every* $\mathbf{x} \in \mathbf{R}^n$, *the set* $\{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in Z\}$ *has* $\mathbf{R}^m$*-measure zero.*

*Proof.* Using (3.8), select a set $H$ of type $G_\delta$ such that $Z \subset H$ and $|H| = 0$. If $H = \bigcap G_k$, we may assume that $G_1$ has finite measure, so that by (6.4),

$$\int\left[\int \chi_H(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x} = \iint \chi_H(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = 0.$$

Therefore, by (5.11), $|\{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in H\}| = \int \chi_H(\mathbf{x},\mathbf{y})\,d\mathbf{y} = 0$ for almost every $\mathbf{x}$. If $\{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in H\}$ has $\mathbf{R}^m$-measure zero, so does $\{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in Z\}$ since $Z \subset H$. It follows that for almost every $\mathbf{x}$, $\chi_Z(\mathbf{x},\mathbf{y})$ is measurable in $\mathbf{y}$ and $\int \chi_Z(\mathbf{x},\mathbf{y})\,d\mathbf{y} = 0$. Hence, $\int[\int \chi_Z(\mathbf{x},\mathbf{y})\,d\mathbf{y}]\,d\mathbf{x} = 0$, which proves the lemma since $\int\int \chi_Z(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = |Z| = 0$.

**(6.6) Lemma** *Let* $E \subset \mathbf{R}^{n+m}$. *If* $E$ *is measurable with finite measure, then* $\chi_E$ *has property* $\mathscr{F}$.

*Proof.* Using (3.28), write $E = H - Z$, where $H$ is of type $G_\delta$ and $Z$ has measure zero. If $H = \bigcap G_k$, choose $G_1$ with finite measure [see the proof of (3.28)]. Since $\chi_E = \chi_H - \chi_Z$, the lemma follows from (6.4), (6.5), and (6.2).

*Proof of Fubini's theorem.* We must show that every $f \in L(d\mathbf{x}\,d\mathbf{y})$ has property $\mathscr{F}$. Since $f = f^+ - f^-$, we may assume by (6.2) that $f \geq 0$. Then by (4.13), there are simple measurable $f_k \nearrow f$, $f_k \geq 0$. Each $f_k \in L(d\mathbf{x}\,d\mathbf{y})$, and by (6.3), it is enough to show that these have property $\mathscr{F}$. Hence, we may assume that $f$ is simple and integrable, say $f = \sum_{j=1}^{N} v_j \chi_{E_j}$. Since each $E_j$ must have finite measure, the result follows from (6.6) and (6.2).

If $f \in L(\mathbf{R}^{n+m})$, then by Fubini's theorem, $f(\mathbf{x},\mathbf{y})$ is a measurable function of $\mathbf{y}$ for almost every $\mathbf{x} \in \mathbf{R}^n$. We now show that the same conclusion holds if $f$ is merely measurable.

**(6.7) Theorem** *Let $f(\mathbf{x},\mathbf{y})$ be a measurable function on $\mathbf{R}^{n+m}$. Then for almost every $\mathbf{x} \in \mathbf{R}^n$, $f(\mathbf{x},\mathbf{y})$ is a measurable function of $\mathbf{y} \in \mathbf{R}^m$.*

*In particular, if $E$ is a measurable subset of $\mathbf{R}^{n+m}$, then the set*

$$E_{\mathbf{x}} = \{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in E\}$$

*is measurable in $\mathbf{R}^m$ for almost every $\mathbf{x} \in \mathbf{R}^n$.*

*Proof.* Note that if $f$ is the characteristic function $\chi_E$ of a measurable $E \subset \mathbf{R}^{n+m}$, then the two statements above are equivalent. To prove the result in this case, write $E = H \cup Z$, where $H$ is of type $F_\sigma$ in $\mathbf{R}^{n+m}$ and $|Z|_{n+m} = 0$. Then $E_{\mathbf{x}} = H_{\mathbf{x}} \cup Z_{\mathbf{x}}$, $H_{\mathbf{x}}$ is of type $F_\sigma$ in $\mathbf{R}^m$, and for almost every $\mathbf{x} \in \mathbf{R}^n$, $|Z_{\mathbf{x}}|_m = 0$ by (6.5). Therefore, $E_{\mathbf{x}}$ is measurable for almost every $\mathbf{x}$.

If $f$ is any measurable function on $\mathbf{R}^{n+m}$, consider the set $E(a) = \{(\mathbf{x},\mathbf{y}) : f(\mathbf{x},\mathbf{y}) > a\}$. Since $E(a)$ is measurable in $\mathbf{R}^{n+m}$, the set $E(a)_{\mathbf{x}} = \{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in E(a)\}$ is measurable in $\mathbf{R}^m$ for almost every $\mathbf{x} \in \mathbf{R}^n$. The exceptional set of $\mathbf{R}^n$-measure zero depends on $a$. The union $Z$ of these exceptional sets for all rational $a$ still has $\mathbf{R}^n$-measure zero. If $\mathbf{x} \notin Z$, then $\{y : f(\mathbf{x},\mathbf{y}) > a\}$ is measurable for all rational $a$, and so for all $a$ by (4.4). This completes the proof.

We will now extend Fubini's theorem to functions defined on subsets of $\mathbf{R}^{n+m}$.

**(6.8) Theorem** *Let $f(\mathbf{x},\mathbf{y})$ be a measurable function defined on a measurable subset $E$ of $\mathbf{R}^{n+m}$, and let $E_{\mathbf{x}} = \{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in E\}$.*

(i) *For almost every $\mathbf{x} \in \mathbf{R}^n$, $f(\mathbf{x},\mathbf{y})$ is a measurable function of $\mathbf{y}$ on $E_{\mathbf{x}}$.*
(ii) *If $f(\mathbf{x},\mathbf{y}) \in L(E)$, then for almost every $\mathbf{x} \in \mathbf{R}^n$, $f(\mathbf{x},\mathbf{y})$ is integrable on $E_{\mathbf{x}}$ with respect to $\mathbf{y}$; moreover, $\int_{E_{\mathbf{x}}} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}$ is an integrable function of $\mathbf{x}$ and*

$$\iint_E f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int_{\mathbf{R}^n}\left[\int_{E_{\mathbf{x}}} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x}.$$

*Proof.* Let $\bar{f}$ be the function equal to $f$ in $E$ and to zero elsewhere in $\mathbf{R}^{n+m}$. Since $f$ is measurable on $E$, $\bar{f}$ is measurable on $\mathbf{R}^{n+m}$. Therefore, by (6.7), $\bar{f}(\mathbf{x},\mathbf{y})$ is a measurable function of $\mathbf{y}$ for almost every $\mathbf{x} \in \mathbf{R}^n$. Since $E_{\mathbf{x}}$ is measurable for almost every $\mathbf{x} \in \mathbf{R}^n$, it follows that $f(\mathbf{x},\mathbf{y})$ is measurable on almost every $E_{\mathbf{x}}$. This proves (i).

If $f \in L(E)$, then $\bar{f} \in L(\mathbf{R}^{n+m})$ and

$$\iint_E f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \iint_{\mathbf{R}^{n+m}} \bar{f}(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int_{\mathbf{R}^n}\left[\int_{\mathbf{R}^m} \bar{f}(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x}.$$

Since $E_{\mathbf{x}}$ is measurable for almost every $\mathbf{x}$, we obtain by theorem (5.24) that

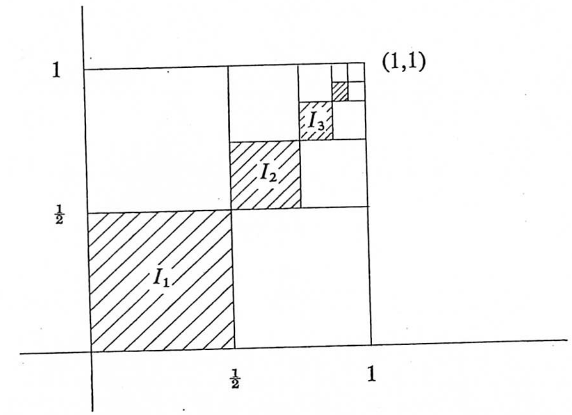


$\int_{\mathbf{R}^m} \bar{f}(\mathbf{x},\mathbf{y})\,d\mathbf{y} = \int_{E_{\mathbf{x}}} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}$ for almost every $\mathbf{x} \in \mathbf{R}^n$. Part (ii) follows by combining equalities.

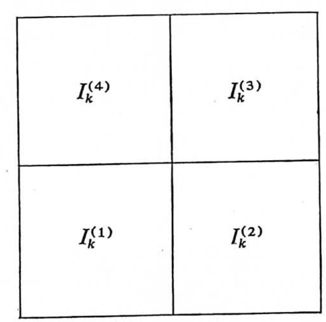
## 2. Tonelli's Theorem

By Fubini's theorem, the finiteness of a multiple integral implies that of the corresponding iterated integrals. The converse is not true, even if all the iterated integrals are equal, as shown by the following example.

**(6.9) Example** Let $n = m = 1$, let $I$ be the unit square and $\{I_k\}$ be the infinite sequence of subsquares shown below:




Subdivide each $I_k$ into four equal subsquares by lines parallel to the $x$ and axes.



$I_k$

For each $k$, let $f = 1/|I_k|$ on the interiors of $I_k^{(1)}$ and $I_k^{(3)}$, and let $f = -1/|$ on the interiors of $I_k^{(2)}$ and $I_k^{(4)}$. Let $f = 0$ on the rest of $I$, that is, outsi

$\bigcup I_k$ and on the boundaries of all the subsquares. Clearly, $\int_0^1 f(x,y)\,dy = 0$ for all $x$, and $\int_0^1 f(x,y)\,dx = 0$ for all $y$. Therefore,

$$\int_0^1\left[\int_0^1 f(x,y)\,dy\right]dx = \int_0^1\left[\int_0^1 f(x,y)\,dx\right]dy = 0.$$

However,

$$\iint_I |f(x,y)|\,dx\,dy = \sum_k \iint_{I_k} |f(x,y)|\,dx\,dy = \sum_k 1 = +\infty.$$

Hence, finiteness of the iterated integrals does not in general imply that of the multiple integral. However, for nonnegative $f$, we have the following basic result.

**(6.10) Theorem (*Tonelli's Theorem*)** *Let $f(\mathbf{x},\mathbf{y})$ be nonnegative and measurable on an interval $I = I_1 \times I_2$ of $\mathbf{R}^{n+m}$. Then, for almost every $\mathbf{x} \in I_1$, $f(\mathbf{x},\mathbf{y})$ is a measurable function of $\mathbf{y}$ on $I_2$. Moreover, as a function of $\mathbf{x}$, $\int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}$ is measurable on $I_1$, and*

$$\iint_I f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int_{I_1}\left[\int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x}.$$

*Proof.* This is actually a corollary of Fubini's theorem. For $k = 1, 2, \dots$, let $f_k(\mathbf{x},\mathbf{y}) = 0$ if $|(\mathbf{x},\mathbf{y})| > k$ and $f_k(\mathbf{x},\mathbf{y}) = \min\{k, f(\mathbf{x},\mathbf{y})\}$ if $|(\mathbf{x},\mathbf{y})| \le k$. Then $f_k \ge 0$, $f_k \nearrow f$ on $I$ and $f_k \in L(I)$ ($f_k$ is bounded and vanishes outside a compact set). Hence, Fubini's theorem applies to each $f_k$. The statement concerning the measurability of $\int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}$ then follows from its analogue for $f_k$; in fact, by the monotone convergence theorem, $\int_{I_2} f_k(\mathbf{x},\mathbf{y})\,d\mathbf{y} \nearrow \int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}$. [The measurability of $f(\mathbf{x},\mathbf{y})$ as a function of $\mathbf{y}$ was proved in (6.8).] By the monotone convergence theorem again,

(a) $$\iint_I f_k(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} \to \iint_I f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y},$$

(b) $$\int_{I_1}\left[\int_{I_2} f_k(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x} \to \int_{I_1}\left[\int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x}.$$

Since $f_k \in L$, the left-hand sides of (a) and (b) are equal. Therefore, so are the right-hand sides, and the theorem follows.

An extension of Tonelli's theorem to functions defined over arbitrary measurable sets $E$ is straightforward.

Since the roles of $\mathbf{x}$ and $\mathbf{y}$ can be interchanged above, it follows that if $f$ is nonnegative and measurable, then

$$\iint_I f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int_{I_1}\left[\int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x} = \int_{I_2}\left[\int_{I_1} f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\right]d\mathbf{y}.$$

In particular, we obtain the important fact that *for $f \ge 0$, the finiteness of*



*any one of Fubini's three integrals implies that of the other two*. Hence, for an measurable $f$, the finiteness of one of these integrals for $|f|$ implies that $f$ i integrable and that all three Fubini integrals of $f$ are equal.

An easy consequence of Tonelli's theorem is that the conclusion

$$\iint_I f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int_{I_1}\left[\int_{I_2} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x}$$

of Fubini's theorem holds for measurable $f$ even if $\iint_I f = \pm\infty$ (that is, holds if $\iint_I f$ merely *exists*). In fact, if $\iint_I f = +\infty$, we have $\iint_I f^+ = +$c and $f^- \in L(I)$. By Tonelli's theorem,

$$\iint_I f^+ = \int_{I_1}\left(\int_{I_2} f^+\,d\mathbf{y}\right)d\mathbf{x}, \qquad \iint_I f^- = \int_{I_1}\left(\int_{I_2} f^-\,d\mathbf{y}\right)d\mathbf{x}.$$

Since $\iint_I f^-$ is finite, the desired formula follows by subtraction.

## 3. Applications of Fubini's Theorem

We shall derive several important results as corollaries of Fubini's ar Tonelli's theorems. The first one is the necessity of the condition in theore (5.1). Using the notation of Chapter 5, we will prove the following result.

**(6.11) Theorem** *Let $f$ be a nonnegative function defined on a measurable s $E \subset \mathbf{R}^n$. If $R(f,E)$, the region under $f$ over $E$, is a measurable subset $\mathbf{R}^{n+1}$, then $f$ is measurable.*

*Proof.* For $0 \le y < +\infty$, we have

$$\{\mathbf{x} \in E : f(\mathbf{x}) \ge y\} = \{\mathbf{x} : (\mathbf{x},y) \in R(f,E)\}.$$

Since $R(f,E)$ is measurable, it follows from (6.7) that $\{\mathbf{x} \in E : f(\mathbf{x}) \ge y\}$ measurable (in $\mathbf{R}^n$) for almost all (linear measure) such $y$. In particul $\{\mathbf{x} \in E : f(\mathbf{x}) \ge y\}$ is measurable for all $y$ in a dense subset of $(0,\infty)$. If $y$ negative, then $\{\mathbf{x} \in E : f(\mathbf{x}) \ge y\} = E$, which is measurable. We conclu that $f$ is measurable [cf. (4.4)].

As a second application of Fubini's theorem, we will prove a result abc the convolution of two functions. More general results of this kind will proved in Chapter 9. If $f$ and $g$ are measurable in $\mathbf{R}^n$, their *convoluti* $(f * g)(\mathbf{x})$ is defined by

$$(f * g)(\mathbf{x}) = \int_{\mathbf{R}^n} f(\mathbf{x} - \mathbf{t})g(\mathbf{t})\,d\mathbf{t},$$

provided the integral exists.

We first claim that $f * g = g * f$, that is, that

**(6.12)** $$\int_{\mathbf{R}^n} f(\mathbf{x} - \mathbf{t})g(\mathbf{t})\,d\mathbf{t} = \int_{\mathbf{R}^n} f(\mathbf{t})g(\mathbf{x} - \mathbf{t})\,d\mathbf{t}.$$

This is actually a special case of results dealing with changes of variable. In this simple case, however, it amounts to the statement that if $\mathbf{x} \in \mathbf{R}^n$, then

$$\int_{\mathbf{R}^n} r(\mathbf{t})\, d\mathbf{t} = \int_{\mathbf{R}^n} r(\mathbf{x} - \mathbf{t})\, d\mathbf{t} \tag{6.13}$$

when $r(\mathbf{t}) = f(\mathbf{x} - \mathbf{t})g(\mathbf{t})$. For fixed $\mathbf{x}$, $\mathbf{x} - \mathbf{t}$ ranges over $\mathbf{R}^n$ as $\mathbf{t}$ does. Therefore, for any measurable $r \geq 0$, (6.13) follows from the geometric interpretation of the integral. [See (5.1) and the definition of the integral of a nonnegative function.] For any measurable $r$, it follows by writing $r = r^+ - r^-$. (For the effect of a *linear* change of variables, see Exercise 20 of Chapter 5.)

The result we wish to prove for convolutions is the following.

**(6.14) Theorem** *If $f \in L(\mathbf{R}^n)$ and $g \in L(\mathbf{R}^n)$, then $(f * g)(\mathbf{x})$ exists for almost every $\mathbf{x} \in \mathbf{R}^n$. Moreover, $f * g \in L(\mathbf{R}^n)$ and*

$$\int_{\mathbf{R}^n} |f * g|\, d\mathbf{x} \leq \left(\int_{\mathbf{R}^n} |f|\, d\mathbf{x}\right)\left(\int_{\mathbf{R}^n} |g|\, d\mathbf{x}\right).$$

In order to prove this, we need a lemma.

**(6.15) Lemma** *If $f(\mathbf{x})$ is measurable in $\mathbf{R}^n$, then the function $F(\mathbf{x},\mathbf{t}) = f(\mathbf{x} - \mathbf{t})$ is measurable in $\mathbf{R}^n \times \mathbf{R}^n = \mathbf{R}^{2n}$.*

*Proof.* Let $F_1(\mathbf{x},\mathbf{t}) = f(\mathbf{x})$. Since $f$ is measurable, it follows from (5.2) that $F_1(\mathbf{x},\mathbf{t})$ is measurable in $\mathbf{R}^{2n}$: in fact, the set $\{(\mathbf{x},\mathbf{t}) : F_1(\mathbf{x},\mathbf{t}) > a\}$, which equals $\{(\mathbf{x},\mathbf{t}) : f(\mathbf{x}) > a,\ \mathbf{t} \in \mathbf{R}^n\}$, is a cylinder set with measurable base $\{\mathbf{x} : f(\mathbf{x}) > a\}$ in $\mathbf{R}^n$. For $(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbf{R}^{2n}$, consider the transformation $\mathbf{x} = \boldsymbol{\xi} - \boldsymbol{\eta}$, $\mathbf{t} = \boldsymbol{\xi} + \boldsymbol{\eta}$. This is a nonsingular linear transformation of $\mathbf{R}^{2n}$, and therefore, by (3.33) (see Exercise 4 of Chapter 4), the function $F_2$ defined by $F_2(\boldsymbol{\xi},\boldsymbol{\eta}) = F_1(\boldsymbol{\xi} - \boldsymbol{\eta}, \boldsymbol{\xi} + \boldsymbol{\eta})$ is measurable in $\mathbf{R}^{2n}$. Since $F_2(\boldsymbol{\xi},\boldsymbol{\eta}) = f(\boldsymbol{\xi} - \boldsymbol{\eta})$, the lemma follows.

*Proof of Theorem (6.14).* Suppose first that both $f$ and $g$ are nonnegative on $\mathbf{R}^n$. By (6.15), $f(\mathbf{x} - \mathbf{t})g(\mathbf{t})$ is measurable on $\mathbf{R}^n \times \mathbf{R}^n$ since it is the product of two such functions. Hence, the integral

$$I = \iint f(\mathbf{x} - \mathbf{t})g(\mathbf{t})\, d\mathbf{t}\, d\mathbf{x}$$

is well-defined. By Tonelli's theorem and (6.13),

$$I = \int\left[\int f(\mathbf{x} - \mathbf{t})g(\mathbf{t})\, d\mathbf{t}\right] d\mathbf{x}$$

$$= \int g(\mathbf{t})\left[\int f(\mathbf{x} - \mathbf{t})\, d\mathbf{x}\right] d\mathbf{t} = \left[\int f(\mathbf{x})\, d\mathbf{x}\right]\left[\int g(\mathbf{t})\, d\mathbf{t}\right].$$

The first of these equations can be written $I = \int (f * g)(\mathbf{x})\, d\mathbf{x}$, so that

$$\int (f * g)(\mathbf{x})\, d\mathbf{x} = \left[\int f(\mathbf{x})\, d\mathbf{x}\right]\left[\int g(\mathbf{x})\, d\mathbf{x}\right].$$



This proves the theorem for $f \geq 0$ and $g \geq 0$. The general case can be reduced to the nonnegative case by observing that $|f * g| \leq |f| * |g|$. Hence,

$$\int |f * g|\, d\mathbf{x} \leq \int (|f| * |g|)\, d\mathbf{x} = \left(\int |f|\, d\mathbf{x}\right)\left(\int |g|\, d\mathbf{x}\right).$$

In the proof above, we showed the following useful fact.

**(6.16) Corollary** *If $f$ and $g$ are nonnegative and measurable on $\mathbf{R}^n$, then*

$$\int_{\mathbf{R}^n} (f * g)\, d\mathbf{x} = \left(\int_{\mathbf{R}^n} f\, d\mathbf{x}\right)\left(\int_{\mathbf{R}^n} g\, d\mathbf{x}\right).$$

Our last application of Fubini's theorem is an important theorem o Marcinkiewicz concerning the structure of closed sets. For simplicity, w will restrict our attention to the one-dimensional case. Given a closed subse $F$ of $\mathbf{R}^1$ and a point $x$, let

$$\delta(x) = \delta(x,F) = \inf\{|x - y| : y \in F\}$$

denote the *distance of $x$ from $F$*. Thus, $\delta(x) = 0$ if and only if $x \in F$. By (1.10) the complement of $F$ is a union $\bigcup_k (a_k,b_k)$ of disjoint open intervals. At mos two of these intervals can be infinite. The graph of $\delta(x)$ is thus an irregula sawtooth curve: over any finite interval $(a_k,b_k)$, the graph is the sides of th isosceles triangle with base $(a_k,b_k)$ and altitude $\frac{1}{2}(b_k - a_k)$; outside the termina points of $F$ the graph is linear. If we move from a point $x$ to a point $y$, th distance from $F$ cannot increase by more than $|x - y|$. Hence,

$$|\delta(x) - \delta(y)| \leq |x - y|;$$

that is, $\delta$ satisfies a Lipschitz condition.

We shall prove the following theorem. [See also Exercises 7–9, an theorem (9.19).]

**(6.17) Theorem (*Marcinkiewicz*)** *Let $F$ be a closed subset of a bound open interval $(a,b)$, and let $\delta(x) = \delta(x;F)$ be the corresponding distan function. Then, given $\lambda > 0$, the integral*

$$M_\lambda(x) = M_\lambda(x;F) = \int_a^b \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}}\, dy$$

*is finite a.e. in $F$. Moreover, $M_\lambda \in L(F)$ and*

$$\int_F M_\lambda \leq 2\lambda^{-1}|G|,$$

*where $G = (a,b) - F$.*

Before the proof, we note that what makes the finiteness of $M_\lambda(x)$ r markable is the singular behavior of $\delta^\lambda(y)/|x - y|^{1+\lambda}$ as $y \to x$. Since $\delta(y)$

$\delta(x)$ as $y \to x$, it follows that $M_\lambda(x) = +\infty$ if $x \notin F$ (see Exercise 9). If $x \in F$, then $\delta(x) = 0$, but the mere Lipschitz character of $\delta$ in the estimate

$$\frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} = \frac{[\delta(y)-\delta(x)]^\lambda}{|x-y|^{1+\lambda}} \le \frac{|x-y|^\lambda}{|x-y|^{1+\lambda}} = \frac{1}{|x-y|}$$

is not enough to imply that $M_\lambda(x)$ is finite since $\int_a^b dy/|x-y| = +\infty$. The key to the convergence of $M_\lambda$ at a point $x$ is the fact that $\delta(y)$ vanishes not only at $x$ but also at every $y \in F$. Thus, roughly speaking, the finiteness of $M_\lambda(x)$ means that $F$ is "very dense" near $x$.

*Proof.* Since $\delta = 0$ in $F$, integration in the integral defining $M_\lambda$ can be restricted to the set $G = (a,b) - F$ without changing $M_\lambda$. Thus,

$$\int_F M_\lambda(x)\,dx = \int_G \delta^\lambda(y)\left(\int_F \frac{dx}{|x-y|^{1+\lambda}}\right) dy,$$

the change in the order of integration being justified by Tonelli's theorem. To estimate the inner integral, fix $y \in G$ and note that for any $x \in F$, we have $|x-y| \ge \delta(y) > 0$. Thus,

$$\int_F \frac{dx}{|x-y|^{1+\lambda}} \le \int_{|x-y|\ge\delta(y)} \frac{dx}{|x-y|^{1+\lambda}} = 2\int_{\delta(y)}^\infty \frac{dt}{t^{1+\lambda}} = 2\lambda^{-1}\delta(y)^{-\lambda}.$$

In particular,

$$\int_F M_\lambda(x)\,dx \le \int_G \delta^\lambda(y)[2\lambda^{-1}\delta(y)^{-\lambda}]\,dy = 2\lambda^{-1}|G| < +\infty.$$

## Exercises

1. (a) Let $E$ be a measurable subset of $\mathbf{R}^2$ such that for almost every $x \in \mathbf{R}^1$, $\{y : (x,y) \in E\}$ has $\mathbf{R}^1$-measure zero. Show that $E$ has measure zero, and that for almost every $y \in \mathbf{R}^1$, $\{x : (x,y) \in E\}$ has measure zero.
   (b) Let $f(x,y)$ be nonnegative and measurable in $\mathbf{R}^2$. Suppose that for almost every $x \in \mathbf{R}^1$, $f(x,y)$ is finite for almost every $y$. Show that for almost every $y \in \mathbf{R}^1$, $f(x,y)$ is finite for almost every $x$.
2. If $f$ and $g$ are measurable in $\mathbf{R}^n$, show that the function $h(\mathbf{x},\mathbf{y}) = f(\mathbf{x})g(\mathbf{y})$ is measurable in $\mathbf{R}^n \times \mathbf{R}^n$. Deduce that if $E_1$ and $E_2$ are measurable subsets of $\mathbf{R}^n$, then their Cartesian product $E_1 \times E_2 = \{(\mathbf{x},\mathbf{y}) : \mathbf{x} \in E_1, \mathbf{y} \in E_2\}$ is measurable in $\mathbf{R}^n \times \mathbf{R}^n$, and $|E_1 \times E_2| = |E_1||E_2|$.
3. Let $f$ be measurable on $(0,1)$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L(0,1)$.
4. Let $f$ be measurable and periodic with period $1: f(t+1) = f(t)$. Suppose that there is a finite $c$ such that

$$\int_0^1 |f(a+t) - f(b+t)|\,dt \le c$$



for all $a$ and $b$. Show that $f \in L(0,1)$. (Set $a = x$, $b = -x$, integrate with respe to $x$, and make the change of variables $\xi = x + t$, $\eta = -x + t$.)

5. (a) If $f$ is nonnegative and measurable on $E$ and $\omega(y) = |\{\mathbf{x} \in E : f(\mathbf{x}) > y\}$ $y > 0$, use Tonelli's theorem to prove that $\int_E f = \int_0^\infty \omega(y)\,dy$. [By defin tion of the integral, we have $\int_E f = |R(f,E)| = \iint_{R(f,E)} d\mathbf{x}\,dy$. Use tl observation in the proof of (6.11) that $\{\mathbf{x} \in E : f(\mathbf{x}) \ge y\} = \{\mathbf{x} : (\mathbf{x},y)$ $R(f,E)\}$, and recall that $\omega(y) = |\{\mathbf{x} \in E : f(\mathbf{x}) \ge y\}|$ unless $y$ is a point discontinuity of $\omega$.]
   (b) Deduce from this special case the general formula

$$\int_E f^p = p\int_0^\infty y^{p-1}\omega(y)\,dy \qquad (f \ge 0,\ 0 < p < \infty).$$

6. For $f \in L(\mathbf{R}^1)$, define the *Fourier transform* $\hat{f}$ of $f$ by

$$\hat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{-ixt}\,dt \qquad (x \in \mathbf{R}^1).$$

(For a complex-valued function $F = F_0 + iF_1$ whose real and imagina parts $F_0$ and $F_1$ are integrable, we define $\int F = \int F_0 + i\int F_1$.) Show that $f$ and $g$ belong to $L(\mathbf{R}^1)$, then

$$(f*g)^\wedge(x) = \hat{f}(x)\hat{g}(x).$$

7. Let $F$ be a closed subset of $\mathbf{R}^1$ and let $\delta(x) = \delta(x,F)$ be the correspondi distance function. If $\lambda > 0$ and $f$ is nonnegative and integrable over the comp ment of $F$, prove that the function

$$\int_{\mathbf{R}^1} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}}\,dy$$

is integrable over $F$, and so is finite a.e. in $F$. [In case $f = \chi_{(a,b)}$, this redu to (6.17).]

8. Under the hypotheses of (6.17) and assuming that $b - a < 1$, prove that t function

$$M_0(x) = \int_a^b [\log 1/\delta(y)]^{-1}|x-y|^{-1}\,dy$$

is finite a.e. in $F$.

9. Show that $M_\lambda(x;F) = +\infty$ if $x \notin F$, $\lambda > 0$.
10. Let $v_n$ be the volume of the unit ball in $\mathbf{R}^n$. Show by using Fubini's theorem tl

$$v_n = 2v_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}\,dt.$$

(We also observe that the integral can be expressed in terms of the $\Gamma$-functi $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$, $s > 0$.]

11. Use Fubini's theorem to prove that

$$\int_{\mathbf{R}^n} e^{-|\mathbf{x}|^2}\,d\mathbf{x} = \pi^{n/2}.$$

[For $n = 1$, write $(\int_{-\infty}^{+\infty} e^{-x^2}\,dx)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy$ and use pc coordinates. For $n > 1$, use the formula $e^{-|\mathbf{x}|^2} = e^{-x_1^2}\cdots e^{-x_n^2}$ and Fubi theorem to reduce to the case $n = 1$.]