

# MAT 562: Symplectic Geometry

## Problem Set 1

**Due by 09/11, in class**

*(if you have not passed the orals yet)*

Two of the exercises from Section 3.1 of the main book and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

### Problem A (counts as 2 exercises)

The  $\mathbb{C}^*$ -action on  $\mathbb{C}^n$  by the coordinate multiplication restricts to a  $\mathbb{C}^*$ -action on  $\mathbb{C}^n - \{0\}$  and  $S^1$ -actions on  $\mathbb{C}^n$  and the unit sphere  $S^{2n-1} \subset \mathbb{C}^n$ . Show that

- (a) the quotient topologies on  $\mathbb{C}P^{n-1}$  given by  $(\mathbb{C}^n - \{0\})/\mathbb{C}^*$  and  $S^{2n-1}/S^1$  are the same (i.e. the map  $S^{2n-1}/S^1 \rightarrow (\mathbb{C}^n - \{0\})/\mathbb{C}^*$  induced by inclusions is a homeomorphism);
- (b)  $\mathbb{C}P^{n-1}$  is a compact topological  $2(n-1)$ -manifold that admits a complex structure so that the quotient projections

$$q: \mathbb{C}^n - \{0\} \rightarrow \mathbb{C}P^{n-1} = (\mathbb{C}^n - \{0\})/\mathbb{C}^* \quad \text{and} \quad p: S^{2n-1} \rightarrow \mathbb{C}P^{n-1} = S^{2n-1}/S^1$$

are a holomorphic submersion and a smooth submersion, respectively;

- (c) the  $S^1$ -action on  $\mathbb{C}^n$  preserves the standard symplectic form  $\omega_{\mathbb{C}^n}$  on  $\mathbb{C}^n$ ;
- (d) the orbits of the restriction of this action to  $S^{2n-1}$  are compact connected one-dimensional submanifolds of  $S^{2n-1}$ ;
- (e) for each  $z \in S^{2n-1}$  the  $\omega_{\mathbb{C}^n}$ -symplectic complement of  $T_z S^{2n-1}$ ,

$$(T_z S^{2n-1})^{\omega_{\mathbb{C}^n}} \equiv \{v \in T_z \mathbb{C}^n : \omega_{\mathbb{C}^n}(v, w) = 0 \ \forall w \in T_z S^{2n-1}\},$$

is the tangent space to the  $S^1$ -orbit at  $z$ ;

- (f) there is a unique 2-form  $\omega_{\mathbb{C}P^{n-1}}$  on  $\mathbb{C}P^{n-1}$  such that  $p^* \omega_{\mathbb{C}P^{n-1}} = \omega_{\mathbb{C}^n}|_{T S^{2n-1}}$ , and this form  $\omega_{\mathbb{C}P^{n-1}}$  is symplectic.

### Problem B (counts as 1 exercise)

Let  $Y$  be a smooth manifold,  $\lambda_{T^*Y}$  be the canonical 1-form on  $T^*Y$ , and  $\omega_{T^*Y} \equiv -d\lambda_{T^*Y}$  be the canonical symplectic form on  $T^*Y$ .

- (a) Suppose  $y \in Y$ . Show that there is a canonical decomposition  $T_y(T^*Y) = T_y Y \oplus T_y^* Y$  and

$$\omega_{T^*Y}|_y(v, w) = \begin{cases} 0, & \text{if } v, w \in T_y Y \text{ or } v, w \in T_y^* Y; \\ w(v), & v \in T_y Y \text{ and } w \in T_y^* Y. \end{cases}$$

Suppose instead that  $\alpha \in \Omega^1(Y)$ . Show that

- (b)  $\alpha^* \lambda_{T^*Y} = \alpha$ ;
- (c) the map  $\phi_\alpha : T^*Y \rightarrow T^*Y$ ,  $\phi_\alpha(\theta) = \alpha_{\pi(\theta)} - \theta$ , is a smooth involution (i.e.  $\phi_\alpha \circ \phi_\alpha = \text{id}_{T^*Y}$ ) satisfying  $\phi_\alpha^* \lambda_{T^*Y} = \pi^* \alpha - \lambda_{T^*Y}$ , where  $\pi : T^*Y \rightarrow Y$  is the bundle projection;
- (d) the involution  $\phi_\alpha$  above is anti-symplectic (i.e.  $\phi_\alpha^* \omega_{T^*Y} = -\omega_{T^*Y}$ ) if and only if  $d\alpha = 0$ .