A Comparison Theorem for Gromov-Witten Invariants in the Symplectic Category

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Abstract

We exploit the geometric approach to the virtual fundamental class, due to Fukaya-Ono and Li-Tian, to compare Gromov-Witten invariants of a symplectic manifold and a symplectic submanifold whenever all constrained stable maps to the former are contained in the latter to first order. Various special cases of the comparison theorem in this paper have long been used in the algebraic category; some of them have also appeared in the symplectic setting. Combined with the inherent flexibility of the symplectic category, the main theorem leads to a confirmation of Pandharipande's Gopakumar-Vafa prediction for GW-invariants of Fano classes in 6-dimensional symplectic manifolds. The proof of the main theorem uses deformations of the Cauchy-Riemann equation that respect the submanifold and Carleman Similarity Principle for solutions of perturbed Cauchy-Riemann equations. In a forthcoming paper, we apply a similar approach to relative Gromov-Witten invariants and the absolute/relative correspondence in genus 0.

Contents

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1 Introduction

Gromov-Witten invariants are certain counts of pseudo-holomorphic curves in symplectic manifolds that play prominent roles in symplectic topology, algebraic geometry, and string theory. These are usually rational numbers, and their precise relations with some sort of integer enumerative counts of curves are rarely clear. However, it is well-known that genus 0 GW-invariants of Fano manifolds are precisely counts of rational curves; this observation is key to enumerating rational curves in projective space in [15, Section 5] and [30, Section 10]. String theory predicts an amazing integral structure for GW-invariants of Calabi-Yau threefolds. These predictions originate in [2], [7], and [8] and are extended to all threefolds in [27].

GW-invariants of a symplectic manifold X are obtained by evaluating natural cohomology classes on the virtual fundamental class (VFC) of the space of stable J-holomorphic maps to X. The main statement of this paper, Theorem 1.2, compares GW-invariants counting stable maps meeting specified constraints in the ambient manifold with analogous counts of such maps to a submanifold containing the images of all such constrained maps to first order. In light of Theorem 1.2, [18] immediately yields Corollary 1.4, concerning GW-invariants of Kahler surfaces. With a bit more work, Theorem 1.2 leads to Theorem 1.5, which confirms the "Fano case" of the Gopakumar-Vafa prediction of [27, Section 0.2]. Theorem 1.2 is obtained by deforming the Cauchy-Riemann equation in two stages so that the first stage respects the submanifold. Carleman Similarity Principle is used to take advantage of properties of solutions of Cauchy-Riemann equations that are preserved by a large class of perturbations of the equations. In a forthcoming paper [39], we will apply similar geometric principles to study relative GW-invariants and the absolute/relative correspondence in genus 0 with applications to birational geometry in the spirit of Hu-Li-Ruan ([9], [10], [19]) and McDuff ([23]).

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1.1 A comparison theorem for GW-invariants

We will denote by $\bar{\mathbb{Z}}^+$ the set of non-negative integers. Let (X,ω) be a compact symplectic manifold. If $g\in\bar{\mathbb{Z}}^+$, S is a finite set, $\beta\in H_2(X;\mathbb{Z})$, and J is an ω-tame[1] almost complex structure on X, denote by $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ the moduli space of equivalence classes of stable S-marked genus g degree β J-holomorphic maps to X. For each $j\in S$, there is a well-defined evaluation map

$$\mathrm{ev}_j : \overline{\mathfrak{M}}_{g,S}(X,\beta;J) \longrightarrow X. \tag{1.1}$$

As standard in GW-theory, we will denote by

$$\psi_j \in H^2\big(\overline{\mathfrak{M}}_{g,S}(X,\beta;J)\big)$$

the first chern class of the universal cotangent line bundle for the j-th marked point. The space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ carries a natural VFC, which is independent of J and will be denoted by

[1] an almost complex structure on (X,ω) is ω-tame if $\omega(v,Jv)>0$ for all $v\in TX$ with $v\neq 0$

$[\overline{\mathfrak{M}}_{g,S}(X,\beta)]^{vir}$. If the (real) dimension of X is $2n$, then

$$\dim\big[\overline{\mathfrak{M}}_{g,S}(X,\beta)\big]^{vir} = \dim_{g,S}(X,\beta) \equiv 2\big(\langle c_1(TX),\beta\rangle + (n-3)(1-g) + |S|\big). \tag{1.2}$$

If J is regular[2], then $\overline{\mathfrak{M}}_{0,S}(X,\beta;J)$ is a topological manifold with a preferred choice of orientation and

$$\big[\overline{\mathfrak{M}}_{0,S}(X,\beta)\big]^{vir} = \big[\overline{\mathfrak{M}}_{0,S}(X,\beta;J)\big].$$

If $a_j \in \bar{\mathbb{Z}}^+$ and $\kappa_j \in H_*(X;\mathbb{Z})$ for each $j \in S$, let

$$\big((\tau_{a_j}\kappa_j)_{j\in S}\big)^X_{g,\beta} \equiv \Big\langle \prod_{j\in S}\big(\psi_j^{a_j}\mathrm{ev}_j^*(\mathrm{PD}_X\kappa_j)\big), \big[\overline{\mathfrak{M}}_{g,S}(X,\beta)\big]^{vir}\Big\rangle, \tag{1.3}$$

where $\mathrm{PD}_X\kappa_j \in H^*(X;\mathbb{Z})$ is the Poincare dual of κ_j in X.[3] In order to avoid any sign ambiguities, we define the number in (1.3) to be 0 if the dimension of κ_j is odd for some j. By (1.2), this number is zero unless

$$\sum_{j\in S}(2a_j + 2n - \dim\kappa_j) = \dim_{g,S}(X,\beta). \tag{1.4}$$

The number (1.3) can be expressed as an integral on a "smaller" moduli space as follows. Choose cobordism representatives $f_j\colon M_j \longrightarrow X$ for κ_j, with $j\in S$.[4] Let

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J) = \Big\{\big([u],(w_j)_{j\in S}\big) \in \overline{\mathfrak{M}}_{g,S}\big(X,\beta;J\big)\times\prod_{j\in S}M_j\colon \mathrm{ev}_j([u])=f_j(w_j)\ \forall\, j\in S\Big\}. \tag{1.5}$$

The space $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ of constrained stable maps also carries a virtual fundamental class and

$$\big((\tau_{a_j}\kappa_j)_{j\in S}\big)^X_{g,\beta} = \Big\langle \prod_{j\in S}\psi_j^{a_j}, \big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)\big]^{vir}\Big\rangle.$$

The subject of this section is a reduction of this GW-invariant of X to a combination of GW-invariants for its submanifolds.

Definition 1.1 *Let Y be a submanifold of X. A smooth map $f\colon M \longrightarrow X$ intersects Y* properly *if $f^{-1}(Y)\subset M$ is a smooth orientable even-dimensional submanifold of M and*

$$d_wf\big(T_w\big(f^{-1}(Y)\big)\big) = d_w(TM)\cap T_{f(w)}Y$$

for every $w\in f^{-1}(Y)$.

[2] an almost complex structure J is genus 0 regular if for every J-holomorphic map $u\colon\Sigma\longrightarrow X$, where Σ is a tree of Riemann spheres, the linearization $D_{J;u}$ of the $\bar\partial_J$-operator at u is surjective

[3] In the descriptions of Sections 3.3 and 3.4, $\big[\overline{\mathfrak{M}}_{g,S}(X,\beta)\big]^{vir}$ is a homology class in an arbitrarily small neighborhood of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ in the space of equivalence classes of L^p_1-maps to X; there are well-defined evaluation maps ev_j and cohomology classes ψ_j on this space as well.

[4] We can assume that this is possible, since each κ_j can be replaced by a multiple for our purposes.

If $f: M \longrightarrow X$ intersects $Y \subset X$ transversally and M, X, and Y are orientable of even total dimension, then f intersects Y properly. However, a proper intersection need not be transverse. For example, any two real lines in $\mathbb{R}^n$ intersect properly, but not transversally if $n \geq 3$. Two curves that are tangent to each other do not intersect properly.

If $f: M \longrightarrow X$ intersects $Y \subset X$ properly and $NY \longrightarrow Y$ is the normal bundle of Y in X, the homomorphisms

$$d_w^{NY} f : T_w M \longrightarrow N_{f(w)} Y, \quad v \longrightarrow d_w f(v) + T_{f(w)} Y, \qquad w \in f^{-1}(Y),$$

have constant rank; the kernel of $d_w^{NY} f$ is $T_w(f^{-1}(Y))$. If M, X, and Y are oriented, an orientation on $f^{-1}(Y)$ then induces an orientation on the vector bundle

$$N^f Y \equiv f^* NY / (\operatorname{Im} d^{NY} f) \longrightarrow f^{-1}(Y).$$

Note that

$$\operatorname{rk} N^f Y = \big(\dim X - \dim M\big) - \big(\dim Y - \dim f^{-1}(Y)\big). \tag{1.6}$$

Let Y be a compact symplectic submanifold of X and

$$\iota_{Y*} : H_*(Y; \mathbb{Z}) \longrightarrow H_*(X; \mathbb{Z})$$

the homomorphism induced by the inclusion $\iota_Y : Y \longrightarrow X$. If $\beta_Y \in H_2(Y; \mathbb{Z})$ and J is an ω-tame almost complex structure on X which preserves $TY \subset TX|_Y$, then ι_Y induces an embedding

$$\overline{\mathfrak{M}}_{g,S}(Y, \beta_Y; J) \hookrightarrow \overline{\mathfrak{M}}_{g,S}(X, \iota_{Y*}\beta_Y; J).$$

If $f_j : M_j \longrightarrow X$, $j \in S$, are smooth maps as above, let

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y, \beta_Y; J) = \big\{ \big([u], (w_j)_{j \in S}\big) \in \overline{\mathfrak{M}}_{g,\mathbf{f}}(X, \iota_{Y*}\beta_Y; J) : \ [u] \in \overline{\mathfrak{M}}_{g,S}(Y, \beta_Y; J) \big\}.$$

If in addition $u : \Sigma_u \longrightarrow Y$ is a J-holomorphic map from a nodal Riemann surface (see Section 2.1), let $\mathcal{H}_u$ denote the space of deformations of the complex structure on Σ_u. The linearization of the $\bar{\partial}_J$-operator for maps to X,

$$D_{J;u}^X : \mathcal{H}_u \oplus L_1^p(\Sigma_u; u^* TX) \longrightarrow L^p(\Sigma_u; T^* \Sigma_u^{0,1} \otimes_{\mathbb{C}} u^* TX), \quad p > 2,$$

induces a generalized Cauchy-Riemann operator

$$D_{J;u}^{NY} : L_1^p(\Sigma_u; u^* NY) \longrightarrow L^p(\Sigma_u; T^* \Sigma_u^{0,1} \otimes_{\mathbb{C}} u^* NY).$$

For each $j \in S$, define

$$\widetilde{\operatorname{ev}}_j : \ker D_{J;u}^{NY} \longrightarrow N_{\operatorname{ev}_1(u)} Y \qquad \text{by} \quad \xi \longrightarrow \xi(z_j(u)) + T_{\operatorname{ev}_1(u)} Y,$$

where $z_j(u) \in \Sigma_u$ is the j-th marked point; this homomorphism is the composition of the differential of the evaluation map (1.1) with the projection to the normal bundle.

Theorem 1.2 *Suppose (X,ω) is a compact symplectic $2n$-manifold, $g\in\bar{\mathbb{Z}}^+$, S is a finite set, $\beta\in H_2(X;\mathbb{Z})$, $a_j\in\bar{\mathbb{Z}}^+$ for each $j\in S$, and $f_j: M_j\longrightarrow X$ is a cobordism representative for $\kappa_j\in H_*(X;\mathbb{Z})$ for each $j\in S$. If J is an ω-tame almost complex structure on X, Y is a compact almost complex submanifold of (X,J), and $\beta_Y\in H_2(Y;\mathbb{Z})$ are such that*

(a) $\iota_{Y*}(\beta_Y)=\beta$ and f_j intersects Y properly for each $j\in S$;

(b) for every $([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$, the homomorphism

$$\ker(D_{J;u}^{NY})\longrightarrow\bigoplus_{j\in S}N_{w_j}^{f_j}Y,\qquad \xi\longrightarrow\big(\widetilde{\mathrm{ev}}_j(\xi)+(\mathrm{Im}\,d_{w_j}^{NY}f_j)\big)_{j\in S}\,,\tag{1.7}$$

is an isomorphism,

then

(1) the space $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ carries a natural VFC (dependent on the orientations of $f_j^{-1}(Y)$) with

$$\begin{split}\dim\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big]^{vir}=\dim_{g,S}(X,\beta)-\sum_{j\in S}\big(2n-\dim\kappa_j\big)\\+\sum_{j\in S}\mathrm{rk}\,N^{f_j}Y-2\big(\langle c_1(NY),\beta_Y\rangle+\mathrm{rk}_{\mathbb{C}}NY\cdot(1-g)\big);\end{split}\tag{1.8}$$

(2) the vector spaces $\mathrm{cok}(D_{J;u}^{NY})$ form a natural oriented vector orbi-bundle

$$\mathrm{cok}\big(D_J^{NY}\big)\longrightarrow\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$$

with

$$\mathrm{rk}_{\mathbb{R}}\mathrm{cok}(D_{J;u}^{NY})=\sum_{j\in S}\mathrm{rk}\,N^{f_j}Y-2\big(\langle c_1(NY),\beta_Y\rangle+\mathrm{rk}_{\mathbb{C}}NY\cdot(1-g)\big);\tag{1.9}$$

(3) $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ is a union of connected components of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ and its contribution to the number (1.3) is given by

$$\mathbf{C}_{g,\mathbf{f}}(Y,\beta_Y)=\Big\langle e\big(\mathrm{cok}(D_J^{NY})\big)\prod_{j\in S}\psi_j^{a_j},\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big]^{vir}\Big\rangle.\tag{1.10}$$

Example A Suppose (X,J) is a Calabi-Yau 3-fold and $Y\subset X$ is a smooth isolated rational curve with $NY\approx\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. We can then apply Theorem 1.2 with $S=\emptyset$, $g=0$, and $\beta=d\iota_{Y*}([Y])$ for any $d\in\mathbb{Z}^+$. The assumption on the normal bundle implies that $\ker(D_{J;u}^{NY})$ is trivial and thus Condition (b) is satisfied. The right-hand side of (1.10) is then the famous multiple-cover contribution of $1/d^3$ ([2], [26, Section 27.5], [36]).

Example B If the image of each map f_j in Theorem 1.2 lies in Y, the second part of Condition (a) is automatically satisfied. Condition (b) is equivalent to the homomorphisms

$$\bigoplus_{j\in S}\widetilde{\mathrm{ev}}_j\colon \ker(D^{NY}_{J;u})\longrightarrow \bigoplus_{j\in S}N_{\mathrm{ev}_1(u)}Y, \qquad ([u],w)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J),$$

being isomorphisms. For example, this is the case if $X=\mathbb{P}^n$, $Y=\mathbb{P}^1\subset X$, $S=\{1,2\}$, $g=0$, $\beta=\iota_{Y*}([Y])$ is the homology class of a line, $a_1,a_2=0$, and $f_1,f_2\colon pt\longrightarrow Y$ are maps to two distinct points. In this particular case,

$$\overline{\mathfrak{M}}_{0,\mathbf{f}}(X,\beta;J)=\overline{\mathfrak{M}}_{0,\mathbf{f}}(Y,\beta_Y;J),$$

where $\beta_Y=[Y]$, and $\mathrm{cok}(D^{NY}_J)$ is the zero vector bundle. Thus,

$$\big(pt,pt\big)^{\mathbb{P}^n}_{0,\beta}=\big((\tau_{a_j}\kappa_j)_{j\in S}\big)^X_{0,\beta}=\mathbf{C}_{0,\mathbf{f}}(Y,\beta_Y)=\,^{\pm}\big|\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big|=\big(pt,pt\big)^{\mathbb{P}^1}_{0,\beta_Y}=1,$$

as expected.[5]

Example C If each map f_j in Theorem 1.2 is transverse to Y, the second part of Condition (a) is again automatically satisfied. Condition (b) is equivalent to the injectivity of the operators $D^{NY}_{J;u}$ whenever $([u],w)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$. For example, this is the case if X is the blowup of $\mathbb{P}^n$, with $n\geq 2$, at a point, $Y\approx\mathbb{P}^{n-1}$ is the exceptional divisor, $S=\{1,2\}$, $g=0$, $\beta_Y\in H_2(Y;\mathbb{Z})$ is the homology class of a line in the exceptional divisor, $\beta=\iota_{Y*}(\beta_Y)$, $a_1,a_2=0$, and $f_1,f_2\colon\mathbb{P}^1\longrightarrow X$ are parametrizations of proper transforms of two distinct lines in $\mathbb{P}^n$ passing through the center of the blowup. In this particular case,

$$\overline{\mathfrak{M}}_{0,\mathbf{f}}(X,\beta;J)=\overline{\mathfrak{M}}_{0,\mathbf{f}}(Y,\beta_Y;J)$$

and $\mathrm{cok}(D^{NY}_J)$ is the zero vector bundle. Thus, if $\bar\ell$ denotes the homology class of f_1 and f_2,

$$\big(\bar\ell,\bar\ell\big)^X_{0,\beta}=\big((\tau_{a_j}\kappa_j)_{j\in S}\big)^X_{0,\beta}=\mathbf{C}_{0,\mathbf{f}}(Y,\beta_Y)=\,^{\pm}\big|\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big|=\big(\bar\ell\cap Y,\bar\ell\cap Y\big)^Y_{0,\beta_Y}=1;$$

see Footnote 5.

Various special cases of Theorem 1.2, such as those in Examples A-C, are standard in the algebraic setting and are used in [3], [14], and [27], for example. Some special cases of Theorem 1.2 have appeared in the symplectic setting as well, including in [17], [25], and [35]. Examples B and C generalize Example A in two opposite directions. By [18], Corollary 1.4 below is yet another special case of Example C. The full statement of Theorem 1.2 mixes the two extreme cases of Examples B and C.

Remark 1.3 A referee brought to the author's attention [29, Lemma 1]. In this case, $Y\approx\mathbb{P}^1$ is an exceptional curve in a complex surface (X,J), $\beta_Y=[Y]$, $\beta=\iota_{Y*}[Y]$, $S=\{1\}$, $a_1=0$, and $f=f_1\colon E\longrightarrow X$ is the inclusion. Thus,

$$\mathrm{ev}_1\colon\overline{\mathfrak{M}}_{0,1}(X,\iota_{Y*}[Y];J)=\overline{\mathfrak{M}}_{0,1}(Y,[Y];J)\longrightarrow Y, \qquad \mathrm{ev}_1\circ\pi_1=f\circ\pi_2\colon\overline{\mathfrak{M}}_{0,f}(Y,[Y];J)\longrightarrow Y$$

[5] This is the number of lines through 2 points in $\mathbb{P}^n$. In this particular case, each operator $D^{NY}_{J;u}$ is $\mathbb{C}$-linear and its zero-dimensional kernel is positively oriented. In general, this need not be the case; see [18, Sections 9,10] for explicit sign computations.

are biholomorphic maps, f intersects Y properly, but

$$\ker(D^{NY}_{J;u})=\{0\} \longrightarrow N^f_{\mathrm{ev}_1(u)}Y \approx \gamma_{\mathrm{ev}_1(u)}, \qquad \xi \longrightarrow \widetilde{\mathrm{ev}}_1(\xi)+(\mathrm{Im}\, d^{NY}_{\mathrm{ev}_1(u)}f),$$

where $\gamma \longrightarrow Y$ is the tautological line bundle, is not an isomorphism for any pair $([u],w)$ in $\overline{\mathfrak{M}}_{0,f}(Y,[Y];J)$. So, as stated, Theorem 1.2 is not applicable in this case, but its assumptions can be weakened with little effect on the proof. As explained at the beginning of Section 3,

- Condition (b) is not necessary at for (1) in Theorem 1.2;
- if Condition (a) holds and the dimension $\ker D^{NY}_{J;u}$ does not depend on a pair $([u],(w_j)_{j\in S})$ in $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$, the vector spaces $\mathrm{cok}(D^{NY}_{J;u})$ still form a natural orbi-bundle of the rank given by (1.9) with the first term on RHS replaced by $\ker D^{NY}_{J;u}$, but it may no longer be orientable;
- if Condition (a) holds and the homomorphism (1.7) is injective, the first part of (3) holds.

Thus, most claims in Theorem 1.2 continue to hold if Condition (b) is relaxed to requiring that the homomorphism (1.7) is an isomorphism onto the fiber $V_{([u],(w_j)_{j\in S})}$ of a subbundle V of the vector bundle

$$N^{\mathbf{f}}Y \equiv \bigoplus_{j\in S}\pi_j^*N^{f_j}Y \longrightarrow \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J),$$

where $\pi_j\colon \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\longrightarrow f_j^{-1}(Y)$ is the projection map. The bundle

$$\mathit{Obs} \equiv \mathrm{cok}(D^{NY}_J)\oplus (N^{\mathbf{f}}Y/V)$$

then has a canonical orientation, its rank is given by RHS of (1.9), and (1.10) holds with $\mathrm{cok}(D^{NY}_J)$ replaced by *Obs*.

The striking conclusion of [18] is that all GW-invariants of a Kahler surface X of general type localize to a canonical divisor. The situation is particularly beautiful if X admits a smooth canonical divisor $\mathcal{K}_X$. If X is minimal, the GW-invariants of X in degrees other than multiples of $\mathcal{K}_X$ vanish. The GW-invariants of X in degrees $\mathcal{K}_X$ and $2\mathcal{K}_X$ are computed in [13] via an algebraic reformulation of [18] and shown to satisfy a conjecture of [20]. In the next paragraph we review the relevant statements from [18].

Let (X,J_0) be a minimal Kahler surface of general type and α the real part of a non-zero holomorphic $(2,0)$-form such that $Y\equiv\alpha^{-1}(0)$ is smooth (and reduced). Since X is minimal, Y is connected. With $\langle\cdot,\cdot\rangle$ denoting the Riemannian metric on X, define

$$\begin{aligned}
&K_\alpha \in \Gamma\big(X;\mathrm{Hom}_{\mathbb{R}}(TX,TX)\big), \quad R_\alpha\in\Gamma\big(Y;\mathrm{Hom}_{\mathbb{R}}(TY\otimes_{\mathbb{C}}NY,NY)\big), \qquad \text{by}\\
&\qquad \langle v_1,K_\alpha v_2\rangle=\alpha(v_1,v_2) \quad \forall\, v_1,v_2\in T_xX,\ x\in X;\\
&\qquad R_\alpha(v_1,v_2)=J_0\big\{\nabla_{v_2}K_\alpha\big\}(v_1)+T_xY \quad \forall\, v_1\in T_xY,\ v_2\in T_xX,\ x\in X.
\end{aligned} \tag{1.11}$$

By [18, Lemmas 2.1,8.2], R_α is well-defined. The almost complex structure J_α on X described in [18, Section 2] agrees with J_0 along the smooth complex curve Y. By [18, Lemma 2.3], every non-constant J_α-holomorphic map $u\colon\Sigma_u\longrightarrow X$ is in fact a J_0-holomorphic map to Y and so lies in the homology class dY for some $d\in\mathbb{Z}^+$. By [18, Section 8], the operator on the normal bundle NY of Y induced by the linearization of the $\bar\partial_{J_\alpha}$-operator for maps to X at such a map u is given by

$$D^{NY}_{J_\alpha;u}=\bar\partial_{u^*NY}+R_\alpha(du,\cdot)\colon L^p_1(\Sigma_u;u^*NY)\longrightarrow L^p\big(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*NY\big), \tag{1.12}$$

where $\bar{\partial}_{u^*NY}$ is the $\bar{\partial}$-operator in the holomorphic bundle $u^*(NY,J_0)\longrightarrow\Sigma_u$. By [18, Proposition 8.6], $D^{NY}_{J_\alpha;u}$ is injective. Thus, the assumptions of Theorem 1.2 are satisfied, and it gives the following corollary.

Corollary 1.4 *Suppose (X,J_0) is a minimal Kahler surface of general type, α is the real part of a non-zero holomorphic $(2,0)$-form such that $Y\equiv\alpha^{-1}(0)$ is smooth, $g\in\bar{\mathbb{Z}}^+$, $d\in\mathbb{Z}^+$, S is a finite set, $S_2\subset S$, $a_j\in\bar{\mathbb{Z}}^+$ for each $j\in S$, and $\kappa_j\in H_2(X;\mathbb{Z})$ for each $j\in S_2$. If R_α is defined by (1.11), then the cokernels of the operators (1.12) form a natural oriented vector orbi-bundle*

$$\mathrm{cok}\big(D^{NY}_\alpha\big)\longrightarrow\overline{\mathfrak{M}}_{g,S}(Y,dY)$$

and

$$\begin{aligned}&\big((\tau_{a_j}\kappa_j)_{j\in S_2},(\tau_{a_j}1)_{j\in S-S_2}\big)^X_{g,d\mathcal{K}_X}\\&\quad=\Big(\prod_{j\in S_2}\langle c_1(T^*X),\kappa_j\rangle\Big)\Big\langle e\big(\mathrm{cok}(D^{NY}_\alpha)\big)\prod_{j\in S_2}\big(\mathrm{ev}_j^*\mathrm{PD}_Y(pt)\big)\prod_{j\in S}\psi_j^{a_j},\big[\overline{\mathfrak{M}}_{g,S}(Y,dY)\big]^{vir}\Big\rangle.\ [6]\end{aligned}$$

1.2 The Fano case of the Gopakumar-Vafa prediction

GW-invariants are generally not integers. On the other hand, at least in the case of smooth projective 3-folds (all of which are symplectic 6-manifolds), certain combinations of them are believed to be integers. Ideally these combinations would be precisely counts of curves of fixed genus and degree and passing through appropriate constraints. A projective 3-fold X is never ideal in this sense, but one might hope that X becomes ideal if its Kahler complex structure is replaced with a generic almost complex one. We show that this is indeed the case in the "Fano" case.

If (X,ω) is a compact symplectic manifold, $g\in\bar{\mathbb{Z}}^+$, S is a finite set, $\beta\in H_2(X;\mathbb{Z})$, and J is an ω-tame almost complex structure on X, let

$$\mathfrak{M}^*_{g,S}(X,\beta;J)\subset\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$$

denote the subspace consisting of simple maps, i.e. J-holomorphic maps $u\colon\Sigma_u\longrightarrow X$ such that Σ_u is a smooth (connected) Riemann surface and $u^{-1}(u(z))=\{z\}$ and $d_zu\neq0$ for some $z\in\Sigma_u$. These conditions imply that u does not factor through a d-fold cover $\Sigma_u\longrightarrow\Sigma$, with $d>1$; see [24, Section 2.5]. If $f_j:M_j\longrightarrow X$, $j\in S$, are smooth maps from compact oriented manifolds of even dimensions, let

$$\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)=\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)\cap\Big(\mathfrak{M}^*_{g,S}(X,\beta;J)\times\prod_{j\in S}M_j\Big),$$

with $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ defined by (1.5). If $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ is a finite set consisting of regular pairs $([u],(w_j)_{j\in S})$, we will denote its signed cardinality by $E^X_{g,\beta}(J,\mathbf{f})$.

[6] If the cobordisms f_j representing κ_j are transverse to Y,

$$\pi_*\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,dY)\big]^{vir}=\Big(\prod_{j\in S_2}\langle c_1(T^*X),\kappa_j\rangle\Big)\big[\overline{\mathfrak{M}}_{g,S}(Y,dY)\big]^{vir},$$

where $\pi\colon\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,dY)\longrightarrow\overline{\mathfrak{M}}_{g,S}(Y,dY)$ is the projection to the first component in (1.5).

If the (real) dimension of X is 6, the expected dimension of the moduli space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ is independent of the genus g; see (1.2). Thus, one can mix curve counts of different genera passing through the same constraints. Furthermore, if $\beta\in H_2(X;\mathbb{Z})$ and $\langle c_1(TX),\beta\rangle<0$, all degree β GW-invariants are zero, since the moduli space of unmarked maps has negative expected dimension. This leaves the "Calabi-Yau" case, $\langle c_1(TX),\beta\rangle = 0$, and the "Fano" case, $\langle c_1(TX),\beta\rangle > 0$. If $g,h\in\bar{\mathbb{Z}}^+$, define $C^X_{h,\beta}(g)\in\mathbb{Q}$ by

$$\sum_{g=0}^{\infty} C^X_{h,\beta}(g)t^{2g} = \left(\frac{\sin(t/2)}{t/2}\right)^{2h-2+\langle c_1(TX),\beta\rangle}. \tag{1.13}$$

For example,

$$C^X_{h,\beta}(0)=1, \qquad C^X_{h,\beta}(1)=\frac{2-2h-\langle c_1(TX),\beta\rangle}{24}.$$

Theorem 1.5 *Suppose (X,ω) is a compact symplectic 6-fold, $\beta\in H_2(X;\mathbb{Z})$, $g\in\bar{\mathbb{Z}}^+$, S is a finite set, and $\kappa_j\in H_*(X;\mathbb{Z})$ for $j\in S$ are such that (1.4) is satisfied with $a_j=0$. If $\langle c_1(TX),\beta\rangle>0$,*

(1) there exists a dense open subset $\mathcal{J}_{\mathrm{reg}}(g,\beta)$ of the space of smooth ω-tame almost complex structures on X such that for all $h\le g$:

- *the moduli space $\mathfrak{M}^*_{h,S}(X,\beta;J)$ consists of regular maps;*
- *for a generic choice of pseudocycle representatives[7] $f_j: M_j\longrightarrow X$ for κ_j, $\mathfrak{M}^*_{h,\mathbf{f}}(X,\beta;J)$ is a finite set of regular pairs $([u],(w_j)_{j\in S})$ such that u is an embedding;*

(2) the numbers $E^X_{h,\beta}(\mathbf{f},J)$, with $h\le g$, are independent of the choice of $J\in\mathcal{J}_{\mathrm{reg}}(g,\beta)$ and f_j and can thus be denoted $E^X_{h,\beta}((\kappa_j)_{j\in S})$;

(3) if $C^X_{g,\beta}(h)$ is defined by (1.13),

$$\big((\kappa_j)_{j\in S}\big)^X_{g,\beta} = \sum_{h=0}^{h=g} C^X_{h,\beta}(g-h)E^X_{h,\beta}\big((\kappa_j)_{j\in S}\big). \tag{1.14}$$

For $g=0,1$, (1.14) gives

$$\begin{aligned}\big((\kappa_j)_{j\in S}\big)^X_{0,\beta} &= E^X_{0,\beta}\big((\kappa_j)_{j\in S}\big),\\ \big((\kappa_j)_{j\in S}\big)^X_{1,\beta} &= E^X_{1,\beta}\big((\kappa_j)_{j\in S}\big)+\frac{2-\langle c_1(TX),\beta\rangle}{24}E^X_{0,\beta}\big((\kappa_j)_{j\in S}\big).\end{aligned} \tag{1.15}$$

The first identity expresses the well-known fact that the genus 0 GW-invariants of a Fano manifold are enumerative. The second identity in (1.15) is the $n=3$ case of the relation between the standard genus 1 GW-invariants and the reduced genus 1 GW-invariants constructed in [38] for all symplectic manifolds.

[7]After replacing κ_j by a multiple, M_j can be taken to be a smooth compact manifold.

We now deduce Theorem 1.5 from Theorem 1.2. By the proof of [24, Theorem 3.1.5], for a generic almost complex structure J on X all moduli spaces $\mathfrak{M}^*_{h,\emptyset}(X,\beta';J)$ are smooth and of the expected dimension, $2\langle c_1(TX),\beta'\rangle$. In particular,

$$\langle c_1(TX),\beta'\rangle<0 \qquad\Longrightarrow\qquad \mathfrak{M}^*_{h,S}(X,\beta';J),\overline{\mathfrak{M}}_{h,S}(X,\beta';J)=\emptyset. \tag{1.16}$$

By a similar argument, for a generic J on X the evaluation maps

$$\mathrm{ev}_1,\mathrm{ev}_2\colon \mathfrak{M}^*_{g,\{1,2\}}(X,\beta;J)\longrightarrow X$$

are transverse, while the bundle section

$$\mathfrak{M}^*_{g,\{1\}}(X,\beta;J)\longrightarrow L_1^*\otimes\mathrm{ev}_1^*TX, \qquad [u]\longrightarrow d_{z_1(u)}u\,,$$

where $L_1\longrightarrow\mathfrak{M}^*_{g,\{1\}}(X,\beta;J)$ is the universal tangent line bundle at the marked point and $z_1(u)\in\Sigma_u$ is the marked point of u, is transverse to the zero set. Thus,

$$\mathfrak{M}^{sing}_{g,S}(X,\beta;J)\equiv\big\{[u]\in\mathfrak{M}^*_{g,S}(X,\beta;J)\colon\ u \text{ is not an embedding}\big\}$$

is the image of a smooth map from a smooth manifold of (real) dimension two less than the dimension of $\mathfrak{M}^*_{g,S}(X,\beta;J)$. It follows that for a generic choice of pseudocycle representatives $f_j\colon M_j\longrightarrow X$ for κ_j, $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ is a 0-dimensional oriented submanifold of

$$\big(\mathfrak{M}^*_{g,S}(X,\beta;J)-\mathfrak{M}^{sing}_{g,S}(X,\beta;J)\big)\times\prod_{j\in S}M_j.$$

We next show that $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ is a finite set. If not, there is a sequence $([u_r],(w_{r,j})_{j\in S})$ in $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ converging to some

$$\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)-\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J).$$

The image of u is a connected J-holomorphic curve in X of genus $h\le g$ with $k\ge1$ irreducible components of degrees $\beta_1,\ldots,\beta_k\in H_2(X;\mathbb{Z})$ such that

$$d_1\beta_1+\ldots+d_k\beta_k=\beta \qquad\text{for some}\quad d_1,\ldots,d_k\in\mathbb{Z}^+.$$

By (1.16), $\langle c_1(TX),\beta_i\rangle\ge0$ for all $i=1,\ldots,k$. Thus,

$$\sum_{i=1}^{i=k}\langle c_1(TX),\beta_i\rangle\le\langle c_1(TX),\beta\rangle.$$

The dimension-counting argument of [24, Section 6.6] then shows that $k=1$ and $d_1=1$. It then follows that the image of u is an irreducible J-holomorphic curve of degree β and genus $h<g$ that meets each of the maps f_j with $j\in S$.

While degree β genus $h<g$ J-holomorphic curves meeting the maps f_j can certainly exist for a generic J, they cannot be limits of other degree β curves meeting the maps f_j by Proposition 3.2 for the following reason. If

$$\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)-\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J),$$

the domain Σ_u of u consists of two or more irreducible components. Furthermore, by the previous paragraph, the restriction of u to all irreducible components of Σ_u, except for one, is constant; let u_{eff} denote the effective part of u, i.e. the non-constant restriction. The domain $\Sigma_{u_{\mathrm{eff}}}$ of u_{eff} is a smooth curve of genus $h<g$ with distinct points $(z_j(u_{\mathrm{eff}}))_{j\in S}$ that are mapped to $(\mathrm{ev}_j(u))_{j\in S}$ by u_{eff}. Thus,

$$\big([u_{\mathrm{eff}}],(w_j)_{j\in S}\big)\in\mathfrak{M}^*_{h,\mathbf{f}}(X,\beta;J);$$

by the previous paragraph, u_{eff} is an embedding onto a smooth J-holomorphic curve Y of genus h degree β meeting the maps f_j. This implies that removing a node from $\Sigma_{u_{\mathrm{eff}}}$ disconnects Σ_u.[8] Since the total evaluation map

$$\mathbf{ev}\equiv\prod_{j\in S}\mathrm{ev}_j:\mathfrak{M}^*_{h,S}(X,\beta;J)\longrightarrow X^S$$

is transverse to $\mathbf{f}$,

$$\ker(D^{NY}_{J;u_{\mathrm{eff}}})\longrightarrow\bigoplus_{j\in S}N^{f_j}_{w_j}Y,\qquad \xi\longrightarrow\big(\xi(z_j(u_{\mathrm{eff}}))+(\mathrm{Im}\,d^{NY}_{w_j}f_j)\big)_{j\in S}\,, \tag{1.17}$$

is surjective; see Section 1.1 for the notation. Since u_{eff} is a regular map,

$$\begin{aligned}\dim\ker(D^{NY}_{J;u_{\mathrm{eff}}})=\mathrm{ind}\big(D^{NY}_{J;u_{\mathrm{eff}}}\big)&=2\big(\langle c_1(NY),Y\rangle+2(1-h)\big)=2\langle c_1(TX),\beta\rangle\\&=\sum_{j\in S}(4-\dim\,M_j)\le\sum_{j\in S}\dim\,N^{f_j}_{f_j(w)}Y;\end{aligned}$$

the second-to-last equality holds by (1.4). Thus, the homomorphism in (1.17) is an isomorphism. On the other hand, $D^{NY}_{J;u}$ is the restriction of the operator $\bigoplus_i D^{NY}_{J;u_i}$ to

$$L^p_1(\Sigma_u;u^*NY)\subset\bigoplus_i L^p_1\big(\Sigma_{u;i};u_i^*NY\big),$$

where $\{\Sigma_{u;i}\}$ are the irreducible components of Σ_u and $u_i=u|_{\Sigma_{u;i}}$. If u_i is a constant map, then $D^{NY}_{J;u_i}$ is the usual $\bar\partial$-operator on the space of functions on Σ_{u_i} with values in $N_{u_i(\Sigma_{u;i})}Y\approx\mathbb{C}^2$. Since Σ_u is a connected nodal Riemann containing $\Sigma_{u_{\mathrm{eff}}}$ as a component, $u|_{\Sigma_{\mathrm{eff}}}=u_{\mathrm{eff}}$, and u is constant on each of the irreducible components of $\Sigma_u-\Sigma_{u_{\mathrm{eff}}}$, it follows that the projection homomorphism

$$\ker D^{NY}_{J;u}\longrightarrow\ker D^{NY}_{J;u_{\mathrm{eff}}},\qquad \xi\longrightarrow\xi|_{\Sigma_{u_{\mathrm{eff}}}}, \tag{1.18}$$

is an isomorphism. Thus, the homomorphism

$$\ker(D^{NY}_{J;u})\longrightarrow\bigoplus_{j\in S}N^{f_j}_{w_j}Y,\qquad \xi\longrightarrow\big(\xi(z_j(u))+(\mathrm{Im}\,d^{NY}_{w_j}f_j)\big)_{j\in S}\,,$$

is an isomorphism, since the homomorphism (1.17) is. Therefore, by Proposition 3.2 there is no sequence in

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)-\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,[Y];J)\supset\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$$

converging to $([u],(w_j)_{j\in S})$.[9]

[8] This observation implies that the homomorphism (1.18) is surjective.

[9] Since $\bar\partial u_r=0$ for such a sequence, the second condition in (3.7) is satisfied for any choice of J-regularized tubular neighborhood of Y in X.

We have thus shown that $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ is a compact oriented 0-dimensional manifold and its signed cardinality $E^X_{g,\beta}(\mathbf{f},J)$ is well-defined. The independence of $E^X_{g,\beta}(\mathbf{f},J)$ of the choices of J and f_j follows from (1.14), with $E^X_{h,\beta}((\kappa_j)_{j\in S})$ replaced by $E^X_{h,\beta}(\mathbf{f},J)$. In turn, this identity follows from Theorem 1.2 and the proof of [27, Theorem 3]. Let Y be a degree β J-holomorphic curve of genus $h\le g$ meeting each f_j. By the above, the assumptions of Theorem 1.2 are satisfied. By definition (see Section 2.4), the orbi-bundle $\mathrm{cok}(D^{NY}_J)$ is dual to the bundle $\ker((D^{NY}_J)^*)$ of kernels of the dual operators $(D^{NY}_J)^*$. For each

$$\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,[Y];J)\subset\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J),$$

the operator $(D^{NY}_{J;u})^*$ is the natural extension of the operator $\bigoplus_i(D^{NY}_{J;u_i})^*$ to $(1,0)$-forms on Σ_u with poles at the nodes such that the residues at each node sum up to 0. Since $(D^{NY}_{J;u_{\mathrm{eff}}})^*$ is injective by the regularity of u_{eff}, the projection

$$\eta\longrightarrow\bigoplus_{\Sigma_{u;i}\neq\Sigma_{u_{\mathrm{eff}}}}\eta|_{\Sigma_{u;i}}$$

to the contracted components is injective. Since $(D^{NY}_{J;u_i})^*=\bar\partial^*$ if u_i is constant, the image of this homomorphism is determined by Σ_u and is independent of $D^{NY}_{J;u_{\mathrm{eff}}}$ (as long as $D^{NY}_{J;u_{\mathrm{eff}}}$ is surjective). Thus, $\mathrm{cok}(D^{NY}_J)$ is isomorphic to the restriction to $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,[Y];J)$ of the obstruction bundle in [27, Section 3], i.e. the bundle of cokernels of the operators $D^{NY}_{J;u}$ as above, but for a holomorphic vector bundle NY. Thus,

$$\begin{split}\mathbf{C}_{g,\mathbf{f}}(Y,\beta_Y)&=\Big\langle e\big(\mathrm{cok}(D^{NY}_J)\big),\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,[Y];J)\big]^{vir}\Big\rangle\\&=C^X_{h,\beta}(g-h)\,\mathrm{sgn}\big([u_{\mathrm{eff}}],(w_j)_{j\in S}\big)\end{split}\tag{1.19}$$

by (1.10) and [27, Theorem 3]. Since

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)=\bigsqcup_{h=0}^{h=g}\ \bigsqcup_{([u],(w_j)_{j\in S})\in\mathfrak{M}^*_{h,\mathbf{f}}(X,\beta;J)}\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathrm{Im}\,u,[\mathrm{Im}\,u];J),$$

the identity (1.14) follows from (1.19).

Theorem 1.5 confirms (a stronger version of) the "Fano" case of [28, Conjecture 2(i)], i.e. that the numbers $E^X_{h,\beta}((\kappa_j)_{j\in S})$ *defined from GW-invariants by (1.14)* are integers. The Calabi-Yau case is fundamentally more difficult as it involves multiple covers of curves.[10] On the other hand, it might be possible to approach [28, Conjecture 2(ii)], i.e. that $E^X_{h,\beta}((\kappa_j)_{j\in S})=0$ for a fixed β and all sufficiently large g if X is projective, by studying possible limits of J_t-holomorphic curves with $J_t\in\mathcal{J}_{\mathrm{reg}}(g,\beta)$ as J_t approaches the standard complex structure on $X\subset\mathbb{P}^n$ and using the Castelnuovo bound [1, p116].

An algebro-geometric approach to Theorem 1.5 has recently been proposed in [12], at least in the usual, more narrow, meaning of *Fano* in algebraic geometry. The stable-map style invariants of

[10] Theorem 1.5 and its proof also apply to the cases when $\langle c_1(TX),\beta\rangle=0$, but β is not a non-trivial integer multiple of another element of $H_2(X;\mathbb{Z})$.

smooth projective varieties defined in [12] are a priori integers in the case of Fano varieties, just like the numbers $E^X_{h,\beta}((\kappa_j)_{j\in S})$. In addition, in this Fano case, they are non-negative integers and satisfy the vanishing prediction of [28, Conjecture 2(ii)]. However, it remains to be shown that they are related to the GW-invariants in the required way, i.e. as in (1.14).

2 Analytic Preliminaries

In this section, we collect a number of background statements concerning solutions of perturbed Cauchy-Riemann equations. For the rest of the paper, fix a real number $p>2$. If Σ is a 2-dimensional manifold, this condition implies that any L^p_1-map $\Sigma\longrightarrow\mathbb{R}$ is continuous and in particular has a well-defined value at each point.

2.1 Nodal Riemann surfaces

Let $(E,\mathfrak{i})\longrightarrow\Sigma$ be an L^p_1-complex vector bundle over a smooth Riemann surface, i.e. a one-dimensional complex manifold. If $z\in\Sigma$ and

$$A_z\in \mathrm{Hom}_{\mathbb{R}}(E_z, T^*_z\Sigma^{0,1}\otimes_{\mathbb{C}}E_z),$$

we define

$$\begin{aligned} &A^*_z\in \mathrm{Hom}_{\mathbb{R}}(T^*_z\Sigma^{1,0}\otimes_{\mathbb{C}}E^*_z, T^*_z\Sigma^{1,1}\otimes_{\mathbb{C}}E^*_z) \qquad \text{by}\\ &\mathrm{Re}\big(v\wedge(A^*_zw)\big)=\mathrm{Re}\big((A_zv)\wedge w\big)\in\Lambda^2_{\mathbb{R}}(T^*_z\Sigma) \qquad \forall\, v\in E_z,\ w\in T^*_z\Sigma^{1,0}\otimes_{\mathbb{C}}E^*_z\,.\end{aligned}$$

Since $\Lambda^2_{\mathbb{R}}(T^*_z\Sigma)$ is one-dimensional, A^*_z is well-defined. If

$$A\in L^p\big(\Sigma;\mathrm{Hom}_{\mathbb{R}}(E,T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)\big),$$

this construction gives rise to an element

$$\begin{aligned} &A^*\in L^p\big(\Sigma;\mathrm{Hom}_{\mathbb{R}}(T^*\Sigma^{1,0}\otimes_{\mathbb{C}}E^*, T^*\Sigma^{1,1}\otimes_{\mathbb{C}}E^*)\big) \qquad \text{s.t.}\\ &\langle\!\langle\xi,A^*\eta\rangle\!\rangle\equiv\mathrm{Re}\Big(\int_\Sigma\xi\wedge(A^*\eta)\Big)=\mathrm{Re}\Big(\int_\Sigma(A\xi)\wedge\eta\Big)\equiv\langle\!\langle A\xi,\eta\rangle\!\rangle \end{aligned} \tag{2.1}$$

for all $\xi\in L^p_1(\Sigma;E)$ and $\eta\in L^p_1(\Sigma;T^*\Sigma^{1,0}\otimes E^*)$.

Let $E\longrightarrow\Sigma$ be as above. If S is a finite subset of Σ, denote by

$$L^p_k\big(\Sigma;E(S)\big)\subset L^p_{k,loc}(\Sigma-S;E)$$

the subspace of sections η of E such that for every $z_0\in S$ there exist a neighborhood U of z_0 in Σ and a coordinate $w\colon U\longrightarrow\mathbb{C}$ such that

$$w(z_0)=0 \qquad \text{and} \qquad w\cdot\eta|_U\in L^p_k(U;E).$$

If $k\geq1$, an element η of $L^p_k(\Sigma;T^*\Sigma^{1,0}\otimes_{\mathbb{C}}E(S))$ has a well-defined residue at $z_0\in S$ given by

$$\mathrm{Res}_{z=z_0}\eta=\xi(z_0)\in E_{z_0} \qquad \text{if} \qquad \eta(z)=\frac{dw}{w(z)}\otimes\xi(z)\ \ \forall\ z\in U,\ \xi\in L^p_1(U;E).$$ [11]

If ϱ is a function assigning to each element $z_0 \in S$ a real subspace $E'_{z_0} \subset E_{z_0}$, let

$$L^p_1\big(\Sigma; T^*\Sigma^{1,0} \otimes_{\mathbb{C}} E(\varrho)\big) = \big\{\eta \in L^p_1\big(\Sigma; E(S)\big) \colon \operatorname{Res}_{z=z_0}\eta \in E'_{z_0} \ \forall\, z_0 \in S\big\}.$$

By a Riemann surface Σ we will mean a compact complex one-dimensional manifold with pairs of distinct points identified. In other words,

$$\Sigma = \widetilde{\Sigma}/\sim, \qquad \text{where} \quad x_i^{(1)} \sim x_i^{(2)} \quad i = 1, \ldots, m, \tag{2.2}$$

for some smooth compact Riemann surface $\widetilde{\Sigma}$ and distinct points $x_i^{(1)}, x_i^{(2)} \in \widetilde{\Sigma}$. The quotient map

$$\sigma \colon \widetilde{\Sigma} \longrightarrow \Sigma$$

is determined by Σ up to an isomorphism. We will denote by

$$\Sigma_{\text{sing}} \equiv \big\{\sigma(x_i^{(1)}) \colon i=1, \ldots, m\big\} \subset \Sigma \qquad \text{and} \qquad \widetilde{\Sigma}_{\text{sing}} \equiv \big\{x_i^{(1)}, x_i^{(2)} \colon i=1, \ldots, m\big\} \subset \widetilde{\Sigma}$$

the subset of singular points of Σ and its preimage under σ, respectively. Let $\Sigma^* \subset \Sigma$ be the subspace of smooth points, i.e. the complement of Σ_{sing}.

If Y is a smooth manifold and Σ is a Riemann surface as above, an L^p_1-map $u \colon \Sigma \longrightarrow Y$ is an L^p_1-map

$$\widetilde{u} \colon \widetilde{\Sigma} \longrightarrow Y \qquad \text{s.t.} \quad \widetilde{u}\big(x_i^{(1)}\big) = \widetilde{u}\big(x_i^{(2)}\big) \ \ \forall\, i = 1, \ldots, m.$$

By a vector bundle $E \longrightarrow \Sigma$, we will mean a topological complex vector bundle such that $\sigma^*E \longrightarrow \widetilde{\Sigma}$ is an L^p_1-complex vector bundle. Let

$$\begin{aligned} L^p_1(\Sigma; E) &= \big\{\xi \in L^p_1(\widetilde{\Sigma}; \sigma^*E) \colon \ \xi(x_i^{(1)}) = \xi(x_i^{(2)}) \ \forall\, i=1, \ldots, m\big\}; \\ L^p\big(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} E\big) &= L^p\big(\widetilde{\Sigma}; T^*\widetilde{\Sigma}^{0,1} \otimes_{\mathbb{C}} \sigma^*E\big). \end{aligned}$$

If S is a finite subset of Σ^*, let $\widetilde{S} = \sigma^{-1}(S)$ and define

$$\begin{aligned} L^p_1\big(\Sigma; \mathcal{K}_\Sigma \otimes_{\mathbb{C}} E(S)\big) &= \Big\{\eta \in L^p_1\big(\widetilde{\Sigma}; T^*\widetilde{\Sigma}^{1,0} \otimes_{\mathbb{C}} \sigma^*E(\widetilde{S} \cup \widetilde{\Sigma}_{\text{sing}})\big) \colon \\ &\qquad\qquad \sum_{\widetilde{z}_0 \in \sigma^{-1}(z_0)} \operatorname{Res}_{z=\widetilde{z}_0}\eta(\widetilde{z}_0) = 0 \ \ \forall\, z_0 \in \Sigma_{\text{sing}}\Big\}, \\ L^p\big(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} \mathcal{K}_\Sigma \otimes_{\mathbb{C}} E(S)\big) &= L^p\big(\widetilde{\Sigma}; T^*\widetilde{\Sigma}^{0,1} \otimes_{\mathbb{C}} T^*\widetilde{\Sigma}^{1,0} \otimes_{\mathbb{C}} \sigma^*E(\widetilde{S} \cup \widetilde{\Sigma}_{\text{sing}})\big). \end{aligned} \tag{2.3}$$

If ϱ is a function assigning to each element $z_0 \in S$ a real subspace $E'_{z_0} \subset E_{z_0}$, let

$$L^p_1\big(\Sigma; \mathcal{K}_\Sigma \otimes_{\mathbb{C}} E(\varrho)\big) = \big\{\eta \in L^p_1\big(\Sigma; \mathcal{K}_\Sigma \otimes_{\mathbb{C}} E(S)\big) \colon \operatorname{Res}_{z=\sigma^{-1}(z_0)}\eta \in E'_{z_0} \ \forall\, z_0 \in S\big\}. \tag{2.4}$$

[11] If $\eta \in L^p_k(\Sigma; T^*\Sigma^{1,0} \otimes_{\mathbb{C}} E(S - z_0))$, then $\operatorname{Res}_{z=z_0}\eta = 0$. The converse is not true; for example, the residue of $\eta = \bar{z}\, dz/z$ is zero at $z=0$, but η is not even continuous at $z=0$. On the other hand, the converse is true if η lies in the kernel of a generalized CR-operator as in Section 2.2.

Similarly, we define

$$L^p_1\big(\Sigma; E(-S)\big) = \big\{\xi\in L^p_1(\Sigma;E)\colon\ \xi(z_0)=0\ \forall\, z_0\in S\big\},$$
$$L^p_1\big(\Sigma; E^*(-\varrho)\big) = \big\{\xi\in L^p_1(\Sigma;E^*)\colon\ \xi(z_0)\in \mathrm{Ann}(E'_{z_0})\ \forall\, z_0\in S\big\},$$

where $\mathrm{Ann}(E'_{z_0})\subset \mathrm{Hom}_{\mathbb{R}}(E_{z_0},\mathbb{R})$ is the annihilator of $E'_{z_0}\subset E_{z_0}$. The real pairings in (2.1) extend to pairings

$$L^p_1\big(\Sigma;E\big)\otimes L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{K}_\Sigma\otimes_{\mathbb{C}}E^*(S)\big) \longrightarrow \mathbb{R},$$
$$L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E\big)\otimes L^p_1\big(\Sigma;\mathcal{K}_\Sigma\otimes_{\mathbb{C}}E^*(S)\big) \longrightarrow \mathbb{R}.$$

Furthermore, the equality in (2.1) holds for all $\eta \in L^p_1\big(\Sigma;\mathcal{K}_\Sigma\otimes_{\mathbb{C}}E^*(S)\big)$.

2.2 Generalized Cauchy-Riemann operators

Definition 2.1 *Let (Y,J) be an almost complex manifold and $(N,\mathfrak{i})\longrightarrow(Y,J)$ a smooth complex vector bundle.*

(1) A $\bar\partial$-operator on $(N,\mathfrak{i})$ is a $\mathbb{C}$-linear map

$$\bar\partial\colon \Gamma(Y;N)\longrightarrow \Gamma^{0,1}(Y;N)\equiv \Gamma\big(Y;T^*Y^{0,1}\otimes_{\mathbb{C}}N\big)$$

such that

$$\bar\partial\big(f\xi\big) = (\bar\partial f)\otimes\xi + f(\bar\partial\xi) \qquad \forall\ f\in C^\infty(Y),\ \xi\in\Gamma(Y;N).$$

(2) A smooth generalized Cauchy-Riemann operator (or smooth CR-operator) on $(N,\mathfrak{i})$ is a differential operator of the form

$$D = \bar\partial + A\colon \Gamma(Y;N)\longrightarrow \Gamma^{0,1}(Y;N), \tag{2.5}$$

where $\bar\partial$ is a $\bar\partial$-operator on $(N,\mathfrak{i})$ and

$$A\in \Gamma\big(Y;\mathrm{Hom}_{\mathbb{R}}(N,T^*Y^{0,1}\otimes_{\mathbb{C}}N)\big).$$

If ∇ is an affine connection in $(N,\mathfrak{i})$, the operator

$$\Gamma(Y;N)\longrightarrow \Gamma^{0,1}(Y;N), \qquad \xi\longrightarrow \frac{1}{2}\big(\nabla\xi + \mathfrak{i}\nabla\xi\circ J\big), \tag{2.6}$$

is a $\bar\partial$-operator on $(N,\mathfrak{i})$. Furthermore, any $\mathbb{C}$-linear CR-operator on $(N,\mathfrak{i})$ is a $\bar\partial$-operator, and any $\bar\partial$-operator on $(N,\mathfrak{i})$ is of the form (2.6) for some (not unique) connection ∇ in $(N,\mathfrak{i})$. In particular, A in the decomposition (2.5) can be assumed to be $\mathbb{C}$-anti-linear.

Let ∇^J be the J-linear connection in TY obtained from a Levi-Civita connection ∇ on Y and $A_Y(\cdot,\cdot)$ the Nijenhuis tensor of J:

$$\left.\begin{aligned}\nabla^J_{\xi_1}\xi_2 &= \frac{1}{2}\Big(\nabla_{\xi_1}\xi_2 - J\nabla_{\xi_1}(J\xi_2)\Big)\\ A_Y(\xi_1,\xi_2) &= \frac{1}{4}\Big([\xi_1,\xi_2]+J[\xi_1,J\xi_2]+J[J\xi_1,\xi_2]-[J\xi_1,J\xi_2]\Big)\end{aligned}\right. \qquad \forall\,\xi_1,\xi_2\in\Gamma(Y;TY).$$

We identify A_Y with the element

$$A_Y \in \Gamma\big(Y; \mathrm{Hom}_{\mathbb{R}}(TY, T^*Y^{0,1}\otimes_{\mathbb{C}}TY)\big), \qquad v \longrightarrow A_Y(\cdot, v).$$

Then,

$$\bar{\partial}_Y \equiv \frac{1}{2}\Big(\nabla^J + J\nabla^J \circ J\Big),\ D_Y \equiv \bar{\partial}_Y + A_Y : \Gamma(Y;TY) \longrightarrow \Gamma^{0,1}(Y;TY)$$

are a $\bar{\partial}$-operator on TY and a smooth CR-operator on TY, respectively.

Definition 2.2 *Let $(E, \mathfrak{i})$ be an L^p_1 complex vector bundle over a Riemann surface $(\Sigma, \mathfrak{j})$.*

(1) A $\bar{\partial}$-operator on $(E, \mathfrak{i})$ is a $\mathbb{C}$-linear map

$$\bar{\partial}\colon L^p_1(\Sigma; E) \longrightarrow L^p\big(\Sigma; T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E\big)$$

such that

$$\bar{\partial}\big(f\xi\big) = (\bar{\partial}f)\otimes\xi + f(\bar{\partial}\xi) \qquad \forall\ f\in C^\infty(\Sigma),\ \xi\in\Gamma(\Sigma; E).$$

(2) A generalized Cauchy-Riemann operator (or CR-operator) on $(E, \mathfrak{i})$ is a differential operator of the form

$$D = \bar{\partial} + A\colon L^p_1(\Sigma; E) \longrightarrow L^p\big(\Sigma; T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E\big), \tag{2.7}$$

where $\bar{\partial}$ is a $\bar{\partial}$-operator on $(E, \mathfrak{i})$ and

$$A \in L^p\big(\Sigma; \mathrm{Hom}_{\mathbb{R}}(E, T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)\big). \tag{2.8}$$

If ∇ is an affine connection in $(E, \mathfrak{i})$, the operator

$$L^p_1(\Sigma; E) \longrightarrow L^p(\Sigma; T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E), \qquad \xi \longrightarrow \frac{1}{2}\big(\nabla\xi + \mathfrak{i}\nabla\xi\circ\mathfrak{j}\big), \tag{2.9}$$

is the usual $\bar{\partial}$-operator for a unique holomorphic structure in $(E, \mathfrak{i})$. Furthermore, any $\mathbb{C}$-linear CR-operator is of the form (2.9).

If Σ and $N \longrightarrow Y$ are as above, an L^p_1-map $u\colon \Sigma \longrightarrow Y$ pulls back a smooth CR-operator D on N to a CR-operator D_u on $u^*N \longrightarrow \Sigma$ as follows. Suppose D is presented as in (2.5) with $\mathbb{C}$-anti-linear A and ∇ is a connection in $(N, \mathfrak{i})$ inducing the corresponding $\bar{\partial}$-operator. Let $\widetilde{u}\colon \widetilde{\Sigma} \longrightarrow Y$ be the map corresponding to u as in Section 2.1 and

$$\widetilde{\nabla}\colon L^p_1(\widetilde{\Sigma}; \widetilde{u}^*N) \longrightarrow L^p(\widetilde{\Sigma}; T^*\widetilde{\Sigma}\otimes_{\mathbb{R}}\widetilde{u}^*N)$$

the connection induced by ∇. Then,

$$D_{\widetilde{u}} = \frac{1}{2}\Big(\widetilde{\nabla} + \mathfrak{i}\widetilde{\nabla}\circ\mathfrak{j}\Big) + A\circ\partial_J\widetilde{u}, \qquad \text{where} \quad \partial_J\widetilde{u} = \frac{1}{2}\Big(du - Jd\widetilde{u}\circ\mathfrak{j}\Big),$$

is a generalized CR-operator on $\widetilde{u}^*(N, \mathfrak{i})$; $D_{\widetilde{u}}$ is independent of the choice of ∇ if u is $(J, \mathfrak{j})$-holomorphic.

Suppose (Y,J) is an almost complex manifold and D_Y is as above. If $(\Sigma,\mathfrak{j})$ is a Riemann surface and $u\colon\Sigma\longrightarrow Y$ is a $(J,\mathfrak{j})$-holomorphic L^p_1-map, then $D_{J;u}\equiv u^*D_Y$ is the linearization of the $\bar\partial_J$-operator on the space of L^p_1-maps from Σ, with complex structure fixed, to Y; see [24, Section 3.1]. If in addition, (Y,J) is an almost complex submanifold of an almost complex manifold (X,J), then

$$D_{J;u}\equiv D^Y_{J;u}\equiv u^*D_Y\colon L^p_1(\Sigma;u^*TY)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*TY\big)$$

is the restriction of

$$D^X_{J;u}\equiv u^*D_X\colon L^p_1(\Sigma;u^*TX)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*TX\big).$$

Thus, $D^X_{J;u}$ induces a CR-operator

$$D^{NY}_{J;u}\colon L^p_1(\Sigma;u^*NY)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*NY\big),$$

where $NY\equiv TX|_Y/TY$ is the complex normal bundle of Y in X.

The next lemma extends Serre duality from $\bar\partial$-operators to CR-operators. If D is as in (2.7), let

$$D^*=\bar\partial-A^*\colon L^p_1(\Sigma;\mathcal{K}_\Sigma\otimes_{\mathbb{C}}E^*)\longrightarrow L^p(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{K}_\Sigma\otimes_{\mathbb{C}}E^*);$$

see (2.1) and (2.3) for notation. If $S\subset\Sigma$ is a finite subset of smooth points of Σ and ϱ is a function assigning to $z_0\in S$ a complex subspace of $E^*_{z_0}$, D^* extends to an operator

$$D^*_\varrho\colon L^p_1\big(\Sigma;\mathcal{K}_\Sigma\otimes_{\mathbb{C}}E^*(\varrho)\big)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{K}_\Sigma\otimes_{\mathbb{C}}E^*(S)\big);$$

see (2.4). Let D_ϱ be the restriction of D to the closed subspace $L^p_1(\Sigma;E(-\varrho))$ of $L^p_1(\Sigma;E)$.

Lemma 2.3 *Let D be a CR-operator on a complex vector bundle $(E,\mathfrak{i})$ over a Riemann surface $(\Sigma,\mathfrak{j})$. If S is a finite subset of smooth points of Σ and ϱ is a function assigning to $z_0\in S$ a real subspace of $E^*_{z_0}$, the homomorphism*

$$\operatorname{cok}D_\varrho\longrightarrow\operatorname{Hom}_{\mathbb{R}}(\ker D^*_\varrho,\mathbb{R}),\qquad \eta\longrightarrow\langle\!\langle\eta,\cdot\rangle\!\rangle, \tag{2.10}$$

is an isomorphism.

Proof: If Σ is smooth and $S=\emptyset$, this is [11, Lemma 2.3.2]. Furthermore, by the twisting construction of [34, Lemma 2.4.1], the elements z_0 of S for which $\varrho(z_0)=E^*_{z_0}$ can be omitted from S.[12] In the general case, the proof of [11, Lemma 2.3.2] shows that the homomorphisms

$$\ker D^*_\varrho\longrightarrow\operatorname{Hom}_{\mathbb{R}}(\operatorname{cok}D_\varrho,\mathbb{R}),\qquad \ker D_\varrho\longrightarrow\operatorname{Hom}_{\mathbb{R}}(\operatorname{cok}D^*_\varrho,\mathbb{R}), \tag{2.11}$$

induced by the pairings (2.1) are well-defined and injective. It follows that

$$\operatorname{ind}D_\varrho+\operatorname{ind}D^*_\varrho\le 0$$

[12] This construction extends the usual procedure of twisting a holomorphic vector bundle by a divisor to generalized CR-operators; it can be seen as a manifestation of Carleman Similarity Principle [5, Theorem 2.2]. In this particular case, the bundle E, which is holomorphic with respect to $\bar\partial$, can be replaced by $E(-S_0)$, where $S_0\subset S$ is the subset of elements z_0 such that $\varrho(z_0)=E^*_{z_0}$.

and equality holds if and only if the homomorphisms (2.11) are isomorphisms. On the other hand, if $\widetilde{D}_\varrho$ and $\widetilde{D}^*_\varrho$ are the operators corresponding to D_ϱ and D^*_ϱ over the normalization $\sigma: \widetilde{\Sigma} \longrightarrow \Sigma$, dropping any matching conditions at the nodes and the other restricting conditions at the points of S, then

$$\begin{aligned} \operatorname{ind} D_\varrho &= \operatorname{ind} \widetilde{D}_\varrho - 2km - \|\varrho\|, \\ \operatorname{ind} D^*_\varrho &= \operatorname{ind} \widetilde{D}^*_\varrho - 2km - 2k|S| + \|\varrho\|, \end{aligned}$$

where k is the complex rank of E, m is the number of nodes in Σ, and

$$\|\varrho\| = \sum_{z_0 \in S} \dim_{\mathbb{R}} \varrho(z_0).$$

Since the kernel and cokernel of $\widetilde{D}^*_\varrho$ are isomorphic to the kernel and cokernel of a CR-operator on $T^*\widetilde{\Sigma} \otimes \sigma^* E^*$ twisted by the preimages of the nodes and the elements of S,

$$\operatorname{ind} \widetilde{D}^*_\varrho = -\operatorname{ind} \widetilde{D}_\varrho + 4km + 2k|S|.$$

It follows that $\operatorname{ind} D^*_\varrho = -\operatorname{ind} D_\varrho$ and thus the injective homomorphisms in (2.11) are in fact isomorphisms.

2.3 Families of nodal Riemann surfaces

By a stratified space (of dimension k), we will mean a topological space $\overline{\mathfrak{M}}$ together with a partition

$$\overline{\mathfrak{M}} = \bigsqcup_{l=0}^{l=k} \mathfrak{M}^{(l)}$$

such that $\mathfrak{M}^{(l)}$ is a smooth manifold of (real) dimension $k-l$ and

$$\overline{\mathfrak{M}}^{(l)} - \mathfrak{M}^{(l)} \subset \bigsqcup_{l'=l+1}^{l=k} \mathfrak{M}^{(l')}.$$

If U is an open subspace of a stratified space $\overline{\mathfrak{M}}$ as above, then

$$U = \bigsqcup_{l=0}^{l=k} (\mathfrak{M}^{(l)} \cap U)$$

is also a stratified space. If $\overline{\mathfrak{M}}_1$ and $\overline{\mathfrak{M}}_2$ are stratified spaces, $\overline{\mathfrak{M}}_1 \times \overline{\mathfrak{M}}_2$ is a stratified space with the strata given by unions of the products of the strata of $\overline{\mathfrak{M}}_1$ and $\overline{\mathfrak{M}}_2$. A continuous map $\pi: \overline{\mathfrak{M}}_1 \longrightarrow \overline{\mathfrak{M}}_2$ between stratified spaces will be called a stratified map if the restriction of π to each stratum of $\overline{\mathfrak{M}}_1$ is a smooth map to a stratum of $\overline{\mathfrak{M}}_2$. A stratified map $\pi_V: V \longrightarrow \overline{\mathfrak{M}}$ will be called a stratified vector bundle if π_V is a topological vector bundle with fiber $\mathbb{C}^k$ and the transition maps from open subsets of $\overline{\mathfrak{M}}$ to $\mathrm{GL}_k\mathbb{C}$ are stratified.

For the purposes of Definition 2.4 below, we set

$$\pi_{\mathrm{std}} \equiv \pi_1 : \mathfrak{U}_{\mathrm{std}} \equiv \big\{(t,u,v) \in \mathbb{C}^3 :\ uv = t\big\} \longrightarrow \mathbb{C}$$

to be the projection to the first component. This is a stratified map with respect to the stratifications

$$\mathbb{C}=\mathbb{C}^*\sqcup\{0\}, \qquad \mathfrak{U}_{\mathrm{std}}=\pi_{\mathrm{std}}^{-1}(\mathbb{C}^*)\sqcup\big(\pi_{\mathrm{std}}^{-1}(0)-0\big)\sqcup\{0\}.$$

For each $t\in\mathbb{C}^*$, define

$$\rho_t\colon \Sigma_t\equiv\pi_{\mathrm{std}}^{-1}(t)\longrightarrow\mathbb{R}^+ \qquad \text{by} \quad \rho_t(t,u,v)=u^2+v^2\,.$$

If in addition $\epsilon\in\mathbb{R}^+$, let

$$\Sigma_{t,\epsilon}=\big\{(t,u,v)\in\Sigma_t\colon\ |u|^2+|v|^2<\epsilon\big\}.$$

If $E\longrightarrow\Sigma_t$ is a normed vector bundle and $\eta\in L^p(\Sigma_t;E)$, let

$$\|\eta\|_{t,\epsilon}=\bigg(\int_{\Sigma_{t,\epsilon}}|\eta|^p\bigg)^{1/p}+\bigg(\int_{\Sigma_{t,\epsilon}}\rho_t^{-\frac{p-2}{p}}|\eta|^2\bigg)^{1/2}.$$

Definition 2.4 *A stratified map $\pi\colon\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ is a flat stratified family of Riemann surfaces if*

- *each fiber $\Sigma_u\equiv\pi^{-1}(u)$ is a (possibly nodal) Riemann surface;*
- *if $z_0\in\Sigma_{u_0}$ is a smooth point, there are neighborhoods U_{z_0} of u_0 in $\overline{\mathfrak{M}}$ and $\widetilde{U}_{z_0}$ of z_0 in $\mathfrak{U}$ and a stratified isomorphism of fiber bundles*

  $$\widetilde{\phi}_{z_0}\colon\widetilde{U}_{z_0}\longrightarrow U_{z_0}\times(\Sigma_{u_0}\cap\widetilde{U}_{z_0})$$

  over U_{z_0} such that the restriction of $\widetilde{\phi}_{z_0}$ to each fiber of π is holomorphic and the restriction of $\widetilde{\phi}_{z_0}$ to $\Sigma_{u_0}\cap\widetilde{U}_{z_0}$ is the identity;
- *if $z_0\in\Sigma_{u_0}$ is a node, there are neighborhoods U_{z_0} of u_0 in $\overline{\mathfrak{M}}$ and $\widetilde{U}_{z_0}$ of z_0 in $\mathfrak{U}$, a stratified space U'_{z_0}, and stratified embeddings*

  $$\phi_{z_0}\colon U_{z_0}\longrightarrow U'_{z_0}\times\mathbb{C} \qquad \textit{and} \qquad \widetilde{\phi}_{z_0}\colon\widetilde{U}_{z_0}\longrightarrow U'_{z_0}\times\mathfrak{U}_{\mathrm{std}}$$

  such that the diagram

  $$\begin{array}{ccc} \widetilde{U}_{z_0} & \xrightarrow{\widetilde{\phi}_{z_0}} & U'_{z_0}\times\mathfrak{U}_{\mathrm{std}} \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\mathrm{id}\times\pi_{\mathrm{std}}} \\ U_{z_0} & \xrightarrow{\phi_{z_0}} & U'_{z_0}\times\mathbb{C} \end{array}$$

  commutes and the restriction of $\widetilde{\phi}_{z_0}$ to each fiber of π is holomorphic.

Definition 2.5 *If S is a finite set, a stratified map $\pi\colon\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ with stratified sections $z_j\colon\overline{\mathfrak{M}}\longrightarrow\mathfrak{U}$, $j\in S$, is a flat stratified family of S-marked Riemann surfaces if*

- *$\pi\colon\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ is a flat stratified family of Riemann surfaces;*
- *$z_j(u)\in\Sigma_u$ is a smooth point for every $u\in\overline{\mathfrak{M}}$ and $j\in S$;*
- *$z_{j_1}(u)\neq z_{j_2}(z)$ for every $u\in\overline{\mathfrak{M}}$, $j_1,j_2\in S$ with $j_1\neq j_2$.*

Definition 2.6 *If* $\pi:\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ *is a flat stratified family of* S*-marked Riemann surfaces and* Y *is a smooth manifold, a continuous map* $F:\mathfrak{U}\longrightarrow Y$ *is a* flat family of S-marked maps *if*

- *for every* $u\in\overline{\mathfrak{M}}$*, the restriction of* F *to* $\Sigma_u\equiv\pi^{-1}(u)$ *is an* L^p_1*-map;*
- *if* $z_0\in\Sigma_{u_0}$ *is a smooth point and* U_{z_0}*,* $\widetilde{U}_{z_0}$*, and* $\widetilde{\phi}_{z_0}$ *are as in Definition 2.4, there exists a compact neighborhood* $K_{z_0}(F)$ *of* z_0 *in* $\Sigma_{u_0}\cap\widetilde{U}_{z_0}$ *such that* $F\circ\widetilde{\phi}_{z_0}^{-1}|_{u\times K_{z_0}(F)}$ *converges to* $F|_{K_{z_0}(F)}$ *in the* L^p_1*-norm as* $u\in U_{z_0}$ *approaches* u_0*;*
- *if* $z_0\in\Sigma_{u_0}$ *is a node and* U_{z_0}*,* $\widetilde{U}_{z_0}$*,* ϕ_{z_0}*, and* $\widetilde{\phi}_{z_0}$ *are as in Definition 2.4,*

$$\lim_{\epsilon\longrightarrow 0}\lim_{\substack{(u',t)\longrightarrow\phi_{z_0}(u)\\ (u',t)\in\phi_{z_0}(U_{z_0})}}\big\|d(F\circ\widetilde{\phi}_{z_0}^{-1}|_{u'\times\Sigma_t})\big\|_{t,\epsilon}=0\,.$$

In the case of interest to us, $\overline{\mathfrak{M}}$ will be a family of S-marked stable maps to a smooth manifold Y. The fiber of $\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ over a point $u:\Sigma_u\longrightarrow Y$ will be the Riemann surface Σ_u.

2.4 Families of generalized CR-operators

Let D be a smooth CR-operator on a vector bundle $(N,\mathfrak{i})$ over an almost complex manifold (Y,J). Suppose $\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ is a flat stratified family of S-marked Riemann surfaces, $F:\mathfrak{U}\longrightarrow Y$ is a flat family of maps, $S_0\subset S$, and ϱ is a function assigning to each $z_0\in S_0$ a real subbundle of $\mathrm{ev}_j^*N^*$. For each $u\in\overline{\mathfrak{M}}$ and $z_0\in S$, let $\varrho_u(z_0)$ be the fiber of $\varrho(z_0)$ over u. Denote by $\ker^F_{\varrho;u}(D)$ and $\ker^F_{\varrho;u}(D^*)$ the kernels of the operators

$$\begin{aligned}\big\{(F|_{\Sigma_u})^*D\big\}_{\varrho_u}&:L^p_1\big(\Sigma_u;\{F|^*_{\Sigma_u}N\}(-\varrho_u)\big)\longrightarrow L^p\big(\Sigma_u;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}F|^*_{\Sigma_u}N\big),\\ \big\{(F|_{\Sigma_u})^*D\big\}^*_{\varrho_u}&:L^p_1\big(\Sigma_u;\mathcal{K}_{\Sigma_u}\otimes_{\mathbb{C}}\{F|^*_{\Sigma_u}N\}(\varrho_u)\big)\\ &\qquad\longrightarrow L^p\big(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}\mathcal{K}_{\Sigma_u}\otimes_{\mathbb{C}}\{F|^*_{\Sigma_u}N\}(\{z_j(u)\}_{j\in S_0})\big),\end{aligned}$$

respectively.

We topologize the sets

$$\ker^F_\varrho(D)\equiv\bigsqcup_{u\in\overline{\mathfrak{M}}}\ker^F_{\varrho;u}(D)\qquad\text{and}\qquad\ker^F_\varrho(D^*)\equiv\bigsqcup_{u\in\overline{\mathfrak{M}}}\ker^F_{\varrho;u}(D^*)$$

by point-wise convergence on compact subsets of the complement of the special (nodal and marked) points of the fiber. In other words, suppose $u_r\in\overline{\mathfrak{M}}$, $r\in\mathbb{Z}^+$, is a sequence converging to $u_0\in\overline{\mathfrak{M}}$ and $\xi_r\in\ker^F_{\varrho;u_r}(D')$ for $r\in\bar{\mathbb{Z}}^+$, where $D'=D,D^*$ and $\bar{\mathbb{Z}}^+=\{0\}\sqcup\mathbb{Z}$. The sequence $\{\xi_r\}$ converges to ξ_0 if for every smooth point $z_0\in\Sigma_{u_0}$, with $z_0\neq z_j(u)$ for $j\in S$, there exists a compact neighborhood $K_{z_0}(F)$ as in Definition 2.6 such that $\xi_r\circ\widetilde{\phi}_{z_0}^{-1}|_{u_r\times K_{z_0}(F)}$ converges pointwise to $\xi_0|_{K_{z_0}(F)}$.

By Carleman Similarity Principle [5, Theorem 2.2], if the restriction of an element ξ of $\ker^F_{\varrho;u}(D')$ to an open subset of a component $\Sigma_{u;i}$ of Σ_u vanishes, then the restriction of ξ to $\Sigma_{u;i}$ is zero as well. This implies that the above convergence topology on $\ker^F_\varrho(D)$ is the topology inherited from the convergence topology on the bundle over $\overline{\mathfrak{M}}$ with fibers $L^p_1(\Sigma_u;u^*N)$ described in [16, Section 3].[13] Furthermore, if the dimension of $\ker^F_{\varrho;u}(D)$ is independent of u, then $\ker^F_{\varrho}(D)\longrightarrow\overline{\mathfrak{M}}$ is a vector bundle. By [30, Section 6], the analogous statement holds for $\ker^F_{\varrho;u}(D^*)$.[14] Lemma 2.3 then implies that $\ker^F_{\varrho}(D^*)\longrightarrow\overline{\mathfrak{M}}$ is a vector bundle if the dimension of $\ker^F_{\varrho;u}(D)$ is independent of $u\in\overline{\mathfrak{M}}$. If in addition, the vector bundles $\ker^F_{\varrho}(D)\longrightarrow\overline{\mathfrak{M}}$ and $\varrho(z_0)$, $z_0\in S$, are oriented (and S is ordered if any of the bundles $\varrho(z_0)$ is of odd rank), then the vector bundle

$$\ker^F_{\varrho}(D^*)\longrightarrow\overline{\mathfrak{M}} \tag{2.12}$$

has a canonical induced orientation, since $\ker^F_{\mathbf{0};u}(D)$ and $(\ker^F_{\mathbf{0};u}(D^*))^*$ are the kernel and cokernel of an operator obtained by a zeroth-order deformation from a first-order complex-linear Fredholm operator; the determinant line of such an operator has a canonical orientation defined via a homotopy of Fredholm operators (see the proof of [24, Theorem 3.1.5]).

3 Proof of Theorem 1.2

The first claim of Theorem 1.2 is immediate from the assumption that $f_j^{-1}(Y)$ is a smooth oriented manifold. Thus,

$$\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big]^{vir}=\bigg(\prod_{j\in S}\big\{\mathrm{ev}_j\times(f_j\circ\pi_j)\big\}^*\big(\mathrm{PD}_{Y^2}(\Delta_Y)\big)\bigg)\cap\bigg[\overline{\mathfrak{M}}_{g,S}(Y,\beta_Y;J)\times\prod_{j\in S}f_j^{-1}(Y)\bigg]^{vir},$$

where $\Delta_Y\subset Y^2$ is the diagonal and $\pi_j:\prod_{j\in S}f_j^{-1}(Y)\longrightarrow f_j^{-1}(Y)$ is the projection onto the j-th component; the identity (1.8) now follows from (1.6). Sections 2.2 and 2.4 imply the second claim of Theorem 1.2. Since the vector spaces

$$\ker\big((D^{NY}_{J;u})^*\big)\approx\mathrm{cok}\big(D^{NY}_{J;u}\big)^* \tag{3.1}$$

have constant rank and are oriented via the isomorphism (1.7), they form natural oriented bundles over the uniformizing charts for $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ described in [16, Section 3]. These bundles glue together to form an oriented vector orbi-bundle over $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$.[15] In the notation of Sections 2.2 and 2.4, this is also the bundle of the cokernels of the injective operators $D^{NY}_{J,\varrho;\mathbf{u}}\equiv(D^{NY}_{J;u})_{\varrho}$, where

$$[\mathbf{u}]\equiv\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J) \tag{3.2}$$

and ϱ is the function assigning to each element $j\in S$ the subbundle $\mathrm{Ann}(\mathrm{ev}_j^*(\mathrm{Im}\,d^{NY}f_j),\mathbb{R})$ of $\mathrm{ev}_j^*NY^*$. The identity (1.9) is immediate from (1.7) and the Index Theorem. The first part of the third claim follows immediately from Proposition 3.2 below in light of assumption (b) in Theorem 1.2.

We note that the second part of the third claim of Theorem 1.2 is consistent with the divisor relation for GW-invariants [26, Section 26.3] in the following sense. Let $f_0: M_0\longrightarrow X$ be a cobordism

[13] While [16, Section 3] concerns only the case $N=TY$, it applies to any vector bundle $N\longrightarrow Y$.

[14] While [30, Section 6] concerns only the case $N=TY$ and $S_0=\emptyset$, the argument applies to any vector bundle $N\longrightarrow Y$. Furthermore, the twisting construction of [34, Lemma 2.4.1] reduces the situation to the case $S_0=\emptyset$. By [33, Chapter 4], which builds on [32], there are Fredholm operators defining these vector spaces that form a continuous family over $\overline{\mathfrak{M}}$ and thus define a K-theory class; however, this statement is stronger than needed here.

[15] Neither the topologies of the bundles over the uniformizing charts nor the isomorphisms (3.1) depend on the Riemannian metrics over the uniformizing charts of [16, Section 3].

representative for some $\kappa_0\in H_{2n-2}(X;\mathbb{Z})$ so that f_0 is transverse to Y and to $f_j: M_j\longrightarrow X$ for every $j\in S$ and $f_0\colon f_0^{-1}(Y)\longrightarrow Y$ is transverse to $f_j\colon f_j^{-1}(Y)\longrightarrow Y$ (as maps to Y) for every $j\in S$. By the first assumption on f_0, $N^{f_0}_{w_0}Y=\{0\}$ for all $w_0\in f_0^{-1}(Y)$. By the second assumption,

$$M_j^0\equiv\big\{(w_0,w_j)\in M_0\times M_j\colon\; f_0(w_0)=f_j(w_j)\big\}$$

is a smooth compact oriented manifold and $f_j^0\equiv f_j\circ\pi_2\colon M_j^0\longrightarrow X$ is a cobordism representative for $\mathrm{PD}_X(\kappa_0)\cap\kappa_j$ for every $j\in S$. The three assumptions together imply that f_j^0 intersects Y properly and the bundle homomorphism

$$N^{f_j^0}Y\longrightarrow \pi_2^*N^{f_j}Y$$

over $f_j^{0\,-1}(Y)$ induced by the identity on $f_j^{0*}TX=\pi_2^*f_j^*TX$ is an isomorphism. Thus, for each $j\in S$, the S-tuple $\mathbf{f}_j$ obtained from $\mathbf{f}$ by replacing its j-th coordinate f_j with f_j^0 satisfies Condition (b) in Theorem 1.2 if $\mathbf{f}$ does. Let

$$\pi_0\colon\overline{\mathfrak{M}}_{g,\{0\}\sqcup S}(Y,\beta_Y;J)\longrightarrow\overline{\mathfrak{M}}_{g,S}(Y,\beta_Y;J)$$

be the forgetful map dropping the 0-th marked point and

$$\widetilde{\pi}_0\colon\overline{\mathfrak{M}}_{g,f_0\sqcup\mathbf{f}}(Y,\beta_Y;J)\longrightarrow\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$$

the map induced by π_0 (dropping the M_0-coordinate). If $[u]\in\overline{\mathfrak{M}}_{g,\{0\}\sqcup S}(Y,\beta_Y;J)$ and π_0 contracts a component $\Sigma_{u;i_0}$ of Σ_u, then

- $u|_{\Sigma_{u;i_0}}$ is constant and
- $\Sigma_{u;i_0}$ is $\mathbb{P}^1$ and contains the 0-th marked point and either
  - precisely two nodes and no other marked points, or
  - precisely one node and only one of the other marked points.

Therefore, if $[u']=\pi_0([u])$ and χ_u is the set of components of Σ_u, then the homomorphisms

$$\ker(D_{J;u}^{NY})\longrightarrow\ker(D_{J;u'}^{NY}),\qquad (\xi_i)_{i\in\chi_u}\longrightarrow(\xi_i)_{i\in\chi_u-i_0}\,,\tag{3.3}$$

$$\ker\big((D_{J;u}^{NY})^*\big)\longrightarrow\ker\big((D_{J;u'}^{NY})^*\big),\qquad (\eta_i)_{i\in\chi_u}\longrightarrow(\eta_i)_{i\in\chi_u-i_0}\,,\tag{3.4}$$

are well-defined and are in fact isomorphisms. Since (3.3) is an isomorphism, $f_0\sqcup\mathbf{f}$ satisfies the assumptions of Theorem 1.2 if and only if $\mathbf{f}$ does. Since the total spaces of the cokernel bundles are topologized using convergence of elements of $\ker(D_{J;u}^{NY})^*$ on compact subsets of smooth points, (3.4) induces an isomorphism of orbi-bundles

$$\mathrm{cok}(D_J^{NY})\longrightarrow\widetilde{\pi}_0^*\mathrm{cok}(D_J^{NY})\tag{3.5}$$

over $\overline{\mathfrak{M}}_{g,f_0\sqcup\mathbf{f}}(Y,\beta_Y;J)$; it extends over a neighborhood of $\overline{\mathfrak{M}}_{g,f_0\sqcup\mathbf{f}}(Y,\beta_Y;J)$ in the space of L^p_1-maps via the construction described at the end of Section 3.1. Thus, by the standard divisor relation,

$$\begin{split}\Big\langle e\big(\mathrm{cok}(D_J^{NY})\big)\prod_{j\in S}\psi_j^{a_j},\big[\overline{\mathfrak{M}}_{g,f_0\sqcup\mathbf{f}}(Y,\beta_Y;J)\big]^{vir}\Big\rangle\\ =\big\langle\mathrm{PD}_X\kappa_0,\beta\big\rangle\cdot\Big\langle e\big(\mathrm{cok}(D_J^{NY})\big)\prod_{j\in S}\psi_j^{a_j},\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big]^{vir}\Big\rangle\\ +\sum_{j^*\in S}\Big\langle e\big(\mathrm{cok}(D_J^{NY})\big)\psi_{j^*}^{a_{j^*}-1}\prod_{j\in S-j^*}\psi_j^{a_j},\big[\overline{\mathfrak{M}}_{g,\mathbf{f}_j}(Y,\beta_Y;J)\big]^{vir}\Big\rangle,\end{split}\tag{3.6}$$

with $\psi_j^{-1}\equiv 0$. In particular, it is sufficient to verify (1.10) under the assumption that $2g+|S|\geq 3$; this slightly simplifies the presentation.[16]

For the remainder of the paper, we assume that $2g+|S|\geq 3$. Section 3.1 sets up notation for the configuration spaces that play a central role in [6] and [16]. The main geometric observation used in the proof of Theorem 1.2 is Proposition 3.2, stated and proved in Section 3.2. Our approach to (1.10) is illustrated in Section 3.3, where (1.10) is verified in some cases, including the case of Theorem 1.5. The general case is the subject of Section 3.4.

3.1 Configuration spaces

Let X be a compact manifold, $\beta\in H_2(X;\mathbb{Z})$, g a non-negative integer, and S a finite set. We denote by $\mathfrak{X}_{g,S}(X,\beta)$ the space of equivalence classes of stable L^p_1-maps $u\colon\Sigma_u\longrightarrow X$ from genus g Riemann surfaces with S-marked points, which may have simple nodes, to X of degree β, i.e.

$$u_*[\Sigma_u]=\beta\in H_2(X;\mathbb{Z}).$$

Let $\mathfrak{X}^0_{g,S}(X,\beta)$ be the subset of $\mathfrak{X}_{g,S}(X,\beta)$ consisting of the stable maps with smooth domains. The space $\mathfrak{X}_{g,S}(X,\beta)$ is topologized in [16, Section 3] using L^p_1-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The space $\mathfrak{X}_{g,S}(X,\beta)$ is stratified by subspaces $\mathfrak{X}_{\mathcal{T}}(X)$ of stable maps from domains of the same geometric type and with the same degree distribution between the components of the domain. Each stratum is the quotient of a smooth Banach manifold $\widetilde{\mathfrak{X}}_{\mathcal{T}}(X)$ by a finite-dimensional Lie group $G_{\mathcal{T}}$; the restriction of the $G_{\mathcal{T}}$-action to any finite-dimensional submanifold of $\widetilde{\mathfrak{X}}_{\mathcal{T}}(X)$ consisting of smooth maps and preserved by $G_{\mathcal{T}}$ is smooth. The closure of the main stratum, $\mathfrak{X}^0_{g,S}(X,\beta)$, is $\mathfrak{X}_{g,S}(X,\beta)$.

If J is an almost complex structure on X, let

$$\Gamma^{0,1}_{g,S}(X,\beta;J)\longrightarrow\mathfrak{X}_{g,S}(X,\beta)$$

be the family of (TX,J)-valued $(0,1)$ L^p-forms. In other words, the fiber of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ over a point $[u]$ in $\mathfrak{X}_{g,S}(X,\beta)$ is the space

$$\Gamma^{0,1}_{g,S}(X,\beta;J)\big|_{[u]}=\Gamma^{0,1}(X,u;J)\big/\mathrm{Aut}(u),\quad\text{where}\quad\Gamma^{0,1}(X,u;J)=L^p\big(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*TX\big).$$

The total space of this family is topologized in [16, Section 3] using L^p-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The restriction of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ is the quotient of a smooth Banach vector bundle $\widetilde{\Gamma}^{0,1}_{\mathcal{T}}(X;J)$ over $\widetilde{\mathfrak{X}}_{\mathcal{T}}(X)$ by $G_{\mathcal{T}}$. The smooth sections of the bundles $\widetilde{\Gamma}^{0,1}_{\mathcal{T}}(X;J)\longrightarrow\widetilde{\mathfrak{X}}_{\mathcal{T}}(X)$ given by

$$\bar\partial_J\big([\Sigma_u,\mathfrak{j}_u;u]\big)=\bar\partial_{J,\mathfrak{j}_u}u=\frac{1}{2}\big(du+J\circ du\circ\mathfrak{j}_u\big)$$

induce sections of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ over $\mathfrak{X}_{\mathcal{T}}(X)$, which define a continuous section $\bar\partial_J$ of the family

$$\Gamma^{0,1}_{g,S}(X,\beta;J)\longrightarrow\mathfrak{X}_{g,S}(X,\beta).$$

[16] If $\beta\neq0$ and $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)\neq\emptyset$, $\langle\mathrm{PD}_X\kappa_0,\beta\rangle\neq0$ for some $\kappa_0\in H_2(X;\mathbb{Z})$ by [4, Theorem 1]. The last term in (3.6) vanishes after adding the divisor constraint κ_0 sufficiently many times.

The zero set of this section is the moduli space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ of equivalence classes of stable J-holomorphic degree β maps from genus-g curves with S-marked points into X. The section $\bar{\partial}_J$ over $\widetilde{\mathfrak{X}}_{\mathcal{T}}(X)$ is Fredholm, i.e. its linearization has finite-dimensional kernel and cokernel at every point of the zero set. The index of the linearization $D_{J;u}$ of $\bar{\partial}_J$ at $u\in\widetilde{\mathfrak{X}}_{\mathcal{T}}(X)$ such that

$$[u]\in\mathfrak{M}_{g,S}(X,\beta;J)\equiv\overline{\mathfrak{M}}_{g,S}(X,\beta;J)\cap\mathfrak{X}^0_{g,S}(X,\beta)$$

is the expected dimension $\dim_{g,S}(X,\beta)$ of the moduli space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$.

If $f_j\colon M_j\longrightarrow X$ for $j\in S$ are smooth maps, $Y\subset X$ is a submanifold, $\beta_Y\in H_2(X;\mathbb{Z})$ is such that $\iota_{Y*}\beta_Y=\beta$, and $\mathcal{T}$ is any combinatorial type of maps to X or Y of degree β or β_Y, respectively, let

$$\begin{aligned}\mathfrak{X}_{g,\mathbf{f}}(X,\beta)&=\Big\{\big([u],(w_j)_{j\in S}\big)\in\mathfrak{X}_{g,S}(X,\beta)\times\prod_{j\in S}M_j\colon\ \mathrm{ev}_j([u])=f_j(w_j)\,\forall\,j\in S\Big\},\\ \mathfrak{X}_{g,\mathbf{f}}(Y,\beta_Y)&=\mathfrak{X}_{g,\mathbf{f}}(X,\beta)\cap\bigg(\mathfrak{X}_{g,S}(Y,\beta_Y)\times\prod_{j\in S}M_j\bigg),\\ \mathfrak{X}_{\mathcal{T},\mathbf{f}}(X)&=\mathfrak{X}_{g,\mathbf{f}}(X,\beta)\cap\bigg(\mathfrak{X}_{\mathcal{T}}(X)\times\prod_{j\in S}M_j\bigg),\\ \mathfrak{X}_{\mathcal{T},\mathbf{f}}(Y)&=\mathfrak{X}_{g,\mathbf{f}}(Y,\beta_Y)\cap\bigg(\mathfrak{X}_{\mathcal{T}}(Y)\times\prod_{j\in S}M_j\bigg).\end{aligned}$$

With $\pi\colon\mathfrak{X}_{g,\mathbf{f}}(X,\beta)\longrightarrow\mathfrak{X}_{g,S}(X,\beta)$ denoting the projection map, let

$$\begin{aligned}\Gamma^{0,1}_{g,\mathbf{f}}(X,\beta;J)&=\pi^*\Gamma^{0,1}_{g,S}(X,\beta;J)\longrightarrow\mathfrak{X}_{g,\mathbf{f}}(X,\beta);\\ \Gamma^{0,1}_{g,\mathbf{f}}(Y,\beta_Y;J)&=\pi^*\Gamma^{0,1}_{g,S}(Y,\beta_Y;J)\longrightarrow\mathfrak{X}_{g,\mathbf{f}}(Y,\beta_Y).\end{aligned}$$

With a_j, $j\in S$, as in Theorem 1.2, let

$$\mathbb{L}_{\mathbf{a},\mathbf{f}}\equiv\bigoplus_{j\in S}a_j\pi^*L_j^*\longrightarrow\mathfrak{X}_{g,\mathbf{f}}(X,\beta),$$

where $L_j\longrightarrow\mathfrak{X}_{g,S}(X,\beta)$ is the tautological line bundle for the j-th marked point.

If J is an almost complex structure on X preserving Y, let g_J be a J-invariant metric on X, ∇^J the J-linear connection of g_J induced by the Levi-Civita connection of g_J, $TY^{\mathrm{v}}\subset TX|_Y$ the g_J-orthogonal complement of TY, and $\pi^{\mathrm{h}}\colon TX|_Y\longrightarrow TY$ the orthogonal projection map. Define

$$\begin{gathered}\widetilde{\nabla}^J\colon\Gamma(Y;TX)\longrightarrow\Gamma(Y;T^*Y\otimes_{\mathbb{R}}TX)\qquad\text{by}\\ \widetilde{\nabla}^J_v(\xi^{\mathrm{h}}+\xi^{\mathrm{v}})=\pi^{\mathrm{h}}\big(\nabla^J_v\xi^{\mathrm{h}}\big)+\nabla^J_v\xi^{\mathrm{v}}\qquad\forall\ v\in TY,\ \xi^{\mathrm{h}}\in\Gamma(Y;TY),\ \xi^{\mathrm{v}}\in\Gamma(Y;TY^{\mathrm{v}}).\end{gathered}$$

This connection in $TX|_Y$ gives rise to a $\mathbb{C}$-linear connection $\nabla^\perp$ on NY and thus to a $\bar{\partial}$-operator $\bar{\partial}^\perp$ on NY. Define

$$D^{NY}\colon\Gamma(Y;NY)\longrightarrow\Gamma(Y;T^*Y^{0,1}\otimes_{\mathbb{C}}NY)\qquad\text{by}\qquad D^{NY}\xi=\bar{\partial}^\perp\xi+A^\perp_X(\cdot,\xi),$$

where $A^\perp_X$ is the composition of the Nijenhuis tensor of J on X with the projection to NY. If $[u]\in\mathfrak{X}_{g,S}(Y,\beta_Y)$, let

$$D^{NY}_{J;u}\colon L^p_1(\Sigma_u;u^*NY)\longrightarrow L^p\big(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*NY\big)$$

be the pull-back of D^{NY} by u with respect to the connection $\nabla^{\perp}$ as in Section 2.2. If $[u]$ is an element of $\overline{\mathfrak{M}}_{g,S}(Y,\beta_Y;J)$, this definition agrees with the one in Section 1.1. Thus, under the assumptions of Theorem 1.2, the dimension of $\mathrm{cok}(D^{NY}_{J;u})$ is fixed on a neighborhood of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ in $\mathfrak{X}_{g,\mathbf{f}}(Y,\beta_Y)$. By Section 2.4, the vector spaces $\mathrm{cok}(D^{NY}_{J;u})$ form a vector orbi-bundle over such a neighborhood.

3.2 Symplectic submanifolds and pseudo-holomorphic maps

Definition 3.1 *If (X,J) is an almost complex manifold and $Y\subset X$ is an almost complex submanifold, a tuple $(\pi_Y\colon U_Y\longrightarrow Y, TU_Y^{\mathrm{h}})$ is a J-regularized tubular neighborhood of Y in X if*

- *U_Y is a tubular neighborhood of Y in X;*
- *$\pi_Y\colon U_Y\longrightarrow Y$ is a vector bundle such that $\pi_Y|_Y=\mathrm{id}_Y$ and $\ker d_y\pi_Y$ is a complex subspace of (T_yX,J) for every $y\in Y$;*
- *$TU_Y^{\mathrm{h}}\longrightarrow U_Y$ is a complex subbundle of (TU_Y,J) such that $d_x\pi_Y\colon TU_Y^{\mathrm{h}}\longrightarrow T_{\pi_Y(x)}Y$ is an isomorphism of real vector spaces for every $x\in U_Y$ and is the identity for every $x\in Y$.*

Every embedded almost complex submanifold Y of an almost complex manifold (X,J) admits a J-regularized tubular neighborhood. Let g be a J-invariant Riemannian metric on X and $\exp^g\colon TX\longrightarrow X$ the exponential map with respect to the Levi-Civita connection of the metric g. Identifying NY with the g-orthogonal complement of TY in $TX|_Y$, we obtain a smooth map

$$\exp^Y\colon NY\longrightarrow X$$

by restricting $\exp^g$. Since Y is an embedded submanifold of X, there exist tubular neighborhoods U_Y' and U_Y of Y in NY and in Y, respectively, such that the map

$$\exp\equiv\exp^Y|_{U_Y'}\colon U_Y'\longrightarrow U_Y$$

is a diffeomorphism. Furthermore, $\exp|_Y=\mathrm{id}_Y$ and $d_y\exp\colon T_yNY\longrightarrow T_yX$ is $\mathbb{C}$-linear for every $y\in Y$. Thus,

$$\pi_Y=\pi_{NY}\circ\exp|_{U_Y'}^{-1}\colon U_Y\longrightarrow Y,$$

where $\pi_{NY}\colon NY\longrightarrow Y$ is the bundle projection map, satisfies the middle condition in Definition 3.1.[17] Furthermore, if $(\ker d\pi_Y)^{\perp}$ is the g-orthogonal complement of $\ker d\pi_Y$ in TU_Y,

$$d_x\pi_Y\colon(\ker d_x\pi_Y)^{\perp}\longrightarrow T_{\pi_Y(x)}Y$$

is an isomorphism and induces a complex structure J_Y in the vector bundle $(\ker d\pi_Y)^{\perp}\longrightarrow U_Y$ (which may differ from J). Let

$$T_xU_Y^{\mathrm{h}}=\big\{v-JJ_Yv\colon v\in(\ker d_x\pi_Y)^{\perp}\big\}.$$

[17] Strictly speaking, $\pi_Y\colon U_Y\longrightarrow Y$ just defined is a neighborhood of the zero section in a vector bundle. This is sufficient for the purposes of Proposition 3.2 below and thus of the entire paper. However, U_Y can be given the structure of a vector bundle by composing $\exp^Y|_{U_Y'}$ with a diffeomorphism $NY\longrightarrow U_Y'$ which preserves the fibers and restricts to the identity around the zero section.

Note that $T_xU_Y^{\mathrm{h}}$ is a complex linear subspace of (T_xU_Y, J_x) for each $x\in U_Y$. Since $(\ker d_y\pi_Y)^{\perp}=T_yY$ and $J_Y|_y=J|_{T_yY}$ for every $y\in Y$,

$$d_y\pi_Y = \mathrm{id}\colon T_yU_Y^{\mathrm{h}} \longrightarrow T_{\pi_Y(y)}Y$$

for every $y\in Y$. Thus,

$$d_x\pi_Y\colon T_xU_Y^{\mathrm{h}} \longrightarrow T_{\pi_Y(x)}Y$$

is an isomorphism for every $x\in U_Y$ if U_Y is sufficiently small. We conclude that TU_Y^{h} satisfies the final condition in Definition 3.1.

Proposition 3.2 *Suppose (X,ω) is a compact symplectic manifold, $g\in\bar{\mathbb{Z}}^+$, S is a finite set, $\beta\in H_2(X;\mathbb{Z})$, and $f_j: M_j \longrightarrow X$ is a smooth map for each $j\in S$ intersecting Y properly. Let J be an ω-tame almost complex structure on X, Y a compact almost complex submanifold of (X,J), and $(\pi_Y: U_Y\longrightarrow Y, TU_Y^{\mathrm{h}})$ a J-regularized tubular neighborhood of Y in X. If $([u_r],(w_{r,j})_{j\in S})\in \mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ is a sequence such that*

$$\begin{gathered} u_r(\Sigma_{u_r})\not\subset Y, \qquad \bar\partial_J u_r\big|_{u_r^{-1}(U_Y)} \in L^p\big(u_r^{-1}(U_Y);T^*(u_r^{-1}(U_Y))^{0,1}\otimes_{\mathbb{C}} u_r^*TU_Y^{\mathrm{h}}\big), \\ \lim_{r\longrightarrow\infty}\big([u_r],(w_{r,j})_{j\in S}\big) = \big([u],(w_j)_{j\in S}\big)\in \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\subset \mathfrak{X}_{g,\mathbf{f}}(X,\beta) \end{gathered} \tag{3.7}$$

for some $\beta_Y\in H_2(Y;\mathbb{Z})$, then

$$\exists\ \ \xi\in\ker D_{J;u}^{NY},\ \ v_j\in T_{w_j}M_j\ \forall\, j\in S \qquad s.t. \qquad \xi\neq 0, \qquad \xi\big(z_j(u)\big)=d_{w_j}f_j(v_j) \quad \forall\, j\in S.$$

In the rest of this section, we prove this proposition by adopting a now-standard rescaling argument. It is sufficient to consider the case $X=NY$ as smooth manifolds and $\pi_Y: NY\longrightarrow Y$ is the bundle projection map. After passing to a subsequence, it can be assumed that the topological types of the domains Σ_{u_r} of u_r are the same (but not necessarily the same as the topological type of Σ_u). The desired vector field ξ and tangent vectors v_j will be constructed by re-scaling u_r in the normal direction to Y and then taking the limit.

For each $j\in S$, let $N_jY\subset T_{w_j}M$ be a complement of $T_{w_j}(f_j^{-1}(Y))$ and

$$\exp_j\colon T_{w_j}M_j \longrightarrow M_j$$

a diffeomorphism onto a neighborhood of w_j in M_j such that

$$\exp_j(0)=w_j, \qquad d_0\exp_j=\mathrm{Id}, \qquad \exp_j(v)\in f_j^{-1}(Y)\ \ \forall\, v\in T_{w_j}(f_j^{-1}(Y)).$$

For each $r\in\mathbb{Z}^+$, define

$$v_{r,j}^{\mathrm{h}}\oplus v_{r,j}^{\perp}\in T_{w_j}(f_j^{-1}(Y))\oplus N_jY = T_{w_j}M_j \qquad \text{by} \quad \exp_j\big(v_{r,j}^{\mathrm{h}}+v_{r,j}^{\perp}\big)=w_{r,j}\,.$$

Choose metrics on NY and N_jY, $j\in S$. By our assumptions,

$$\epsilon_r\equiv \sup_{z\in\Sigma_{u_r}}\big|u_r(z)\big|\in\mathbb{R}^+, \quad \lim_{r\longrightarrow\infty}\epsilon_r=0, \quad \lim_{r\longrightarrow\infty}v_{r,j}^{\mathrm{h}}=0\ \ \forall\, j\in S, \quad \big|v_{r,j}^{\perp}\big|\le C\epsilon_r\ \ \forall\, r\in\mathbb{Z}^+,\, j\in S,$$

for some $C\in\mathbb{R}^+$ independent of r and j (because f_j intersects Y properly). By the last condition, for each $j\in S$ (a subsequence of) the sequence

$$\widetilde{v}_{r,j}^{\perp} = \epsilon_r^{-1} v_{r,j}^{\perp}, \quad r\in\mathbb{Z}^+,$$

converges to some $v_j\in N_jY\subset T_{w_j}M_j$.

For each $r\in\mathbb{Z}^+$, we define

$$\begin{aligned}
&m_r\colon NY \longrightarrow NY && \text{by} \quad m_r(x)=\epsilon_r\cdot x;\\
&J_r\in\Gamma\big(NY;\mathrm{Hom}(T(NY),T(NY))\big) && \text{by} \quad J_r|_x=\big\{d_x m_r\big\}^{-1}\circ J_{\epsilon_r x}\circ d_x m_r;\\
&\widetilde{u}_r\colon \Sigma_{u_r}\longrightarrow NY && \text{by} \quad \widetilde{u}_r(z)=\epsilon_r^{-1}\cdot u_r(z);\\
&\eta_r\in L^p(\Sigma_{u_r};T^*\Sigma_{u_r}^{0,1}\otimes_{\mathbb{C}}\widetilde{u}_r^*T(NY)) && \text{by} \quad \eta_r=\big\{d_{\widetilde{u}_r(\cdot)}m_r\big\}^{-1}\circ\bar\partial_J u_r.
\end{aligned}$$

If in addition $j\in S$, define $\widetilde{f}_{r,j}\colon T_{w_j}M_j\longrightarrow NY$ by

$$\widetilde{f}_{r,j}\big(v^{\mathrm{h}}+v^{\perp}\big)=\epsilon_r^{-1}\cdot f_j\big(\exp_j(v^{\mathrm{h}}+\epsilon_r v^{\perp})\big) \quad \forall\, v^{\mathrm{h}}\in T_{w_j}(f_j^{-1}(Y)),\ v^{\perp}\in N_jY.$$

Then, for all $r\in\mathbb{Z}^+$,

$$\bar\partial_{J_r}\widetilde{u}_r=\eta_r, \quad \sup_{z\in\Sigma_{\widetilde{u}_r}}\big|\widetilde{u}_r(z)\big|=1, \quad \widetilde{u}_r(z_j(u_r))=\widetilde{f}_{r,j}\big(v_{r,j}^{\mathrm{h}}+\widetilde{v}_{r,j}^{\perp}\big) \ \ \forall\, j\in S. \tag{3.8}$$

By the following paragraph, the sequence of almost complex structures J_r C^∞-converges on compact subsets of NY to an almost complex structure $\widetilde{J}$ such that $\widetilde{J}|_{TY}=J|_{TY}$ and

$$\bar\partial_{\widetilde{J}}\xi=0 \iff D_{J;u}^{NY}\xi=0 \qquad \forall\, \xi\in\Gamma\big(\Sigma_u;u^*NY\big).$$

Furthermore, the sequence η_r converges to 0. Thus, by (3.8), $\widetilde{u}_r$ converges to some

$$\begin{gathered}
[\widetilde{u}]\in\overline{\mathfrak{M}}_{g,S}(NY,\beta;\widetilde{J})\subset\mathfrak{X}_{g,S}(NY,\beta) \qquad \text{s.t.}\\
\widetilde{u}(\Sigma_{\widetilde{u}})\not\subset Y, \quad \widetilde{u}(z_j(\widetilde{u}))=d_{w_j}f_j(v_j)\in N_{f_j(w_j)}Y \ \ \forall\, j\in S.
\end{gathered}$$

Since $\pi_Y\circ\widetilde{u}=u$, $\widetilde{u}$ corresponds to a section ξ of $u^*NY\longrightarrow\Sigma_u$ as needed.

It remains to prove the two local claims made above. It is sufficient to assume that

$$\pi_Y=\pi_1\colon NY=Y\times\mathbb{C}^k\longrightarrow Y$$

as vector bundles over Y, and there exists

$$\begin{gathered}
\alpha\in\Gamma\big(Y\times\mathbb{C}^k;\mathrm{Hom}_{\mathbb{R}}(\pi_1^*TY,\pi_2^*T\mathbb{C}^k)\big) \qquad \text{s.t.}\\
\alpha|_{Y\times 0}=0, \qquad T_{(y,w)}U_Y^{\mathrm{h}}=\big\{\big(y',\alpha_{(y,w)}(y')\big)\colon y'\in T_yY\big\} \qquad \forall\,(y,w)\in Y\times\mathbb{C}^k.
\end{gathered} \tag{3.9}$$

Thus, by assumption on u_r,

$$\bar\partial_J u_r=(\nu^{\mathrm{h}},\alpha_{u_r}\nu^{\mathrm{h}}) \qquad \text{for some} \quad \nu^{\mathrm{h}}\in L^p\big(\Sigma_{u_r};T^*\Sigma_{u_r}\otimes_{\mathbb{R}}u_r^{\mathrm{h}*}TY\big),$$

where $u_r^{\mathrm{h}}=\pi_1\circ u_r$. Let

$$J=\begin{pmatrix} J^{\mathrm{hh}} & J^{\mathrm{hv}} \\ J^{\mathrm{vh}} & J^{\mathrm{vv}} \end{pmatrix}: TU_Y=\pi_1^*TY\oplus\pi_2^*T\mathbb{C}^k \longrightarrow \pi_1^*TY\oplus\pi_2^*T\mathbb{C}^k$$

be the almost complex structure. By Definition 3.1, $J^{\mathrm{hv}}|_{Y\times 0}=0$ and $J^{\mathrm{vh}}|_{Y\times 0}=0$; we can also assume that $J^{\mathrm{vv}}|_{Y\times 0}=\mathfrak{i}$ is the standard complex structure on $\mathbb{C}^k$. If $\vec{\nabla}$ is the gradient with respect to the standard coordinates on $\mathbb{C}^k$, it follows that

$$\alpha_{(y,w)}=\widetilde{\alpha}_{(y,w)}w, \qquad J^{\mathrm{vh}}_{(y,w)}=\widetilde{J}^{\mathrm{vh}}_{(y,w)}w, \qquad J^{\mathrm{vv}}_{(y,w)}=\mathfrak{i}+\widetilde{J}^{\mathrm{vv}}_{(y,w)}w, \qquad \text{where}$$
$$\widetilde{\alpha}_{(y,w)}=\int_0^1\vec{\nabla}\alpha_{(y,tw)}\,dt, \qquad \widetilde{J}^{\mathrm{vh}}_{(y,w)}=\int_0^1\vec{\nabla}J^{\mathrm{vh}}_{(y,tw)}\,dt, \qquad \widetilde{J}^{\mathrm{vv}}_{(y,w)}=\int_0^1\vec{\nabla}J^{\mathrm{vv}}_{(y,tw)}\,dt.$$

This gives

$$\eta_r=\begin{pmatrix} \nu^{\mathrm{h}} \\ \epsilon_r^{-1}\{\widetilde{\alpha}_{u_r}u_r\}\nu_r^{\mathrm{h}} \end{pmatrix}\longrightarrow 0\,,$$
$$J_r|_{(y,w)}=\begin{pmatrix} J^{\mathrm{hh}}_{(y,\epsilon_r w)} & \epsilon_r J^{\mathrm{hv}}_{(y,\epsilon_r w)} \\ \epsilon_r^{-1}J^{\mathrm{vh}}_{(y,\epsilon_r w)} & J^{\mathrm{vv}}_{(y,\epsilon_r w)} \end{pmatrix}\longrightarrow \begin{pmatrix} J_{T_yY} & 0 \\ \widetilde{J}^{\mathrm{vh}}_{(y,0)}w & \mathfrak{i} \end{pmatrix}\equiv \widetilde{J}_{(y,w)}\,,$$
$$D_{J;u}\begin{pmatrix} \xi^{\mathrm{h}} \\ \xi^{\mathrm{v}} \end{pmatrix}=\begin{pmatrix} \bar{\partial}\xi^{\mathrm{h}} \\ \bar{\partial}\xi^{\mathrm{v}}+\frac{1}{2}\{\widetilde{J}^{\mathrm{vh}}_{(y,0)}\xi^{\mathrm{v}}\}du\circ\mathfrak{j} \end{pmatrix};$$

the last identity is a special case of [24, (3.1.4)]. This concludes the proof of Proposition 3.2.

3.3 Geometric motivation for (1.10)

In this section we give a rough argument for (1.10) before translating it into the virtual setting of [6] and [16] in Section 3.4. As explained at the end of this section, this argument suffices in some cases. We continue with the notation of Theorem 1.2 and Section 3.1. For the remainder of the paper, we assume that (1.4) holds; otherwise, the left-hand side of (1.10) vanishes by definition, while the right-hand side vanishes by (1.8) and (1.9). Our assumption implies that

$$\dim_{g,\mathbf{f}}(Y,\beta_Y)\equiv\dim\left[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\right]^{vir}=2\sum_j a_j+\mathrm{rk}_{\mathbb{R}}\mathrm{cok}(D_J^{NY}). \tag{3.10}$$

We also assume that $a_j\geq 0$ for every $j\in S$.

If ν is a sufficiently small multi-section of $\Gamma^{0,1}_{g,\mathbf{f}}(X,\beta;J)$ over $\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$, the space

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu)=\{\bar{\partial}_J+\nu\}^{-1}(0)\subset\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$$

is compact, because $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ is. If in addition ν is smooth and generic in the appropriate sense, $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu)$ is stratified by smooth branched orbifolds of even dimensions. If φ is a multi-section of the orbi-bundle $\mathbb{L}_{\mathbf{a},\mathbf{f}}\longrightarrow\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$, let

$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J,\nu)=\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu)\cap\varphi^{-1}(0). \tag{3.11}$$

If ν is sufficiently small and generic and φ is generic, the left-hand side of (1.10) is the number of elements of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J,\nu)$ counted with appropriate multiplicities that lie in a small neighborhood of

$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J) \equiv \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\cap\varphi^{-1}(0)$$

in $\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$.

In order to verify (1.10), fix a J-regularized tubular neighborhood $(\pi_Y\colon U_Y\longrightarrow Y, TU_Y^{\mathrm{h}})$. We will take $\nu=\nu_Y+\nu_X$ so that

- for every $\mathbf{u}=([u],(w_j)_{j\in S})\in\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ with $[u]\in\mathfrak{X}_{g,S}(U_Y,\beta_Y)$,
$$\nu_Y(\mathbf{u}) \in L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}TU_Y^{\mathrm{h}});$$
- $\nu_Y|_{\mathfrak{X}_{g,\mathbf{f}}(Y,\beta_Y)}$ is generic, so that $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is stratified by smooth branched manifolds of the expected dimensions and the dimension of the main stratum
$$\mathfrak{M}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y) \equiv \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\Big(\mathfrak{X}^0_{g,S}(Y,\beta_Y)\times\prod_{j\in S}M_j\Big)$$
is $\dim_{g,\mathbf{f}}(Y,\beta_Y)$;
- ν_X is generic and small relative to ν_Y.

Using π_Y, $d\pi_Y|_{TU_Y^{\mathrm{h}}}^{-1}$, and a bump function around Y with support in U_Y, any section of

$$\pi^*\Gamma^{0,1}_{g,S}(Y,\beta_Y;J) \longrightarrow \mathfrak{X}_{g,S}(Y,\beta_Y)\times\prod_{j\in S}M_j$$

can be extended to a section of $\Gamma^{0,1}_{g,\mathbf{f}}(X,\beta;J)$ over $\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ satisfying the first condition above. In light of Proposition 3.2, this condition implies that there exists an open neighborhood $\mathcal{U}(\nu_Y)$ of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ in $\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ such that

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu_Y)\cap\mathcal{U}(\nu_Y) = \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y).$$

In addition, choose a multi-section φ of the bundle $\mathbb{L}_{\mathbf{f},\mathbf{a}}\longrightarrow\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ so that φ is transverse to the zero set on every stratum of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ and every stratum of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu)$. This implies that the dimension of every stratum of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is at most the rank (1.9) of the bundle $\mathrm{cok}(D_J^{NY})$ over $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ and the equality holds only for the main stratum.

By the first assumption on ν_Y above, for every element $[u]$ of $\overline{\mathfrak{M}}_{g,S}(Y,\beta_Y;J,\nu_Y)$ the linearization

$$D^X_{J,\nu_Y;u}\colon\mathcal{H}_u\oplus L^p_1(\Sigma_u;u^*TX)\longrightarrow L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*TX)$$

of the section $\bar\partial_J+\nu_Y$ for maps to X restricts to the linearization

$$D^Y_{J,\nu_Y;u}\colon\mathcal{H}_u\oplus L^p_1(\Sigma_u;u^*TY)\longrightarrow L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*TY)$$

of the section $\bar\partial_J+\nu_Y$ for maps to Y. Thus, $D^X_{J,\nu_Y;u}$ descends to a Fredholm operator

$$D^{NY}_{J,\nu_Y;u}\colon L^p_1(\Sigma_u;u^*NY)\longrightarrow L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*NY).$$

If ν_Y is sufficiently small, by the last assumption in Theorem 1.2 the operator

$$D^{NY}_{J,\nu_Y,\varrho;\mathbf{u}} \equiv \big(D^{NY}_{J,\nu_Y;u}\big)_\varrho\colon \big\{\xi\in L^p_1(\Sigma_u;u^*NY)\colon\ \xi(z_j(u))\in \mathrm{Im}\, d^{NY}_{w_j} f_j\ \forall\, j\in S\big\} \\ \longrightarrow L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*NY)$$

is injective for every $[\mathbf{u}]\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ as in (3.2). Thus, the cokernels of these operators still form an oriented vector orbi-bundle over $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ of rank (1.9), which will be denoted by $\mathrm{cok}(D^{NY}_{J,\nu_Y,\varrho})$. Furthermore, $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is compact if ν_Y is sufficiently small (because $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ is) and is a union of connected components of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu_Y)$ by Proposition 3.2.

The left-hand side of (1.10) is the number of elements of

$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J,\nu_Y+\nu_X)\subset \mathfrak{X}_{g,S}(X,\beta)\times\prod_{j\in S}M_j$$

that lie in a small neighborhood of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ for any sufficiently small and generic ν_X. The map component of any such element must be of the form $\exp_{u_\upsilon}\xi$, where

- $([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is an element of a fixed stratum, i.e. the topological structure of Σ_u is fixed;
- υ is a small gluing parameter for Σ_u consisting of the smoothings of the nodes of Σ_u;
- $u_\upsilon\colon\Sigma_{u_\upsilon}\longrightarrow Y$ is the approximately (J,ν_Y)-map corresponding to υ as in [37, Section 3];
- $\xi\in L^p_1(\Sigma_\upsilon;u_\upsilon^*TX)$ is small with respect to the $\|\cdot\|_{\upsilon,p,1}$-norm of [16, Section 3] and satisfies

$$\begin{aligned}&\{\bar\partial_J+\nu_Y\}u_\upsilon + D_{J,\nu_Y;u_\upsilon}\xi+\nu_X(u_\upsilon)+N_\upsilon(\xi)=0,\\ &\xi(z_j(u_\upsilon))\in\mathrm{Im}\,(d^{NY}_{w_j}f_j)+T_{f_j(w_j)}Y\quad\forall\, j\in S,\end{aligned}\tag{3.12}$$

  where N_υ is a combination of a term quadratic in ξ and a term which is linear in ξ and ν_X.

Projecting (3.12) to NY, we obtain

$$\begin{aligned}&D^{NY}_{J,\nu_Y;u_\upsilon}\zeta+\nu_X^{\perp}(u_\upsilon)+N_\upsilon^{\perp}(\zeta)=0,\\ &\zeta\in L^p_1(\Sigma_{u_\upsilon};u_\upsilon^*NY),\quad \zeta(z_j(u_\upsilon))\in\mathrm{Im}\,(d^{NY}_{w_j}f_j)\quad\forall\, j\in S.\end{aligned}\tag{3.13}$$

This equation has no small solutions in $\varphi^{-1}(0)$ away from the subset of elements

$$\mathbf{u}\equiv([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$$

for which $\nu_X^{\perp}(\mathbf{u})$ lies in the image of $D^{NY}_{J,\nu_Y,\varrho;\mathbf{u}}$, i.e. the projection $\bar\nu_X(\mathbf{u})$ of $\nu_X(\mathbf{u})$ to $\mathrm{cok}(D^{NY}_{J,\nu_Y,\varrho;\mathbf{u}})$ is zero. For dimensional reasons, all zeros of $\bar\nu_X$ lie in the main stratum

$$\mathfrak{M}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\equiv\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\mathfrak{M}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y).$$

Thus, only $\mathfrak{M}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ contributes to the left-hand side in (1.10). In this case equation (3.13) no longer involves υ and thus $u_\upsilon=u$. Since φ vanishes transversally on $\mathfrak{M}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ and $\bar\nu_X$

on $\mathfrak{M}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$, the contribution of the main stratum to the left-hand side is the signed cardinality of the oriented zero-dimensional orbifold

$$\mathfrak{M}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\bar{\nu}_X^{-1}(0).$$

As $\bar{\nu}_X$ extends to a continuous multi-section of the orbi-bundle

$$\mathrm{cok}(D^{NY}_{J,\nu_Y,\varrho})\longrightarrow\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y), \tag{3.14}$$

which is transverse to the zero set over every stratum, the left-hand side of (1.10) is the euler class of the bundle (3.14) evaluated on $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$. While the operators $D^{NY}_{J,\nu_Y;\mathbf{u}}$ and $D^{NY}_{J;u}$ are not the same, they are homotopic through operators keeping the dimension of the cokernels fixed and thus define orbi-bundles with the same euler class, as needed.

The above argument requires some notion of smoothness for the strata of $\mathfrak{X}_{\mathcal{T},\mathbf{f}}(X)$ or at least $\mathfrak{X}_{\mathcal{T},\mathbf{f}}(Y)$. If the domain curve Σ_u of $[u]$ with its marked points is stable for every element $([u],(w_j)_{j\in S})$ of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$, then every stratum $\mathfrak{X}_{\mathcal{T},\mathbf{f}}(X)$ meeting $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ is a smooth Banach orbifold. The topological aspects of the resulting setting are sorted out in [21], and the above argument suffices in such cases. These include the cases of Theorem 1.5 (with $2g+|S|\geq 3$, which can be assumed) and Corollary 1.4 (since the genus of $Y=\alpha^{-1}(0)$ is at least 2), but not of Example A or the specific cases of Examples B or C.

In general, $\mathfrak{X}_{\mathcal{T}}(X)$ is a subspace of a product of main strata $\mathfrak{X}^0_{g_i,S_i}(X,\beta_i)$ for some g_i, S_i, and β_i and the restriction of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ is the direct sum of the pull-backs of the corresponding bundles over the components of the product. If for every $([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ and every unstable component $\Sigma_{u;i}$ of Σ_u the restriction of u to $\Sigma_{u;i}$ is regular in the appropriate sense, then ν can be taken to be a smooth section of the components of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ coming from the "stable parts" of $\mathcal{T}$; as in the previous paragraph there is a well-defined notion of smoothness over these components. This is done explicitly in [31, Section 2]. The resulting extension of the previous paragraph then covers the specific cases of Examples B and C.

Finally, for an arbitrary symplectic manifold (X,ω), the notion of "smoothness" is described by introducing smooth finite-dimensional approximations to $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$. This is done in the next section.

3.4 Virtual setting

Continuing with the notation of Section 3.1, we now recall the virtual fundamental class setup of [6] and [16] and then reformulate the argument of Section 3.3 for (1.10) in the general case.

An atlas for $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ is a collection $\{(\mathcal{U}_\alpha,E_\alpha)\}_{\alpha\in\mathcal{A}}$, where

- $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ in $\mathfrak{X}_{g,S}(X,\beta)$ and $E_\alpha\subset\Gamma^{0,1}_{g,S}(X,\beta;J)|_{\mathcal{U}_\alpha}$ is a topological (finite-rank) vector orbi-bundle over $\mathcal{U}_\alpha$;
- $\bar{\partial}_J^{-1}(E_\alpha)$ is a smooth orbifold and $\bar{\partial}_J^{-1}(E_\alpha)\cap\mathfrak{X}_T(X)$ is a smooth sub-orbifold of $\bar{\partial}_J^{-1}(E_\alpha)$ of the codimension corresponding to $\mathcal{T}$ (twice the number of nodes) for every stratum $\mathfrak{X}_T(X)$;

- the restriction of E_α to $\bar\partial_J^{-1}(E_\alpha)$ is a smooth vector orbi-bundle and the restriction of $\bar\partial_J$ to $\bar\partial_J^{-1}(E_\alpha)$ is a smooth section of $E_\alpha|_{\bar\partial_J^{-1}(E_\alpha)}$;
- for every $[u]\in\overline{\mathfrak{M}}_{g,S}(X,\beta;J)\cap\bar\partial_J^{-1}(E_\alpha)\cap\bar\partial_J^{-1}(E_{\alpha'})$, there exists $\gamma\in\mathcal{A}$ such that
$$[u]\in\mathcal{U}_\gamma\subset\mathcal{U}_\alpha\cap\mathcal{U}_{\alpha'}\,,\qquad E_\alpha,E_{\alpha'}\big|_{U_\gamma}\subset E_\gamma\,,$$
the restrictions of E_α and $E_{\alpha'}$ to $\bar\partial_J^{-1}(E_\gamma)\cap\mathfrak{X}_T(X)$ are smooth orbifold subbundles of the restriction of E_γ, and the restriction of $\bar\partial_J$ to $\bar\partial_J^{-1}(E_\gamma)\cap\mathfrak{X}_T(X)$ is transverse to E_α and $E_{\alpha'}$;
- for every $[u]\in\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$,
$$\Gamma^{0,1}(X,u;J)=\big\{D_{J;u}\xi\colon\,\xi\in L^p_1(\Sigma_u;u^*TX)\big\}+\widetilde{E}_\alpha|_u\,,\tag{3.15}$$
where $\widetilde{E}_\alpha|_u\subset\widetilde{\Gamma}^{0,1}_{\mathcal{T}}(X;J)|_u$ is the preimage of $E_\alpha|_u$ under the quotient map
$$\widetilde{\Gamma}^{0,1}_{\mathcal{T}}(X;J)|_u\longrightarrow\Gamma^{0,1}_{g,S}(X,\beta;J)|_{[u]}\,.$$

Such collections $\{(\mathcal{U}_\alpha,E_\alpha)\}_{\alpha\in\mathcal{A}}$ are described in [6, Section 12] and [16, Section 3]. An atlas for $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ is defined similarly, with the domain of $D_{J;u}$ in (3.15) replaced by
$$\big\{\xi\in L^p_1(\Sigma_u;u^*TX)\colon\,\xi(z_j(u))\in\operatorname{Im}d_{w_j}f_j\ \forall\,j\in S\big\}$$
for an element $([u],(w_j)_{j\in S})$ of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$. Such an atlas induces a compatible atlas for the total space of the restriction of the bundle $\mathbb{L}_{\mathbf{a},\mathbf{f}}$ to $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$.

A multi-section ν of $\Gamma^{0,1}_{g,\mathbf{f}}(X,\beta;J)$ for an atlas $\{(\mathcal{U}_\alpha,E_\alpha)\}_{\alpha\in\mathcal{A}}$ is a continuous multi-section such that the restriction of ν to $\bar\partial_J^{-1}(E_\alpha)$ is a smooth section of E_α. Similarly, a multi-section φ of $\mathbb{L}_{\mathbf{a},\mathbf{f}}$ for $\{(\mathcal{U}_\alpha,E_\alpha)\}_{\alpha\in\mathcal{A}}$ is a continuous multi-section such that the restriction of φ to $\bar\partial_J^{-1}(E_\alpha)$ is smooth. A multi-section ν as above is regular if the restriction of ν to $\bar\partial_J^{-1}(E_\alpha)\cap\mathfrak{X}_{\mathcal{T},\mathbf{f}}(X)$ is transverse to the zero set in E_α for every α and $\mathcal{T}$. If $(\{(\mathcal{U}_\alpha,E_\alpha)\}_{\alpha\in A},\nu)$ is regular, $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu)$ is stratified by smooth branched orbifolds of even dimensions. The existence of regular multi-sections for a refinement of a subatlas is the subject of [6, Chapter 1] and [22, Section 4].[18] If ν is sufficiently small and regular and φ is generic, the left-hand side of (1.10) is again the weighted number of elements of
$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J,\nu)\equiv\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu)\cap\varphi^{-1}(0)$$
that lie in a small neighborhood of
$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J)\equiv\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\cap\varphi^{-1}(0)$$
in $\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$.

By [6, Chapter 3] and [16, Section 3], pairs $(\mathcal{U}_{Y;\alpha},E_{Y;\alpha})$ for an atlas for
$$\overline{\mathfrak{M}}_{g,S}(Y,\beta_Y;J)\times\prod_{j\in S}M_j$$
that restrict to an atlas for $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ can be obtained in the following way. Given $\mathbf{u}=([u],(w_j)_{j\in S})$, choose

[18] It is also shown in [6] and [22] that a regular multi-section ν determines a rational homology class; however, this notion of virtual fundamental class is not necessary for defining GW-invariants or comparing the two sides of (1.10).

- a neighborhood $V_{Y;u}$ of $u(\Sigma_u)$ in Y;
- a representative $u\colon \Sigma_u \longrightarrow Y$ for $[u]$;
- universal family of deformations $\mathcal{W}_u \longrightarrow \Delta_u$ of Σ_u with its marked points (thus $\Sigma_u \subset \mathcal{W}_u$);
- a finite-dimensional subspace

$$\mathcal{E}_{Y;\mathbf{u}} \subset \Gamma_c\big(\mathcal{W}_u^* \times V_{Y;u}; \pi_1^*(T^*\mathcal{W}_u^v)^{0,1} \otimes_{\mathbb{C}} \pi_2^* TY\big),$$

where $\mathcal{W}_u^* \subset \mathcal{W}_u$ is the subspace of smooth points of the fibers, $T\mathcal{W}_u^v \subset T\mathcal{W}_u$ is the vertical tangent space, and Γ_c denotes the space of smooth compactly supported bundle sections, such that

$$\Gamma(\Sigma_u; T^*\Sigma_u^{0,1} \otimes_{\mathbb{C}} u^*TY) = \big\{D_u\xi \colon \xi \in \Gamma(\Sigma_u; u^*TY),\ \xi(z_i(u)) \in \mathrm{Im}\, d_{w_j} f_j \ \forall\, j \in S\big\} + \{\mathrm{id}\times u\}^* \mathcal{E}_{Y;\mathbf{u}}$$

if $\mathbf{u} \in \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y, \beta_Y; J)$; if $\mathbf{u} \notin \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y, \beta_Y; J)$, the point-wise condition on ξ is omitted.

If $\mathbf{u}' = ([u'], (w'_j)_{j \in S})$ with $[u'] \in \mathfrak{X}_{g,S}(V_{Y;u}, \beta_Y)$ and $\Sigma_{u'} \in \Delta_u$, let

$$\widetilde{E}_{Y;\mathbf{u}}|_{\mathbf{u}'} = \{\mathrm{id}\times u'\}^* \mathcal{E}_{Y;\mathbf{u}}\,.$$

By [6, Chapter 3] and [16, Section 3], $\mathcal{U}_{Y;\alpha}$ can be taken to be the image of a sufficiently small neighborhood $\widetilde{\mathcal{U}}_{Y;\alpha}$ of $\mathbf{u}$ in the space of L^p_1-maps from the fibers of $\mathcal{W}_u \longrightarrow \Delta_u$ to X under the equivalence relation and $E_{Y;\alpha}$ the image of the bundle formed by the spaces $\widetilde{E}_{Y;\mathbf{u}}|_{\mathbf{u}'}$ over $\widetilde{\mathcal{U}}_{Y;\alpha}$. With these choices, $\bar{\partial}_J^{-1}(E_{Y;\alpha})$ consists of equivalence classes of smooth maps to Y.

Fix a J-regularized tubular neighborhood $(\pi_Y \colon U_Y \longrightarrow Y, TU_Y^{\mathrm{h}})$ of Y in X. Using π_Y and $d\pi_Y|_{TU_Y^{\mathrm{h}}}^{-1}$, each $\mathcal{E}_{Y;\mathbf{u}}$ can be extended to a finite-dimensional subspace

$$\mathcal{E}_{X|Y;\mathbf{u}} \subset \Gamma_c\big(\mathcal{W}_u^* \times V_{X;u}; \pi_1^*(T^*\mathcal{W}_u^v)^{0,1} \otimes_{\mathbb{C}} \pi_2^* TU_Y^{\mathrm{h}}\big) \subset \Gamma_c\big(\mathcal{W}_u^* \times V_{X;u}; \pi_1^*(T^*\mathcal{W}_u^v)^{0,1} \otimes_{\mathbb{C}} \pi_2^* TX\big)$$

for a neighborhood $V_{X;u}$ of $V_{Y;u}$ in $U_Y \subset Y$. A larger subspace

$$\mathcal{E}_{X;\mathbf{u}} \subset \Gamma_c\big(\mathcal{W}_u^* \times V_{X;u}; \pi_1^*(T^*\mathcal{W}_u^v)^{0,1} \otimes_{\mathbb{C}} \pi_2^* TX\big)$$

can then be chosen so that

$$\Gamma(\Sigma_u; T^*\Sigma_u^{0,1} \otimes_{\mathbb{C}} u^*TX) = \big\{D_u\xi \colon \xi \in \Gamma(\Sigma_u; u^*TX),\ \xi(z_i(u)) \in \mathrm{Im}\, d_{w_j} f_j \ \forall\, j \in S\big\} + \{\mathrm{id}\times u\}^* \mathcal{E}_{X;\mathbf{u}}\,,$$

whenever $[u] \in \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y, \beta_Y; J)$. This gives rise to a pair $(\mathcal{U}_{X;\alpha}, E_{X;\alpha})$ for an atlas for $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X, \beta; J)$; the union of such pairs covers $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y, \beta_Y; J)$. Since $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y, \beta_Y; J)$ is a union of components of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X, \beta; J)$, this sub-collection of an atlas is sufficient for determining the left-hand side of (1.10). Similarly, using π_Y, $d\pi_Y|_{TU_Y^{\mathrm{h}}}^{-1}$, and a bump function around Y with support in U_Y, any multi-section of

$$\pi_1^*\Gamma_{g,S}^{0,1}(Y, \beta_Y; J) \longrightarrow \mathfrak{X}_{g,S}(Y, \beta_Y) \times \prod_{j \in S} M_j$$

for the atlas $(\{(\mathcal{U}_{Y;\alpha}, E_{Y;\alpha})\}_{\alpha \in \mathcal{A}})$ gives rise to a multi-section ν of

$$\Gamma_{g,\mathbf{f}}^{0,1}(X, \beta; J) \longrightarrow \mathfrak{X}_{g,\mathbf{f}}(X, \beta)$$

for the atlas $(\{(\mathcal{U}_{X;\alpha}, E_{X;\alpha})\}_{\alpha\in\mathcal{A}})$ such that

$$\nu([\mathbf{u}]) \in L^p\big(\Sigma_u; T^*\Sigma_u^{0,1}\otimes u^*TU_Y^{\mathrm{h}}\big)$$

for every $\mathbf{u}=([u],(w_j)_{j\in S})\in\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ with $[u]\in\mathfrak{X}_{g,S}(U_Y,\beta_Y)$.

Let $\nu=\nu_Y+\nu_X$ be a regular multi-section of $\Gamma^{0,1}_{g,\mathbf{f}}(X,\beta)$ for atlas for $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ as above so that

- for every $\mathbf{u}=([u],(w_j)_{j\in S})\in\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ with $[u]\in\mathfrak{X}_{g,S}(U_Y,\beta_Y)$,
$$\nu_Y(\mathbf{u}) \in L^p(\Sigma_u; T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*TU_Y^{\mathrm{h}});$$
- $\nu_Y|_{\mathfrak{X}_{g,\mathbf{f}}(Y,\beta_Y)}$ is a regular multi-section of $\Gamma^{0,1}_{g,\mathbf{f}}(Y,\beta_Y)$ so that $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is stratified by smooth branched orbifolds of the expected dimensions and the dimension of the main stratum
$$\mathfrak{M}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\equiv\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\left(\mathfrak{X}^0_{g,S}(Y,\beta_Y)\times\prod_{j\in S}M_j\right)$$
is $\dim_{g,\mathbf{f}}(Y,\beta_Y)$;
- ν_X is small relative to ν_Y.

The previous paragraph implies that such multi-sections ν_Y exist. By Proposition 3.2, the first condition implies that there exists an open neighborhood $\mathcal{U}(\nu_Y)$ of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ in $\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ such that
$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu_Y)\cap\mathcal{U}(\nu_Y)=\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y).$$
In addition, choose a multi-section φ of the bundle $\mathbb{L}_{\mathbf{f},\mathbf{a}}\longrightarrow\mathfrak{X}_{g,\mathbf{f}}(X,\beta)$ for the above atlas so that φ is transverse to the zero set on every stratum of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ and every stratum of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu)$.

For each $\alpha\in\mathcal{A}$ and $\mathbf{u}\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\mathcal{U}_{Y;\alpha}$, let
$$\mathcal{D}_{\nu_Y,\alpha;\mathbf{u}}\colon T_{\mathbf{u}}\bar\partial_J^{-1}(E_{X;\alpha})\longrightarrow E_{X;\alpha}$$
be the linearization of the section $\bar\partial_J+\nu_Y$ over $\bar\partial_J^{-1}(E_{X;\alpha})$ along the zero set. The kernel of $\mathcal{D}_{\nu_Y,\alpha;\mathbf{u}}$ is the tangent space of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ at $\mathbf{u}$. If α and γ are as in the overlap condition in the definition of an atlas above, then

$$\begin{gathered}E_{X;\alpha}\cap\operatorname{Im}\mathcal{D}_{\nu_Y,\gamma;\mathbf{u}}=\operatorname{Im}\mathcal{D}_{\nu_Y,\alpha;\mathbf{u}}\quad\forall\,\mathbf{u}\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\mathcal{U}_{Y;\gamma}\,,\\ \dim\bar\partial_J^{-1}(E_{X;\gamma})-\dim\bar\partial_J^{-1}(E_{X;\alpha})=\operatorname{rk}E_{X;\gamma}-\operatorname{rk}E_{X;\alpha}.\end{gathered}$$

Thus, the inclusion $T\bar\partial_J^{-1}(E_{X;\alpha})\longrightarrow T\bar\partial_J^{-1}(E_{X;\gamma})$ over $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\mathcal{U}_{Y;\gamma}$ induces isomorphisms
$$\operatorname{cok}(\mathcal{D}_{\nu_Y,\alpha;\mathbf{u}})\longrightarrow\operatorname{cok}(\mathcal{D}_{\nu_Y,\gamma;\mathbf{u}}).$$
It follows that these vector spaces form an orbi-bundle $\operatorname{cok}(\mathcal{D}_{\nu_Y})$ over $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$. By the last requirement in the definition of an atlas and condition (b) in Theorem 1.2, the homomorphism

$$\operatorname{cok}(\mathcal{D}_{\nu_Y,\alpha;\mathbf{u}})\longrightarrow\operatorname{cok}(D^{NY}_{J;\mathbf{u}})$$

induced by the inclusion $E_{X;\alpha} \longrightarrow \Gamma^{0,1}_{g,\mathbf{f}}(X,\beta;J)$ followed by the projections to NY and the cokernel is surjective for all $\mathbf{u}\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\mathcal{U}_{Y;\alpha}$, if ν_Y is sufficiently small. A dimension count then shows that this homomorphism is an isomorphism (the injectivity also follows from Proposition 3.2). Thus, the orbi-bundles

$$\mathrm{cok}(\mathcal{D}_{\nu_Y}), \mathrm{cok}(D_J^{NY}) \longrightarrow \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$$

are isomorphic.

The left-hand side of (1.10) is the number of elements of

$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J,\nu_Y+\nu_X) \subset \mathfrak{X}_{g,S}(X,\beta) \times \prod_{j\in S} M_j$$

that lie in a small neighborhood of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ for a small generic multi-section ν_X. The number of such elements near $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\mathcal{U}_{Y;\alpha}$ is the number of solutions of

$$\mathcal{D}_{\nu_Y,\alpha;\mathbf{u}}\xi + \nu_X(\mathbf{u}) + N_\alpha(\xi) = 0, \qquad \xi \in T_{\mathbf{u}}\bar{\partial}_J^{-1}(E_{X;\alpha}),$$

with small ξ, where N_α is a combination of a term quadratic in ξ and a term which is linear in ξ and ν_X. This equation has no solutions in $\varphi^{-1}(0)$ away from the subset of elements

$$\mathbf{u} \in \overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$$

for which $\nu_X(\mathbf{u})$ lies in the image of $\mathcal{D}_{\nu_Y,\alpha;\mathbf{u}}$, i.e. the projection $\bar{\nu}_X(\mathbf{u})$ to $\mathrm{cok}(\mathcal{D}_{\nu_Y,\alpha;\mathbf{u}})$ is zero. Since φ vanishes transversally on $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ and $\bar{\nu}_X$ on $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$, the left-hand side of (1.10) is the signed cardinality of oriented zero-dimensional orbifold

$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y) \cap \varphi^{-1}(0).$$

By the definition, this is also the euler class of $\mathrm{cok}(\mathcal{D}_{\nu_Y})$ evaluated on $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$, which by the above isomorphism of cokernel bundles equals to the right-hand side of (1.10).

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