

MAT 562: Symplectic Geometry

Problem Set 7

Due by 12/09, in class

(if you have not passed the orals yet)

Two of the exercises from Chapter 2 of Ana's *Seminar on Symplectic Toric Manifolds*, Exercises IV.12-14 from Audin's book, and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

For $N \in \mathbb{Z}^+$, define $[N] = \{1, \dots, N\}$ and

$$\tilde{H}_{\mathbb{C}^N} : \mathbb{C}^N \longrightarrow \mathbb{R}^N, \quad \tilde{H}_{\mathbb{C}^N}(z_1, \dots, z_N) = \pi(|z_1|^2, \dots, |z_N|^2), \quad \tilde{\mu}_{\mathbb{C}^N} = \sum_{j=1}^N (-y_j dx_j + x_j dy_j).$$

For a $n \times N$ \mathbb{Z} -matrix A , let

$$\iota_A : \mathbb{T}_A \equiv \ker(A : \mathbb{T}^N \equiv \mathbb{R}^N / \mathbb{Z}^N \longrightarrow \mathbb{T}^n \equiv \mathbb{R}^n / \mathbb{Z}^n) \longrightarrow \mathbb{T}^N$$

be the inclusion, $\iota_A^* : \mathbb{R}^N = T_{\mathbb{1}}^* \mathbb{T}^N \longrightarrow T_{\mathbb{1}}^* \mathbb{T}_A$ be the restriction homomorphism induced by ι_A , and

$$\tilde{H}_A \equiv \iota_A^* \circ \tilde{H}_{\mathbb{C}^N} : \mathbb{C}^N \longrightarrow T_{\mathbb{1}}^* \mathbb{T}_A \tag{1}$$

be a moment map for the restriction of the standard coordinate-wise \mathbb{T}^N -action on \mathbb{C}^N to $\mathbb{T}_A \subset \mathbb{T}^N$. Let $(\mathbb{T}_A)_{\mathbb{C}} \subset \mathbb{T}_{\mathbb{C}}^N$ be the complexification of $\mathbb{T}_A \subset \mathbb{T}^N$ and $(\mathbb{T}_A)_i \subset (\mathbb{T}_A)_{\mathbb{C}}$ be the purely imaginary subgroup (it corresponds to a subgroup of $(\mathbb{R}^*)^N \subset (\mathbb{C}^*)^N$ via $e^{2\pi i \cdot}$). The group $\mathbb{T}_{\mathbb{C}}^N \approx (\mathbb{C}^*)^N$ acts on \mathbb{C}^N by the coordinate-wise multiplication in the usual way. If in addition $\alpha \in T_{\mathbb{1}}^* \mathbb{T}_A$, let

$$\begin{aligned} P_A^\alpha &= \{s \in (\mathbb{R}^{\geq 0})^N : \iota_A^*(s) = \alpha\}, \quad \tilde{Z}_A^\alpha = \tilde{H}_A^{-1}(\alpha) = \tilde{H}_{\mathbb{C}^N}^{-1}(P_A^\alpha) \subset \mathbb{C}^N, \\ \mathcal{V}_A^\alpha &= \{J \subset [N] : |J| = n, P_A^\alpha \cap (\mathbb{R}^{\geq 0})^{[N]-J} \neq \emptyset\}, \quad \tilde{M}_A^\alpha = \mathbb{C}^N - \bigcup_{\substack{J \subset [N] \\ P_A^\alpha \cap (\mathbb{R}^{\geq 0})^J = \emptyset}} \mathbb{C}^J, \quad M_A^\alpha \equiv \tilde{M}_A^\alpha / (\mathbb{T}_A)_{\mathbb{C}}. \end{aligned} \tag{2}$$

For $J \subset [N]$, let A_J be the $n \times |J|$ submatrix of A consisting of the columns indexed by J .

Problem P (counts as 2 exercises)

Let $P \equiv \{r \in \mathbb{R}^n : v_k \cdot r \geq c_k \ \forall k \in [N]\}$ be a Delzant polytope with the inward normals v_1, \dots, v_N to the facets (codimension 1 faces) meeting at each vertex of P forming a \mathbb{Z} -basis for \mathbb{Z}^N . Define

$$A = (v_1 \ \dots \ v_N) : (\mathbb{R}^N, \mathbb{Z}^N) \longrightarrow (\mathbb{R}^n, \mathbb{Z}^n), \quad c = (c_1, \dots, c_N) \in \mathbb{R}^N, \quad \alpha = -\iota_A^*(c) \in T_{\mathbb{1}}^* \mathbb{T}_A.$$

Thus, $\tilde{Z}_A^\alpha \subset \mathbb{C}^N$ is the preimage of the regular value $\alpha \in T_{\mathbb{1}}^* \mathbb{T}^*$ of \tilde{H}_A , \mathbb{T}_A acts freely on \tilde{Z}_A^α , and

$$(M_P, \omega_P) \equiv (\tilde{Z}_A^\alpha, \omega_{\mathbb{C}^N}|_{T\tilde{Z}_A^\alpha}) / \mathbb{T}_A$$

is the compact connected symplectic manifold obtained from P via the Delzant construction in class. Show that

- (a) the open subspace $\tilde{M}_A^\alpha \subset \mathbb{C}^N$ contains \tilde{Z}_A^α and is path-connected, simply connected, preserved by the $\mathbb{T}_{\mathbb{C}}^N$ -action, and acted on freely by $(\mathbb{T}_A)_{\mathbb{C}} \subset \mathbb{T}_{\mathbb{C}}^N$;
- (b) the smooth map $(\mathbb{T}_A)_i \times \tilde{Z}_A^\alpha \longrightarrow \mathbb{C}^N$, $(g, z) \longrightarrow g \cdot z$, is a diffeomorphism onto \tilde{M}_A^α ;
- (c) the inclusions $\tilde{Z}_A^\alpha \longrightarrow \tilde{M}_A^\alpha$ and $\mathbb{T}_A \longrightarrow (\mathbb{T}_A)_{\mathbb{C}}$ induce a homeomorphism

$$M_A^\alpha \equiv \tilde{M}_A^\alpha / (\mathbb{T}_A)_{\mathbb{C}} \longrightarrow \tilde{Z}_A^\alpha / \mathbb{T}_A \equiv M_P$$

with respect to the quotient topologies;

- (d) the smooth manifold $M_A^\alpha = M_P$ is simply connected and admits a complex manifold structure, compatible with the smooth and symplectic structures, so that the quotient projection $q : \tilde{M}_A^\alpha \longrightarrow M_A^\alpha$ is a holomorphic submersion and $(\mathbb{T}_A)_{\mathbb{C}}$ acts on M_A^α by biholomorphisms.

Problem Q (counts as 2 exercises)

Let A be a $n \times N$ \mathbb{Z} -matrix, $\mathcal{R}_A \subset T_{\mathbb{1}}^* \mathbb{T}_A$ be the subset of regular values of the map \tilde{H}_A defined in (1), and $\alpha \in T_{\mathbb{1}}^* \mathbb{T}_A$. Show that

- (a) the subset $P_A^\alpha \subset \mathbb{C}^N$ defined in (2) is bounded if and only if $P_A^0 = \{0\}$;
- (b) $\alpha \in \mathcal{R}_A$ if and only if $\alpha \notin \iota_A^*((\mathbb{R}^{\geq 0})^J)$ for any $J \subset [N]$ with $|J| < N - \text{rk } A$;
- (c) the subset $\mathcal{R}_A \subset T_{\mathbb{1}}^* \mathbb{T}_A$ of regular values of \tilde{H}_A is open.

Suppose in addition $\alpha \in T_{\mathbb{1}}^* \mathbb{T}_A$ is regular value of \tilde{H}_A . Let $\mathcal{R}_A^\alpha \subset \mathcal{R}_A$ be the connected component containing α . Show that

- (d) $\mathcal{R}_A^\alpha \subset T_{\mathbb{1}}^* \mathbb{T}_A$ is preserved by the multiplication by \mathbb{R}^+ ;
- (e) \mathbb{T}_A acts freely on \tilde{Z}_A^α if and only if $\det A_J \in \{\pm 1\}$ for every $J \in \mathcal{V}_A^\alpha$;
- (f) $(\mathbb{T}_A)_{\mathbb{C}}$ acts freely on \tilde{M}_A^α if and only if $\det A_J \in \{\pm 1\}$ for every $J \in \mathcal{V}_A^\alpha$;
- (g) $\mathcal{V}_A^{\alpha'} = \mathcal{V}_A^\alpha$ and $\tilde{M}_A^{\alpha'} = \tilde{M}_A^\alpha$ if $\alpha' \in \mathcal{R}_A^\alpha$;
- (h) $(\mathbb{T}_A)_{\mathbb{C}}$ acts freely on $\tilde{M}_A^{\alpha'}$ and $M_A^{\alpha'} = M_A^\alpha$ as complex manifolds if $\alpha' \in \mathcal{R}_A^\alpha$ and $(\mathbb{T}_A)_{\mathbb{C}}$ acts freely on \tilde{M}_A^α .

Problem R (counts as 2 exercises)

Let M be a smooth manifold. A connection in a (smooth) complex line bundle $L \rightarrow M$ is a \mathbb{C} -linear map

$$\nabla: \Gamma(M; L) \rightarrow \Gamma(M; T^*M \otimes L) \quad \text{s.t.} \quad \nabla(fs) = df \otimes s + f \nabla s \quad \forall s \in \Gamma(M; L), f \in C^\infty(M; \mathbb{C}).$$

Such a map extends to L -valued p -forms by

$$\nabla: \Gamma(M; \Lambda^p(T^*M) \otimes L) \rightarrow \Gamma(M; \Lambda^{p+1}(T^*M) \otimes L), \quad \nabla(\eta \otimes s) = (d\eta) \otimes s + (-1)^p \eta \otimes (\nabla s).$$

A connection ∇ in L is compatible with a Hermitian inner-product $\langle \cdot, \cdot \rangle$ on L if

$$w(\langle s_1, s_2 \rangle) = \langle \nabla_w s_1, s_2 \rangle + \langle s_1, \nabla_w s_2 \rangle \quad \forall s_1, s_2 \in \Gamma(M; L), w \in TM.$$

A connection 1-form on a (smooth) principal S^1 -bundle $S \rightarrow M$ is an S^1 -invariant 1-form μ on (the total space of) S such that

$$\mu(\zeta_S) = 2\pi, \quad \text{where} \quad \zeta_S \in \Gamma(S; TS), \quad \zeta_S(\tilde{x}) = \left. \frac{d}{dt} (e^{2\pi i t} \cdot \tilde{x}) \right|_{t=0} \quad \forall \tilde{x} \in S.$$

- (a) Suppose ∇ is a connection in a complex line bundle $L \rightarrow M$. Show that there exists a \mathbb{C} -valued 2-form κ_∇ on M so that

$$\nabla(\nabla \tilde{\eta}) = \kappa_\nabla \wedge \tilde{\eta} \quad \forall \tilde{\eta} \in \Gamma(M; \Lambda^p(T^*M) \otimes L), p \in \mathbb{Z}^{\geq 0}.$$

- (b) Show that the S^1 -invariance condition on a connection 1-form μ in a principal S^1 -bundle $\pi_S: S \rightarrow M$ can be equivalently replaced by the condition $\iota_{\zeta_S} d\mu = 0$. Furthermore, for any such μ , there exists an \mathbb{R} -valued 2-form κ_μ on M so that $d\mu = \pi_S^* \kappa_\mu$.
- (c) Show that a complex line bundle $L \rightarrow M$ with a Hermitian inner-product $\langle \cdot, \cdot \rangle$ corresponds to a principal S^1 -bundle $S \rightarrow M$ (and vice versa), while a connection ∇ in L compatible with $\langle \cdot, \cdot \rangle$ corresponds to a connection 1-form μ in the associated principal S^1 -bundle S so that $\kappa_\nabla = i\kappa_\mu$.

Note. The 2-forms κ_∇ and κ_μ above are called the curvature forms of ∇ and α , respectively. By a Čech cohomology computation (p141 in Griffiths&Harris) and (c),

$$c_1(L) = \frac{i}{2\pi} [\kappa_\nabla] = -\frac{1}{2\pi} [\kappa_\mu] \in H_{\text{deR}}^2(M),$$

if ∇ is a connection in a complex line bundle $L \rightarrow M$ and α is a connection 1-form in an associated principal S^1 -bundle.

Problem S (counts as 2 exercises)

Let A be a $n \times N$ \mathbb{Z} -matrix, $c \equiv (c_1, \dots, c_N) \in \pi\mathbb{Z}^N$, $\alpha = -t_A^*(c) \in T_{\mathbb{1}}^*\mathbb{T}_A$, and

$$\zeta_{\mathbb{C}} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \in \Gamma(\mathbb{C}; T\mathbb{C}).$$

For $r \equiv (r_1, \dots, r_N) \in \mathbb{R}^N$, let $\zeta_r \in \Gamma(\mathbb{C}^N; T\mathbb{C}^N)$ be the vector field given by

$$\zeta_r(z) = \left. \frac{d}{dt} (e^{2\pi i r_1 t} z_1, \dots, e^{2\pi i r_N t} z_N) \right|_{t=0} \quad \forall z \equiv (z_1, \dots, z_N) \in \mathbb{C}^n.$$

- (a) Show that the 1-form $\tilde{\mu} \equiv (-\tilde{\mu}_{\mathbb{C}^N}) \oplus \tilde{\mu}_{\mathbb{C}}$ on $\mathbb{C}^N \times \mathbb{C}$ is invariant under the \mathbb{T}^N -action on $\mathbb{C}^N \times \mathbb{C}$ given by

$$(e^{2\pi i r_1}, \dots, e^{2\pi i r_N}) \cdot (z, z') = ((e^{-2\pi i r_1} z_1, \dots, e^{-2\pi i r_N} z_N), e^{2i(c_1 r_1 + \dots + c_N r_N)} z') \quad (3)$$

and $\tilde{\mu}(\zeta_{-r}, 2\pi a \zeta_{\mathbb{C}}) = 2r \cdot \tilde{H}_{\mathbb{C}^N} + 2a \tilde{H}_{\mathbb{C}}$ for every $a \in \mathbb{R}$.

Suppose in addition \mathbb{T}_A acts freely on \tilde{Z}_A^α . Show that

- (b) the quotient $\tilde{Z}_A^\alpha \times_{\mathbb{T}_A} S^1$ of $\tilde{Z}_A^\alpha \times S^1$ by the restriction of the action of (3) to \mathbb{T}_A is a principal S^1 -bundle over $\tilde{Z}_A^\alpha / \mathbb{T}_A$;
- (c) $\tilde{\mu}$ descends to a connection 1-form μ_α on the above principal S^1 -bundle with $p^* \kappa_{\mu_\alpha} = -2\omega_{\mathbb{C}^N} |_{\tilde{Z}_A^\alpha}$, where $p: \tilde{Z}_A^\alpha \rightarrow \tilde{Z}_A^\alpha / \mathbb{T}_A$ is the quotient projection;
- (d) the quotient L_α of $\tilde{M}_A^\alpha \times \mathbb{C}$ by the restriction of the complexification of the action of (3) to $(\mathbb{T}_A)_{\mathbb{C}}$ is a holomorphic line bundle over M_A^α with

$$c_1(L_\alpha) = \frac{1}{\pi} [\omega_A^\alpha] \in H_{\text{deR}}^2(\tilde{M}_A^\alpha),$$

where ω_A^α is the symplectic form on $\tilde{Z}_A^\alpha / \mathbb{T}_A = M_A^\alpha$ determined by $p^* \omega_A^\alpha = \omega_{\mathbb{C}^N} |_{\tilde{Z}_A^\alpha}$.

- (e) Let $P \subset \mathbb{C}^N$, A , and $\alpha \in T_{\mathbb{1}}^*\mathbb{T}_A$ be as in Problem P. Show that the complex manifold M_A^α is projective.