SYMPLECTIC GEOMETRY

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These are lecture notes for two courses, taught at the University of Toronto in Spring 1998 and in Fall 2000. Our main sources have been the books "Symplectic Techniques" by Guillemin-Sternberg and "Introduction to Symplectic Topology" by McDuff-Salamon, and the paper "Stratified symplectic spaces and reduction", Ann. of Math. 134 (1991) by Sjamaar-Lerman.

Contents

CHAPTER 1

Linear symplectic algebra

1. Symplectic vector spaces

Let E be a finite-dimensional, real vector space and E^* its dual. The space $\wedge^2 E^*$ can be identified with the space of skew-symmetric bilinear forms $\omega : E \times E \to \mathbb{R}$, $\omega(v,w) = -\omega(w,v)$.

DEFINITION 1.1. The pair (E,ω) is called a symplectic vector space if $\omega \in \wedge^2 E^*$ is non-degenerate, that is, if the kernel

$$\ker \omega := \{v \in E |\, \omega(v,w) = 0 \text{ for all } w \in E\}$$

is trivial. Two symplectic vector spaces (E_1,ω_1) and (E_2,ω_2) are called *symplectomorphic* if there is an isomorphism $A : E_1 \to E_2$ with $A^*\omega_2 = \omega_1$. The group of symplectomorphisms of (E,ω) is denoted $\mathrm{Sp}(E)$.

Since $\mathrm{Sp}(E)$ is a closed subgroup of $\mathrm{Gl}(E)$, it is (by a standard theorem of Lie group theory) a Lie subgroup.

EXAMPLE 1.2. Let $E = \mathbb{R}^{2n}$ with basis $\{e_1,\ldots,e_n,f_1,\ldots,f_n\}$. Then

$$\omega(e_i,e_j) = 0,\ \omega(f_i,f_j) = 0,\ \omega(e_i,f_j) = \delta_{i,j}. \tag{1}$$

defines a symplectic structure on E. Examples of symplectomorphisms are $A(e_j) = f_j,\ A(f_j) = -e_j$ or $A(e_j) = e_j + f_j,\ A(f_j) = f_j$. Also

$$A(e_j) = \sum_k B_{jk} e_k,\ A(f_j) = \sum_k (B^{-1})_{kj} f_k,$$

for any invertible $n \times n$-matrix B, is a symplectomorphism.

EXAMPLE 1.3. Let V be a real vector space of dimension n, and V^* its dual space. Then $E = V \oplus V^*$ has a natural symplectic structure: $\omega((v,\alpha),(v',\alpha')) = \alpha'(v) - \alpha(v')$. If $B : V \to V$ is any isomorphism and $B^* : V^* \to V^*$ the dual map, $B \oplus (B^*)^{-1} : E \to E$ is a symplectomorphism.

EXAMPLE 1.4. Let E be a complex vector space of complex dimension n, with complex, positive definite inner product (=Hermitian metric) $h : E \times E \to \mathbb{C}$. Then E, viewed as a real vector space, with bilinear form the imaginary part $\omega = \mathrm{Im}(h)$ is a symplectic vector space. Every unitary map $E \to E$ preserves h, hence also ω and is therefore symplectic.

EXERCISE 1.5. Show that these three examples of symplectic vector spaces are in fact symplectomorphic.

2. Subspaces of a symplectic vector space

DEFINITION 2.1. Let (E, ω) be a symplectic vector space. For any subspace $F \subseteq E$, we define the ω-perpendicular space F^ω by

$$F^\omega = \{v \in E,\ \omega(v, w) = 0 \text{ for all } w \in F\}$$

With our assumption that E is finite dimensional, ω is non-degenerate if and only if the map

$$\omega^\flat : E \to E^*,\ \langle \omega^\flat(v), w\rangle = \omega(v, w)$$

is an isomorphism. F^ω is the pre-image of the annihilator $\mathrm{ann}(F) \subset E^*$ under $\omega^\flat$. From this it follows immediately that

$$\dim F^\omega = \dim E - \dim F$$

and

$$(F^\omega)^\omega = F.$$

DEFINITION 2.2. A subspace $F \subseteq E$ of a symplectic vector space is called

(a) isotropic if $F \subseteq F^\omega$,
(b) co-isotropic if $F^\omega \subseteq F$
(c) Lagrangian if $F = F^\omega$,
(d) symplectic if $F \cap F^\omega = \{0\}$.

The set of Lagrangian subspaces of E is called the *Lagrangian Grassmannian* and denoted $\mathrm{Lag}(E)$.

Notice that F is isotropic if and only if F^ω is co-isotropic. For example, every 1-dimensional subspace is isotropic and every codimension 1 subspace is co-isotropic.

EXAMPLE 2.3. In the above example $E = \mathbb{R}^{2n}$, let $L = \mathrm{span}\{g_1, \ldots, g_n\}$ where for all i, $g_i = e_i$ or $g_i = f_i$. Then L is a Lagrangian subspace.

LEMMA 2.4. *For any symplectic vector space* (E, ω) *there exists a Lagrangian subspace* $L \in \mathrm{Lag}(E)$.

PROOF. Let L be an isotropic subspace of E, which is maximal in the sense that it is not contained in any isotropic subspace of strictly larger dimension. Then L is Lagrangian: For if $L^\omega \neq L$, then choosing any $v \in L^\omega \backslash L$ would produce a larger isotropic subspace $L \oplus \mathrm{span}(v)$. $\square$

An immediate consequence is that any symplectic vector space E has even dimension: For if L is a Lagrangian subspace, $\dim E = \dim L + \dim L^\omega = 2 \dim L$.

Lemma 2.4 can be strengthened as follows.

LEMMA 2.5. *Given any finite collection of Lagrangian subspaces $M_1, \ldots, M_r$, one can find a Lagrangian subspace L with $L \cap M_j = \{0\}$ for all j.*

PROOF. Let L be an isotropic subspace with $L \cap M_j = \{0\}$ and not properly contained in a larger isotropic subspace with this property. We claim that L is Lagrangian. If not, L^ω is a coisotropic subspace properly containing L. Let $\pi: L^\omega \to L^\omega/L$ be the quotient map. Choose any 1-dimensional subspace $F \subset L^\omega/L$, such that both F is transversal to all $\pi(M_j \cap L^\omega)$. This is possible, since $\pi(M_j \cap L^\omega)$ is isotropic and therefore has positive codimension. Then $L' = \pi^{-1}(F)$ is an isotropic subspace with $L \subset L'$ and $L' \cap M_j = \{0\}$. This contradiction shows $L = L^\omega$. □

3. Symplectic bases

THEOREM 3.1. *Every symplectic vector space (E, ω) of dimension $2n$ is symplectomorphic to $\mathbb{R}^{2n}$ with the standard symplectic form from Example 1.2.*

PROOF. Pick two transversal Lagrangian subspaces $L, M \in \mathrm{Lag}(E)$. The pairing

$$L \times M \to \mathbb{R},\ (v, w) \mapsto \omega(v, w)$$

is non-degenerate. In other words, the composition

$$M \hookrightarrow E \xrightarrow{\omega^\flat} E^* \to L^*$$

(where the last map is dual to the inclusion $L \hookrightarrow E$) is an isomorphism. Let $e_1, \ldots, e_n$ be a basis for L and $f_1, \ldots, f_n$ the dual basis for $L^* \cong M$. By definition of the pairing, ω is given in this basis by (1). □

DEFINITION 3.2. A basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of (E, ω) for which ω has the standard form (1) is called a *symplectic basis*.

Our proof of Theorem 3.1 has actually shown a little more:

COROLLARY 3.3. *Let (E_i, ω_i), $i = 1, 2$ be two symplectic vector spaces of equal dimension, and $L_i, M_i \in \mathrm{Lag}(E_i)$ such that $L_i \cap M_i = \{0\}$. Then there exists a symplectomorphism $A\colon E_1 \to E_2$ such that $A(L_1) = L_2$ and $A(M_1) = M_2$.*

(Compare with isometries of inner product spaces, which are much more rigid!) In the following section we give an alternative proof of Theorem 3.1 using complex structures.

4. Compatible complex structures

Recall that a complex structure on a vector space V is an automorphism $J: V \to V$ such that $J^2 = -\mathrm{Id}$.

DEFINITION 4.1. A complex structure J on a symplectic vector space (E, ω) is called ω-compatible if

$$g(v, w) = \omega(v, Jw)$$

defines a positive definite inner product. This means in particular that J is a symplectomorphism:

$$(J^*\omega)(v,w) = \omega(Jv,Jw) = g(Jv,w) = g(w,Jv) = \omega(w,J^2v) = -\omega(w,v) = \omega(v,w).$$

We denote by $\mathcal{J}(E,\omega)$ the space of compatible complex structures.

We equip $\mathcal{J}(E,\omega)$ with the subset topology induced from $\mathrm{End}(E)$. Later we will see that it is in fact a smooth submanifold.

EXAMPLE 4.2. In Example 1.2, a compatible almost complex structure J is given by $Je_i = f_i,\ Jf_i = -e_i$. This identifies $(\mathbb{R}^{2n},\omega,J)$ with $\mathbb{C}^n$.

A compatible complex complex structure makes E into a Hermitian vector space (= complex inner product space), with Hermitian metric

$$h(v,w) = g(v,w) + \sqrt{-1}\omega(v,w).$$

That is, h is complex-linear with respect to the second entry and complex-antilinear with respect to the first entry,

$$h(v,Jw) = \sqrt{-1}h(v,w),\ h(Jv,w) = -\sqrt{-1}h(v,w),$$

and $h(v,v) > 0$ for $v \neq 0$. We will show below that $\mathcal{J}(E,\omega) \neq \emptyset$. Assuming this for a moment, let $J \in \mathcal{J}(E,\omega)$ and pick an orthonormal complex basis $e_1 \ldots e_n$. Let $f_i = Je_i$. Then $e_1,\ldots,e_n,f_1,\ldots,f_n$ is a *symplectic* basis:

$$\omega(e_i,f_j) = \mathrm{Im}(h(e_i,Je_j)) = \mathrm{Im}(\sqrt{-1}h(e_i,e_j)) = \delta_{ij},\quad \omega(e_i,e_j) = \mathrm{Im}(h(e_i,e_j)) = 0$$

and similarly $\omega(f_i,f_j) = 0$. This is the promised alternative proof of Theorem 3.1.

The next Theorem gives a convenient method for constructing compatible complex structures. For any vector space V let $\mathrm{Riem}(V)$ denote the convex open subset of the space S^2V^* of symmetric bilinear forms, consisting of positive definite inner products.

THEOREM 4.3. *Let (E,ω) be a symplectic vector space. There is a canonical continuous surjective map*

$$F:\ \mathrm{Riem}(E) \to \mathcal{J}(E,\omega).$$

The map $G:\ \mathcal{J}(E,\omega) \to \mathrm{Riem}(E), J \mapsto g$ associating to each compatible complex structure the corresponding Riemannian structure is a section, i.e., $F \circ G(J) = J$.

PROOF. Given $k \in \mathrm{Riem}(E)$ let $A \in \mathrm{Gl}(E)$ be defined by

$$k(v,w) = \omega(v,Aw)$$

Since ω is skew-symmetric, A is skew-adjoint (with respect to k): $A^T = -A$. It follows that in the polar decomposition $A = J|A|$ with $|A| = (A^TA)^{1/2} = (-A^2)^{1/2}$, J and $|A|$ commute. Therefore $J^2 = -\mathrm{Id}$. The equation

$$\omega(v,Jw) = \omega(v,A|A|^{-1}w) = k(v,|A|^{-1}w) = k(|A|^{-1/2}v,|A|^{-1/2}w)$$

shows that $g(v,w) = \omega(v, Jw)$ defines a positive definite inner product. We thus obtain a continuous map $F : \mathrm{Riem}(E) \to \mathcal{J}(E,\omega)$. By construction it satisfies $F \circ G = \mathrm{id}$, in particular it is surjective. $\square$

COROLLARY 4.4. *The space $\mathcal{J}(E,\omega)$ is contractible. (In particular, any two compatible complex structures can be deformed into each other.)*

PROOF. Let $X = \mathrm{Riem}(E)$ and $Y = \mathcal{J}(E,\omega)$. The space X is contractible since it is a convex subset of a vector space. Choose a contraction $\phi : \ I \times X \to X$, where $\phi_0 = \mathrm{Id}_X$ and ϕ_1 is the map onto some point in X. Then $\psi = F \circ \phi \circ (\mathrm{Id} \times G)$ is the required retraction of Y. $\square$

Given a Lagrangian subspace L of E, any orthonormal basis $e_1, \ldots, e_n$ of L is a an orthornormal basis for E viewed as a complex Hermitian vector space. The map taking this to an orthonormal basis $e_1', \ldots, e_n'$ of $L' \in \mathrm{Lag}(E)$ is unitary. Hence $\mathrm{U}(E)$ acts transitively on the set of Lagrangian subspaces. The stabilizer in $\mathrm{U}(E)$ of $L \in \mathrm{Lag}(E)$ is canonically identified with the orthogonal group $\mathrm{O}(L)$. This shows:

COROLLARY 4.5. *Any choice of $L \in \mathrm{Lag}(E)$ and $J \in \mathcal{J}(E,\omega)$ identifies the set of Lagrangian subspaces of E with the homogeneous space*

$$\mathrm{Lag}(E) = \mathrm{U}(E)/\,\mathrm{O}(L).$$

Hence $\mathrm{Lag}(E)$ is a manifold of dimension $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

5. The group Sp(E) of linear symplectomorphisms

Let (E,ω) be a symplectic vector space of dimension $\dim E = 2n$, and $\mathrm{Sp}(E) \subset \mathrm{Gl(E)}$ the Lie group of symplectomorphisms $A : \ E \to E$, $A^*\omega = \omega$. Its dimension can be found as follows: Since any two symplectic vector spaces of the same dimension are symplectomorphic, the general linear group $\mathrm{Gl}(E)$ acts transitively on the open subset of $\wedge^2 E^*$ consisting of non-degenerate 2-forms. The stabilizer at ω is $\mathrm{Sp}(E)$. It follows that

$$\dim \mathrm{Sp}(E) = \dim \mathrm{Gl}(E) - \dim \wedge^2 E^* = (2n)^2 - \frac{(2n)(2n-1)}{2} = 2n^2 + n.$$

The Lie algebra $\mathfrak{sp}(E)$ of $\mathrm{Sp}(E)$ consists of all $\xi \in \mathfrak{gl}(E)$ such that $\omega(\xi v, w) + \omega(v, \xi w) = 0$.

The following example (really a repetition of example 1.3) shows in particular that $\mathrm{Sp}(E)$ is not compact.

EXAMPLE 5.1. Let $L, M \in \mathrm{Lag}(E)$ be transversal Lagrangian subspaces, and identify $M = L^*$ so that $E = L \oplus L^*$. Given $B \in \mathrm{Gl}(L)$ let $B^* \in \mathrm{Gl}(L^*)$ the dual map. Then $A = B \oplus (B^{-1})^*$ is a symplectomorphism. Thus for any splitting $E = L \oplus M$ there is a canonical embedding

$$\mathrm{Gl}(L) \to \mathrm{Sp}(E).$$

as a closed subgroup. Note that any $A \in \mathrm{Sp}(E)$ preserving L, M is of this type.

EXAMPLE 5.2. Another natural subgroup of $\mathrm{Sp}(E)$ is the group $U(E) \subset \mathrm{Sp}(E)$ of automorphisms preserving the Hermitian structure for a given compatible complex structure $J \in \mathcal{J}(E,\omega)$.

Let us now fix a compatible complex structure $J \in \mathcal{J}(E,\omega)$. Let $\mathrm{U}(E) \subset \mathrm{Sp}(E)$ denote the unitary group and g the inner product defined by J, ω. If J' is another compatible complex structure, the map $A : E \to E$ taking an orthonormal basis with respect to (J,g) into one for (J',g') is symplectic and satisfies $A^*J' = J$. This shows:

COROLLARY 5.3. *The action of the symplectic group* $\mathrm{Sp}(E)$ *on the space* $\mathcal{J}(E,\omega)$ *of compatible complex structures is transitive, with stabilizer at* $J \in \mathcal{J}(E,\omega)$ *equal to the unitary group* $U(E)$. *That is,* $\mathcal{J}(E,\omega)$ *may be viewed as a homogeneous space*

$$\mathcal{J}(E,\omega) = \mathrm{Sp}(E)/U(E). \tag{2}$$

This shows in particular that $\mathrm{Sp}(E)$ is *connected*. We see that $\mathcal{J}(E,\omega)$ is a non-compact smooth manifold of dimension $(2n^2+n) - n^2 = n^2 + n$. We will show below that the choice of J actually identifies $\mathcal{J}(E,\omega)$ with a vector space. Let $(\ )^T$ denote the transpose of an endomorphism with respect to g.

LEMMA 5.4. *An automorphism* $A \in \mathrm{Gl}(E,\omega)$ *is in* $\mathrm{Sp}(E)$ *if and only if*

$$A^T = J\,A^{-1}J^{-1}$$

where A^T *is the transpose of* A *with respect to* g. *An endomorphism* $\xi \in \mathfrak{gl}(E)$ *is in* $\mathfrak{sp}(E)$ *if and only if*

$$\xi^T = J\xi J.$$

PROOF. $A \in \mathrm{Sp}(E)$ if and only if for all $v, w \in E$, $\omega(Av, Aw) = \omega(v,w)$, or equivalently $g(JAv, Aw) = g(Jv, w)$, i.e. $A^TJA = J$. The other identity is checked similarly. □

EXERCISE 5.5. For $E = \mathbb{R}^{2n}$ with the standard symplectic basis and the standard symplectic structure, J is given by a matrix in block form, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, verify that $A \in \mathrm{Sp}(E)$ is symplectic if and only if $a^Tc,\ b^Td$ are symmetric and $a^Td - b^Tc = I$. In particular, for $n = 1$ we have $\mathrm{Sp}(\mathbb{R}^2) = \mathrm{Sl}(2,\mathbb{R})$.

THEOREM 5.6 (Symplectic eigenvalue Theorem). *Let* $A \in \mathrm{Sp}(E)$. *Then* $\det(A) = 1$, *and all eigenvalues of* A *other than* $1, -1$ *come in either pairs*

$$\lambda, \overline{\lambda}, \quad |\lambda| = 1$$

or quadruples

$$\lambda, \overline{\lambda},\ \lambda^{-1},\ \overline{\lambda}^{-1}, |\lambda| \neq 1.$$

The members of each multiplet all appear with the same multiplicity. The multiplicities of eigenvalues -1 *and* $+1$ *are even.*

PROOF. Since $\det(J) = 1$, the Lemma shows that $\det(A)^2 = 1$. Hence $\det(A) = 1$ since $\mathrm{Sp}(E)$ is connected. For any $A \in \mathrm{Gl}(E)$ the eigenvalues appear in complex-conjugate pairs of equal multiplicity. For $A \in \mathrm{Sp}(A)$, the eigenvalues λ, λ^{-1} have equal multiplicity since by the Lemma, the matrices A and A^{-1} are similar: $A^T = J\, A^{-1} J^{-1}$. The multiplicities of eigenvalues -1 and $+1$ have to be even since the product of all eigenvalues equals $\det A = 1$. □

LEMMA 5.7. *Suppose $A \in \mathrm{Sp}(E)$ is symmetric, $A = A^T$ so that A is diagonalizable and all eigenvalues are real. Let $E_\lambda = \ker(A - \lambda)$ denote the eigenspace. Then*

$$E_\lambda^\omega = \oplus_{\lambda\mu \neq 1} E_\mu.$$

In particular, all E_λ for eigenvalues $\lambda \notin \{1, -1\}$ are isotropic while the eigenspaces for $\lambda \in \{1, -1\}$ are symplectic. Moreover, $E_\lambda \oplus E_{\lambda^{-1}}$ is symplectic.

PROOF. For $v \in E_\lambda, w \in E_\mu$ we have

$$\omega(v, w) = \omega(Av, Aw) = \lambda\mu\omega(v, w).$$

This, together with a check of dimensions proves the Lemma. □

6. Polar decomposition of symplectomorphisms

We will use Lemma 5.7 to obtain the polar decomposition of symplectomorphisms. Recall that for any $A \in \mathrm{Gl}(E)$, the polar decomposition is the unique decomposition $A = CB$ into an orthogonal matrix C and a positive definite symmetric matrix B. Explicitly, $B = |A| := (A^T A)^{1/2}$ and $C = A|A|^{-1}$. Since the exponential map defines a diffeomorphism from the space of symmetric matrices onto positive definite symmetric matrices, this shows that $\mathrm{Gl}(E)$ is diffeomorphic to a product of $\mathrm{O}(E)$ and a vector space. We want to show that if $A \in \mathrm{Sp}(E)$, then both factors in the polar decomposition are in $\mathrm{Sp}(E)$. Thus let

$$\mathfrak{p} = \{\xi \in \mathfrak{sp}(E) |\ \xi = \xi^T\}, \quad P = \{A \in \mathrm{Sp}(E) |\ A = A^T,\ A > 0\}$$

be the intersection of $\mathrm{Sp}(E)$ with the set of positive definite automorphism and of $\mathfrak{sp}(E)$ with the space of symmetric endomorphisms.

LEMMA 6.1. *The exponential map restricts to a diffeomorphism* $\exp : \mathfrak{p} \to P$.

PROOF. Since clearly $\exp(\mathfrak{p}) \subset P$, it suffices to show that $\exp : \mathfrak{p} \to P$ is onto. Given $A \in P$, let $\xi \in \mathfrak{gl}(E)$ the unique symmetric endomorphism with $\exp(\xi) = A$. We have to show $\xi \in \mathfrak{sp}(E)$, or equivalently that $A^s = \exp(s\xi) \in \mathrm{Sp}(E)$ for all $s \in \mathbb{R}$. The power A^s acts on $v \in E_\lambda$ as $\lambda^s\,\mathrm{Id}$. Let $v \in E_\lambda$, $w \in E_\mu$. If $\lambda\mu \neq 1$ then $\omega(v, w) = 0$ and also $\omega(A^s v,\ A^s w) = (\lambda\mu)^s \omega(v, w) = 0$. If $\lambda\mu = 1$ then $\omega(A^s v,\ A^s w) = (\lambda\mu)^s \omega(v, w) = \omega(v, w)$. This shows $A^s \in \mathrm{Sp}(E)$. □

THEOREM 6.2 (Polar decomposition). *The map* $\mathrm{U}(E) \times P \to \mathrm{Sp}(E),\ (C, B) \mapsto A = CB$ *is a diffeomorphism.*

Proof. We have to show that the map is onto. Let $A \in \mathrm{Sp}(E)$. By Lemma 5.4, $A^T = JAJ^{-1} \in \mathrm{Sp}(E)$. Therefore, using Lemma 6.1, $|A| = (A^TA)^{1/2} \in \mathrm{Sp}(E)$ and $A|A|^{-1} \in \mathrm{Sp}(E) \cap \mathrm{O}(E) = U(E)$. $\square$

Since $\mathcal{J}(E,\omega) = \mathrm{Sp}(E)/\mathrm{U}(E)$, we have thus shown:

Corollary 6.3. *Any fixed $J \in \mathcal{J}(E,\omega)$ defines a canonical diffeomorphism between $\mathcal{J}(E,\omega)$ and the vector space $\mathfrak{p}$.*

In particular, we see once again that $\mathcal{J}(E,\omega)$ is contractible.

Corollary 6.4. $\mathrm{Sp}(E)$ *is homotopically equivalent to* $\mathrm{U}(E)$. *In particular, it is connected and has fundamental group $\pi_1(\mathrm{Sp}(E)) = \mathbb{Z}$.*

Remark 6.5. Let $\mathfrak{g} = \mathfrak{sp}(E)$ the Lie algebra of the symplectic group and $\mathfrak{k} = \mathfrak{u}(E) = \mathfrak{sp}(E) \cap \mathfrak{o}(E)$ the Lie algebra of the orthogonal group. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as vector spaces, and

$$[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}.$$

The Killing form on $\mathfrak{g}$ is positive definite on $\mathfrak{k}$ and negative definite on $\mathfrak{p}$. From these facts it follows that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of the semi-simple Lie algebra $\mathfrak{sp}(E)$. In particular, $K = \mathrm{U}(E)$ is a maximal compact subgroup of $G = \mathrm{Sp}(E)$.

Remark 6.6. The symplectic group $\mathrm{Sp}(E)$ should not be confused with the symplectic group $\mathrm{Sp}(n)$ from the theory of compact Lie groups. They are however two different real forms of the same complex Lie group.

7. Maslov indices and the Lagrangian Grassmannian

Let E, ω, J, g as before, $\dim E = 2n$. The determinant map $\det : \mathrm{U}(E) \to S^1$ induces an isomorphism of fundamental groups $\pi_1(\mathrm{U}(E)) \to \pi_1(S^1) = \mathbb{Z}$. Composing with the identification $\pi_1(\mathrm{U}(E)) \cong \pi_1(\mathrm{Sp}(E))$, we obtain an isomorphism

$$\mu : \pi_1(\mathrm{Sp}(E)) \to \mathbb{Z}$$

called the *Maslov index* of a loop of symplectomorphisms. It is independent of the choice of J, since any two choices are homotopic. If A, B are two loops and AB there pointwise product, $\mu(AB) = \mu(A) + \mu(B)$. Dually, we can pull-back the generator $\alpha \in H^1(S^1,\mathbb{Z}) \cong \mathbb{Z}$ by det to find a canonical class $\hat{\mu} \in H^1(\mathrm{Sp}(E),\mathbb{Z})$, called the *Maslov class*.

There are other "Maslov indices" related to the geometry of the Lagrangian Grassmannian

$$\mathrm{Lag}(E) = \mathrm{U}(E)/\mathrm{O}(L).$$

Since $\det : U(E) \to S^1$ takes values ± 1 on $\mathcal{O}(L)$, its square descends to a well-defined function $\det^2 : \mathrm{Lag}(E) \to S^1$.

THEOREM 7.1 (Arnold). *The map* $\det^2 : \mathrm{Lag}(E) \to S^1$ *defines an isomorphism of fundamental groups,* $\mu : \pi_1(\mathrm{Lag}(E)) \cong \pi_1(S^1) = \mathbb{Z}$ *(independent of the choice of* L *or* J*). It is called the* Maslov index of a loop of Lagrangian subspaces.

PROOF. Choose an orthonormal basis for L to identify $L = \mathbb{R}^n \subset \mathbb{C}^n$ and $E \cong \mathbb{C}^n$ so that $\mathrm{Lag}(E) = \mathrm{U}(n)/\mathcal{O}(n)$. For $t \in [0,1]$ let $A_k(t) \in \mathrm{U}(n)$ be the diagonal matrix with entries $(\exp(\sqrt{-1}k\pi t), 1, \ldots, 1)$. Since $A_k(1) \in \mathcal{O}(n)$, we obtain a loop $L_k(t) = A_k(t)\mathbb{R}^n$. This loop has Maslov index k, which shows that μ is surjective.

To show that μ is injective, it is enough to show that any loop $L(t)$ with $L(0) = L(1) = \mathbb{R}^n$ can be deformed into one of the loops L_k. To see this lift $L(t)$ to a path $A(t) \in \mathrm{U}(t)$ (not necessarily closed) with endpoints $A(0) = 1$ and $A(1)$ equal to a diagonal matrix with entries $(\pm 1, 1, \ldots, 1)$. We have to show that $A(t)$ can be deformed into one of the paths $A_k(t)$ while keeping the endpoints fixed. If the endpoint is the identity matrix $\mathrm{diag}(1, 1, \ldots, 1)$ so that $A(t)$ is actually a loop, this is clear because $\pi_1(U(n)) \to \pi_1(S^1) = \mathbb{Z}$ is an isomorphism; in this case k must be even. If $A(1) = \mathrm{diag}(-1, 1, \ldots, 1)$ the path $B(t) = A(t)\, A_{-1}(t)$ (pointwise product) is a loop, i.e. can be deformed into A_{2l} for some l. Thus $A(t) = B(t)A_1(t)$ can be deformed into $A_{2l}\, A_1 = A_{2l+1}$. $\square$

Pulling back the generator $\alpha \in H^1(S^1, \mathbb{Z})$ by $\det^2$ we find an integral cohomology class $\hat{\mu} \in H^1(\mathrm{Lag}(E), \mathbb{Z})$ called the Maslov class.

PROPOSITION 7.2. *Let* $A : S^1 \to \mathrm{Sp}(E)$ *and* $L : S^1 \to \mathrm{Lag}(E)$ *be loops of symplectomorphisms resp. of Lagrangian subspaces. Then*

$$\mu(A(L)) = \mu(L) + 2\mu(A).$$

PROOF. Using the notation from the previous proof we may assume that A takes values in $\mathrm{U}(n)$ since $\mathrm{Sp}(E) = U(E) \times \mathfrak{p}$. Any such A is homotopic to a loop A_{2l} where $l = \mu(A)$. The Proposition follows since $A_{2l} \circ A_k = A_{k+2l}$. $\square$

Given $M \in \mathrm{Lag}(E)$ consider the subset

$$\mathrm{Lag}(E; M) = \{L \in \mathrm{Lag}(E) |\, L \cap M = \{0\}\}.$$

LEMMA 7.3. *For any fixed* $L \in \mathrm{Lag}(E; M)$ *one has a canonical diffeomorphism*

$$\mathrm{Lag}(E; M) \cong S^2(L^*),\ \ N \mapsto S_N$$

with the space of symmetric bilinear forms on L*. In particular it is contractible. The kernel* $\ker(S_N) \subset L$ *is the intersection* $N \cap L$*.*

PROOF. Recall that the non-degenerate pairing $L \times M \to \mathbb{R}$ defined by ω identifies $M = L^*$. Any n-dimensional subspace N transversal to M is of the form $N = \{v + \bar{S}_N(v) |\, v \in L\}$ for some linear map $\bar{S}_N : L \to M$. One has $N \cap L = \ker(\bar{S}_N)$. Since $L^* = M$ we may view S as a bilinear form $S_N \in L^* \otimes L^*$. The condition that N is Lagrangian is equivalent to

$$0 = \omega(v + \bar{S}_N(v), w + \bar{S}_N(w)) = \omega(v, \bar{S}_N(w)) - \omega(w, \bar{S}_N(v))$$

for all $v, w \in L$. In terms of S_N this says precisely that $S_N \in S^2 L^*$ is symmetric. $\square$

We note in passing that this Lemma defines coordinate charts on the Lagrangian Grassmannian.

REMARK 7.4. The coordinate version of this Lemma is as follows: Let $L \in \mathrm{Lag}(\mathbb{R}^{2n})$ such that L is transversal to the span of $f_1, \ldots, f_n$. Then L has a unique basis of the form $g_i = e_i + \sum_j S_{ij} f_j$. The condition $L^\omega = L$ translates into S being a symmetric matrix. If L, L' are two such Lagrangian subspaces and g_j, g_j' the corresponding bases, the pairing $\omega : L \times L' \to \mathbb{R}$ is given by

$$\omega(g_i, g_j') = S_{ij} - S_{ij}'.$$

That is, the dimension of the intersection $L \cap L' = L^\omega \cap L'$ equals the nullity of $S - S'$.

The fact that $\mathrm{Lag}(E; M)$ is contractible can be used to generalize the Maslov index to paths $L : [0, 1] \to \mathrm{Lag}(E)$ which are not loops but satisfy the boundary conditions $L(0), L(1) \in \mathrm{Lag}(E; M)$. Indeed, we can complete L to a loop $\tilde{L} : S^1 = \mathbb{R}/\mathbb{Z} \to \mathrm{Lag}(E)$ with $\tilde{L}(t) = L(2t)$ for $0 \leq t \leq 1/2$ and $\tilde{L}(t) \in \mathrm{Lag}(E; M)$ for $1/2 \leq t \leq 1$ such that $\tilde{L}(1) = L(0)$. The *Maslov intersection index* is defined as

$$[L : M] := \mu(\tilde{L}) \in \mathbb{Z}$$

which is independent of the choice of $\tilde{L}$.

REMARK 7.5. Maslov's index can be interpreted as a (signed) intersection number of L with the "singular cycle" $\mathrm{Lag}(E) \setminus \mathrm{Lag}(E; M)$. It was in this form that Maslov originally introduced his index. The difficulty of this approach is that the singular cycle is not a smooth submanifold of $\mathrm{Lag}(E)$. Given a path in $\mathrm{Lag}(E)$, one perturbs this path until it intersects only the smooth part of the singular cycle and all intersections are transversal. It is then necessary to prove that the index is independent of the choice of perturbation.

Maslov invented his index in the context of geometrical optics ("high frequency asymptotics") and quantum mechanics "semi-classical approximation". It appears physically as a phase shift when a light ray passes through a focal point; a phenomenon discovered in the 19th century. Mathematically Maslov's theory gave rise to Hörmander's theory of Fourier integral operators in PDE.

Maslov's index can be generalized to paths L that are not necessarily transversal to M at the end points. This was first done by Dazord in a 1979 paper and re-discovered several times since then. We will describe one such construction in the following section.

8. The index of a Lagrangian triple

In this section we describe a different approach towards Maslov indices, using the Hörmander-Kashiwara index of a Lagrangian triple. As a motivation, consider the action of $\mathrm{Sp}(E)$ on $\mathrm{Lag}(E)$. We have seen that this action is transitive. Moreover, any two ordered pairs of transversal Lagrangian subspaces can be carried into each other by some symplectomorphism. An analogous statement is true if one fixes the dimension of the intersection $\dim(L_1 \cap L_2)$.

EXERCISE 8.1. Show that for any $L_1, L_2 \in \mathrm{Lag}(E)$ there exists a symplectic basis in which L_1 is spanned by the $e_1, \ldots, e_n$ and L_2 by $e_1, \ldots, e_k, f_{k+1}, \ldots, f_n$. where $k = \dim(L_1 \cap L_2)$. It follows that the action of $\mathrm{Sp}(E)$ on $\mathrm{Lag}(E) \times \mathrm{Lag}(E)$ has $n+1$ orbits, labeled by the dimension of intersections.

Is this true also for *triples* of Lagrangian sub-spaces?

EXERCISE 8.2. Let $E = \mathbb{R}^2$ with symplectic basis e, f. Let $L_1 = \mathrm{span}\{e\}$, $L_2 = \mathrm{span}\{f\}$. What is the form of a matrix of the most general symplectomorphism preserving L_1, L_2? Let $L_3 = \mathrm{span}\{e + f\}$, and (L_1', L_2', L_3') a second triple of Lagrangian subspaces with $L_1' = L_1, L_2' = L_2$. Show by direct computation that there exists a symplectic transformation $A \in \mathrm{Sp}(E)$ with $L_j' = A(L_j)$ for all $j = 1, 2, 3$, if and only if $L_3' = \mathrm{span}\{e + \lambda f\}$ with $\lambda > 0$.

Thus, specifying the dimensions of intersections is insufficient for describing the orbit of a Lagrangian triple L_1, L_2, L_3. There is another invariant called the Hörmander-Kashiwara index of a Lagrangian triple.

Before we define the index, let us recall that the signature $\mathrm{Sig}(B) \in \mathbb{Z}$ of a symmetric matrix B is defined to be the number of its positive eigenvalues, minus the number of its negative eigenvalues. More abstractly, letting $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$ denote the sign function

$$\mathrm{sign}(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ +1 & \text{if } t > 0 \end{cases}$$

and defining $\mathrm{sign}(B)$ by functional calculus, we have $\mathrm{Sig}(B) = \mathrm{tr}(\mathrm{sign}(B))$. The signature has the property $\mathrm{Sig}(ABA^t) = \mathrm{Sig}(B)$ for any invertible matrix A. If $k \in S^2(V^*)$ is a symmetric bi-linear form (equivalently, a quadratic form) on a vector space V, one defines

$$\mathrm{Sig}(k) := \mathrm{Sig}(B)$$

where B is the matrix of k in a given basis of V. The signature and the nullity are the only invariants of a symmetric bilinear form: That is, the action of $\mathrm{Gl}(V)$ on $S^2(V^*)$ has a finite number of orbits labeled by $\dim(\ker(V))$ and $\mathrm{Sig}(k)$.

Given three Lagrangian subspaces (not necessarily transversal) consider the symmetric bilinear form $Q(L_1, L_2, L_3)$ on their direct sum $L_1 \oplus L_2 \oplus L_3$, given by

$$Q(L_1, L_2, L_3)((v_1, v_2, v_3), (v_1, v_2, v_3)) = \omega(v_1, v_2) + \omega(v_2, v_3) + \omega(v_3, v_1).$$

The *index of the the Lagrangian triple* (L_1, L_2, L_3) is the signature,

$$s(L_1, L_2, L_3) := \mathrm{Sig}(Q(L_1, L_2, L_3)) \in \mathbb{Z}.$$

It is due to Hörmander (in his famous 1971 paper on Fourier integral operators) and, in greater generality, Kashiwara (according to the book by Lion-Vergne). Clearly s is invariant under the action of $\mathrm{Sp}(E)$ on $\mathrm{Lag}(E)^3$.

Choosing bases for L_1, L_2, L_3, the definition gives $Q(L_1, L_2, L_3)$ as a symmetric $3n \times 3n$-matrix. One can reduce to signatures of $n \times n$-matrices as follows. Choose a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of E, such that L_1, L_2, L_3 are transversal to the span of $f_1, \ldots, f_n$. Let S_j denote the symmetric bilinear forms on the span of $e_1, \ldots, e_n$ corresponding to S_j. In terms of the basis, S_j is just a matrix, and $Q(L_1, L_2, L_3)$ is given by a symmetric matrix,

$$Q(L_1, L_2, L_3) = \frac{1}{2} \begin{pmatrix} 0 & S_1 - S_2 & S_3 - S_1 \\ S_1 - S_2 & 0 & S_2 - S_3 \\ S_3 - S_1 & S_2 - S_3 & 0 \end{pmatrix}.$$

Lemma 8.3. $s(L_1, L_2, L_3) = \mathrm{Sig}(S_1 - S_2) + \mathrm{Sig}(S_2 - S_3) + \mathrm{Sig}(S_3 - S_1)$.

Proof. (Brian Feldstein) Let

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

An elementary calculation shows that $\det(T) \neq 0$ and that $T\, Q(L_1, L_2, L_3)\, T^t$ is the symmetric matrix,

$$\begin{pmatrix} S_3 - S_1 & 0 & 0 \\ 0 & S_2 - S_3 & 0 \\ 0 & 0 & S_1 - S_2. \end{pmatrix}.$$

From this the Lemma is immediate. □

Theorem 8.4. *The signature* $s : \mathrm{Lag}(E)^3 \to \mathbb{Z}$ *of a Lagrangian triple has the following properties:*

(a) *s is anti-symmetric under permutations of L_1, L_2, L_3.*
(b) *(Cocycle Identity) For all quadruples $L_1, L_2, L_3, L_4 \in \mathrm{Lag}(E)$,*

$$s(L_1, L_2, L_3) - s(L_2, L_3, L_4) + s(L_3, L_4, L_1) - s(L_4, L_1, L_2) = 0.$$

(c) *If $M(t)$ is a continuous path of Lagrangian subspaces such that $M(t)$ is always transversal to $L_1, L_2 \in \mathrm{Lag}(E)$, then $s(L_1, L_2, M(t))$ is constant as a function of t.*
(d) *Any ordered triple of Lagrangian subspaces is determined up to symplectomorphism by the five numbers*

$$\dim(L_1 \cap L_2), \dim(L_2 \cap L_3), \dim(L_3 \cap L_1), \dim(L_1 \cap L_2 \cap L_3),\ s(L_1, L_2, L_3).$$

Proof. The first property is immediate from the definition, while the second and third property follow from the Lemma. The fourth property is left as a non-trivial exercise. (Perhaps try it first for the case that the L_j are pairwise transversal.) □

LEMMA 8.5. *Suppose $L_1(t), L_2(t) \in \mathrm{Lag}(E)$ are two paths of Lagrangian subspaces, $a \leq t \leq b$. Suppose there exists $M \in \mathrm{Lag}(E)$ transversal to $L_1(t)$ and $L_2(t)$ for all $t \in [a,b]$. then the difference*

$$[L_1 : L_2] := \tfrac{1}{2}(s(L_1(a), L_2(a), M) - s(L_1(b), L_2(b), M))$$

is independent of the choice of such M.

PROOF. Let M, M' be two choices. By the cocycle identity, the first term changes by

$$s(L_1(a), L_2(a), M) - s(L_1(a), L_2(a), M') = s(L_1(a), M, M') - s(L_2(a), M, M').$$

We have to show that this equals the change of the second term,

$$s(L_1(b), L_2(b), M) - s(L_1(b), L_2(b), M') = s(L_1(b), M, M') - s(L_2(b), M, M').$$

But $s(L_1(t), M, M')$ and $s(L_2(t), M, M')$ are independent of t, since L_i stay transversal to M, M'. □

We define the Maslov intersection index for two arbitrary paths $L_1, L_2 : [a,b] \to \mathrm{Lag}(E)$ as follows: Choose a subdivision $a = t_0 < t_1 < \cdots t_k = b$ such that, for all $j = 1, \ldots, k$, there is a Lagrangian subspace M^j transversal to $L_1(t), L_2(t)$ for $t \in [t_{j-1}, t_j]$. Then put

$$[L_1 : L_2] = \frac{1}{2}\sum_{j=1}^{k}(s(L_1(t_{j-1}), L_2(t_{j-1}), M^j) - s(L_1(t_j), L_2(t_j), M^j)).$$

Clearly, this is independent of the choice of subdivision and of the choice of the M^j. Note that this definition does not require transversality at the endpoints. The intersection is additive under concatenation of paths. It is anti-symmetric $[L_1 : L_2] = -[L_2 : L_1]$.

EXERCISE 8.6. Show that for any path of symplectomorphisms $A : [a,b] \to \mathrm{Sp}(E)$, $[A(L_1) : A(L_2)] = [L_1 : L_2]$.

EXERCISE 8.7. Let $E = \mathbb{R}^2$, and let $L_1, L_2 : [a,b] \to \mathrm{Lag}(E)$ be defined by $L_1(t) = \mathrm{span}(f + te)$ and $L_2(t) = \mathrm{span}(f)$. Find $[L_1 : L_2]$. How does it depend on a, b?

EXERCISE 8.8. Let $L_1, L_2, L_3 : [a,b] \to \mathrm{Lag}(E)$ be three paths of Lagrangian subspaces. Show that

$$[L_1 : L_2] + [L_2 : L_3] + [L_3 : L_1] = \tfrac{1}{2}(s(L_1(a), L_2(a), L_3(a)) - s(L_1(b), L_2(b), L_3(b)).$$

The approach can also be used to define Maslov indices of paths (not necessarily loops) of symplectomorphisms. Let E^- denote E with minus the symplectic form, and let $E \oplus E^-$ be equipped with the symplectic form $\mathrm{pr}_1^* \omega - \mathrm{pr}_2^* \omega$ where pr_i are the projections to the first and second factor.

PROPOSITION 8.9. *For any symplectomorphism $A \in \mathrm{Sp}(E)$, the graph*

$$\Gamma_A := \{(Av, v) |\, v \in E\} \subset E \oplus E^-$$

is a Lagrangian subspace.

PROOF. Let pr_1, pr_2 denote the projections from $E \oplus E^-$ to the respective factor. For all $v_1, v_2 \in E$, we have

$$(\mathrm{pr}_1^* \omega - \mathrm{pr}_2^* \omega)((Av_1, v_1), (Av_2, v_2)) = -\omega(v_1, v_2) + \omega(Av_1, Av_2) = 0.$$

□

In this sense Lagrangian subspaces of $E^- \oplus E$ may be viewed as generalized symplectomorphisms. If $A : [a, b] \to \mathrm{Sp}(E)$ is a path of symplectomorphisms, one can define a Maslov index $[\Gamma_A : \Delta]$ where $\Delta \subset E \oplus E^-$ is the diagonal. For loops based at the identity this reduces (up to a factor of 2) to the index $\mu(A)$ introduced earlier.

9. Linear Reduction

Suppose $F \subseteq E$ is a subspace. Then the kernel of the restriction of ω to F is just $F \cap F^\omega$ (by the very definition of F^ω). It follows that the quotient space $E_F = F/(F \cap F^\omega)$ inherits a natural symplectic form ω_F: Letting $\pi : F \to F/(F \cap F^\omega)$ be the quotient map we have

$$\omega_F(\pi(v), \pi(w)) = \omega(v, w)$$

for all $v, w \in F$. The space E_F is called the *reduced space* or *symplectic quotient*.

PROPOSITION 9.1. *Suppose $F \subseteq E$ is co-isotropic and $L \in \mathrm{Lag}(E)$ Lagrangian. Let L_F be the image of $L \cap F$ under the reduction map $\pi : F \to F/F^\omega = E_F$. Then $L_F \in \mathrm{Lag}(E_F)$.*

PROOF. Since $L \cap F$ is isotropic, it is immediate that L_F is isotropic. To verify that L_F is Lagrangian we just have to count dimensions: Using $(F_1 \cap F_2)^\omega = F_1^\omega + F_2^\omega$ for any $F_1, F_2 \subseteq E$ we compute

$$\begin{aligned}
\dim(L \cap F^\omega) &= \dim E - \dim(L \cap F^\omega)^\omega \\
&= \dim E - \dim(L + F) \\
&= \dim E - \dim L - \dim F + \dim(L \cap F) \\
&= \dim(L \cap F) - \dim F + \dim L.
\end{aligned}$$

This shows that

$$\dim L_F = \dim(L \cap F) - \dim(L \cap F^\omega) = \dim F - \dim L,$$

on the other hand

$$\dim E_F = \dim F - \dim F^\omega = 2 \dim F - \dim E = 2 \dim F - 2 \dim L.$$

□

For any symplectic vector space (E, ω) let E^- denote E with symplectic form $-\omega$. Suppose E_1, E_2, E_3 are symplectic vector spaces. and let

$$E = E_3 \oplus E_2^- \oplus E_2 \oplus E_1^-.$$

Then the diagonal $\Delta \subset E$, consisting of vectors (v_3, v_2, v_2, v_1) is co-isotropic. Given Lagrangian sub-spaces $\Gamma_1 \in \mathrm{Lag}(E_2 \oplus E_1^-)$ and $\Gamma_2 \in \mathrm{Lag}(E_3 \oplus E_2^-)$, the direct product is a Lagrangian subspace of E. The *composition* of Γ_1, Γ_2 is defined as

$$\Gamma_2 \circ \Gamma_1 = (\Gamma_2 \times \Gamma_1)_\Delta \in \mathrm{Lag}(E_3 \oplus E_1^-).$$

This is really the composition of relations:

$$\Gamma_2 \circ \Gamma_1 = \{(v_3, v_1) | \, \exists v_2 \in E_2 \text{ with } (v_3, v_2) \in \Gamma_2, \; (v_2, v_1) \in \Gamma_1\}.$$

If A_1, A_2 are symplectomorphisms,

$$\Gamma_{A_2 \circ A_1} = \Gamma_{A_2} \circ \Gamma_{A_1}.$$

Similarly, for $L \in \mathrm{Lag}(E)$ we have $\Gamma_A \circ L = A(L)$.

Exercise 9.2. Let $F \subseteq E$ be co-isotropic. Show that

$$\Gamma_F = \{(v, w) \in E_F \oplus E^- | w \in F, \; v = \pi(w)\}$$

(where $\pi : F \to E_F$ is the projection) is Lagrangian. It satisfies $\Gamma_F \circ L = L_F$ for all $L \in \mathrm{Lag}(E)$.

CHAPTER 2

Review of Differential Geometry

1. Vector fields

We assume familiarity with the definition of a manifold (charts, smooth maps etc.). We will always take *paracompactness* as part of the definition – this condition ensures that every open cover U_α of M admits a subordinate partition of unity $f_\alpha \in C^\infty(M)$. That is, f_α is a non-negative function supported in U_α, near every point only a finite number of f_α's are non-zero, and $\sum_\alpha f_\alpha = 1$.

Let $\mathfrak{X}(M)$ denote the vector space of derivations $X : C^\infty(M) \to C^\infty(M)$. That is $X \in \mathfrak{X}(M)$ if and only is

$$X(fg) = X(f)g + f\,X(g)$$

for all $f, g \in C^\infty(M)$. From this definition it follows easily that the value of $X(f)$ at $m \in M$ depends only on the behavior of f in an arbitrarily small neighborhood of m (i.e. on the "germ" of f at m). One can show that in any chart $U \subset M$, with local coordinates $x_1, \ldots, x_n$ every $X \in \mathfrak{X}(M)$ has the form

$$X(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i},$$

where a_i are smooth functions. Elements of $\mathfrak{X}(M)$ are called *vector fields*. The space $\mathfrak{X}(M)$ is a $C^\infty(M)$-module (that is, for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ one has $fX \in \mathfrak{X}(M)$), and it is a Lie algebra with bracket

$$[X, Y] = X \circ Y - Y \circ X.$$

In local coordinates, if $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_i b_i \frac{\partial}{\partial x_i}$ then

$$[X, Y] = \sum_i \Big(\sum_j a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\Big) \frac{\partial}{\partial x_i}.$$

A tangent vector at $m \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = v(f)\,g(m) + f(m)\,v(g)$$

for all smooth f, g. The space of tangent vectors at m is denoted T_mM, and the union

$$TM := \coprod_{m \in M} T_mM$$

is the tangent bundle. In local coordinated, every $v \in T_mM$ is of the form

$$v(f) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(m).$$

One can use this to define a manifold structure on TM, with local coordinates $x_1, \ldots, x_n, v_1, \ldots, v_n$. The natural projection map $\tau : TM \to M$ is smooth.

Clearly, every vector field X defines a tangent vector $v = X_m$ by $X_m(f) := X(f)(m)$. Conversely, every smooth map $X : M \to TM$ with $\tau \circ X = \mathrm{id}_M$ defines a vector field. Thus vector fields can be viewed as sections of the tangent bundle TM.

For any smooth function $F : M \to N$ one has a linear *pull-back* map

$$F^* : C^\infty(N) \to C^\infty(M), \quad F^*g = g \circ F.$$

This defines a smooth push-forward map

$$F_* : TM \to TN, \quad v \mapsto F_*(v)$$

where $F_*(v)(g) = v(F^*g)$. This map is fiberwise linear, but does not in general carry vector fields to vector fields. (There are two problems: If F is not surjective we have no candidate for the section $N \to TN$ away from the image of F. If F is not injective, we may have more than one candidate over some points in the image of F.) Vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called F-related (write $X \sim_F Y$) if for all $g \in C^\infty(N)$,

$$X(F^*g) = F^*(Y(g)).$$

This is equivalent to $F_*(X_m) = Y_{F(m)}$ for all $m \in M$. From the definition one sees easily the important fact,

$$X_1 \sim_F Y_1,\, X_2 \sim_F Y_2 \Rightarrow [X_1, X_2] \sim_F [Y_1, Y_2].$$

If F is a diffeomorphism, we denote by F_*X the unique vector field that is F-related to X.

A (global) *flow* on M is a smooth map

$$\phi : \mathbb{R} \times M \to M, (t, m) \mapsto \phi_t(m)$$

with

$$\phi_0 = \mathrm{id}_M, \quad \phi_t \circ \phi_s = \phi_{t+s}.$$

Every global flow on M defines a vector field $X \in \mathfrak{X}(M)$ by

$$X(f) = \frac{\partial}{\partial t}\Big|_{t=0} \phi_t^* f.$$

For compact manifold M every vector field arises in this way. For non-compact M, one has to allow for "incomplete" flows, that is one has to restrict the definition of ϕ to some open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, and require $\phi_t(\phi_s(m)) = \phi_{t+s}(m)$ whenever these terms are defined. Suppose ϕ_t is the flow defined by $X \in \mathfrak{X}(M)$. We define the *Lie derivative* of vector fields $Y \in \mathfrak{X}(M)$ by

$$L_X(Y) = \frac{\partial}{\partial t}\Big|_{t=0} (\phi_{-t})_*(Y) \in \mathfrak{X}(M).$$

It is a very important fact that for all vector fields X, Y,

$$L_X(Y) = [X, Y].$$

2. Differential forms

Let E be a vector space of dimension n over $\mathbb{R}$. A k-form on E is a multi-linear map

$$\alpha : \underbrace{E \times \ldots \times E}_{k \text{ times}} \to \mathbb{R},$$

anti-symmetric in all entries. The space of k-forms is denoted $\wedge^k E^*$. One has $\wedge^k E^* = \{0\}$ for $k > n$, $\wedge^0 E^* = \mathbb{R}$ and

$$\dim \wedge^k E^* = \begin{pmatrix} n \\ k \end{pmatrix}$$

for $0 \leq k \leq n$. The space $\wedge E^* = \bigoplus_{k=0}^n \wedge^k E^*$ is an algebra with product given by anti-symmetrization of $\alpha \otimes \beta$:

$$(\alpha \wedge \beta)(X_1, \ldots, X_{k+l}) = \sum_{\sigma \in S_{k+l}} \frac{(-1)^\sigma}{k!l!} \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})$$

for $\alpha \in \wedge^k E^*$ and $\beta \in \wedge^l E^*$. Given $v \in E$ there is a natural *contraction map* $\iota_v : \wedge^k(E^*) \to \wedge^{k-1} E^*$ given by

$$\iota_v \alpha = \alpha(v, \cdot, \ldots, \cdot).$$

This operator is a graded derivation:

$$\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^k \alpha \wedge \iota_v \beta.$$

REMARK 2.1. In general, given a $\mathbb{Z}$-graded algebra $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}^k$ over a commutative ring $\mathcal{R}$, a derivation of degree r of $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ taking $\mathcal{A}^k$ into $\mathcal{A}^{k+r}$ and satisfying the graded Leibniz rule,

$$D(ab) = D(a)b + (-1)^{kr} a\, D(b)$$

for $a\mathcal{A}^k,\ b \in \mathcal{A}^l$. Let $\mathrm{Der}(A) = \bigoplus_{r \in \mathbb{Z}} \mathrm{Der}^r(A)$ denote the graded space of graded derivations of r. Then $\mathrm{Der}(A)$ is a graded Lie algebra over $\mathcal{R}$: That is, for $D_j \in \mathrm{Der}^{r_j}(\mathcal{A})$, $j = 1, 2$ one has

$$[D_1, D_2] := D_1 D_2 + (-1)^{r_1 r_2} D_2 D_1 \in \mathrm{Der}^{r_1 + r_2}(\mathcal{A})$$

Suppose now that M is a manifold. Then we can define vector bundles

$$\wedge^k T^*M := \coprod_{m\in M} \wedge^k (T_m M)^*.$$

Smooth sections of $\wedge^k T^*M$ are called k-forms, and the space of k-forms is denoted $\Omega^k(M)$. The space $\Omega^\star(M) = \bigoplus_k \Omega^k(M)$ is a graded algebra over $C^\infty(M)$. Equivalently, k-forms can be viewed as $C^\infty(M)$-multilinear anti-symmetric maps

$$\alpha : \underbrace{\mathfrak{X}(M) \times \cdots \mathfrak{X}(M)}_{k \text{ times}} \to C^\infty(M).$$

Note $\Omega^0(M) = C^\infty(M)$. For any vector field $X \in \mathfrak{X}(M)$ there is a contraction operator $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ which is a graded derivation.

For any function $f \in C^\infty(M)$, denote by $\mathrm{d}f \in \Omega^1(M)$ the 1-form such that $\mathrm{d}f(X) = X(f)$.

THEOREM 2.2. *The map $d : \Omega^0(M) \to \Omega^1(M)$ extends uniquely to a graded derivation $d : \Omega(M) \to \Omega(M)$ of degree* 1*, in such a way that $d^2 = 0$.*

The quotient space $H^k(M) = \ker(\mathrm{d}|\Omega^k)/\operatorname{im}(\mathrm{d}|\Omega^{k-1})$ is called the kth de Rham cohomology group of M. Wedge product gives $H^\star(M)$ the structure of a graded algebra. For M compact, the de Rham cohomology groups are always finite-dimensional vector spaces.

Suppose now that $F : M \to N$ is smooth. Then F defines a unique pull-back map $F^* : \Omega^k(N) \to \Omega^k(M)$, by

$$F^*\beta(v_1, \dots, v_k) = \beta(F_*(v_1), \dots, F_*(v_k))$$

for all $v_j \in T_m M$. The exterior differential respects F, that is

$$\mathrm{d}F^*\beta = F^*\mathrm{d}\beta.$$

Suppose ϕ_t is the flow of $X \in \mathfrak{X}$. The Lie derivative of $\alpha \in \Omega^k(M)$ with respect to X is defined as follows:

$$L_X\alpha = \frac{\partial}{\partial t}\Big|_{t=0} \phi_t^* \alpha.$$

Since pull-backs commute with the exterior differential, $L_X \circ \mathrm{d} = \mathrm{d} \circ L_X$. The operator L_X is a graded derivation of degree 0:

$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta).$$

One has the following relations between the derivations d, L_X, ι_X.

$$[\mathrm{d}, \mathrm{d}] = 2\mathrm{d}^2 = 0, \quad [\mathrm{d}, L_X] = 0, \ [L_X, L_Y] = L_{[X,Y]},$$

$$[\iota_X, \iota_Y] = 0, \quad [L_X, \iota_Y] = \iota_{[X,Y]}, \ [\mathrm{d}, \iota_X] = L_X.$$

The last formula

$$\mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d} = L_X$$

is known as Cartan's formula. These relations show that the linear subspace of $\mathrm{Der}(\Omega(M))$ spanned by ι_X, L_X, d is in fact a subalgebra of the graded Lie algebra $\mathrm{Der}(\Omega(M))$.

Let us re-express some of the above in local coordinates $x_1, \ldots, x_n$ for a chart $U \subset M$. At every point $m \in M$, the 1-forms $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ are a basis of T_m^*M dual to the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ of T_mM. Every 1-form α is given on U by an expression

$$\alpha = \sum_{i=1}^{n} \alpha_i \mathrm{d}x_i$$

where α_i are smooth functions. For $X = \sum_i X_i \frac{\partial}{\partial x_i}$ we have

$$\alpha(X) = \sum_{i=1}^{n} \alpha_i X_i.$$

More generally, for every ordered tuple $I = (i_1, \ldots, i_k)$ of cardinality $|I| = k$ put

$$\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

Then every $\alpha \in \Omega^k(M)$ has the coordinate expression

$$\alpha = \sum_I \alpha_I \mathrm{d}x_I$$

where α_I are smooth functions and the sum is over all *ordered* tuples $i_1 < \ldots < i_k$. One has

$$\mathrm{d}\alpha = \sum_{j=1}^{n} \sum_I \frac{\partial \alpha_I}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_I,$$

indeed, this formula is forced on us by the derivation property of d and the conditions d^2=0.

EXERCISE 2.3. Verify that this formula indeed gives $\mathrm{d}^2 = 0$, using the equality of mixed partials.

EXERCISE 2.4. Work out the coordinate expressions for Lie derivatives L_X and contractions ι_X.

A volume form on a manifold M of dimension n is an $n - form$ $\Lambda \in \Omega^n(M)$ with $\Lambda_m \neq 0$ for all m. If M admits a volume form it is called orientable. The choice of an equivalence class of volume forms, where Λ_1, Λ_2 are equivalent if $\Lambda_2 = f\Lambda_1$ with $f > 0$ everywhere, is called an orientation.

EXERCISE 2.5. For any volume form Λ and $X \in \mathfrak{X}(M)$, the divergence of X with respect to Λ is the function $\mathrm{div}_\Lambda(X)$ such that $L_X\Lambda = \mathrm{div}_\Lambda(X)\Lambda$. Find a formula for $L_X\Lambda$ in local coordinates.

CHAPTER 3

Foundations of symplectic geometry

1. Definition of symplectic manifolds

DEFINITION 1.1. A symplectic manifold is a pair (M, ω) consisting of a manifold M together with a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$. Given two symplectic manifolds (M_i, ω_i), a symplectomorphism is a diffeomorphism $F\, M_1 \to M_2$ such that $F^*\omega_2 = \omega_1$. The group of symplectomorphism of M onto itself is denoted $\mathrm{Symp}(M, \omega)$. The space of vector fields X with $L_X\omega = 0$ is denoted $\mathfrak{X}(M, \omega)$.

Non-degeneracy means that for each $m \in M$, the form ω_m is a symplectic form on T_mM, in particular $\dim M = 2n$ is even. Equivalently, the top exterior power ω_m^n is non-zero.

DEFINITION 1.2. The volume form $\Lambda = \exp(\omega)_{[\dim M]} = \frac{1}{n!}\omega^n$ is called the Liouville form.

DEFINITION 1.3. For any $H \in C^\infty(M, \mathbb{R})$, the corresponding Hamiltonian vector field X_H is the unique vector field satisfying

$$\iota_{X_H}\omega = \mathrm{d}H$$

The space of vector fields X of the form $X = X_H$ is denoted $\mathfrak{X}_{\mathrm{Ham}}(M, \omega)$.

PROPOSITION 1.4. *Every Hamiltonian vector field is a symplectic vector field. That is,*

$$\mathfrak{X}_{\mathrm{Ham}}(M, \omega) \subseteq \mathfrak{X}(M, \omega).$$

PROOF. Suppose $X = X_H$, that is $\iota_X\omega = \mathrm{d}H$. Then

$$L_X\omega = \mathrm{d}\iota_X\omega = \mathrm{d}\mathrm{d}H = 0.$$

□

2. Examples

2.1. Example: open subsets of $\mathbb{R}^{2n}$.

EXAMPLE 2.1. The basic example are open subsets $U \subseteq \mathbb{R}^{2n}$. Let $q_1, \dots, q_n, p_1, \dots, p_n$ be coordinates with respect to a symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ for $\mathbb{R}^{2n}$. This identifies $e_j = \frac{\partial}{\partial q_j}$ and $f_j = \frac{\partial}{\partial p_j}$. In terms of the dual 1-forms $\mathrm{d}q_1, \ldots, \mathrm{d}p_n$, the symplectic form is given by

$$\omega = \sum_{j=1}^{n} \mathrm{d}q_j \wedge \mathrm{d}p_j$$

and the Liouville form reads

$$\Lambda = \mathrm{d}q_1 \wedge \mathrm{d}p_1 \wedge \ldots \wedge \mathrm{d}q_n \wedge \mathrm{d}p_n.$$

Given a smooth function H on U, we have

$$X_H = \sum_{j=1}^{n} \Big(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \Big).$$

Hence the ordinary differential equation defined by X_H is

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

Note that Hamiltonians $H(q,p) = p_j$ generate translation in q_j-direction while $H(q,p) = q_j$ generate translation in minus p_j-direction.

Definition 2.2. Let (M, ω) be a symplectic manifold. We denote by $\mathfrak{X}_{\mathrm{Ham}}(M, \omega)$ the space of Hamiltonian vector fields on M, and by $\mathfrak{X}(M, \omega)$ the space of vector fields X on M preserving ω, i.e. $L_X \omega = 0$.

2.2. Example: Surfaces. Let Σ be an orientable 2-manifold, and $\omega \in \Omega^2(\Sigma)$ a volume form. Then ω is non-degenerate (since $\omega^n = \omega \neq 0$ everywhere) and closed (since every top degree form is obviously closed). A symplectomorphism is just a volume-preserving diffeomorphism in this case. By a result of Moser, any two volume forms on a compact manifold M, defining the same orientation and having the same total volume, are related by some diffeomorphism of M. In particular, every closed symplectic 2-manifold is determined up to symplectomorphism by its genus and total volume.

2.3. Example: Cotangent bundles. A very important example for symplectic manifolds are cotangent bundles. Let Q be a manifold, $M = T^*Q$ its cotangent bundle. There is a *canonical* 1-form $\theta \in \Omega^1(T^*Q)$ given as follows: Let $\pi : M = T^*Q \to Q$ the bundle projection, $\mathrm{d}\pi : TM \to TQ$ its tangent map. Then for vectors $X_m \in T_m M$,

$$\langle \theta_m, X_m \rangle := \langle m, \mathrm{d}_m \pi(X_m) \rangle$$

(since $m \in T^*_{\pi(m)} Q$ is a covector at $\pi(m)$, we can pair it with the projection of X_m to the base!) An alternative characterization of the form θ is as follows.

Proposition 2.3. *θ is the unique 1-form $\theta \in \Omega^1(T^*Q)$ with the property that for any 1-form $\alpha \in \Omega^1(Q)$ on the base,*

$$\alpha = \alpha^* \theta$$

where on the right hand side, α is viewed as a section $\alpha : Q \to T^*Q = M$.

PROOF. To check the property let $w \in T_qQ$. Then $\mathrm{d}_q\alpha(Y)$ projects to w. We have, therefore,

$$\langle(\alpha^*\theta)_q, w\rangle = \langle\theta_{\alpha_q}, d_q\alpha(w)\rangle = \langle\alpha_q,\, w\rangle.$$

Uniqueness follows since every tangent vector $v \in T_mM$ (except the vertical ones, i.e. those in the kernel of $\mathrm{d}_m\pi$) can be written in the form $\mathrm{d}_q\alpha(w)$, where $\alpha \in \Omega 1(Q)$, with $\alpha(q) = m$ and $w \in T_qQ$. (Non-vertical vectors span all of T_mM). □

It is useful to work out the form θ in local coordinates.

For any vector bundle $\pi : E \to Q$ of rank k, one obtains local coordinates over a bundle chart $W \subset Q$ by choosing local coordinates $q_1, \dots, q_n$ on Q and a basis $\epsilon_1, \dots, \epsilon_k : Q \to E$ for the $C^\infty(W)$-module of sections of $E|_W$. Any point $m \in E$ is then given by the coordinates q_i of its base point $q = \pi(m)$ and the coordinates $p_1, \dots, p_k$ such that $m = \sum_j p_i\epsilon_i(q)$. Thus if $\sigma = \sum_i \sigma_i\epsilon_i : W \to E|_W$ is any section over W, the pull-backs of q_i, p_i viewed as functions on $E|_W$ are we have $\sigma^*p_i = \sigma_i$ and $\sigma^*q_i = q_i$.

In our case, $E = T^*Q,\ k = n$ and a natural basis for the space of sections is given by the 1-forms $\epsilon_i = \mathrm{d}q_i$. The corresponding coordinates $q_1,\, \dots,\, q_n,\, p_1, \dots,\, p_n$ on $T^*Q|_W$ are called *cotangent coordinates*.

LEMMA 2.4. *In local cotangent coordinates $q_1,\, \dots,\, q_n,\, p_1, \dots,\, p_n$ on T^*Q, the canonical 1-form θ is given by*

$$\theta|(T^*W) = \sum_j p_j\, dq_j.$$

PROOF. Let $\alpha = \sum_j \alpha_j\mathrm{d}q_j$ be a 1-form on W. Then $\alpha^*p_j = \alpha_j, \quad \alpha^*q_j = q_j$, thus

$$\alpha^*\sum_j p_j\,\mathrm{d}q_j = \sum_j \alpha_j\mathrm{d}q_j = \alpha.$$

□

THEOREM 2.5. *Let $M = T^*Q$ be a cotangent bundle and θ its canonical 1-form. Then $\omega = -d\theta$ is a symplectic structure on M.*

PROOF. In local cotangent coordinates, $\omega = \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$. □

We will now describe some natural symplectomorphisms and Hamiltonian vector fields on $M = T^*Q$.

Let $f : Q_1 \to Q_2$ be a diffeomorphism. Then the tangent map $\mathrm{d}f$ is a diffeomorphism $TQ_1 \to TQ_2$, and dually there is a diffeomorphism

$$F = (\mathrm{d}f^{-1})^* : T^*Q_1 \to T^*Q_2$$

(called cotangent lift of f) covering f. For all $\alpha \in \Omega^1(Q_1)$ one has a commutative diagram,

$$\begin{array}{ccc} T^*Q_1 & \xrightarrow{F} & T^*Q_2 \\ \uparrow_{\alpha} & & \uparrow_{(f^{-1})^*\alpha} \\ Q_1 & \xrightarrow{f} & Q_2 \end{array}.$$

PROPOSITION 2.6 (Naturality of the canonical 1-form). *Let $F : T^*Q_1 \to T^*Q_2$ be the cotangent lift of f. Then F preserves the canonical 1-form, $F^*\theta_2 = \theta_1$, hence F is a symplectomorphism: $F^*\omega_2 = \omega_1$,*

PROOF. This is clear since our definition of the canonical 1-form was coordinate-free. For the sceptic, check the property $\alpha^*\theta_1 = \alpha$: We have

$$\alpha^*(F^*\theta_2) = (F \circ \alpha)^*\theta_2 = ((f^{-1})^*\alpha \circ f)^*\theta_2 = f^*((f^{-1})^*\alpha)^*\theta_2 = f^*(f^{-1})^*\alpha = \alpha.$$

□

This gives a natural group homomorphism

$$\mathrm{Diff}(Q) \to \mathrm{Symp}(T^*Q, \omega), \;\; f \mapsto (df^{-1})^*. \tag{3}$$

Another subgroup of $\mathrm{Symp}(T^*Q)$ is obtained from the space of closed 1-forms $Z^1(Q) \subset \Omega^1(Q)$. For any $\alpha \in \Omega^1(Q)$ let $G_\alpha : T^*Q \to T^*Q$ be the diffeomorphism obtained by adding α.

PROPOSITION 2.7. *For all $\alpha \in \Omega^1(Q)$,*

$$G_\alpha^*\theta - \theta = \pi^*\alpha$$

Thus G_α is a symplectomorphism if and only if $d\alpha = 0$, that is $\alpha \in Z^1(Q)$.

PROOF. Let $\beta \in \Omega^1(Q)$. Then

$$\beta^*G_\alpha^*\theta = (G_\alpha \circ \beta)^*\theta = (\alpha + \beta)^*\theta = \alpha + \beta = \beta^*\pi^*\alpha + \beta.$$

By the characterizing property of θ this proves the Proposition. □

We thus find a group homomorphism

$$Z^1(Q) \to \mathrm{Symp}(T^*Q) \tag{4}$$

Recall that for any representation of a group G on a vector space V, one defines $G \ltimes V$ to be the group whose underlying set is $G \times V$ and with product structure,

$$(g_1, v_1)(g_2, v_2) = (g_1g_2, v_1 + g_1.v_2).$$

In this case, we can let $\mathrm{Diff}(Q)$ act on $Z^1(Q)$ by $f \cdot \alpha = (f^{-1})^*\alpha$. It is easy to check that the homomorphisms (3) and (4) combine into a group homomorphism

$$\mathrm{Diff}(Q) \ltimes Z^1(Q) \to \mathrm{Symp}(T^*Q).$$

The semi-direct product $\mathrm{Diff}(Q) \ltimes Z^1(Q)$ may be viewed as an infinite-dimensional generalization of the Euclidean group of motions $\mathrm{O}(n) \ltimes \mathbb{R}^n$.

We will now show that the generators of the action of $\mathrm{Diff}(Q)$ are Hamiltonian vector fields. Let Y be a vector field on Q. Then there is a unique vector field $X = \mathrm{Lift}(Q)$ on T^*Q with the property that the flow of X is the cotangent lift of the flow of Y. We call the map

$$\mathrm{Lift}: \mathfrak{X}(Q) \to \mathfrak{X}(T^*Q)$$

the cotangent lift of a vector field. Note that X projects onto Y, that is $X \sim_\pi Y$. Let $H \in C^\infty(T^*Q, \mathbb{R})$ be defined as the contraction $H = \iota_X \theta$.

Lemma 2.8. *The cotangent lift X of Y is a Hamiltonian vector field, with Hamiltonian $H = \iota_X\theta$.*

Proof. Since the flow of X preserves θ, $L_X\theta = 0$. Therefore,

$$\mathrm{d}H = \mathrm{d}\iota_X\theta = -\iota_X \mathrm{d}\theta + L_X\theta = -\iota_X\mathrm{d}\theta = \iota_X\omega.$$

□

Suppose Y is given in local coordinates by $Y = \sum_j Y_j \frac{\partial}{\partial q_j}$. What are X and H in these coordinates? Since X projects to Y under π, we know that $X - \sum_j Y_j \frac{\partial}{\partial q_j}$ is a vertical vector field, i.e. a linear combination of $\frac{\partial}{\partial p_j}$. The vertical part does not contribute to $\iota_X\theta$ since $\theta = \sum_j p_j \mathrm{d}q_j$ is a *horizontal* 1-form. Hence

$$H(q,p) = \iota_X\theta = \sum_j Y_j(q)\iota_{\frac{\partial}{\partial q_j}}\theta = \sum_j Y_j(q)p_j.$$

From this we recover,

$$X = \sum_{j=1}^n Y_j \frac{\partial}{\partial q_j} - \sum_{j,k=1}^n p_j \frac{\partial Y_j}{\partial q_k}\frac{\partial}{\partial p_k}.$$

From these equations, we see that the Hamiltonians corresponding to cotangent lifts are those which are *linear along the fibers* of T^*Q. Other interesting flows are generated by Hamiltonians that are *constant along the fibers* of T^*Q, i.e. of the form $H = \pi^* f$, with $f \in C^\infty(Q)$. The flow generated by such an H is given, in terms of the notation G_α introduced above, by $\phi_t = G_{-t\ df}$. The Hamiltonian vector field corresponding to H is, in local cotangent coordinates,

$$X_{\pi^* f} = -\sum_j \frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j}.$$

Exercise 2.9. Verify these claims!

On the total space of any vector bundle $E \to Q$ there is a canonical vector field $\mathcal{E} \in \mathfrak{X}(E)$, called the Euler vector field; its flow Φ_t is fiberwise multiplication by e^t. In our case $E = T^*Q$, we have in local cotangent coordinates

$$\mathcal{E} = \sum_j p_j \frac{\partial}{\partial p_j}.$$

PROPOSITION 2.10. *The Euler vector field satisfies*

$$L_{\mathcal{E}}\omega = \omega,\ \iota_{\mathcal{E}}\omega = -\theta.$$

In particular, $\mathcal{E}$ is the vector field corresponding to $-\theta$ under the isomorphism $\omega^\flat : TQ \to T^*Q$.

PROOF. The 1-form θ is homogeneous of degree 1 along the fibers. That is, under fiberwise multiplication by e^t it transforms according to $(e^t)^*\theta = e^t\theta$. Taking the derivative this shows

$$L_{\mathcal{E}}\theta = \theta.$$

Applying d gives the first formula in the Proposition. The second formula is obtained using the Cartan formula for the Lie derivative, together with $\iota_{\mathcal{E}}\theta = 0$. □

REMARK 2.11. A symplectic manifold M, together with a free $\mathbb{R}$-action whose generating vector field $\mathcal{E}$ satisfies such that $L_{\mathcal{E}}\omega = \omega$ is called a *symplectic cone*. Thus, cotangent bundles minus their zero section are examples of symplectic cones. Another example is $\mathbb{R}^{2n} - \{0\}$.

PROPOSITION 2.12. *For any closed 2-form $\sigma \in \Omega^2(Q)$, the sum $\omega+\pi^*\sigma$ is a symplectic form on T^*Q. The Liouville form of $\omega + \pi^*\sigma$ equals that for ω.*

PROOF. Since $\pi^*\sigma$ vanishes on tangent vectors to fibers of π, its kernel at $m \in T^*Q$ contains a Lagrangian subspace, i.e. its kernel is co-isotropic. The claim now follows from the following Lemma. □

LEMMA 2.13. *Let (E, ω) be a symplectic vector space and $\tau \in \wedge^2 E^*$ a 2-form such that* $\ker\tau$ *is co-isotropic. Then $(\omega + \tau)^n = \omega^n$. In particular, $\omega + \tau$ is non-degenerate.*

PROOF. Since $\ker\tau$ is co-isotropic it contains a Lagrangian subspace L. Let $e_1, \ldots, e_n$ a basis for L. For $k < n$ we have $\iota(e_1)\ldots\iota(e_n)\omega^k\tau^{n-k} = 0$. Therefore $\omega^k\tau^{n-k} = 0$, and it follows that $(\omega + \tau)^n = \omega^n$. In particular, $\omega + \tau$ is symplectic since its top power is a volume form. □

This has the following somewhat silly corollary: For any manifold Q with a closed 2-form σ there exists a symplectic manifold (M, ω) and an embedding $\iota : Q \to M$ such that $\iota^*\omega = \sigma$. (Proof: Take $M = T^*Q$ with symplectic form $\omega = -\mathrm{d}\theta + \pi^*\sigma$.)

2.4. Example: Kähler manifolds. An *almost complex manifold* is a manifold Q together with a smoothly varying complex structure on each tangent space; i.e. a smooth section $J : Q \to \mathrm{Gl}(TQ)$ satisfying $J^2 = -\mathrm{id}$. A *complex manifold* is a manifold, together with an atlas consisting of open subsets of $\mathbb{C}^n$, in such a way that the transition functions are holomorphic maps. Every complex manifold is almost complex, the automorphism J given by multiplication by $\sqrt{-1}$ in complex coordinate charts.

The Newlander-Nirenberg theorem (see e.g. the book by Kobayashi-Nomizu) gives a necessary and sufficient criterion (vanishing of the *Nijenhuis tensor*) for when an almost complex structures is integrable, i.e. comes from a complex manifold.

An almost complex structure J on a symplectic manifold (M,ω) is called ω-compatible if it is ω-compatible on every tangent space T_mM. We denote by $\mathcal{J}(M,\omega)$ the space of ω-compatible almost complex structures on M. The constructions from linear symplectic algebra can be carried out fiberwise: Letting $\mathrm{Riem}(M)$ denote the space of Riemannian metrics (the space of sections $g:\, M\to S^2(T^*M)$ such that each g_m is an inner product on T_mM) we have a canonical surjective map

$$\mathrm{Riem}(M)\to\mathcal{J}(M,\omega)$$

which is a left inverse to the map $\mathcal{J}(M,\omega)\to\mathrm{Riem}(M)$ associating to J_m the corresponding inner products on T_mM. In particular $\mathcal{J}(M,\omega)$ is non-empty. Similar to the linear case, one finds that any two $J_0,J_1\in\mathcal{J}(M,\omega)$ can be smoothly deformed within $\mathcal{J}(M,\omega)$. More precisely, there exists a smooth map $J:\,[0,1]\times M\to\mathrm{Gl}(TM)$ such that $J(t,m)\in\mathcal{J}(T_mM,\omega_m)$ and $J(0,\cdot)=J_0,\ J(1,\cdot)=J_1$.

For any $J\in\mathcal{J}(M,\omega)$ the triple (M,ω,J) is called an almost Kähler manifold (sometimes also almost Hermitian manifold). If J comes from an honest complex structure then (M,ω,J) is called a Kähler manifold. An example of a Kähler manifold is $M=\mathbb{C}^n$.

Proposition 2.14. *Let (M,ω,J) be a Kähler manifold. Let (N,J_N) be an complex manifold and $\iota:\,N\to M$ an complex immersion: That is, $J\circ d\iota=d\iota\circ J_N$. Then $(N,\iota^*\omega,J_N)$ is an Kähler manifold. Similar assertions hold for the almost Kähler category and almost complex immersions.*

Proof. Obviously, every complex subspace of a Hermitian vector space is Hermitian. Applying this to each $d\iota_n(T_nN)\subset T_{\iota(n)}M$ we see that the closed 2-form $\iota^*\omega$ is non-degenerate, and $J_N\in\mathcal{J}(N,\iota^*\omega)$. $\square$

This shows in particular that every complex submanifold of $\mathbb{C}^n$ is a symplectic manifold. Notice that if $N\subset\mathbb{C}^n$ is the zero locus of a collection of homogeneous polynomials, such that N is smooth away from $\{0\}$ then $N\backslash\{0\}$ is a symplectic cone.

We next consider complex projective space,

$$\mathbb{C}P(n)=(\mathbb{C}^{n+1}\backslash\{0\})/(\mathbb{C}\backslash\{0\})=S^{2n+1}/S^1.$$

Let $\iota:\,S^{2n+1}\to\mathbb{C}^{n+1}$ the embedding and $\pi:\,S^{2n+1}\to\mathbb{C}P(n)$ the projection. At every point $z\in S^{2n+1}$, we have a canonical splitting of tangent spaces

$$T_z\mathbb{C}^{n+1}=T_{\pi(z)}\mathbb{C}P(n)\oplus\mathrm{span}_{\mathbb{C}}\{z\}$$

as complex vector spaces. Since $T_{\pi(z)}\mathbb{C}P(n)$ is a complex subspace, it is also symplectic. This induces a non-degenerate 2-form ω on $\mathbb{C}P(n)$ which by construction is compatible with the complex structure. Letting $\overline{\omega}$ be the symplectic structure of $\mathbb{C}^n$, we have $\iota^*\overline{\omega}=\pi^*\omega$. Therefore $\pi^*d\omega=\iota^*d\overline{\omega}=0$, showing that ω is closed. This shows that $\mathbb{C}P(n)$ is a Kähler manifold. The 2-form ω is called Fubini-Study form. (Later we will see this construction of ω more systematically as a symplectic reduction.)

By the above proposition, every nonsingular projective variety is a Kähler manifold.

We have seen that every symplectic vector space (E,ω) admits a compatible almost complex structure. It is natural to ask whether it also admits a compatible complex structure, i.e. whether every symplectic manifold is Kähler. The answer is negative: A first counterexample was found by Kodaira and later rediscovered by Thurston. By now there are large families of counterexamples, see e.g. work of McDuff and Gompf. For a discussion of the Kodaira-Thurston counterexample, see the book McDuff-Salamon, "Introduction to Symplectic Topology".

3. Basic properties of symplectic manifolds

3.1. Hamiltonian and symplectic vector fields. We will now study the Lie algebras of Hamiltonian and symplectic vector fields in more detail. Let (M,ω) be a symplectic manifold. By definition, a vector field X is Hamiltonian if $\iota_X\omega = \mathrm{d}H$ for some smooth function H. This means that the isomorphism between vector fields and 1-forms $\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M)$ defined by ω restricts to an isomorphism

$$\omega^\flat : \mathfrak{X}_{\mathrm{Ham}}(M,\omega) \to B^1(M)$$

with the space $B^1(M) = \Omega^1(M) \cap \mathrm{im}(\mathrm{d})$ of exact 1-forms. Similarly a vector field is symplectic if and only if $L_X\omega = 0$, which by Cartan's identity means $\mathrm{d}\iota_X\omega = 0$. Thus we have an isomorphism

$$\omega^\flat : \mathfrak{X}(M,\omega) \to Z^1(M)$$

with the space $Z^1(M) = \Omega^1(M) \cap \ker(\mathrm{d})$ of closed 1-forms. Thus the quotient $\mathfrak{X}(M,\omega)/\mathfrak{X}_{Ham}(M,\omega)$ is just the first deRham cohomology group cohomology $H^1(M) = Z^1(M)/B^1(M)$, and we have an exact sequence of vector spaces

$$0 \to \mathfrak{X}_{Ham}(M,\omega) \to \mathfrak{X}(M,\omega) \to H^1(M) \to 0. \tag{5}$$

We conclude that if $H^1(M) = \{0\}$ (e.g. for simply connected spaces such as $M = \mathbb{C}^n$ or $M = \mathbb{C}P(n)$) *every* symplectic vector field is Hamiltonian.

Proposition 3.1. *For all $Y_1, Y_2 \in \mathfrak{X}(M,\omega)$, one has*

$$[Y_1, Y_2] = -X_{\omega(Y_1,Y_2)}.$$

Proof. Let $Y_1, Y_2 \in \mathfrak{X}(M,\omega)$. Then

$$\begin{aligned} \mathrm{d}(\omega(Y_1,Y_2)) &= \mathrm{d}\iota_{Y_2}\iota_{Y_1}\omega \\ &= L_{Y_2}\iota_{Y_1}\omega - \iota_{Y_2}\mathrm{d}\iota_{Y_1}\omega \\ &= L_{Y_2}\iota_{Y_1}\omega \\ &= \iota(L_{Y_2}Y_1)\omega = -\iota([Y_1,Y_2])\omega. \end{aligned}$$

□

Proposition 3.1 shows that $[\mathfrak{X}(M,\omega), \mathfrak{X}(M,\omega)] \subseteq \mathfrak{X}_{Ham}(M,\omega)$. In particular, $\mathfrak{X}_{Ham}(M,\omega)$ is an ideal in the Lie algebra $\mathfrak{X}(M,\omega)$ and the quotient Lie algebra $\mathfrak{X}(M,\omega)/\mathfrak{X}_{Ham}(M,\omega)$ is abelian. It follows that (5) is an exact sequence of Lie algebras, where $H^1(M)$ carries the trivial Lie algebra structure.

3.2. Poisson brackets. Consider next the surjective map $C^\infty(M) \to \mathfrak{X}_{Ham}(M,\omega), H \mapsto X_H$. Its kernel is the space $Z^0(M) = H^0(M)$ of locally constant functions. We thus have an exact sequence of vector spaces

$$0 \longrightarrow Z^0(M) \longrightarrow C^\infty(M) \longrightarrow \mathfrak{X}_{Ham}(M,\omega) \longrightarrow 0. \tag{6}$$

We will now define a Lie algebra structure on $C^\infty(M)$ to make this into an exact sequence of Lie algebras. Proposition 3.1 indicates what the right definition of the Lie bracket should be.

Definition 3.2. Let (M,ω) be a symplectic manifold. The Poisson bracket of two functions $F, G \in C^\infty(M,\mathbb{R})$ is defined as

$$\{F,G\} = -\omega(X_F, X_G).$$

From the definition it is immediate that the Poisson bracket is anti-symmetric. Using that $\iota(X_G)\omega = \mathrm{d}G$ by definition together with Cartan's identity, one has the alternative formulas

$$\{F,G\} = L_{X_F}G = -L_{X_G}F.$$

These formulas show for example that if F Poisson commutes with a given Hamiltonian G, then F is an *integral of motion* for X_G: That is, F is constant along solution curves of X_G.

Proposition 3.3. *The Poisson bracket defines a Lie algebra structure on* $C^\infty(M,\mathbb{R})$*: That is, it is anti-symmetric and satisfies the Jacobi identity*

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$$

for all F, G, H. *The map* $C^\infty(M) \to \mathfrak{X}(M)$, $F \mapsto X_F$ *is a Lie algebra homomorphism:*

$$X_{\{F,G\}} = [X_F, X_G]. \tag{7}$$

Proof. Equation (7) is just a special case of Proposition 3.1. The first statement follows from the calculation,

$$\begin{aligned}
\{F,\{G,H\}\} &= L_{X_F}\{G,H\} \\
&= -L_{X_F}(\omega(X_G,X_H)) \\
&= -\omega([X_F,X_G],X_H) - \omega(X_G,[X_F,X_H]) \\
&= -\omega(X_{\{F,G\}},X_H) - \omega(X_G,X_{\{F,H\}}) \\
&= -\{H,\{F,G\}\} + \{G,\{F,H\}\}.
\end{aligned}$$

□

An immediate consequence of (7) is:

Corollary 3.4. *If* $F, G \in C^\infty(M)$ *Poisson-commute, the flows of their Hamiltonian vector fields* X_F, X_G *commute.*

DEFINITION 3.5. An algebra A together with a Lie structure $[\cdot,\cdot]$ is called a Poisson algebra if

$$[FG,H] = F[G,H] + [F,H]G.$$

For any algebra A, the canonical Lie bracket $[F,G] = FG - GF$ satisfies this property.

PROPOSITION 3.6. *The algebra* $(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ *is a Poisson algebra.*

PROOF.

$$\{FG,H\} = L_{X_H}(FG) = (L_{X_H}F)G + F(L_{X_H}G) = \{F,H\}\,G + F\{G,H\}.$$

□

PROPOSITION 3.7. *For any compact connected symplectic manifold, Lie algebra extension* (6) *has a canonical splitting. That is, there exists a canonical Lie algebra homomorphism* $\mathfrak{X}_{Ham}(M,\omega) \to C^\infty(M,\mathbb{R})$ *that is a right inverse to the map* $F \mapsto X_F$*.*

PROOF. The required map associates to every $X \in \mathfrak{X}_{Ham}(M,\omega)$ the unique H such that $X_H = X$ and $\int_M H\Lambda = 0$ (where Λ is the Liouville form). The equality

$$\int_M \{F,G\}\,\Lambda = \int_M (L_{X_F}G)\Lambda = \int_M L_{X_F}(G\Lambda) = 0$$

shows that this is indeed a Lie homomorphism. □

Let us give the expression for the Poisson bracket for open subsets $U \subset \mathbb{R}^{2n}$, with symplectic coordinates $q_1,\dots,q_n,p_1,\dots,p_n$. We have

$$X_F = \sum_{j=1}^n \Big(\frac{\partial F}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial}{\partial p_j} \Big),$$

hence $\{F,G\} = X_F(G)$ is given by

$$\{F,G\} = \sum_{j=1}^n \Big(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} \Big).$$

EXERCISE 3.8. Verify directly, in local coordinates, that the right hand side of this formula defines a Lie bracket.

3.3. Review of some differential geometry. As a preparation for the following section we briefly review the notions of submersions, fibrations, and foliations. Let Q be an n-dimensional manifold.

3.3.1. *Submersions.* A *submersion* is a smooth map $f : Q \to B$ to a manifold B such that for all $q \in Q$, the tangent map $\mathrm{d}_q f = f_*(q) : T_qQ \to T_{f(q)}B$ is surjective. For any submersion, the fibers $f^{-1}(a)$ are smooth embedded submanifolds of Q of dimension $k = n - \dim B$. In fact Q is *foliated* by such submanifolds, in the following sense: Every point $q \in Q$ has a coordinate neighborhood U with coordinates $x_1, \ldots, x_n$, (with q corresponding to $x = 0$) and $f(q)$ a coordinate neighborhood with coordinates $y_1, \ldots, y_{n-k}$ (with $f(q)$ corresponding to $y = 0$) such that the map f is given by projection to the first $n - k$ coordinates.

Let $\pi : Q \to B$ be a submersion. Let the space of vertical vector fields

$$\mathfrak{X}_{\mathrm{vert}}(Q) = \{X \in \mathfrak{X}(Q) |\, X \sim_\pi 0\}$$

be the space of all vector fields taking values in $\ker \pi_*$, i.e tangent to the fibers. Let

$$\Omega_{\mathrm{hor}}(Q) = \{\alpha \in \Omega(Q) |\, \iota_X \alpha = 0 \ \ \forall X \in \mathfrak{X}_{\mathrm{vert}}(Q)\}$$

the space of horizontal forms, and

$$\Omega_{\mathrm{basic}}(Q) = \{\alpha \in \Omega(Q) |\, L_X \alpha = 0, \ \iota_X \alpha = 0 \ \ \forall X \in \mathfrak{X}_{\mathrm{vert}}(Q)\}$$

the space of *basic* forms. Notice that $\Omega_{\mathrm{basic}}(Q)$ preserved by the exterior differential d, that is, it is a subcomplex of $\Omega(Q), \mathrm{d})$.

3.3.2. *Fibrations.* A submersion is called a *fibration* if it is surjective and has the local triviality condition: There exists a manifold F (called standard fiber) such that every point in M there is a neighborhood U and a diffeomorphism $\phi : U \to f(U) \times F$, in such a way that f is just projection to the first factor. One can show that every surjective submersion with compact fibers is a fibration. In particular, if Q is compact every submersion from Q is fibrating.

For any fibration, $\pi : Q \to B$, pull-back defines an isomorphism

$$\pi^* : \Omega^k(B) \to \Omega^k_{\mathrm{basic}}(Q).$$

(This is easily verified in bundle charts $\pi : U \times F \to U$.)

3.3.3. *Distributions and foliations.* A vector subbundle $E \subset TQ$ of rank k is called a *distribution*. For example, if $f : Q \to B$ is a submersion (or more generally if the tangent map f_* has constant rank), then $E = \ker(f_*) \subset TQ$ is a distribution. Also, if $X \in \mathfrak{X}(Q)$ is nowhere vanishing vector field, $\mathrm{span}(X)$ is a distribution of rank 1. A submanifold $S \subset Q$ is called an *integral submanifold* if $TS = E|_S$. For example, the integral submanifolds of $\mathrm{span}(X)$ are just integral curves of X. A distribution E of rank k is called *integrable* if through every point $q \in Q$ there passes a k-dimensional integral submanifold. As one can show, this is is equivalent to the condition that every point has a neighborhood U and a submersion $f : U \to \mathbb{R}^{n-k}$ such that $E|_U = \ker(f_*)$. For this reason integrable distributions are also called *foliations*.

Example 3.9. Let $S^3 \subset \mathbb{C}^2$ the unit sphere. For each $z \in S^3$ let $E_z \subset T_zS^2$ denote the orthogonal complement of the complex line through z. The distribution E is not integrable.

A necessary condition for integrability of a distribution is that for every two vector fields X_1, X_2 taking values in E, their Lie bracket $[X_1, X_2]$ takes values in E. This follows because the Lie bracket of two vector fields tangent to a submanifold $S \subset Q$ is also tangent to S. Frobenius' theorem states that this condition is also sufficient:

THEOREM 3.10 (Frobenius' criterion). *A distribution $E \subset TQ$ is integrable if and only if the space of sections of E is closed under the Lie bracket operation.*

EXAMPLE 3.11. Let $Q = \mathbb{R}^3$ with coordinates x, y, z, and let $E \subset TQ$ the vector subbundle spanned by the two vector fields,

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Since $[X_1, X_2] = \frac{\partial}{\partial y}$ is not in E, the distribution E is not integrable.

EXAMPLE 3.12. A smooth map $f : Y_1 \to Y_2$ is called a constant rank map if the image of the tangent map $f_* : TY_1 \to TY_2$ has constant dimension. Examples are submersions (f_* surjective) or immersions (f_* injective). The kernel of every constant rank map defines an integrable distribution. Indeed, two vector fields X_1, X_2 are in the kernel if and only if $X_j \sim_f 0$. Hence also $[X_1, X_2] \sim_f 0$.

(Here we have used that for any vector bundle map $F : E_1 \to E_2$ of constant rank, the kernel and image of F are smooth vector subbundles. In particular $\ker(f_*)$ is a smooth vector subbundle.)

3.4. Lagrangian submanifolds. Let (M, ω) be a symplectic manifold.

DEFINITION 3.13. A submanifold (or, more generally, an immersion) $\iota : N \hookrightarrow M$ is called co-isotropic (resp. isotropic, Lagrangian, symplectic) if at any point $n \in N$, T_nN is a co-isotropic (resp. isotropic, Lagrangian, symplectic) subspace of T_nM.

For example $\mathbb{R}P(n) \subset \mathbb{C}P(n)$ is a Lagrangian submanifold. Also, the fibers of a cotangent bundle T^*Q are Lagrangian. Another important example is:

PROPOSITION 3.14. *The graph $\Gamma_\alpha \subset T^*Q$ of a 1-form $\alpha : Q \to T^*Q$ is Lagrangian if and only if α is closed.*

PROOF.

$$\alpha^*\omega = -\alpha^*\mathrm{d}\theta = -\mathrm{d}\alpha^*\theta = -\mathrm{d}\alpha.$$

□

In local coordinates: If $\alpha = \sum_j \alpha_j \mathrm{d}q_j$, the pull-back of $\omega = \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$ to the graph of α is given by

$$\sum_j \mathrm{d}q_j \wedge \mathrm{d}\alpha_j = \sum_{j,k} \frac{\partial \alpha_j}{\partial q_k} \mathrm{d}q_j \wedge \mathrm{d}q_k = -\mathrm{d}\alpha.$$

Hence α is closed if and only if the pull-back of ω to the graph vanishes.

Let $N \subset Q$ be a submanifold of Q. Dual to the inclusion $\iota_* = \mathrm{d}\iota : TN \to TQ|_N$ there is a surjective vector bundle map,

$$(\mathrm{d}\iota)^* : T^*Q|_N \to T^*N.$$

Its kernel $\ker((\mathrm{d}\iota)^*)$ (i.e. the pre-image of the zero section $N \subset T^*Q|_N$) consists of covectors that vanish on all tangent vectors to N. One calls $\mathrm{Ann}(TN) := \ker((\mathrm{d}\iota)^*)$ the annihilator bundle of TN, or also the *conormal bundle*.

Proposition 3.15. *The conormal bundle to any submanifold $N \subset Q$ is a Lagrangian submanifold of T^*Q. More generally, let $\alpha \in \Omega^1(N)$ be a 1-form on N. Then the pre-image of the graph $\Gamma_\alpha \subset T^*N$ under the map $(d\iota)^* : T^*Q|_N \to T^*N$ is a Lagrangian submanifold of T^*Q.*

Proof. Locally, near any point of N we can choose coordinates $q_1, \ldots, q_n$ on Q such that N is given by equations $q_{k+1} = 0, \ldots, q_n = 0$. In the corresponding cotangent coordinates q_j, p_j on T^*Q, $\mathrm{Ann}(TN)$ is given by equations $q_{k+1} = 0, \ldots, q_n = 0$, $p_1, \ldots, p_k = 0$. Clearly each summand in $\omega = \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$ vanishes on this submanifold. More generally, the pre-image $((\mathrm{d}\iota)^*)^{-1}(\Gamma_\alpha)$ is given by equations

$$q_{k+1} = 0, \ldots, q_n = 0, p_1 = \alpha_1, \ldots, p_k = \alpha_k.$$

Hence the pull-back of ω to this submanifold is given by

$$\sum_{i=1}^k \mathrm{d}q_i \wedge \mathrm{d}\alpha_j = \sum_{i,j=1}^k \frac{\partial \alpha_i}{\partial q_j} \mathrm{d}q_i \wedge \mathrm{d}q_j$$

so that ω vanishes on this submanifold if and only if α is closed. ☐

Proposition 3.16. *Let (M_j, ω_j) be symplectic manifolds, and let M_1^- denote M_1 with symplectic form $-\omega_1$. A diffeomorphism $F : M_1 \to M_2$ is a symplectomorphism if and only if its graph*

$$\Gamma_F := \{(F(m), m) |\, m \in M\} \subset M_2 \times M_1^-$$

is Lagrangian.

Proof. Similar to the linear case. ☐

Here is a nice application of these considerations.

Theorem 3.17 (Tulczyjew). *Let $E \to B$ be a vector bundle, $E^* \to B$ its dual bundle. There is a canonical symplectomorphism $T^*E \cong T^*E^*$.*

Outline of Proof. Consider the vector space direct sum $N = E \oplus E^*$ as a smooth submanifold of $Q = E \times E^*$. The natural pairing between E and E^* defines a function $f : N = E \oplus E^* \to \mathbb{R}$, let $\alpha = \mathrm{d}f$. By the above, α defines a Lagrangian submanifold L of $T^*Q = T^*E \times T^*E^*$. One checks that the projections from L onto both factors T^* and T^*E^* are diffeomorphisms, hence L is the graph of a symplectomorphism. ☐

Tulczyjew only considered the case $E = TQ$. It was pointed out by D. Roytenberg in his thesis that the argument works for any vector bundles. (In fact, he uses a generalization to super-vector bundles.)

3.5. Constant rank submanifolds. A submanifold $\iota : N \to M$ is called a "constant rank submanifold" if the dimension of the kernel of $(\iota^*\omega)_n$ is independent of $n \in N$.

PROPOSITION 3.18. *Let N be a manifold together with a closed 2-form σ of constant rank. Then the subbundle* $\ker(\sigma)$ *is integrable, i.e. defines a foliation.*

PROOF. We use Frobenius' criterion. Suppose $X_1, X_2 \in \mathfrak{X}(N)$ with $\iota_{X_j}\sigma = 0$. Since σ is closed, this implies $L_{X_j}\sigma = \mathrm{d}\iota_{X_j}\sigma = 0$. Hence

$$\iota_{[X_1,X_2]}\sigma = L_{X_1}\iota_{X_2}\sigma - \iota_{X_2}L_{X_1}\sigma = 0.$$

□

If this so-called null-foliation of N is fibrating, i.e. if the leaves of the foliation are the fibers of a submersion $\pi : N \to B$, where B is the space of leaves of the foliation. The form σ is basic for this fibration since $\iota_X\sigma = 0$ and $L_X\sigma = 0$ for all vertical vector fields. It follows that B inherits a unique 2-form ω_B such that

$$\pi^*\omega_B = \sigma$$

DEFINITION 3.19. The symplectic manifold (B, ω_B) is called the symplectic reduction of (N, σ).

REMARK 3.20. Note that the above discussion carries over for any closed differential form of constant rank on N.

In typical applications, N is a constant rank submanifold of a symplectic manifold with $\sigma = \iota^*\omega$. Of course, for a random constant rank submanifold it rarely happens that the null foliation is fibrating, unless additional symmetries are at work.

3.6. Co-isotropic submanifolds. An important special case of constant rank submanifolds of a symplectic manifold (M, ω) are co-isotropic submanifolds, i.e. submanifolds $N \subset M$ with $TN^\omega \subset TN$. For any submanifold $N \subset M$, let

$$C^\infty(M)_N = \{F \in C^\infty(M) |\, F|_N = 0\}$$

denote its vanishing ideal. The tangent bundle $TN \subset TM|_N$ and its annihilator have the following algebraic characterizations:

$$T_nN = \{v \in T_nM |\, v(F) = 0 \text{ for all } F \in C^\infty(M)_N\},$$

$$\mathrm{Ann}(T_nN) = \{\alpha \in T_n^*M |\, \alpha = \mathrm{d}F|_n \text{ for some } F \in C^\infty(M)_N\}.$$

The map $\omega^\flat : T_nM \to T_n^*M$ identifies the annihilator bundle with $TN^\omega \subset TM|_N$, and $\mathrm{d}F$ with X_F. Thus

$$T_nN^\omega = \{X_F(n) |\, F \in C^\infty(M)_N\}. \tag{8}$$

THEOREM 3.21. *The following three statements are equivalent:*

(a) *For all $F \in C^\infty(M)_N$, the Hamiltonian vector field X_F is tangent to N.*
(b) *The space $C^\infty(M)_N$ is a Poisson subalgebra of $C^\infty(M)$.*
(c) *N is a coisotropic submanifold of M.*

PROOF. X_F is tangent to N if and only if $X_F(G) = 0$ for all $G \in C^\infty(M)_N$. Since $X_F(G) = \{F, G\}$ this shows that (a) and (b) are equivalent. We have seen above that TN^ω is spanned by restrictions of Hamiltonian vector fields $X_F|_N$ with $F \in C^\infty(M)_N$. Hence N is coisotropic ($TN^\omega \subset TN$) if and only if every such vector field is tangent to N. This shows that (a) and (c) are equivalent. □

LEMMA 3.22. *Suppose $F : M \to \mathbb{R}^k$ is a submersion. Suppose that the components of F Poisson-commute, $\{F_i, F_j\} = 0$. Then the fibers of F are co-isotropic submanifolds of M of codimension k.*

PROOF. Let $N = F^{-1}(a)$. The vector fields X_{F_j} are tangent to N since $X_{F_j}(F_i) = \{F_j, F_i\} = 0$. Since $\mathrm{d}F_1, \ldots, \mathrm{d}F_k$ span $\mathrm{Ann}(TN)$ at each point of N, the Hamiltonian vector fields X_{F_j} span TN^ω. This shows $TN^\omega \subset TN$. □

REMARK 3.23. (a) In particular, given a submersion $F : M \to \mathbb{R}^n$ where $\dim M = 2n$, with Poisson-commuting components, the fibers of F define a Lagrangian foliation of M. This is the setting for completely integrable systems. We will discuss this case later in much more detail.
(b) The assumptions of the Proposition can be made more geometric, by saying that F should be a Poisson map, for the trivial Poisson structure on $\mathbb{R}^k$.

The following Proposition shows that locally, any coisotropic submanifold is obtained in this way:

PROPOSITION 3.24. *Let $\iota : N \hookrightarrow M$ be a codimension k co-isotropic submanifold. For any $m \in N$ there exists a neighborhood $U \subset M$ containing m, and a smooth submersion $F : U \to \mathbb{R}^k$ such that the components of F Poisson-commute, $\{F_i, F_j\} = 0$ for all i, j, and $N \cap U = F^{-1}(0)$.*

PROOF. The proof is by induction. Suppose that we have constructed Poisson-commuting functions $F_1, \ldots, F_l : U \to \mathbb{R}$, $l < k$ such that $(F_1, \ldots, F_l) : U \to \mathbb{R}^l$ is a submersion and $(F_1, \ldots, F_l)^{-1}(0) \supseteq U \cap N$. By the previous Lemma the fibers of $F' := (F_1, \ldots, F_l)$ define a foliation of U by co-isotropic submanifolds. Let $X_i = X_{F_i}$. The fact that the F_i vanish on N means that X_i is tangent to N, i.e. N is invariant under the flow, and is contained in a unique fiber N' of F'. Choosing U smaller if necessary, we can pick a codimension l submanifold $S \subset U$ transverse to N', and a submersion $F_{l+1} : S \to \mathbb{R}$ such that $N' \cap S \subset F_{l+1}^{-1}(0)$. Choosing U and S even smaller, we can extend F_{l+1} to a function on U, invariant under the flows of the commuting vector fields $X_1, \ldots, X_l$. This means that $F_1, \ldots, F_{l+1}$ all Poisson commute, and the map $(F_1, \ldots, F_{l+1})$ is a submersion near N'. □

Later we will obtain a much better version of this result (local normal form theorem), showing that one can actually take U to be a tubular neighborhood of N.

CHAPTER 4

Normal Form Theorems

1. Moser's trick

Moser's trick was used by Moser in a very short paper (1965) to show that on any compact oriented manifold any two normalized volume forms are diffeomorphism equivalent. (A volume form is a top degree nowhere vanishing differential form.) In order to describe his proof we recall the following fact from differential geometry:

Lemma 1.1. *Let $X_t \in \mathrm{Vect}(Q)$ $(t \in \mathbb{R})$ a time-dependent vector field on a manifold Q, with flow ϕ_t. For every differential form $\alpha \in \Omega^*(Q)$,*

$$\phi_t^* \, L_{X_t}\alpha = \frac{\partial}{\partial t}\,\phi_t^*\alpha$$

(on the region $U \subset Q$ where $\phi_t : U \to Q$ is defined).

Note that if α is a 0-form, this is just the definition of the flow ϕ_t. The general case follows because both sides are derivations of $\Omega(M)$ commuting with d.

Theorem 1.2 (Moser). *Let Q be a compact, oriented manifold, and Λ_0, Λ_1 two volume forms such that $\int_M \Lambda_0 = \int_M \Lambda_1$. Then there exists a smooth isotopy $\phi_t \in \mathrm{Diff}(Q)$ such that $\phi_1^*\Lambda_1 = \Lambda_0$.*

Proof. Moser's argument is as follows. First, note that every $\Lambda_t = (1-t)\Lambda_0 + t\Lambda_t$ is a volume form. Second, since Λ_0 and Λ_1 have the same integral they define the same cohomology class: $\Lambda_1 = \Lambda_0 + \mathrm{d}\beta$ for some $n-1$-form β. Thus

$$\Lambda_t = \Lambda_0 + t\,\mathrm{d}\beta.$$

Since each Λ_t is a volume form, the map $X \mapsto \iota_X\Lambda_t$ from vector fields to $n-1$-forms $(n = \dim Q)$ is an isomorphism. It follows that there is a unique time dependent vector field X_t solving

$$\iota_{X_t}\Lambda_t + \beta = 0.$$

Let ϕ_t denote the flow of X_t. Then

$$\begin{aligned}\frac{\partial}{\partial t}\,\phi_t^*\Lambda_t &= \phi_t^*\Big(\mathrm{d}\beta + L_{X_t}\Lambda_t\Big)\\ &= \mathrm{d}\Big(\beta + \iota_{X_t}\Lambda_t\Big)\\ &= 0.\end{aligned}$$

This shows $\phi_t^*\Lambda_t = \Lambda_0$. Now put $\phi = \phi_1$. □

Mosers theorem shows that volume forms on a given compact oriented manifold Q are classified up to diffeomorphism by their integral. The idea from Moser's proof applies to many similar problems. A typical application in symplectic geometry is as follows.

THEOREM 1.3. *Let ω_t be a family of symplectic 2-forms on a compact manifold M, depending smoothly on $t \in [0,1]$. Suppose that*

$$\omega_t = \omega_0 + d\beta_t$$

for some smooth family of 1-forms $\beta_t \in \Omega^1(M)$. Then there exists a family of diffeomorphisms (i.e. a smooth isotopy) ϕ_t such that

$$\phi_t^* \omega_t = \omega_0$$

for all t.

PROOF. Define a time-dependent vector field X_t by

$$\iota_{X_t}\omega_t + \frac{\partial}{\partial t}\beta_t = 0.$$

Let ϕ_t denote its flow. Then $\phi_t^*\omega_t$ is independent of t:

$$\frac{\partial}{\partial t}\,\phi_t^*\omega_t = \phi_t^*\Big(\frac{\partial}{\partial t}\omega_t + L_{X_t}\omega_t\Big) = \mathrm{d}\Big(\frac{\partial}{\partial t}\beta_t + \iota_{X_t}\omega_t\Big) = 0.$$

□

Alan Weinstein used Moser's argument to give a simple proof of Darboux's theorem, saying that symplectic manifold have no local invariants, and some generalizations. We will present Weinstein's proof below, after some review of homotopy operators in de Rham theory.

2. Homotopy operators

Let Q_1, Q_2 be smooth manifolds and $f_0, f_1 : Q_1 \to Q_2$ two smooth maps. Suppose f_0, f_1 are homotopic, i.e. that they are the boundary values of a continuous map

$$f : [0,1] \times Q_1 \to Q_2.$$

As one can show f can always taken to be smooth. Define the homotopy operator

$$\mathcal{H} : \Omega^k(Q_2) \to \Omega^{k-1}(Q_1)$$

as a composition

$$\mathcal{H}(\alpha) = \int_{[0,1]} f^*\alpha.$$

Here $\int_{[0,1]} : \Omega^k([0,1] \times Q_1) \to \Omega^{k-1}(Q_1)$ is fiber integration, i.e. integrating out the $s \in [0,1]$ variable. (The integral of a form not containing $\mathrm{d}s$ is defined to be 0.)

EXERCISE 2.1. Verify that for any form β on $[0,1]\times Q$,

$$\int_{[0,1]} \mathrm{d}\beta = -\mathrm{d}\int_{[0,1]} \beta + \iota_1^*\beta - \iota_0^*\beta$$

where $\iota_j : Q \to Q\times\{j\}$ are the two inclusions. (Hint: fundamental theorem of calculus!)

As a consequence the map $\mathcal{H}$ has the property,

$$\mathrm{d}\circ\mathcal{H} + \mathcal{H}\circ\mathrm{d} = f_1^* - f_0^* : \Omega^k(Q_2) \to \Omega^k(Q_1)$$

Thus if α is a closed form on Q_2, then $\beta = \mathcal{H}(\alpha)$ solves

$$f_1^*\alpha - f_0^*\alpha = \mathrm{d}\beta.$$

Hence f_0^* and f_1^* induce the same map in cohomology.

EXAMPLE 2.2. (Poincare lemma.) Let $U \subset \mathbb{R}^m$ be an open ball around 0. Let $\iota : \{0\} \to U$ be the inclusion and $\pi : U \to \{0\}$ the projection. Then ι^* induces an isomorphism $H^k(U) = H^k(\mathrm{pt})$, with inverse π^*. That is, every closed form $\alpha \in \Omega^k(U)$ with $k > 0$ is a coboundary: $\alpha = \mathrm{d}\beta$.

PROOF. Since it is obvious that $\iota^*\circ\pi^* = (\pi\circ\iota)^*$ is the identity map, we only need to show that $\pi^*\circ\iota^* = (\iota\circ\pi)^*$ is the identity map in cohomology. Let $f_t : U \to U$ be multiplication by $t \in [0,1]$. The de Rham homotopy operator $\mathcal{H}$ shows that f_1^*, f_0^* induce the same map in cohomology. The claim follows since $f_1 = \mathrm{id}$ and $f_0 = \iota\circ\pi$. □

3. Darboux-Weinstein theorems

THEOREM 3.1 (Darboux). *Let (M,ω) be a symplectic manifold of dimension $\dim M = 2n$ and $m \in M$. Then there exist open neighborhoods U of m and V of $\{0\} \in \mathbb{R}^{2n}$, and a diffeomorphism $\phi : V \to U$ such that $\phi(0) = m$ and $\phi^*\omega = \sum_j dq_j \wedge dp_j$.*

Coordinate charts of this type are called *Darboux charts*.

PROOF. Using any coordinate chart centered at m, we may assume that M is an open ball U around $m = 0 \in \mathbb{R}^{2n}$, with ω some possibly non-standard symplectic form. Let $\omega_1 = \omega$ and ω_0 the standard symplectic form. Since any two symplectic forms on the vector space $T_0\mathbb{R}^{2n}$ are related by a linear transformation, we may assume that ω_1 agrees with ω_0 at the origin. Using the homotopy operator from Example 2.2 let

$$\beta := \mathcal{H}(\omega_1 - \omega_0) \in \Omega^1(U).$$

As in the original Moser trick put

$$\omega_t = \omega_0 + t\mathrm{d}\beta.$$

For all $t \in [0,1]$, ω_t agrees with ω_0 at 0. Hence, taking U smaller if necessary, we may assume ω_t is non-degenerate on U for all $t \in [0,1]$. Define a time-dependent vector field X_t on U by

$$\iota_{X_t}\omega_t = -\beta.$$

The flow of this vector field will not be complete in general. Since $\omega_1 - \omega_0$ vanishes at 0, the 1-form β and therefore the vector field X_t also vanish at 0. Hence we can find a smaller neighborhood U' of 0 such that the flow $\phi_t : U' \to U$ is defined for all $t \in [0,1]$. The flow satisfies

$$\frac{\partial}{\partial t}\phi_t^*\omega_t = \phi_t^*(L_{X_t}\omega_t + \mathrm{d}\beta) = \phi_t^*(\mathrm{d}\iota_{X_t}\omega_t + \mathrm{d}\beta) = 0,$$

hence by integration $\phi_t^*\omega_t = \omega_0$. Darboux's theorem follows by setting $\phi = \phi_1$. □

Again, the proof has shown a bit more: In a sufficiently small neighborhood of $0 \in \mathbb{R}^{2n}$ any two symplectic forms are isotopic.

Darboux's theorem says that *symplectic manifolds have no local invariants*, in sharp contrast to Riemannian geometry where there are many local invariants (curvature invariants). Darboux's theorem can be strengthened to the statement that the symplectic form near any submanifold N of a symplectic manifold M is determined by the restriction of ω to $TM|_N$:

THEOREM 3.2. *Let (M_j, ω_j), $j = 0, 1$ be two symplectic manifolds, and $\iota_j : N_j \to M_j$ given submanifolds. Suppose there exists a diffeomorphism $\psi : N_0 \to N_1$ covered by a symplectic vector bundle isomorphism*

$$\hat{\psi} : TM_0|_{N_0} \to TM_1|_{N_1}.$$

such that $\hat{\psi}$ restricts to the tangent map $\psi_* : TN_0 \to TN_1$. Then ψ extends to a symplectomorphism ϕ from a neighborhood of N_0 in M_0 to a neighborhood N_1 in M_1.

PROOF. Any submanifold N of a manifold M has a "tubular neighborhood" diffeomorphic to the total space of the normal bundle $\nu_N = TM|_N/TN$. Since $\hat{\psi}$ induces an isomorphism $\nu_{N_1} \to \nu_{N_0}$, we may assume that $M_1 = M_0 =: M$ is the total spaces of a vector bundle $\pi : M \to N$ over a given manifold $N_1 = N_0 =: N$, with two given symplectic forms ω_0, ω_1 that agree along N. Let $\mathcal{H} : \Omega^k(M) \to \Omega^{k-1}(M)$ be the standard homotopy operator for the vector bundle $\pi : M \to N$, and put $\beta = \mathcal{H}(\omega_1 - \omega_0)$ and $\omega_t = \omega_0 + t\mathrm{d}\beta$. Since ω_t agrees with ω_0 along N, it is in particular symplectic on a neighborhood of N in M. On that neighborhood we can define a time-dependent vector field X_t with $\iota_{X_t}\omega_t + \beta = 0$. Let ϕ_t be its flow (defined on an even small neighborhood for all $t \in [0,1]$), and put $\phi_1 =: \phi$. By Moser's argument $\phi^*\omega_1 = \omega_0$. □

THEOREM 3.3. *Let (M, ω) be a symplectic manifold, $\iota : L \hookrightarrow M$ a Lagrangian submanifold. There exists a neighborhood U_0 of L in M, a neighborhood U_1 of L in T^*L, and a symplectomorphism from U_0 to U_1 fixing L.*

PROOF. By theorem 3.2 it is enough find a symplectic bundle isomorphism $TM|_L \cong T(T^*L)|_L$. Choose a compatible almost complex structure J on M. Then $J(TL) \subset TM|_L$ is a Lagrangian subbundle complementary to TL, and is therefore isomorphic (by means of the symplectic form) to the dual bundle. It follows that

$$TM|_L \cong TL \oplus T^*L.$$

as a symplectic vector bundle. The same argument applies to M replaced with T^*L. Thus

$$TM|_L \cong TL \oplus T^*L \cong T(T^*L)|_L.$$

□

This type of result generalizes to constant rank submanifolds, as follows. First we need a definition.

DEFINITION 3.4. For any constant rank submanifold $\iota : N \to M$ of a symplectic manifold (M, ω), the *symplectic normal bundle* is the symplectic vector bundle

$$TN^\omega / TN \cap TN^\omega$$

Note that for co-isotropic submanifolds the symplectic normal bundle is just 0. For an isotropic submanifold of dimension k it has rank $2(n-k)$ where $2n = \dim M$. The following theorem is due to Marle, extending earlier results of Weinstein (for the cases N Lagrangian or symplectic) and Gotay (for the case N co-isotropic).

THEOREM 3.5 (Constant rank embedding theorem). *Let $\iota_j : N_j \hookrightarrow M_j$ $(j = 1, 2)$ two constant rank submanifolds of symplectic manifolds (M_j, ω_j). Let*

$$F_j = TN_j^{\omega_j} / (TN_j^{\omega_j} \cap TN_j)$$

be their symplectic normal bundles. Suppose there exists a symplectic bundle isomorphism

$$\hat{\psi} : F_0 \to F_1$$

covering a diffeomorphism $\psi : N_0 \to N_1$ such that

$$\psi^* \iota_1^* \omega_1 = \iota_0^* \omega_0.$$

Then ψ extends to a symplectomorphism ϕ of neighborhoods of N_j in M_j, such that ϕ induces $\hat{\psi}$.

Thus, a neighborhood of a constant rank submanifold $\iota : N \to M$ is characterized up to symplectomorphism by $\iota^*\omega$ together with the symplectic normal bundle. In particular, if N is co-isotropic a neighborhood is completely determined by $\iota^*\omega$.

PROOF. Suppose $\iota : N \to M$ is a compact constant rank submanifold of a symplectic manifold (M, ω). There are three natural symplectic vector bundles over N:

$$\begin{aligned} E &= TN/(TN \cap TN^\omega), \\ F &= TN^\omega/(TN \cap TN^\omega), \\ G &= (TN \cap TN^\omega) \oplus (TN \cap TN^\omega)^* \end{aligned}$$

Identifying E with a complementary subbundle to $TN \cap TN^\omega$ in TN we have

$$TN \cong E \oplus (TN \cap TN^\omega)$$

(isomorphism of bundles with 2-forms) likewise

$$TN^\omega \cong F \oplus (TN \cap TN^\omega),$$

Therefore $TN + TN^\omega = E \oplus F \oplus (TN \cap TN^\omega)$. Let J be an ω-compatible complex structure on $TM|_N$, preserving the two subbundles E, F. Then the isotropic subbundle $J(TN \cap TN^\omega) \subset TM|_N$ is a complement to $TN + TN^\omega$, which by means of ω is identified with $(TN \cap TN^\omega)^*$. This shows

$$TM|_N \cong E \oplus F \oplus G$$

as symplectic vector bundles. To prove the constant rank embedding theorem, choose isomorphisms of this type for both $TM_i|_{N_i}$. Then E_0, E_1 and G_0, G_1 are symplectomorphic since $\psi : N_0 \to N_1$ preserves two-forms, and $F_0 \cong F_1$ by assumption of the theorem. Now apply Theorem 3.2 $\square$

CHAPTER 5

Lagrangian fibrations and action-angle variables

1. Lagrangian fibrations

We had seen that for any submersion $F : M \to \mathbb{R}^k$ from a symplectic manifold M, such that the the components F_i Poisson-commute, the fibers of F are co-isotropic submanifolds of codimension k. In particular, if $k = n = \frac{1}{2}\dim M$, the fibers are Lagrangian submanifolds.

DEFINITION 1.1. Let (M, ω) be a symplectic manifold. A Lagrangian fibration is a fibration $\pi : M \to B$ such that every fiber is a Lagrangian submanifold of M.

This implies in particular $\dim B = n = \frac{1}{2}\dim M$.

EXAMPLES 1.2. (a) The fibers of a cotangent bundle $\pi : M = T^*Q \to Q$ are a Lagrangian fibration.
(b) If $Q = (\mathbb{R}/\mathbb{Z})^n = T^n$ is an n-torus, we have a natural trivialization $T^*(T^n) = (T^n) \times \mathbb{R}^n$ and the map $\pi : T^*(T^n) \to \mathbb{R}^n$ defines a Lagrangian fibration of $T^*(T^n)$.

Is it possible to generalize the second example to compact manifolds Q other than a torus? That is, is it possible to find a Lagrangian fibration of T^*Q such that the zero section $Q \subset T^*Q$ is one of the leaves? We will show in this section that the answer is "no": The leaves of a Lagrangian fibration are *always* diffeomorphic to an open subsets of products of vector spaces with tori.

We will need some terminology group actions on manifold. If G is a Lie group and Q a manifold, a group action is a smooth map

$$\mathcal{A} : G \times Q \to Q,\ (g, q) \mapsto \mathcal{A}(g, q) \equiv g.q$$

such that $e.q = q$ and $g_1 \cdot (g_2.q) = (g_1 g_2).q$. For $q \in Q$ the set $G.q = \mathcal{A}(G, q)$ is called the orbit, the subgroup $G_q = \{g \in G |\, g.q\}$ the stabilizer. Note that $G.q = G/G_q$. If G_q is compact then $G.q$ is a manifold.

LEMMA 1.3. *Let G be a connected Lie group and $\mathcal{A} : G \times Q \to Q$ a group action on a connected manifold Q. Then the following are equivalent:*

(a) *The action is transitive: For some (hence all) $q \in Q$, $G.q = Q$.*
(b) *The action is locally transitive: That is, for all $q \in Q$ there is a neighborhood U such that $(G.q) \cap U = U$.*
(c) *There exists $q \in Q$ such that the tangent map to $G \to Q$, $g \mapsto g.q.$ is surjective.*

PROOF. (a) just means that the orbits are open and closed. Both (b) and (c) imply that condition. The converse is obvious. □

Of particular interest is the case $\dim Q = \dim G$. We will say that Q is an *principal homogeneous G-space* if Q comes equipped with a free, transitive action of G. Any choice of a base point $q \in Q$ identifies $Q = G.q \cong G$. Any two such identifications differ by a translation in G. If G is a torus we call Q an *affine torus*, if G is a vector space we call Q an *affine vector space*.

Suppose V is an n-dimensional vector space acting transitively on an n-dimensional manifold Q. The stabilizer V_q is a discrete subgroup of V, independent of q. Thus Q carries the structure of a principal homogeneous $H = V/V_q$-space. Note that since H is compact, connected and abelian, it is a product of a vector space and a torus. That is, Q is a product of an *affine* torus and an *affine* vector space.

The above considerations also make sense for fiber bundles: If $\mathcal{G} \to B$ is a group bundle (i.e. a fiber bundle where the fibers carry group structures, and admitting bundle charts $\pi^{-1}(U) \cong U \times F$ that are fiberwise group isomorphisms with a fixed group F), one can define actions on fiber bundles $E \to B$ to be smooth maps

$$\mathcal{G} \times_B E \to E$$

that are fiberwise group actions. For example, if E is a vector bundle and $\mathrm{Gl}(E)$ is the bundle of general linear groups, one has a natural action of $\mathrm{Gl}(E)$ on E. In particular, an affine torus bundle $\pi : M \to B$ is a fiber bundle with a fiberwise free, transitive action of a torus bundle $\mathcal{T} \to B$. Thus if $\pi : M \to B$ has compact fibers, then every fiberwise transitive action of a vector bundle $E \to B$ with $\dim E = \dim M$ gives M the structure of an affine torus bundle. The torus bundle $\mathcal{T} \to B$ is the quotient bundle E/Λ, where $\Lambda \to B$ is the bundle of stabilizer groups. Any section $\sigma : B \to M$ makes M into a torus bundle, however in general there are obstructions to the existence of such a section. On the other hand, if $\mathcal{T}$ is trivial then $M \to B$ becomes a T-principal bundle.

Let us now consider a Lagrangian fibration $\pi : M \to B$. The following discussion is based mainly on the paper, J. J. Duistermaat: "*On global action-angle variables*", Comm. Pure Appl. Math. **33** (1980), 687–706. For simplicity we will usually assume that the fibers of π are compact and connected.

THEOREM 1.4. *Let (M, ω) be a symplectic manifold and $\pi : M \to B$ be a Lagrangian fibration with compact, connected fibers. Then there is a canonical, fiberwise transitive vector bundle action $T^*B \times M \to M$. Thus $\pi : M \to B$ has canonically the structure of an affine torus bundle.*

PROOF. We denote by $VM \subset TM$ the vertical tangent bundle, $V_mM = \ker(\mathrm{d}_m\pi)$. By assumption VM is a Lagrangian subbundle of the symplectic vector bundle TM. For any 1-form $\alpha \in \Omega^1(B)$ let $X_\alpha \in \mathfrak{X}(M)$ be the vector field defined by

$$\iota_{X_\alpha}\omega = -\pi^*\alpha.$$

For all vertical vector fields $Y \in \mathfrak{X}(M)$,

$$\omega(X_\alpha, Y) = \iota_Y \iota_{X_\alpha}\omega = -\iota_Y \pi^*\alpha = 0.$$

Since VM is a Lagrangian subbundle, it follows that X_α is a vertical vector field. Note that the value of X_α at $m \in M$ depends only on $\alpha_{\pi(m)}$. Thus we have constructed a linear map $V_m M \cong T^*_{\pi(m)}B$. Clearly this map is an isomorphism. Taking all of these isomorphisms together we have constructed a canonical bundle isomorphism

$$VM \cong \pi^* T^* B.$$

Let F^t_α denote the flow of the vector field X_α, and $F_\alpha = F^1_\alpha : M \to M$ the time one flow. Since the X_α are vertical the flow preserves the fibers, and again $F_\alpha(m)$ depends only on $\alpha_{\pi(m)}$. Thus we obtain a fiber bundle map

$$T^*B \times_B M \to M, \quad (\alpha_b, m) \mapsto F_\alpha(m).$$

To show that this is a vector bundle action, we need to show that the vector fields X_α all commute. Thus let α_1, α_2 be two 1-forms and $X_j = X_{\alpha_j}$ the vector fields they define. We have

$$\begin{aligned}
\iota_{[X_1,X_2]}\omega &= (L_{X_1}\iota_{X_2} - \iota_{X_2}L_{X_1})\omega \\
&= -L_{X_1}\pi^*\alpha_2 - \iota_{X_2}\mathrm{d}\iota_{X_1}\omega \\
&= -L_{X_1}\pi^*\alpha_2 + \iota_{X_2}\pi^*\mathrm{d}\alpha_1 \\
&= 0
\end{aligned}$$

since $\pi^*\alpha_j$ and $\pi^*\mathrm{d}\alpha_j$ are basic forms on $\pi : M \to B$. Since ω is non-degenerate this verifies $[X_1, X_2] = 0$. Since each map $T^*_{\pi(m)}B \to V_m$ is an isomorphism, it follows that the action is fiberwise transitive. □

LEMMA 1.5. *Let $\alpha \in \Omega^1(B)$ be a 1-form, and X_α, F^t_α the corresponding vector field and its flow. Then*

$$(F^t_\alpha)^*\omega = \omega - t\,\pi^* d\alpha.$$

In particular, F_α is a symplectomorphism if and only if α is closed.

PROOF. The Lemma follows by integrating

$$\frac{\partial}{\partial t}(F^t_\alpha)^*\omega = (F^t_\alpha)^* L_X\omega = -(F^t_\alpha)^*\pi^*\mathrm{d}\alpha = -\pi^*\mathrm{d}\alpha$$

from 0 to t. □

Let $\Lambda \subset T^*B$ be bundle of stabilizers for the T^*B-action, and

$$\tau : \mathcal{T} = T^*B/\Lambda \to B$$

the torus bundle.

PROPOSITION 1.6. *Λ is a Lagrangian submanifold of T^*B.*

PROOF. Suppose $U \subseteq B$ is an open subset such that Λ is trivial over U, $\Lambda|_U \cong U \times \mathbb{Z}^n$. Any sheet of $\Lambda|_U \to U$ is given by the graph of a 1-form $\alpha : B \to T^*B$ such that $F_\alpha = \mathrm{id}_{\pi^{-1}(U)}$. By the previous Lemma, this means $\pi^*\mathrm{d}\alpha = 0$. Thus $\mathrm{d}\alpha = 0$ showing that the sheet corresponding to α is a Lagrangian submanifold. $\square$

Consider the Lagrangian fibration $p : T^*B \to B$, with standard symplectic form $-\mathrm{d}\theta$ on T^*B. Any $\alpha \in \Omega^1(B)$ defines a vertical vector field $\hat{X}_\alpha$ on T^*B, and a flow $\hat{F}^t_\alpha$. We had discussed this flow in the section on cotangent bundles, where it was denoted G^t_α. We had proved that $\hat{F}^t_\alpha$ is the diffeomorphism of T^*B given by "adding $t\alpha$". Let us briefly recall the argument: For all 1-forms α the canonical 1-form θ on T^*B transforms according to

$$(\hat{F}^t_\alpha)^*\theta = \theta + t\, p^*\alpha.$$

This follows by integrating

$$\frac{\partial}{\partial t}(\hat{F}^t_\alpha)^*\theta = (\hat{F}^t_\alpha)^* L_{\hat{X}_\alpha}\theta = (\hat{F}^t_\alpha)^*(\iota(\hat{X}_\alpha)\mathrm{d}\theta) = (\hat{F}^t_\alpha)^*\, p^*\alpha = p^*\alpha.$$

from 0 to t. By the property $\beta^*\theta = \beta$ of the canonical 1-form we find,

$$\hat{F}^t_\alpha \circ \beta = (\hat{F}^t_\alpha \circ \beta)^*\theta = \beta^*(\theta + t\, p^*\alpha) = \beta + t\alpha$$

showing that $\hat{F}^t_\alpha$ adds on $t\alpha$.

PROPOSITION 1.7. *The symplectic form on T^*B descends to $\mathcal{T} = T^*B/\Lambda$. The projection $\tau : \mathcal{T} \to B$ is a Lagrangian fibration.*

PROOF. We have seen that local sections of Λ are given by closed 1-forms, but adding a closed 1-form in T^*B is a symplectic transformation. $\square$

To summarize the discussion up to this point: Given any symplectic fibration $\pi : M \to B$ with compact connected fibers, we constructed a torus bundle $\tau : \mathcal{T} \to B$ with a transitive bundle action of

$$\mathcal{T} \times_B M \to M.$$

This shows that M is an *affine* torus bundle. Moreover, $\mathcal{T}$ itself was found to carry a symplectic structure, such that $\tau : \mathcal{T} \to B$ is a Lagrangian fibration! Nonetheless, in general $\mathcal{T}$ is different from M. The affine torus bundle M becomes a torus bundle only after we choose a global section $\sigma : B \to M$, but in general there may be obstructions to the existence of such a section.

Let us assume that these obstructions vanish, and let $\sigma : B \to M$ be a global section. (Any other section differs by the action of a section of $\mathcal{T}$). The choice of σ sets up a unique $\mathcal{T}$-equivariant fiber bundle isomorphism

$$A : \mathcal{T} = T^*B/\Lambda \cong M,$$

sending the identity-section of $\mathcal{T}$ to σ. Equivariance means that

$$A \circ \hat{F}_\alpha = F_\alpha \circ A$$

for all 1-forms $\alpha \in \Omega^1(B)$.

The cohomology class $[\sigma^*\omega] \in H^2(B)$ is independent of the choice of σ. Indeed, if $\tilde\sigma = F_\beta \circ \omega$ is another choice of σ, then

$$\tilde\sigma^*\omega = \sigma^*(F_\beta^*\omega) = \sigma^*(\omega - \pi^*\mathrm{d}\beta) = \sigma^*\omega - \mathrm{d}\beta.$$

Let us assume that $[\sigma^*\omega] = 0$ (of course, this is automatic if ω is exact, e.g. if M is an open subset of a cotangent bundle T^*Q.) Then $\sigma^*\omega = \mathrm{d}\beta$ for some 1-form β, and replacing σ with $F_\beta \circ \sigma$ we can assume $\sigma^*\omega = 0$, i.e. the graph of σ is a Lagrangian submanifold. Note that both the existence of σ and the condition $[\sigma^*\omega] = 0$ are automatic if B is contractible, i.e. always locally.

PROPOSITION 1.8. *The choice of any section $\sigma : B \to M$ with $\sigma^*\omega = 0$ defines a symplectomorphism $A : \mathcal{T} = T^*B/\Lambda \cong M$.*

PROOF. Let $\tilde A : T^*B \to M$ be the map covering A. It suffices to show that $\tilde A$ is a local symplectomorphism, i.e. $\tilde A^*\omega = -\mathrm{d}\theta$. The map $\tilde A$ is uniquely defined by its property

$$\tilde A \circ \alpha = F_\alpha \circ \sigma$$

for every 1-form $\alpha : B \to T^*B$. We have

$$\begin{aligned}
\alpha^*\tilde A^*\omega &= (\tilde A \circ \alpha)^*\omega \\
&= (F_\alpha \circ \sigma)^*\omega \\
&= \sigma^* F_\alpha^*\omega \\
&= \sigma^*(\omega - \mathrm{d}\pi^*\alpha) \\
&= -\mathrm{d}\sigma^*\pi^*\alpha \\
&= -\mathrm{d}\alpha \\
&= -\mathrm{d}\alpha^*\theta \\
&= \alpha^*(-\mathrm{d}\theta)
\end{aligned}$$

Since this is true for any α, we conclude[1] $\tilde A^*\omega = -\mathrm{d}\theta$. □

2. Action-angle coordinates

We are now ready to define *action-angle coordinates*. Recall that up to this point, we have made two assumptions on the fibration $\pi : M \to B$: First, we assume that it admits a section $\sigma : B \to M$, second we assume that the cohomology class $[\sigma^*\omega] \in H^2(M)$ (defined independent of the choice of σ) is zero. Then we choose σ to be a Lagrangian section and this gives a symplectomorphism $\mathcal{T} \to M$.

Let us now also assume that the base B is simply connected. This implies that the group bundle $\Lambda \to B$ is trivial: Choose an isomorphism $\Lambda_b \cong \mathbb{Z}^n$ for some b, and identify

[1] Actually, with a little bit of cheating: A k-form on a fiber bundle over B is determined by its pull-backs under sections of the fiber bundle, if and only if $\dim B \geq k$. Hence we need $\dim B \geq 2$. However the case $\dim B = 1$ is obvious anyway.

$\Lambda_{b'} \cong \Lambda_b$ by choosing a path from b to b'. (This is independent of the choice of path if B is simply connected.) Thus we have an isomorphism

$$\Lambda = B \times \mathbb{Z}^n,$$

and any two such isomorphisms differ by an action of $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{Gl}(n, \mathbb{Z})$, the group of invertible matrices A such that both A and A^{-1} have integer coefficients (this implies $\det A = \pm 1$). Let $\beta_1, \ldots, \beta_n \in \Omega^1(B)$ be the closed 1-forms corresponding to this trivialization. At any point $b \in B$ they define a basis for Λ_b, hence also for $T_b^* B$. Thus we have also trivialized

$$T^*B \cong B \times \mathbb{R}^n.$$

and

$$M \cong \mathcal{T} \cong B \times (\mathbb{R}^n/\mathbb{Z}^n) = B \times T^n.$$

The map $s : M \to (\mathbb{R}/\mathbb{Z})^n = T^n$ given by projection to the second factor are the *angle coordinates*. Since the 1-forms β_i take values in Λ they are closed. They define symplectic vector fields $X_i \in \mathrm{Vect}(\pi^{-1}(U))$ whose flows F_i^t in terms of the angle coordinates are given by

$$s_j \mapsto s_j \text{ if } j \neq i,$$
$$s_i \mapsto s_i + t.$$

Since B is by assumption simply connected the β_i are in fact exact:

$$\beta_i = \mathrm{d} I_i.$$

The I_i's (or their pull-backs to $\pi^{-1}(U)$) are called *action coordinates*. The choice of I_i defines an embedding $B \hookrightarrow \mathbb{R}^n$. That is, B is diffeomorphic to an open subset of $\mathbb{R}^n$!. It is clear that $I_1, s_1, \ldots, I_n, s_n$ lift to the standard cotangent coordinates on $T^*B \subset T^*(R^n)$. Therefore ω takes on the form, in action-angle coordinates,

$$\omega = \sum_i \mathrm{d} I_i \wedge \mathrm{d} s_i.$$

Notice that the choice of action-angle coordinates is very rigid: The β_i's are determined up to the action of a matrix $C \in \mathrm{Gl}(n, \mathbb{Z})$, and the choice of I_i is determined up to a constant. That is, any other set of action-angle coordinates is of the form,

$$s_i' = \sum_j C_{ij} s_j + \pi^* c_i, \quad I_i' = \sum_j (C^{-1})_{ji} I_j + d_i$$

where the c_i are functions on B and d_i are constants. The I_j can be viewed as functions on B; note that this induces, in particular, an affine-linear structure on B.

If we drop the assumption that B is simply connected, there are obstructions to the existence of global action-angle coordinates. (Even if the Lagrangian section $\sigma : B \to M$ exists.) The first, most serious, obstruction is the *monodromy obstruction*: To introduce angle coordinates we have to assume that the bundle Λ is trivial; this however will only be true if the monodromy map

$$\pi_1(B) \to \mathrm{Aut}(\Lambda_b) \cong \mathrm{Gl}(n, \mathbb{Z})$$

is trivial. If the monodromy obstruction vanishes, there can still be an obstruction to the existence of action coordinates: The forms β_i defined by the angle coordinates are closed, but they need not be exact in general.

Of course, all obstructions vanish locally, e.g. over contractible open subsets $U \subset B$. On the other hand, Duistermaat shows that for a very standard integrable system, the spherical pendulum, the monodromy obstruction is non-zero.

EXERCISE 2.1. Suppose (M, ω) is a symplectic manifold such that ω is exact: $\omega = \mathrm{d}\gamma$ for some 1-form γ. Let $\pi : M \to B$ be a Lagrangian fibration with compact connected fibers, with B simply connected. Given $b \in B$ let

$$A_1(b), \ldots, A_n(b) : \mathbb{R}/\mathbb{Z} \to \pi^{-1}(b)$$

be smooth loops in $\pi^{-1}(b)$ generating the fundamental group of the fiber. Suppose that the $A_i(b)$ define continuous functions $A_i : B \times \mathbb{R}/\mathbb{Z} \to M$. Show that the formula

$$I_j(m) := \int_{A_j(\pi(m))} \gamma$$

defines a set of action variables.

For an up-to-date discussion of Lagrangian foliations, including a review of recent developments, see the preprint Nguyen Tien Zung: "*Symplectic Topology of Integrable Hamiltonian Systems, II: Characteristic Classes*", posted as math.DG/0010181.

3. Integrable systems

After this lengthy general discussion let us finally make the connection with the theory of integrable systems. Let (M, ω) be a compact symplectic manifold, $H \in C^\infty(M, \mathbb{R})$ a Hamiltonian and X_H its vector field. In general the flow of X_H can be very complicated, unless there are many "integrals of motion". An integral of motion is a function $G \in C^\infty(M, \mathbb{R})$ such that $X_H(G) = 0$, or equivalently $\{H, G\} = 0$. An integral of motion defines itself a Hamiltonian flow X_G, which commutes with the flow of X_H since $[X_H, X_G] = -X_{\{H,G\}} = 0$.

DEFINITION 3.1. The dynamical system (M, ω, H) is called integrable over if there exists n integrals of motion $G_1, \ldots, G_n \in C^\infty(M, \mathbb{R})$, $\{G_j, H\} = 0$ such that

(a) The G_i are "in involution", i.e. they Poisson-commute: $\{G_i, G_j\} = 0$.
(b) The map $G : M \to \mathbb{R}^n$ is a submersion almost everywhere, i.e. $\mathrm{d}G_1 \wedge \ldots \wedge \mathrm{d}G_n \neq 0$ on an open dense subset.

THEOREM 3.2 (Liouville-Arnold). *Let (M, ω, H) be a completely integrable Hamiltonian dynamical system, with integrals of motion G_j. Suppose G is a proper map. Let $M' \subset M$ be the subset on which G is a submersion, let B be the set of connected components of fibers of $G|_{M'}$, and $\pi : M' \to B$ the induced map. Then $\pi : M \to B$ is a Lagrangian fibration with compact connected fibers, hence it is an affine torus bundle.*

The flow F^t of X_H is vertical and preserves the affine structure; in local action-angle coordinates it is given by

$$I_j(t) = I_j(0),$$
$$s_j(t) = s_j(0) + t\,\frac{\partial H}{\partial I_j}.$$

PROOF. It remains to prove the description of the flow of X_H. Since $\{G_j, H\} = 0$ for all j, $H|_{M'}$ is constant along the fibers of G, i.e. is the pull-back of a function on B. In local action-angle coordinates (I, s) for the fibration, this means that H is a function of the I_i's, and the Hamiltonian vector field becomes

$$X_H = \sum_j \frac{\partial H}{\partial I_j}\frac{\partial}{\partial s_j}.$$

□

4. The spherical pendulum

As one of the simplest non-trivial examples of an integrable system let us briefly discuss the spherical pendulum. We first give the general description of the motion of a particle on a Riemannian manifold Q in a potential $V : Q \to \mathbb{R}$.

One defines the kinetic energy $T \in C^\infty(TQ)$ by $T(v) = \frac{1}{2}||v||^2$. Using the identification $g^\flat : TQ \to T^*Q$ given by the metric, view T as a function on T^*Q. The Hamiltonian is the total energy $H = T + V \in C^\infty(T^*Q)$.

In local coordinates q_i on Q,

$$T(v) = \tfrac{1}{2}\textstyle\sum_{ij} g(q)_{ij}\dot{q}_i\dot{q}_j,$$

where g_{ij} is the metric tensor. The relation between velocities and momenta in local coordinates is $p_i = \sum_j g_{ij}\dot{q}_j$. Thus

$$T(q,p) = \tfrac{1}{2}\textstyle\sum_{ij} h(q)_{ij}\ p_i p_j$$

where $h(q)_{ij}$ is the inverse matrix to $g(q)_{ij}$, and

$$H(q,p) = \tfrac{1}{2}\textstyle\sum_{ij} h(q)_{ij}\ p_i p_j + V(q).$$

The configuration space Q of the spherical pendulum is the 2-sphere, which by an appropriate normalization we can take to be the unit sphere in $\mathbb{R}^3$. Let $\phi \in [0, 2\pi], \psi \in (0, \pi)$ be polar coordinates on S^2, that is

$$x_1 = \sin\psi\cos\phi,\ x_2 = \sin\psi\sin\phi,\ x_3 = \cos\psi.$$

The potential energy is

$$V = \cos\psi$$

and the kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{3}\dot{x}_i^2 = \frac{1}{2}(\dot{\psi}^2 + \sin^2\psi\dot{\phi}^2).$$

Thus

$$H = \frac{1}{2}\big(p_\psi^2 + \frac{1}{\sin^2\psi}\, p_\phi^2\big) + \cos\psi$$

(The apparent singularity at $\psi = 0, \pi$ comes only from the choice of coordinates.) An integral of motion for this system is given by $G = p_\phi$. Indeed,

$$\{H, G\} = 0$$

because H does not depend on ϕ (i.e. because the problem has rotational symmetry around the x_3-axis). Since $\dim T^*S^2 = 4$, it follows that the spherical pendulum is a completely integrable system.

The image of the map (G, H) has the form $H \geq f(G)$ where $f(G)$ is a symmetric function shaped roughly like a parabola. The minimum of f is the point $(G, H) = (0, -1)$, corresponding to the stable equilibrium. The set of singular values of (G, H) consists of the boundary of the region, i.e. the set of points $(G, F(G))$, together with the *unstable* equilibrium $(0, 1)$.

Removing these singular points from the image of (G, H), we obtain a non-simply connected region B, and one can raise the question about existence of global action-angle variables. Duistermaat shows that they do not exist in this system: The lattice bundle $\Lambda \to B$ is non-trivial, i.e. the monodromy obstruction does not vanish.

CHAPTER 6

Symplectic group actions and moment maps

1. Background on Lie groups

A Lie group is a group G with a manifold structure on G such that group multiplication is a smooth map. (This implies that inversion is a smooth map also.) A Lie subgroup $H \subset G$ is a subgroup which is also a submanifold. By a theorem of Cartan, every closed subgroup of a Lie group is an (embedded) Lie subgroup (i.e, smoothness is automatic). In this case the homogeneous space G/H inherits a unique manifold structure such that the quotient map is smooth.

Let $L_g : G \to G,\ a \mapsto ga$ denote left translation and $R_g : G \to G,\ a \mapsto ag$ right translation. A vector field X on G is called left-invariant if it is L_g-related to itself, for all $g \in G$. Any left-invariant vector field is determined by its value $X(e) = \xi$ at the identity element. Thus evaluation at e gives a vector space isomorphism $\mathfrak{X}^L(G) \cong \mathfrak{g} := T_eG$. But $\mathfrak{X}^L(G)$ is closed under the Lie bracket operation of vector fields. The space $\mathfrak{g} = T_eG$ with Lie bracket induced in this way is called the Lie algebra of G. That is, we define the Lie bracket on $\mathfrak{g}$ by the formula

$$[\xi, \eta]^L := [\xi^L, \eta^L],$$

where ξ^L is the left-invariant vector field with $\xi^L(e) = \xi$. For matrix Lie groups (i.e. closed subgroups of $\mathrm{GL}(n, \mathbb{R})$), the Lie bracket coincides with the commutator of matrices. Working with the space of right-invariant vector fields would have produced the *opposite* bracket: We have

$$[\xi, \eta]^R = -[\xi^R, \eta^R].$$

Let $F^t_\xi : G \to G$ be the flow of ξ^L. One defines the exponential map

$$\exp : \mathfrak{g} \to G$$

by $\exp(\xi) = F^1_\xi(e)$. In terms of exp, the flow of ξ^L is $g \mapsto g\exp(t\xi)$, while the flow of ξ^R is $g \mapsto \exp(t\xi)g$. For matrix Lie groups, exp is the usual exponential of matrices.

Let $\mathrm{Ad}_g = L_g \circ R_{g^{-1}}$, i.e. $\mathrm{Ad}_g(a) = gAg^{-1}$. Clearly Ad_g fixes e, so it induces a linear transformation (still denoted Ad_g) of $\mathfrak{g} = T_eG$. One defines

$$\mathrm{ad}_\xi(\eta) = \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Ad}_{\exp(t\xi)}(\eta).$$

It turns out that $\mathrm{ad}_\xi(\eta) = [\xi, \eta]$, which gives an alternative way of defining the Lie bracket on $\mathfrak{g}$.

2. Generating vector fields for group actions

DEFINITION 2.1. Let G be a Lie group. An action of G on a manifold Q is a smooth map

$$\mathcal{A}: G \times Q \to Q,\ (g, q) \mapsto \mathcal{A}_g(q) = g \cdot q$$

such that the map $G \to \mathrm{Diff}(Q),\quad g \mapsto \mathcal{A}_g$ is a group homomorphism.

A manifold Q together with a G-action is called a G-manifold. A map $F: Q_1 \to Q_2$ between two G-manifolds is called equivariant if it intertwines the G-actions, that is, $g.F(q_1) = F(g.q_1)$.

EXAMPLE 2.2.
(a) There are three natural actions of any Lie group G on itself: The left-action $g.a = ga$, the right action $g.a = ag^{-1}$, and the adjoint action $g.a = gag^{-1}$.
(b) Any finite dimensional linear representation of G on a vector space V is a G-action on V.

DEFINITION 2.3. Let $\mathfrak{g}$ be a Lie algebra. A *Lie algebra action* of $\mathfrak{g}$ on Q is a smooth vector bundle map

$$Q \times \mathfrak{g} \to TQ,\ (q, \xi) \mapsto \xi_Q(q)$$

such that the map $\mathfrak{g} \to \mathfrak{X}(Q),\ \xi \mapsto \xi_Q$ is a Lie algebra homomorphism.

Suppose Q is a G-manifold. For $\xi \in \mathfrak{g}$ let the *generating vector field* ξ_Q be the unique vector field with flow

$$q \mapsto \exp(-t\xi).q \quad .$$

If $F: Q_1 \to Q_2$ is an equivariant map, then $\xi_{Q_1} \sim_F \xi_{Q_2}$.

EXAMPLE 2.4. For any $\xi \in \mathfrak{g}$ let ξ^L denote the unique left-invariant vector field with $\xi^L(e) = \xi$. Similarly let ξ^R be the right-invariant vector field with $\xi^R(e) = \xi$. The generating vector field for the left-action is right-invariant (since the left-action commutes with the right-action), and its value at e is $-\xi$. Hence the left-action is generated by $-\xi^R$. Similarly the right-action is generated by ξ^L, and the adjoint action by $\xi^L - \xi^R$.

PROPOSITION 2.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any action of a Lie group G on a manifold Q the map*

$$\mathfrak{g} \to \mathfrak{X}(Q), \quad \xi \mapsto \xi_Q$$

is a Lie algebra action of $\mathfrak{g}$ on Q. For $g \in G$ one has

$$g_* \xi_Q = (\mathrm{Ad}_g\, \xi)_Q.$$

PROOF. The idea of proof is to reduce to the case of the action $R_{g^{-1}}$ of G on itself. Let $\tilde{Q} = G \times Q$ with G-action $g.(a, q) = (ag^{-1}, q)$. The generating vector fields for this action are $\xi_{\tilde{Q}} = (\xi^L, 0)$. Since $[\xi^L, \eta^L] = [\xi, \eta]^L$ by definition of the Lie bracket,

$$[\xi_{\tilde{Q}}, \eta_{\tilde{Q}}] = [\xi, \eta]_{\tilde{Q}}. \tag{9}$$

The map $F : G \times Q \to Q,\ (a, q) \mapsto a^{-1}.q.$ is a G-equivariant fibration. In particular, $\xi_{\tilde{Q}} \sim_F \xi_Q$ and (9) implies $[\xi_Q, \eta_Q] = [\xi, \eta]_Q$. For the second equation, we note that for the action of G on itself by left multiplication,

$$(R_{g^{-1}})_* \xi^L = (\mathrm{Ad}_g)_* \xi^L = (\mathrm{Ad}_g \xi)^L.$$

Hence $g_* \xi_{\tilde{Q}} = (\mathrm{Ad}_g \xi)_{\tilde{Q}}$. Again, since F is G-equivariant, this implies $g_* \xi_Q = (\mathrm{Ad}_g \xi)_Q$. $\square$

If G is simply connected and Q is compact, the converse is true: every $\mathfrak{g}$-action on Q integrates to a G-action.

Any action of G on Q gives rise to an action on TQ and T^*Q. If $q \in Q$ is fixed under the action of G, then these actions induce linear G-actions (i.e. representations) on T_qQ and T_q^*Q. In particular, the conjugation action of G on itself induces an action on the Lie algebra $\mathfrak{g} = T_eG$, called the adjoint action, and on $\mathfrak{g}^*$, called the co-adjoint action. The two actions are related by

$$\langle g.\mu, \xi \rangle = \langle \mu, g^{-1}.\xi \rangle, \quad \mu \in \mathfrak{g}^*, \xi \in \mathfrak{g}.$$

Exercise 2.6. Using the identifications $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ and $T^*\mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g}^*$ determine the generating vector fields for the adjoint and co-adjoint actions.

3. Hamiltonian group actions

A G-action $g \mapsto \mathcal{A}_g$ on a symplectic manifold (M, ω) is called *symplectic* if $\mathcal{A}_g \in \mathrm{Symp}(M, \omega)$ for all g. Similarly a $\mathfrak{g}$-action $\xi \mapsto \xi_M$ is called symplectic if $\xi_M \in \mathfrak{X}(M, \omega)$ for all ξ. Clearly, the $\mathfrak{g}$-action defined by a symplectic G-action is symplectic. The G-action or $\mathfrak{g}$-action is called weakly Hamiltonian if all ξ_M are Hamiltonian vector fields. That is, in the Hamiltonian case there exists a function $\Phi(\xi) \in C^\infty(M)$ for all ξ such that $\xi_M = X_{\Phi(\xi)}$. One can always choose $\Phi(\xi)$ to depend linearly on ξ (define Φ first for a basis of $\mathfrak{g}$ and then extend by linearity). The map $\xi \mapsto \Phi(\xi)$ can then be viewed as a function $\Phi \in C^\infty(M) \otimes \mathfrak{g}^*$.

Definition 3.1. A symplectic G-action on a symplectic manifold (M, ω) is called weakly Hamiltonian if there exists a map (called *moment map*

$$\Phi \in C^\infty(M) \otimes \mathfrak{g}^*$$

such that for all ξ, the function $\langle \Phi, \xi \rangle \in C^\infty(M)$ is a Hamiltonian for ξ_M:

$$\mathrm{d}\langle \Phi, \xi \rangle = \iota(\xi_M)\omega.$$

It is called Hamiltonian if Φ is equivariant with respect to the G-action on M and the co-adjoint action of G on $\mathfrak{g}^*$.

Similarly one defines moment maps for Hamiltonian $\mathfrak{g}$-actions; one requires Φ to be $\mathfrak{g}$-equivariant in this case.

It is obvious that if a group G acts in a Hamiltonian way, and if $H \to G$ is a homomorphism (e.g. inclusion of a subgroup) then the action of H is Hamiltonian; the moment map is the composition of the G-moment map with the dual map $\mathfrak{g}^* \to \mathfrak{h}^*$.

We sometimes write $\Phi^\xi = \langle \Phi, \xi \rangle$ for the ξ-component of Φ. The equivariance condition means that $g^*\Phi = g.\Phi = (\mathrm{Ad}_{g^{-1}})^*\Phi$, or in detail:

$$\langle g^*\Phi, \eta \rangle = \langle \mathrm{Ad}^*_{g^{-1}} \Phi, \eta \rangle = \langle \Phi, \mathrm{Ad}_{g^{-1}}(\eta) \rangle$$

for all $m \in M$, $\eta \in \mathfrak{g}$. Writing $g = \exp(t\xi)$ and taking the derivative at $t = 0$ we find the infinitesimal version of this condition reads,

$$\xi_M \Phi^\eta = \Phi^{[\xi,\eta]}.$$

For an abelian group, the conjugation action is trivial so that equivariance simply means invariance.

One way of interpreting the equivariance condition is as follows. For a weakly Hamiltonian G-action the fundamental vector fields define a Lie homomorphism homomorphism $\mathfrak{g} \to \mathfrak{X}_{Ham}(M, \omega) \subseteq \mathfrak{X}(M, \omega)$. This can always be lifted to a linear map $\mathfrak{g} \to C^\infty(M, \mathbb{R})$. The action is Hamiltonian if it can be lifted to an *equivariant* map, which for a connected group is equivalent to the following:

Lemma 3.2. *For a Hamiltonian G-action, the map*

$$\mathfrak{g} \to C^\infty(M),\ \xi \mapsto \langle \Phi, \xi \rangle$$

is a Lie homomorphism.

Proof.

$$\{\Phi^\xi, \Phi^\eta\} = X_{\Phi^\xi}(\Phi^\eta) = \xi_M \Phi^\eta = \Phi^{[\xi,\eta]}.$$

□

Notice that we defined the the Poisson bracket in such a way that $C^\infty(M) \to \mathfrak{X}(M)$ is a Lie homomorphism.

From now on, we will always assume that the moment map is equivariant unless stated otherwise.

Lemma 3.3. *Any weakly Hamiltonian action of a Lie group G on (M, ω) is Hamiltonian if*

(a) *G is compact, or*
(b) *M is compact.*

Proof. Suppose Φ is any moment map, not necessarily equivariant. Define

$$g \cdot \Phi = (g^{-1})^*(\mathrm{Ad}^*_{g^{-1}})\Phi$$

so that Φ is equivariant if and only if $g \cdot \Phi = \Phi$ for all g. We claim that $g \cdot \Phi$ is also a moment map for the G-action. Indeed,

$$\begin{aligned} \mathrm{d}\langle g \cdot \Phi, \xi \rangle &= (g^{-1})^* \mathrm{d}\langle \Phi, \mathrm{Ad}_g(\xi)\rangle \\ &= (g^{-1})^* g^* \iota(\xi_M)\omega \\ &= \iota_{\xi_M}\omega. \end{aligned}$$

If G is compact, we obtain a G-equivariant moment map by averaging over the group. If M is compact and connected, normalize Φ by the condition $\int_M \Phi = 0$. Then Φ is invariant since $\int_M g \cdot \Phi = 0$. $\square$

We remark that the above Lemma also holds for actions of connected, simply connected semi-simple Lie groups (i.e. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$), or more generally for any connected, simply connected group G for which the first and second Lie algebra cohomology vanish. See Theorem 26.1 in Guillemin-Sternberg, *Symplectic techniques in Physics*, Cambridge University Press 1984, for details.

4. Examples of Hamiltonian G-spaces

4.1. Linear momentum and angular momentum. Recall that for any vector field Y on a manifold Q, the cotangent lift $\hat{Y} \in \mathfrak{X}(T^*Q)$ is a Hamiltonian vector field $\hat{Y} = X_H$, with $H = \iota_{\hat{Y}}\theta$. We leave it as an exercise to check that linear map

$$\mathfrak{X}(Q) \to C^\infty(T^*Q),\ Y \mapsto \iota_{\hat{Y}}\theta$$

is actually a Lie algebra homomorphism (using the Poisson bracket on T^*Q). Hence if Q is a G-manifold, we obtain a moment map for the cotangent lift of the G-action to T^*Q by composing this map with the generating vector fields, $\mathfrak{g} \mapsto \mathfrak{X}(Q)$. Recall that in local cotangent coordinates, if $Y = \sum_j Y_j(q)\frac{\partial}{\partial q_j}$ then the corresponding Hamiltonian is $H(q,p) = \sum_j Y_j(q)p_j$.

For example, let $M = T^*(\mathbb{R}^n) = \mathbb{R}^{2n}$ with standard symplectic coordinates q_j, p_j. Let $G = \mathbb{R}^n$ act on itself by translation. The Lie algebra is $\mathfrak{g} = \mathbb{R}^n$, with exponential map $\exp : \mathfrak{g} \to G$ the identity map of $\mathbb{R}^n$.

The generating vector fields $b_{\mathbb{R}^n}$ for $b \in \mathbb{R}^n$ are obtained from the calculation,

$$(b_{\mathbb{R}^n} f)(q) = \frac{\partial}{\partial t}\Big|_{t=0} \exp(-tb)^* f(q) = \frac{\partial}{\partial t}\Big|_{t=0} f(q - tb) = -\sum_j b_j \frac{\partial f}{\partial q_j}.$$

Thus $b_{\mathbb{R}^n} = -\sum_j b_j \frac{\partial}{\partial q_j}$ and the moment map for the cotangent lift is

$$\langle \Phi, b\rangle = -\sum_j b_j p_j.$$

Using the standard inner product $(b, b') = b \cdot b'$ on $\mathbb{R}^n$ to identify $(\mathbb{R}^n) \cong (\mathbb{R}^n)^*$, we find that

$$\Phi(q,p) = -p.$$

That is, the moment map is just linear momentum (up to an irrelevant sign that just comes from the chosen identification $(\mathbb{R}^n) \cong (\mathbb{R}^n)^*$).

Consider next the cotangent lift of the action of $G = \mathrm{Gl}(n, \mathbb{R})$ on $\mathbb{R}^n$. The Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ is the space of all real matrices, with exponential map the exponential map for matrices. We compute the generating vector field $A_{\mathbb{R}^n}$ for the action of $A \in \mathrm{gl}(n, \mathbb{R})$ as follows:

$$\begin{aligned} (A_{\mathbb{R}^n} f)(q) &= \frac{\partial}{\partial t}\Big|_{t=0} \exp(-tA)^* f(q) \\ &= \frac{\partial}{\partial t}\Big|_{t=0} f(\exp(-tA)q) \\ &= -\sum_j (Aq)_j \frac{\partial f}{\partial q_j} \\ &= -\sum_{j,k} A_{jk} q_k \frac{\partial f}{\partial q_j}, \end{aligned}$$

that is, $A_{\mathbb{R}^n} = -\sum_{j,k} A_{jk} q_k \frac{\partial}{\partial q_j}$. A moment map for the cotangent lift of the action is

$$\langle \Phi, A \rangle = -\sum_{j,k} A_{jk} p_j q_k.$$

Hence, using the non-degenerate bilinear form $(A, A') = \mathrm{tr}(A^t A')$ on $\mathfrak{gl}(n, \mathbb{R})$ to identify $\mathfrak{gl}(n, \mathbb{R}) \cong \mathfrak{gl}(n, \mathbb{R})^*$, we have

$$\Phi(q, p)_{ij} = p_i q_j.$$

Note that the pairing on $\mathfrak{gl}(n, \mathbb{R})$ is not invariant under the adjoint action, i.e. it does not identify the adjoint and coadjoint action. Instead, the coadjoint action becomes the contragredient action $g.A = \mathrm{Ad}((g^{-1})^t)A$.

The pairing does, however, restrict to an invariant pairing on the sub-algebra $\mathfrak{h} = \mathfrak{o}(n)$ of skew-symmetric matrices $A = -A^t$. The corresponding connected Lie subgroup of $\mathrm{Gl}(n, \mathbb{R})$ is the special orthogonal group $H = \mathrm{SO}(n)$. The moment map $\Psi : T^*\mathbb{R}^n \to \mathfrak{o}(n)^*$ for the action of $\mathrm{SO}(n)$ reads,

$$\Psi(q, p)_{ij} = \tfrac{1}{2}(p_i q_j - p_j q_i).$$

For $n = 3$, we can further identify $\mathrm{so}(3)^* \cong \mathbb{R}^3$ with the standard rotation action of $\mathrm{SO}(3)$, and Ψ just becomes just *angular momentum* $\Psi(q, p) = \vec{p} \times \vec{q}$ (up to an irrelevant factor, which again just depends on the chosen identification $\mathfrak{so}(3) \cong \mathfrak{so}(3)^*$).

4.2. Exact symplectic manifolds. The previous examples generalize to cotangent bundles T^*Q of G-manifolds Q, or more generally to what is called *exact* symplectic manifolds. A symplectic manifold (M, ω) is called exact if $\omega = -\mathrm{d}\theta$ for a 1-form θ (which is sometimes called a symplectic potential). Note that a compact symplectic manifold is never exact (unless it is 0-dimensional): Indeed, if $\omega = -\mathrm{d}\theta$ is exact, then also the Liouville form $\omega^n = -\mathrm{d}\theta\omega^{n-1}$ is exact. Hence if M were compact, Stokes' theorem would show that M has zero volume.

PROPOSITION 4.1. *Suppose (M,ω) is an exact symplectic manifold, $\omega = -d\theta$. Then any G-action on M preserving θ is Hamiltonian, with moment map*

$$\langle \Phi, \xi\rangle = \iota(\xi_M)\theta.$$

Note that if G is compact one can construct an invariant θ by averaging. If $H^1(M,\mathbb{R}) = 0$ then Φ is independent of the choice of invariant θ.

PROOF. We calculate $\mathrm{d}\iota(\xi_M)\theta = -\iota(\xi_M)\mathrm{d}\theta + L(\xi_M)\theta = \iota(\xi_M)\omega$. The resulting moment map is equivariant:

$$g^*\langle\Phi,\xi\rangle = g^*(\iota(\xi_M)\theta) = \iota(g_*\xi_M)g^*\theta = \iota(g_*\xi_M)\theta = \iota((\mathrm{Ad}_g\,\xi)_M)\theta = \langle\Phi, \mathrm{Ad}_g\,\xi\rangle.$$

□

The examples considered above were of the form $M = T^*Q$, with G acting by the cotangent lift of a G-action on Q. Another example is provided by the defining action of $\mathrm{U}(n)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$:

4.3. Unitary representations. Introduce complex coordinates $z_j = q_j + ip_j$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ and write the symplectic form as

$$\omega = \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j = \frac{i}{2}\sum_j \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j} = \frac{1}{2i}h(\mathrm{d}z,\mathrm{d}z),$$

where $h(w,w') = \sum_i \overline{w}_i w'_i$ is the Hermitian inner product. Then $\omega = -\mathrm{d}\theta$ with

$$\theta = \frac{i}{4}\sum_j \big(\bar z_j \mathrm{d}z_j - z_j \mathrm{d}\bar z_j\big) = -\frac{1}{2}\,\mathrm{Im}(h(z,\mathrm{d}z)),$$

This choice of θ is preserved under the unitary group. The Lie algebra $\mathfrak{u}(n)$ of $\mathrm{U}(n)$ consists of skew-adjoint matrices ξ. The generating vector fields are

$$\xi_{\mathbb{C}^n} = \sum_{j,k}\big(\xi_{jk}\, z_k \frac{\partial}{\partial z_j} - \xi_{kj}\,\overline{z_k}\frac{\partial}{\partial \overline{z_j}}\big)$$

where $\xi \in \mathfrak{u}(n)$ is a skew-Hermitian matrix. Hence $\langle\Phi,\xi\rangle = -\iota(\xi_{\mathbb{C}^n})\theta$ is given by

$$\langle\Phi(z),\xi\rangle = \frac{i}{2}\sum_{j,k}\xi_{jk}\,\bar z_j\, z_k = \frac{i}{2}\,h(z,\xi z)$$

(where h is the Hermitian metric) defines a moment map. Using the inner product on $\mathrm{u}(n)$, $(\xi,\eta) = -\mathrm{Tr}(\xi\eta)$ to identify the Lie algebra and its dual,

$$\Phi(z)_{kj} = -\frac{1}{4}\big(\bar z_j\, z_k - \bar z_k\, z_j\big).$$

This example also shows that any finite dimensional unitary representation $G \to \mathrm{U}(n)$ defines a Hamiltonian action of G on $\mathbb{C}^n$; the moment map is the composition of the

$\mathrm{U}(n)$ moment map with the projection $\mathfrak{u}(n)^* \to \mathfrak{g}^*$ dual to $\mathfrak{g} \to \mathfrak{u}(n)$. For example, the moment map for the scalar S^1-action is given by $-\frac{1}{2}||z||^2$.

4.4. Projective Representations. The action of $\mathrm{U}(n+1)$ on $\mathbb{C}^{n+1}$ induces an action on $\mathbb{C}P(n)$ which also turns out to be Hamiltonian. In homogeneous coordinates $[z_0 : \ldots : z_n]$, the moment map is

$$\Phi([z], \xi) = \frac{i}{2} \frac{\sum_{j,k} \xi_{jk}\, \overline{z_j}\, z_k}{\sum_j |z_j|^2}.$$

We will verify this fact later in the context of symplectic reduction.

4.5. Symplectic representations. Generalizing the case of unitary representations, consider any *symplectic representation* of G on a symplectic vector space (E, ω). That is, G acts by a homomorphisms $G \to \mathrm{Sp}(E)$ into the symplectic group. A moment map for such an action is given by the formula,

$$\langle \Phi(v), \xi \rangle = \tfrac{1}{2}\omega(v, \xi.v).$$

Indeed, if we identify $T_v E = E$ the generating vector field for $\xi \in \mathfrak{g}$ is just $\xi_E(v) = -\xi.v$. Thus for $w \in E$ we have

$$\omega(\xi_E(v), w) = \omega(w, \xi.v).$$

On the other hand, the map Φ defined above satisfies

$$\begin{aligned} \mathrm{d}_v \Phi^\xi(w) &= \tfrac{1}{2}\tfrac{\partial}{\partial t}\Big|_{t=0} \omega(v + tw, \xi.(v + tw)) \\ &= \tfrac{1}{2}(\omega(w, \xi.v) + \omega(v, \xi.w)) \\ &= \omega(w, \xi.v), \end{aligned}$$

verifying that Φ is a moment map.

4.6. Coadjoint Orbits. As a preparation, let us note that for any Hamiltonian G-space (M, ω, Φ), the moment map determines the pull-back of ω to any G-orbit. Indeed, since ξ_M is the Hamiltonian vector field for Φ^ξ,

$$\omega(\xi_M, \eta_M) = -\{\Phi^\xi, \Phi^\eta\} = -\Phi^{[\xi,\eta]}.$$

In particular if M is a homogeneous Hamiltonian G-manifold (i.e. if M is a single G-orbit), ω is completely determined by Φ!

THEOREM 4.2 (Kirillov-Kostant-Souriau). *Let $\mathcal{O} \subset \mathfrak{g}^*$ be an orbit for the coadjoint action of G on $\mathfrak{g}^*$. There exists a unique symplectic structure on $\mathcal{O}$ for which the action is Hamiltonian and the moment map is the inclusion $\Phi : \mathcal{O} \hookrightarrow \mathfrak{g}^*$.*

PROOF. For any $\mu \in \mathcal{O}$ consider the skew-symmetric bilinear form on $\mathfrak{g}$,

$$B_\mu(\xi, \eta) = \langle \mu, [\xi, \eta] \rangle.$$

Writing $B_\mu(\xi,\eta) = \langle \mu, \mathrm{ad}_\xi \eta\rangle = \langle (\mathrm{ad}_\xi)^*\mu, \eta\rangle$ we see that the kernel of B_μ consists of all ξ with $\mathrm{ad}_\xi^* \mu = 0$, i.e. $\xi_\mathcal{O}(\mu) = 0$. It follows that

$$\omega_\mu(\xi_\mathcal{O}(\mu), \eta_\mathcal{O}(\mu)) = -\langle \mu, [\xi, \eta]\rangle. \tag{10}$$

is a well-defined symplectic 2-form on $T_\mu\mathcal{O}$. The calculation

$$B_{g\mu}(g.\xi, g.\eta) = \langle g.\mu, [g.\xi, g.\eta]\rangle = \langle g.\mu, g.[\xi,\eta]\rangle = \langle \mu, [\xi,\eta]\rangle = B_\mu(\xi,\eta)$$

shows that the resulting 2-form ω on $\mathcal{O}$ is G-invariant (and therefore, smooth!), and equation (10) gives the moment map condition $\iota(\xi_\mathcal{O})\omega = \mathrm{d}\langle \Phi, \xi\rangle$ for the inclusion map $\Phi : \mathcal{O} \hookrightarrow \mathfrak{g}^*$:

$$\iota(\eta_\mathcal{O})\mathrm{d}\langle \Phi, \xi\rangle = L(\eta_\mathcal{O})\langle \Phi, \xi\rangle = \langle (\mathrm{ad}_\eta)^*\Phi, \xi\rangle = -\langle \Phi, [\xi,\eta]\rangle = \omega(\xi_\mathcal{O}, \eta_\mathcal{O}).$$

To check $\mathrm{d}\omega = 0$, we compute:

$$\iota(\xi_\mathcal{O})\mathrm{d}\omega = L(\xi_\mathcal{O})\omega - \mathrm{d}\iota(\xi_\mathcal{O})\omega = 0.$$

As remarked above, the moment map uniquely determines the symplectic form. $\square$

EXAMPLE 4.3. Let $G = \mathrm{SO}(3)$. Identify the Lie algebra $\mathfrak{so}(3)$ with $\mathbb{R}^3$, by identifying the standard basis vectors of $\mathfrak{so}(3)$ as follows:

$$e_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} e_2 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} e_3 \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This identification takes the adjoint action of SO(3) to the standard rotation action on $\mathbb{R}^3$, and takes the invariant inner product $(A, B) \mapsto -\frac{1}{2}\,\mathrm{tr}(AB)$ on $\mathfrak{so}(3)$ to the standard inner product on $\mathbb{R}^3$. The inner product also identifies $\mathfrak{so}(3)^* \cong \mathbb{R}^3$. The coadjoint orbits for SO(3) are the 2-spheres around 0, together with the or-gin $\{0\}$.

THEOREM 4.4 (Kostant-Souriau). *Let G be a Lie group, and (M, ω, Φ) is a Hamiltonian G-space on which G acts transitively. Then M is a covering space of a coadjoint orbit, with 2-form obtained by pull-back of the KKS form on $\mathcal{O}$.*

PROOF. Let $\mathcal{O} = \Phi(M)$. It is clear that the map $\Phi : M \to \mathcal{O}$ is a submersion. We had already seen that the 2-form on M is determined by the moment map condition, and the formula shows that the map $\Phi : M \to \mathcal{O}$ preserves the symplectic form. Hence its tangent map is a bijection everywhere, and so $\Phi : M \to \mathcal{O}$ is a local diffeomorphism. $\square$

Note that while M is a homogeneous space G/G_m its image under the moment map is a homogeneous space $G/G_{\Phi(m)}$, and the covering map is a fibration with (discrete !) fiber $G_{\Phi(m)}/G_m$. Hence, non-trivial coverings can be obtained only if the stabilizer $G_{\Phi(m)}$ is disconnected (for compact connected Lie group do not have proper subgroups of the same dimension).

If G is a *compact, connected* Lie group then it is known that all stabilizer groups G_μ for the coadjoint action are connected, so one has:

THEOREM 4.5. *If G is compact, connected and (M, ω, Φ) is a homogeneous Hamiltonian G-space, the moment map induces a symplectomorphism of M with the coadjoint orbit $\mathcal{O} = \Phi(M)$.*

The connectedness of G_μ can be shown as follows: Using the fibration $G \to G/G_\mu$, and the fact that G and G/G_μ are connected it suffices to show that the base G/G_μ is simply connected. (Given $p_0, p_1 \in G_\mu$ choose a path $\gamma : [0,1] \to G$ connecting them. γ projects to a closed path in G/G_μ, and can be contracted to a constant path. By the homotopy lifting property this homotopy can be lifted to a homotopy with fixed end points of γ. This will then be a path in G_μ connecting p_0, p_1.) The simply-connectedness of G/G_μ in turn follows from Morse theory; this will be explained later.

4.7. Poisson manifolds. Moment maps fit very nicely into the more general category of Poisson manifolds.

DEFINITION 4.6. A Poisson manifold is a manifold M together with a bilinear map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that

(a) $\{\cdot,\cdot\}$ is a Lie algebra structure on $C^\infty(M)$, and
(b) for all $H \in C^\infty(M)$, the map $C^\infty(M) \to C^\infty(M), F \mapsto \{H, F\}$ is a derivation.

A smooth map $\phi : M_1 \to M_2$ between Poisson manifolds is called a Poisson map if

$$\phi^*\{F_1, F_2\} = \{\phi^* F_1, \phi^* F_2\}.$$

A vector field $X \in \mathfrak{X}(M)$ is called Poisson if

$$X\{F_1, F_2\} = \{X(F_1), F_2\} + \{F_1, X(F_2)\}.$$

Since any derivation of $C^\infty(M)$ is given by a vector field, any H defines a so-called Hamiltonian vector field X_H by $X_H(F) = \{H, F\}$.

EXERCISE 4.7. Show that X_H is a Poisson vector field. Show that the flow of any complete Poisson vector field is Poisson.

Examples of Poisson manifolds are of course symplectic manifolds, with the Poisson bracket associated to the symplectic structure. Another important example, due to Kirillov, is the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. For any $\mu \in \mathfrak{g}^*$ and any function $F \in C^\infty(\mathfrak{g}^*)$ identify

$$\mathrm{d}F_\mu \in T^*_\mu \mathfrak{g}^* \cong (\mathfrak{g}^*)^* = \mathfrak{g}.$$

Then define

$$\{F, G\}(\mu) := \langle \mu, [\mathrm{d}F_\mu, \mathrm{d}G_\mu].\rangle$$

EXERCISE 4.8. Verify that this is a Poisson structure on $\mathfrak{g}^*$. Show that the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ of a coadjoint orbit is a Poisson map.

The definition of a moment map carries over to Poisson manifolds: A G-action is Poisson if it preserves Poisson brackets, and any such action is Hamiltonian if there exists an equivariant smooth map $\Phi : M \to \mathfrak{g}^*$ such that

$$\xi_M = X_{\langle \Phi, \xi \rangle}$$

for all $\xi \in \mathfrak{k}$.

EXERCISE 4.9. Show that the coadjoint action of G on $\mathfrak{g}^*$ is Hamiltonian, with moment map the identity map.

EXERCISE 4.10. Let (M, ω) be a symplectic manifold and G a connected Lie group acting symplectically on M. Suppose there exists a moment map $\Phi : M \to \mathfrak{g}^*$ in the weak sense. Show that Φ is equivariant if and only if Φ is a Poisson map for the Kirillov Poisson structure. Conversely, show that for every Poisson map $\Phi : M \to \mathfrak{g}^*$, the equation

$$\iota(\xi_M)\omega = \mathrm{d}\langle \Phi, \xi \rangle$$

defines a symplectic Lie algebra action of $\mathfrak{g}$ on M. More generally, show that for any Poisson manifold M, any Poisson map $\Phi : M \to \mathfrak{g}^*$ defines a Poisson $\mathfrak{g}$-action on M.

4.8. 2d Gauge Theory. Let us now spend some time discussing, somewhat informally, an interesting ∞-dimensional example. We begin with a rapid introduction to what we call (with some exaggeration) gauge theory.

Let Σ be a compact oriented manifold, G a compact Lie group, and $\Omega^k(\Sigma, \mathfrak{g})$ the k-forms on Σ with values in $\mathfrak{g}$. The space $\mathcal{A}(\Sigma) := \Omega^1(\Sigma, \mathfrak{g})$ will be viewed as an infinite dimensional manifold, called the space of *connections*. For any $A \in \mathcal{A}(\Sigma)$ there is a corresponding *covariant derivative*

$$\mathrm{d}_A : \Omega^k(\Sigma, \mathfrak{g}) \to \Omega^{k+1}(\Sigma, \mathfrak{g}),$$

defined

$$\mathrm{d}_A \xi = \mathrm{d}\xi + [A, \xi]$$

(using the wedge product). The *gauge action* of $\mathcal{G}(\Sigma) := C^\infty(\Sigma, G)$ on $\mathcal{A}(\Sigma)$ is defined by

$$g \cdot A = \mathrm{Ad}_g(A) - \mathrm{d}g\, g^{-1}$$

where the first term is the pointwise adjoint action. The second term is written for matrix-groups so that $\mathrm{d}gg^{-1}$ makes sense as a 1-form on Σ with values in $\mathfrak{g}$. More invariantly, it is the pull-back under $g : \Sigma \to G$ of the right-invariant Maurer-Cartan form $\overline{\theta} \in \Omega^1(G, \mathfrak{g})$; i.e. $\overline{\theta}$ is the unique right-invariant form such that for any right-invariant vector field ξ^R, $\iota(\xi^R)\overline{\theta} = \xi$. The gauge action is defined in such a way that the following is true:

LEMMA 4.11. *For all* $\xi \in \Omega^k(\Sigma, \mathfrak{g})$,

$$d_{g \cdot A}\, \mathrm{Ad}_g(\xi) = \mathrm{Ad}_g(d_A \xi).$$

PROOF. Using the definition, and using that for all $\xi \in \Omega^k(\Sigma, \mathfrak{g})$,

$$\begin{aligned}
\mathrm{d}(\mathrm{Ad}_g \xi) &= \mathrm{d}(g\xi g^{-1}) \\
&= \mathrm{d}g\xi g^{-1} + \mathrm{Ad}_g \mathrm{d}\xi + (-1)^k g\xi \mathrm{d}g^{-1} \\
&= \mathrm{d}gg^{-1} \mathrm{Ad}_g \xi - (-1)^k \mathrm{Ad}_g \xi \, \mathrm{d}gg^{-1} + \mathrm{Ad}_g(\mathrm{d}\xi) \\
&= [\mathrm{d}gg^{-1}, \mathrm{Ad}_g \xi] + \mathrm{Ad}_g(\mathrm{d}\xi)
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{d}_{g\cdot A} \mathrm{Ad}_g(\xi) &= \mathrm{d}(\mathrm{Ad}_g(\xi)) + [\mathrm{Ad}_g(A) - \mathrm{d}gg^{-1}, \mathrm{Ad}_g \xi] \\
&= \mathrm{Ad}_g(\mathrm{d}\xi) + [\mathrm{d}gg^{-1}, \mathrm{Ad}_g \xi] + [\mathrm{Ad}_g(A) - \mathrm{d}gg^{-1}, \mathrm{Ad}_g \xi] \\
&= \mathrm{Ad}_g(\mathrm{d}_A\xi).
\end{aligned}$$

□

Due to the presence of the gauge term the square of d_A is usually not zero:

LEMMA 4.12. *Let* $\mathrm{curv}(A) = dA + \frac{1}{2}[A, A] \in \Omega^2(\Sigma, \mathfrak{g})$ *be the curvature of* A. *Then for all* $\xi \in \Omega^k(\Sigma, \mathfrak{g})$,

$$d_A^2\xi = [\mathrm{curv}(A), \xi]$$

The curvature transforms equivariantly:

$$\mathrm{curv}(g \cdot A) = \mathrm{Ad}_g \mathrm{curv}(A).$$

PROOF.

$$\begin{aligned}
\mathrm{d}_A^2\xi &= \mathrm{d} \circ [A, \cdot] + [A, \cdot] \circ \mathrm{d} + [A, [A, \cdot]] \\
&= [\mathrm{d}A, \cdot] + \frac{1}{2}[[A, A], \cdot]
\end{aligned}$$

where the last term is obtained by the Jacobi identity. Equivariance follows by the transformation property for d_A. □

By the lemma, $\mathrm{d}_A^2 = 0$ is precisely if the curvature is zero.

We now view $\mathcal{A}(\Sigma)$ as a $\mathcal{G}(\Sigma)$-space. What are the generating vector fields? The Lie algebra of the gauge group is identified with $\Omega^0(\Sigma, \mathfrak{g})$.

LEMMA 4.13. *The generating vector fields for* $\xi \in \Omega^0(\Sigma, \mathfrak{g})$ *are*

$$\xi_{\mathcal{A}(\Sigma)}(A) = d_A\xi.$$

PROOF.

$$\begin{aligned}
\xi_{\mathcal{A}(\Sigma)}(A) &= \frac{\partial}{\partial t}\Big|_{t=0} \big(\mathrm{Ad}_{\exp(-t\xi)}(A) - \mathrm{d}(\exp(-t\xi)) \exp(t\xi) \big) \\
&= -[\xi, A] + \mathrm{d}\xi = \mathrm{d}_A\xi.
\end{aligned}$$

□

We now specialize to two dimensions: $\dim \Sigma = 2$. Let us fix an invariant inner product on $\mathfrak{g}$ (unique up to scalar if G is simple). Then $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ is an ∞-dimensional symplectic manifold: The 2-form is

$$\omega_A(a, b) = \int_\Sigma a \dot{\wedge} b$$

for all $a, b \in T_A\Omega^1(\Sigma, \mathfrak{g}) \cong \Omega^1(\Sigma, \mathfrak{g})$, using the inner product.

THEOREM 4.14. *The gauge action of $\mathcal{G}(\Sigma)$ on $\mathcal{A}(\Sigma)$ is Hamiltonian, with moment map minus the curvature $A \mapsto \operatorname{curv}(A)$, i.e.*

$$\langle \Phi(A), \xi \rangle = -\int_\Sigma \operatorname{curv}(A) \cdot \xi.$$

PROOF. We calculate: For all $a \in \Omega^1(\Sigma, \mathfrak{g})$, viewed as a constant vector field,

$$\begin{aligned}
\langle \mathrm{d}\langle \Phi, \xi \rangle\Big|_A, a \rangle &= \frac{\partial}{\partial t}\Big|_{t=0} \langle \Phi(A + ta), \xi \rangle \\
&= -\frac{\partial}{\partial t}\Big|_{t=0} \int_\Sigma \operatorname{curv}(A + ta) \cdot \xi \\
&= -\frac{\partial}{\partial t}\Big|_{t=0} \int_\Sigma \big(\mathrm{d}A + t\mathrm{d}a + t[A, a] + \frac{1}{2}([A, A] + t^2[a, a])\big) \cdot \xi \\
&= -\int_\Sigma \mathrm{d}_A a \cdot \xi \\
&= -\int_\Sigma a \cdot \mathrm{d}_A \xi \\
&= \omega(\xi_{\mathcal{A}(\Sigma)}(A), a).
\end{aligned}$$

$\square$

It is interesting to extend this calculation to 2-manifolds with boundary $\partial\Sigma$. Everything carries over, but the partial integration produces an extra boundary term so that the moment map is

$$\langle \Phi(A), \xi \rangle = -\int_\Sigma \operatorname{curv}(A) \cdot \xi - \int_{\partial\Sigma} A \cdot \xi.$$

That is, the (informally) dual space to the Lie algebra $\Omega^0(\Sigma, \mathfrak{g})$ of the gauge group is identified with $\Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g})$ with the natural pairing, and the moment map is

$$\Omega^0(\Sigma, \mathfrak{g}) \to \Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g}), \quad A \mapsto (\operatorname{curv}(A), \iota_{\partial\Sigma}^* A).$$

Notice however that this moment map is no longer equivariant in the usual sense, for the action on the second summand is still the gauge action! This leads one to define a central extension of the gauge group. Define

$$c : \mathcal{G}(\Sigma) \times \mathcal{G}(\Sigma) \to \mathrm{U}(1), \ c(g_1, g_2) = \exp(-i\pi \int_\Sigma g_1^{-1}\mathrm{d}g_1 \wedge \mathrm{d}g_2 g_2^{-1}),$$

and let $\widehat{\mathcal{G}(\Sigma)} = \mathcal{G}(\Sigma) \times \mathrm{U}(1)$ with product

$$(g_1, z_1)(g_2, z_2) := (g_1 g_2, z_1 z_2\, c(g_1, g_2)).$$

One can show that this does indeed define a group structure (i.e. c is a cocycle). The Lie algebra of this new group is $\Omega^0(\Sigma, \mathfrak{g}) \oplus \mathbb{R}$ with defining cocycle $\int_\Sigma \mathrm{d}\xi_1 \mathrm{d}\xi_2 = \int_{\partial\Sigma} \xi_1 \mathrm{d}\xi_2$, and its dual is $\Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g}) \oplus \mathbb{R}$, with action

$$(g, z) \cdot (\alpha, \beta, \lambda) = (\mathrm{Ad}_g\, \alpha, \mathrm{Ad}_g\, \beta - \lambda \mathrm{d}g g^{-1}, \lambda).$$

It follows that the moment map for the action of the extended gauge group (where the extra circle acts trivially) is equivariant, the image of the original moment map is identified with the hyperplane $\lambda = 1$.

5. Symplectic Reduction

5.1. The Meyer-Marsden-Weinstein Theorem. Let (M, ω, Φ) be a Hamiltonian G-space. As usual we assume that Φ is an equivariant moment map. One of the basic properties of the moment map is the following:

LEMMA 5.1. *For all $m \in M$, the kernel and image of the tangent map to Φ are given by*

$$\ker(d_m \Phi) = T_m(G \cdot m)^\omega,$$
$$\mathrm{im}(d_m \Phi) = \mathrm{ann}(\mathfrak{g}_m).$$

PROOF. By the defining condition of the moment map,

$$\omega_m(\xi_M(m), X) = \iota(X) \mathrm{d}\langle \Phi, \xi \rangle \Big|_m = \langle \mathrm{d}_m \Phi(X), \xi \rangle.$$

Therefore $\ker(\mathrm{d}_m \Phi) = \{\xi_M(m) | \xi \in \mathfrak{g}\}^\omega = T_m(G \cdot m)^\omega$.

By the same equation, $\xi \in \mathfrak{g}_m$ implies that $\langle \mathrm{d}_m \Phi(X), \xi \rangle = 0$ for all X, hence $\mathrm{im}(\mathrm{d}_m \Phi) \subseteq \mathrm{ann}(\mathfrak{g}_m)$. Equality follows by dimension count, using non-degeneracy of ω:

$$\begin{aligned} \dim(\mathrm{im}(\mathrm{d}_m \Phi)) &= \dim M - \dim(\ker(\mathrm{d}_m \Phi)) \\ &= \dim M - \dim(T_m(G \cdot m))^\omega \\ &= \dim(G \cdot m) = \dim G - \dim G_m = \dim \mathrm{ann}(\mathfrak{g}_m). \end{aligned}$$

□

THEOREM 5.2. *A point $\mu \in \mathfrak{g}^*$ is a regular value of Φ if and only if for all $m \in \Phi^{-1}(\mu)$, the stabilizer group G_m is discrete. In this case, $\Phi^{-1}(\mu)$ is a constant rank submanifold. The leaf of the null foliation through $m \in \Phi^{-1}(\mu)$ is the orbit $G_\mu.m$.*

PROOF. Since μ is a regular value, $\mathrm{d}_m \Phi$ is surjective for all $m \in M$. By the Lemma, this means that $\mathrm{ann}(\mathfrak{g}_m) = \mathfrak{g}^*$ or equivalently $\mathfrak{g}_m = \{0\}$. This shows that $G_m \subseteq G_\mu$ is discrete. Now let $\iota_\mu : \Phi^{-1}(\mu) \hookrightarrow M$ be the inclusion. Using $T_m\Phi^{-1}(\mu) = \ker(\mathrm{d}_m\Phi)$ the kernel of $\iota_\mu^*\omega$ is

$$\begin{aligned} \ker \iota_\mu^*\omega\Big|_m &= T_m\Phi^{-1}(\mu) \cap T_m\Phi^{-1}(\mu)^\omega \\ &= T_m\Phi^{-1}(\mu) \cap T_m(G\cdot m) \\ &= T_m(G_\mu \cdot m). \end{aligned}$$

□

THEOREM 5.3 (Marsden-Weinstein, Meyer). *Let (M, ω, Φ) be a Hamiltonian G-space. Suppose μ is a regular value of Φ and that the foliation of $\Phi^{-1}(\mu)$ by G_μ-orbits is a fibration. (This assumption is satisfied if G_μ is compact and the G_μ-action is free.) Let*

$$\pi_\mu : \Phi^{-1}(\mu) \to \Phi^{-1}(\mu)/G_\mu =: M_\mu$$

be the quotient map onto the orbit space. There exists a unique symplectic form ω_μ on the reduced space *M_μ such that*

$$\iota_\mu^*\omega = \pi_\mu^*\omega.$$

PROOF. Since the null-foliation is given by the G_μ-orbits, this is a special case of the theorem on reduction of constant rank submanifolds. □

The reduced space at 0 is often denoted $M_0 = M/\!\!/G$, this notation is useful if several groups are involved. The symplectic quotients M_μ depend only on the coadjoint orbit $\mathcal{O} = G.\mu$. Let $\mathcal{O}^-$ be the same G-space but with minus the KKS form as a symplectic form and minus the inclusion as a moment map. The moment map for the diagonal action on $M \times \mathcal{O}^-$ is

$$\tilde{\Phi} : M \times \mathcal{O}^- \to \mathfrak{g}^*,\ (m, \mu) \mapsto \Phi(m) - \mu.$$

PROPOSITION 5.4 (Shifting-trick). *μ is a regular value of Φ if and only if 0 is a regular value of*

$$\tilde{\Phi} : M \times \mathcal{O}^- \to \mathfrak{g}^*,\ (m, \mu) \mapsto \Phi(m) - \mu.$$

Moreover the G_μ-action on $\Phi^{-1}(\mu)$ is free if and only if the G-action on $\tilde{\Phi}^{-1}(0)$ is free. There is a canonical symplectomorphism,

$$M_\mu \cong (M \times \mathcal{O}^-)/\!\!/G.$$

PROOF. Consider the map $\Phi^{-1}(\mathcal{O}) \to \tilde{\Phi}^{-1}(0)$ by $m \mapsto (m, \Phi(m))$. Clearly, this map is a G-equivariant bijection. Since $\Phi^{-1}(\mathcal{O}) = G.\Phi^{-1}(\mu)$, the G-action on $\Phi^{-1}(\mathcal{O})$ has discrete (resp. trivial) stabilizers if and only if the G_μ-action on $\Phi^{-1}(\mu)$ has discrete (resp. trivial) stabilizers. The map $M \to M \times \mathcal{O}^-$, $m \mapsto (m, \mu)$ preserves 2-forms, hence so does the map $\Phi^{-1}(\mu) \to \tilde{\Phi}^{-1}(0), m \mapsto (m, \mu)$. It follows that the maps

$$M_\mu = \Phi^{-1}(\mu)/G_\mu \to (\tilde{\Phi}^{-1}(0) \cap M \times \{\mu\})/G_\mu \to \tilde{\Phi}^{-1}(0)/G$$

are all symplectomorphisms. □

EXAMPLE 5.5. Let $M = \mathbb{C}^{N+1}$, with the standard symplectic form

$$\omega = \frac{i}{2}\sum_{j=0}^{N} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j,$$

and scalar $S^1 = \mathbb{R}/\mathbb{Z}$-action (multiplication by $\exp(2\pi i t)$). We recall that a moment map for this action is given by $\Phi(z) = \pi||z||^2$. The reduced space at level π is $\mathbb{C}P(N)$. The reduced form is known as the Fubini-Study form. Reducing at a different value $\lambda\pi$ amounts to rescaling the symplectic form by λ.

EXAMPLE 5.6. Let (M, ω) be an exact symplectic manifold, $\omega = -\mathrm{d}\theta$, and suppose θ is invariant under some G-action, where G is a compact Lie group. Let Φ be the corresponding moment map $\langle \Phi, \xi \rangle = \iota(\xi_M)\theta$. Then if 0 is a regular value of Φ and the G-action on $\Phi^{-1}(0)$ is free, the pull-back $\iota^*\theta$ is invariant and horizontal, hence descends to a 1-form θ_0 on M_0 such that $\pi^*\theta_0 = \iota^*\theta$, and one has

$$\omega_0 = -\mathrm{d}\theta_0.$$

It follows that the symplectic quotient of an exact Hamiltonian G-space at 0 an exact symplectic manifold.

EXAMPLE 5.7. As a sub-example, consider the case $M = T^*Q$, where G acts by the cotangent lift of some G-action on Q. Let $\rho : T^*Q \to Q$ denote the projection. The moment map is given by

$$\langle \Phi(m), \xi \rangle = \langle m, \, \xi_Q(q) \rangle$$

where $q = \rho(m)$. This shows that the zero level set is the union of covectors orthogonal to orbits:

$$\Phi^{-1}(0) = \coprod_{q \in Q} \mathrm{ann}(T_q(G \cdot q)).$$

Since $\Phi^{-1}(0)$ contains the zero section Q, it is clear that the G-action on $\Phi^{-1}(0)$ is locally free if and only if the action on Q is locally free. If the action is free, we have $(T^*Q)/\!\!/G = T^*(Q/G)$. To see this (at least set-theoretically), note that

$$T(Q/G) = \Big(\coprod_q T_qQ/T_q(G \cdot q)\Big)/G$$

so that

$$T^*(Q/G) = \Big(\coprod_q \mathrm{ann}(T_q(G \cdot q))\Big)/G.$$

To identify the symplectic forms one has to identify the reduced canonical 1-form θ_0 with the canonical 1-form on $T^*(Q/G)$, we leave this as an exercise.

For an arbitrary G-space Q the singular reduced space $(T^*Q)/\!\!/G$ may be viewed as a cotangent bundle for the singular space Q/G.

EXAMPLE 5.8. Returning to our 2-d gauge theory example, the reduction $\mathcal{A}(\Sigma)/\!\!/\mathcal{G}(\Sigma)$ is the moduli space of flat connections on Σ.

5.2. Reduced Hamiltonians. Let G be a compact Lie group. Suppose (M, ω, Φ) is a Hamiltonian G-space, that $\mu \in \mathfrak{g}^*$ is a regular value of the moment map, and that the action of G_μ on the level set $\Phi^{-1}(\mu)$ is free. Then every invariant Hamiltonian $H \in C^\infty(M)^G$ descends to a unique function $H_\mu \in C^\infty(M_\mu)$ with $\pi_\mu^* H_\mu = \iota_\mu^* H$. Passing to the reduced Hamiltonian H_μ is often a first step in solving the equations of motion for H. From H-invariance of X_H it follows that the restriction $(X_H)\big|_{\Phi^{-1}(\mu)} \in \mathfrak{X}(\Phi^{-1}(\mu))$ is π_μ-related to $X_{H_\mu} \in \mathfrak{X}(M_\mu)$, that is its flow projects down to the flow on M_μ. After one has solved the reduced system (i.e. determined its flow $F_\mu(t)$) it is a second step to lift $F_\mu(t)$ up to the level set $\Phi^{-1}(\mu)$.

Example 5.9. Consider the motion of a particle on $\mathbb{R}^2$ in a potential $V(q)$. It is described by the Hamiltonian on $T^*\mathbb{R}^2$,

$$H(q,p) = \frac{||p||^2}{2} + V(q).$$

Suppose the potential has rotational symmetry, i.e. that it depends only on $r = ||q||$. Then H is invariant under the cotangent lift of the rotation action of $G = S^1$. We had seen that the moment map for this action is angular momentum, $\Phi(q,p) = p_2 q_1 - q_2 p_1$. In polar coordinates, (r, θ) on $\mathbb{R}^2$ and corresponding cotangent coordinates on T^*R^2,

$$H(r, \theta, p_r, p_\theta) = \frac{1}{2}(p_r^2 + \frac{1}{r^2} p_\theta^2) + V(r)$$

and $\Phi = p_\theta$. The symplectic form on $T^*\mathbb{R}^2$ is $\omega = \mathrm{d}r \wedge p_r + \mathrm{d}\theta \wedge p_\theta$. Every value $\mu \neq 0.$ is a regular value of Φ (since S^1 acts freely on the set where $p_\theta = r^2 \dot\theta \neq 0$). On $\iota_\mu : \Phi^{-1}(\mu) \hookrightarrow T^*\mathbb{R}^2$ the second term disappears, i.e. $\iota_\mu^* \omega = \mathrm{d}r \wedge p_r$. It follows that $M_\mu \cong T^*\mathbb{R}_{>0}$ symplectically, and the reduced Hamiltonian is

$$H_\mu(r, p_r) = \tfrac{1}{2} p_r^2 + V_{eff}(r)$$

with the effective potential,

$$V_{eff}(r) = V(r) + \frac{\mu^2}{2r^2}.$$

Using conservation of energy

$$\frac{p_r^2}{2} + V_{eff}(r) = \frac{\dot r^2}{2} + V_{eff}(r) = E,$$

i.e. $\dot r^2 = 2(E - V_{eff}(r))$, one obtains the solution in implicit form,

$$t - t_0 = \int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{2(E - V_{eff}(r))}}.$$

Using $r^2 \dot\theta = p_\theta = \mu$, one also obtains a differential equation for the trajectories,

$$\frac{\partial r}{\partial \theta} = \frac{r^2}{\mu} \sqrt{2(E - V_{eff}(r))},$$

with solutions,

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\mu \mathrm{d}r}{r^2\sqrt{2(E - V_{eff}(r))}}.$$

In the special case $V(r) = -\frac{1}{r}$ (Kepler problem) this integral can be solved and leads to conic sections – see any textbook on classical mechanics.

5.3. Reduction in stages. As a special case of "reduced Hamiltonian" one sometimes has a reduced moment map. For the simplest situation, suppose G, H are compact Lie groups and (M, ω) is a Hamiltonian $G \times H$-space, with moment map (Φ, Ψ). Since the two actions commute, $\langle \Phi^\xi, \Psi^\eta \} = 0$ for all $\xi \in \mathfrak{g}$, $\eta \in \mathfrak{h}$. In particular, Φ is H-invariant and Ψ is G-invariant. Let μ be a regular value of Φ, so that the reduced space M_μ is defined. Since Ψ is G-invariant, it descends to a map $\Psi_\mu : M_\mu \to \mathfrak{h}^*$. It is the moment map for the H-action on M_μ induced from the H-action on $\Phi^{-1}(\mu) \subset M$.

LEMMA 5.10 (Reduction in Stages). *Suppose μ is a regular value of Φ and (μ, ν) a regular value for (Φ, Ψ). Then ν is a regular value for Ψ_μ. If G_μ acts freely on $\Phi^{-1}(\mu)$ and $G_\mu \times H_\nu$ acts freely on $\Phi^{-1}(\mu) \cap \Psi^{-1}(\nu)$, then H_ν acts freely on $\Psi_\mu^{-1}(\nu)$, and there is a natural symplectomorphism*

$$(M_\mu)_\nu \cong M_{(\mu,\nu)}.$$

PROOF. Clearly, if G_μ acts with finite (resp. trivial) stabilizers on $\Phi^{-1}(\mu)$ and $G_\mu \times H_\nu$ acts with finite (resp. trivial) stabilizers on $\Phi^{-1}(\mu) \cap \Psi^{-1}(\nu)$, the same is true for the H_ν-action on $\Psi_\mu^{-1}(\nu)$. This proves the first part since a level set having finite stabilizers is equivalent to the level being a regular value. The second part follows because the natural identifications

$$(M_\mu)_\nu = \Psi_\mu^{-1}(\nu)/H_\nu = (\Phi^{-1}(\mu) \cap \Psi^{-1}(\nu))/(G_\mu \times H_\nu) = M_{(\mu,\nu)}$$

all preserve 2-forms. □

5.4. The cotangent bundle of a Lie group. Let G be a compact Lie group. For all $\xi \in \mathfrak{g}$ the left- and right invariant vector fields are related by the adjoint action, as follows:

$$\xi^L(g) = (\mathrm{Ad}_g\, \xi)^R(g).$$

The map

$$G \times \mathfrak{g} \to TG, \quad (g, \xi) \mapsto \xi^L(g)$$

is a vector bundle isomorphism called left trivialization of TG. In left trivialization, ξ^L becomes the constant vector field ξ and ξ^R becomes the vector field $\mathrm{Ad}_{g^{-1}}(\xi)$. Dual to the left-trivialization of TG there is the trivialization $T^*G \cong G \times \mathfrak{g}^*$ by left-invariant 1-forms.

EXERCISE 5.11. Show that in left trivialization of TG, the tangent maps to inversion $\mathrm{Inv} : G \to G,\ g \mapsto g^{-1}$, left action $L_h : G \to G,\ g \mapsto hg$ and right action $R_{h^{-1}} : G \to G,\ g \mapsto gh^{-1}$ are given by

$$\begin{aligned}\mathrm{Inv}_*(g,\xi) &= (g^{-1}, \mathrm{Ad}_g\,\xi),\\ (L_h)_*(g,\xi) &= (hg,\xi),\\ (R_{h^{-1}})_*(g,\xi) &= (gh^{-1}, \mathrm{Ad}_h\,\xi).\end{aligned}$$

Show that the respective cotangent lifts are

$$\begin{aligned}\mathrm{Inv}_*(g,\mu) &= (g^{-1}, (\mathrm{Ad}_{g^{-1}})^*\mu),\\ (L_h)_*(g,\mu) &= (hg,\mu),\\ (R_{h^{-1}})_*(g,\mu) &= (gh^{-1}, (\mathrm{Ad}_{h^{-1}})^*\mu).\end{aligned}$$

Since the generating vector fields for the left-and right action are $-\xi^R, \xi^L$ respectively, we find that the moment maps for these actions are

$$\Phi_L(g,\mu) = -\mathrm{Ad}_g^*(\mu), \quad \Phi_R(g,\mu) = \mu.$$

Note that the cotangent lifts of both the left action and the right action are free. In particular, every $\nu \in \mathfrak{g}^*$ is a regular value for both moment maps.

THEOREM 5.12. *The symplectic reduction $(T^*G)_\nu$ by the right action, with G-action inherited from the left-action, is the coadjoint orbit $G\cdot(-\nu)$.*

PROOF. It suffices to note that the left action of G on the level set $\Phi_R^{-1}(\mu)$ is free and transitive, and the action on the quotient has stabilizer conjugate to G_μ. The moment map induced by Φ_L identifies gives a symplectomorphism onto $G\cdot(-\nu)$. □

Of course, the reduced spaces with respect to the left action are coadjoint orbits $G\cdot(-\nu)$ as well: the cotangent lift of the inversion map $G \to G,\ g \mapsto g^{-1}$ exchanges the roles of the left- and right action.

THEOREM 5.13. *Let (M,ω,Φ) be a Hamiltonian G-space. Let G act diagonally on $T^*G \times M$, where the action on T^*G is the right action. Consider the reduced space at 0 as a Hamiltonian G-space, with G-action induced from the left-G-action on T^*G. Then there is a canonical isomorphism of Hamiltonian G-spaces,*

$$(T^*G \times M)/\!\!/G \cong M$$

PROOF. Use left trivialization $T^*G \cong G \times \mathfrak{g}^*$. The map

$$T^*G \times M \to T^*G \times M,\ (g,\mu,m) \mapsto (g,\mu,g.m)$$

is symplectic, and takes the diagonal action to the right-action on the first factor, and the left-action becomes a diagonal action. Hence it induces a symplectomorphism

$$(T^*G \times M)/\!\!/G \cong T^*G/\!\!/G \times M = M.$$

□

6. Normal forms and the Duistermaat-Heckman theorem

Let (M, ω, Φ) be a Hamiltonian G-space, where G is a compact Lie group. If 0 is a regular value of the moment map then so are nearby values $\mu \in \mathfrak{g}^*$. What is the relation between M_0 and reduced spaces M_μ at nearby values?

To investigate this question we describe the reduction process in terms of a normal form.

Let $Z = \Phi^{-1}(0)$ be the zero level set, $\iota : Z \hookrightarrow M$ the inclusion and $\pi : Z \to M_0$ the projection. Since the G-action on $\Phi^{-1}(0)$ has finite stabilizers, there exists a connection 1-form

$$\alpha \in \Omega^1(Z, \mathfrak{g}),$$

that is, α satisfies the two conditions

$$g^*\alpha = \mathrm{Ad}_g^* \alpha, \quad \iota(\xi_Z)\alpha = \xi.$$

Such a form can be constructed as follows. Consider the embedding $Z \times \mathfrak{g} \to TZ$, $(m, \xi) \mapsto \xi_Z(m)$ as a vector subbundle. Choose a G-invariant Riemannian metric on Z, and let $p : TZ \to Z \times \mathfrak{g}$ be the orthogonal projection with respect to that metric. The 1-form α is defined by $p(v) = (m, \alpha(v))$ for $v \in T_m M$.

Let $\mathrm{pr}_1, \mathrm{pr}_2$ be the projections from $Z \times \mathfrak{g}^*$ to the first and second factor. Define a 2-form on the product $X = Z \times \mathfrak{g}^*$ by

$$\sigma = \mathrm{pr}_1^* \pi^* \omega_0 + \mathrm{d}\langle \mathrm{pr}_2, \alpha \rangle,$$

let G-action act diagonally (using the coadjoint action on $\mathfrak{g}^*$).

Theorem 6.1 (Local normal form near the zero level set). *The 2-form σ is non-degenerate (i.e. symplectic) on some neighborhood of Z, and satisfies*

$$\iota(\xi_X)\sigma = d\langle \mathrm{pr}_2, \xi \rangle$$

for all $\xi \in \mathfrak{g}$ (that is, pr_2 is a moment map). There exists an equivariant symplectomorphism between neighborhoods of Z in M and in X, intertwining the two moment maps.

Proof. Notice that the 2-form

$$\mathrm{pr}_1^* \pi^* \omega_0 + \langle \mathrm{d}\,\mathrm{pr}_2, \alpha \rangle$$

is non-degenerate. Since σ differs from this 2-form by a term $\langle \mathrm{pr}_2, \mathrm{d}\alpha \rangle$ which vanishes along Z, it follows that σ is non-degenerate near Z. For the second part, notice that the tangent bundle to the null foliation $\ker(\iota^*\omega) \subset TZ$ is a trivial bundle $Z \times \mathfrak{g} \to Z$, where G acts on $\mathfrak{g}$ by the adjoint action. By the equivariant version of the co-isotropic embedding theorem, it follows that neighborhoods of Z in M and of Z in X are equivariantly symplectomorphic. Since both moment maps vanish on Z, it is automatic that the symplectomorphism intertwines the moment maps. ∎

We can view $Z \times \mathfrak{g}^*$ also as a quotient $(Z \times T^*G)/G$, using left trivialization to identify $T^*G \cong G \times \mathfrak{g}^*$; the G-action on $Z \times \mathfrak{g}^*$ is induced from the cotangent lift of the right G-action.

It follows that the reduced space at μ close to 0 is symplectomorphic to $(Z \times G \cdot (-\mu))/G$, that is:

Corollary 6.2. *The reduced space M_μ for μ close to 0 fiber over M_0, with fibers coadjoint orbits:*

$$M_\mu \cong (Z \times G \cdot (-\mu))/G.$$

Letting $\Psi : G\cdot(-\mu) \hookrightarrow \mathfrak{g}^*$ be the embedding, the pull-back of the 2-form ω_μ to $Z \times G\cdot(-\mu)$ is

$$\pi^*\omega_0 + d\langle \Psi, \alpha \rangle.$$

Consider in particular the case that G is a torus $T \cong (\mathbb{R}/\mathbb{Z})^k$. Then the coadjoint action is trivial, and the reduced spaces at 0 and at nearby values are diffeomorphic. Notice that $\mathrm{d}\alpha$ descends to a 2-form M_0 which is just the curvature form $F^\alpha \in \Omega^2(M_0, \mathfrak{t})$ of the torus bundle $\Phi^{-1}(0) \to M_0$. As a consequence we find that the symplectic form changes according to

$$\omega_\mu = \omega_0 + \langle \mu, F^\alpha \rangle$$

In particular this change is *linear in* μ!! This result depends on our identification of $M_\mu \cong M_0$. This identification is not natural, but any two identifications are related by an isotopy of M_0. Since cohomology classes are stable under isotopies, it makes sense to compare cohomology classes, and the above discussion proves:

Theorem 6.3 (Duistermaat-Heckman). *The cohomology class of the symplectic form changes according to*

$$[\omega_\mu] = [\omega_0] + \langle \mu, c \rangle$$

where $c \in H^*(M_0) \otimes \mathfrak{t}$ is the first Chern class of the torus bundle $\Phi^{-1}(0) \to M_0$.

In particular this change is linear in μ !!! As an immediate consequence one has:

Corollary 6.4. *Let (M, ω, Φ) be a Hamiltonian T-space, and let U be a connected component of the set of regular values of Φ. Then the volume function $U \to \mathbb{R}$, $\mu \mapsto$ $\mathrm{Vol}(M_\mu)$ is given by a polynomial of degree at most $\frac{1}{2}\dim M - \dim T$.*

(Here the maximum degree is obtained as half the dimension of a reduced space.)

7. The symplectic slice theorem

7.1. The slice theorem for G-manifolds. Let G be a compact Lie group, and H a closed Lie subgroup. Then every G-equivariant vector bundle over the homogeneous space G/H is of the form

$$E = G \times_H W \equiv (G \times W)/H$$

where W is an H-representation, the quotient is taken by the H-action $h.(g, w) = (gh^{-1}, h.w)$ and the G-action is given by $g_1.[g, w] = [g_1 g, w]$. Indeed, given E one defines $W = E_m$ to be the fiber over the identity coset $m = eH$, and the map $G \times_H W \mapsto E$, $[g, w] \mapsto g.w$ is easily seen to be a well-defined, equivariant vector bundle isomorphism.

For example, suppose M is a G-manifold, $m \in M$, and $H = G_m$. Let $W = T_m M / T_m(G.m)$ be the so-called *slice representation* of H. Then $E = G \times_H W$ is the normal bundle $\nu_{\mathcal{O}} = TM|_{\mathcal{O}}/T\mathcal{O}$ to the orbit $\mathcal{O} = G.m = G/H$. Recall now that by the tubular neighborhood theorem, if N is any submanifold of M there is a diffeomorphism of neighborhoods of N in M and in ν_N. (The diffeomorphism is constructed using geodesic flow with respect to a Riemannian metric on M.) If N is G-invariant, this diffeomorphism can be chosen G-equivariant. (Just take the Riemannian metric to be G-invariant.) We conclude:

THEOREM 7.1 (Slice theorem). *Let M be a G-manifold, and $m \in M$ a point with stabilizer $H = G.m$ and slice representation $W = T_m M / T_m(G.m)$. There exists a G-equivariant diffeomorphism from an invariant open neighborhood of the orbit $\mathcal{O} = G.m$ to a neighborhood of the zero section of $E = G \times_H W$.*

COROLLARY 7.2. *There exists a neighborhood U of $\mathcal{O} = G.m$ with the property that all stabilizer groups G_x, $x \in U$ are conjugate to subgroups of $H = G.m$. In particular, if M is compact there are only finitely many conjugacy classes of stabilizer groups.*

PROOF. Identify some neighborhood of the orbit with the model $E = G \times_H W$. Let $x = g.y$ with $y \in W$. Then $G_x = \mathrm{Ad}_g(G_y)$. But G_y is a subgroup of H, since it preserves the fiber $W = E_m$. $\square$

COROLLARY 7.3. *If G is compact abelian, and M is compact, the number of stabilizer subgroups $\{H \subset G |\ H = G_m$ for some $m \in M\}$ is finite.*

PROOF. For an abelian group, conjugation is trivial. $\square$

DEFINITION 7.4. For any subgroup H of G one denotes its conjugacy class by (H), and calls the G-invariant subset

$$M_{(H)} = \{m \in M |\, G_m \text{ is } G\text{-conjugate to } H\},$$

the points *of orbit type* (H). One also defines

$$M^H = \{m \in M |\, G_m \supset H\}, \quad M_H = \{m \in M |\, G_m = H\}.$$

PROPOSITION 7.5. *The connected components of $M_{(H)}, M_H$ and M^H are smooth submanifolds of M.*

PROOF. Any orbit $\mathcal{O} \subset M_{(H)}$ contains a point $m \in M$ with $G_m = H$. In the model $E = G \times_H W$ near $\mathcal{O}$, we have

$$E_{(H)} = G \times_H W^H = G/H \times W^H$$

since W^H is a vector subspace of W, this is clearly a smooth subbundle of E. The connected components of M^H are smooth submanifolds, since for all $m \in M^H$, a neighborhood is H-equivariantly modeled by the H-action on $T_m M$ and $(T_m M)^H$ is a vector subspace. The closure $\overline{M}^H$ is a union of connected components of M^H. Since M_H is open in its closure, it is in particular a submanifold. □

The decomposition $M = \bigcup_{(H)} M_{(H)}$ is called the orbit type stratification of M. Using the local model near orbits, one can show that it is indeed a stratification in the technical sense. Note that since each $M_{(H)}/G$ is a (union of) smooth manifolds, it induces a decomposition (in fact, stratification) of the orbit space M/G.

7.2. The slice theorem for Hamiltonian G-manifolds. In symplectic geometry one can go one step further and try to equip the total space to the normal bundle with a symplectic structure. Thus let (M, ω, Φ) be a Hamiltonian G-space. Assume that $m \in \Phi^{-1}(0)$ is in the zero level set. This implies that the orbit is an isotropic submanifold: Since Φ vanishes on $\mathcal{O}$ we have $T_m\mathcal{O} \subset \ker(\mathrm{d}_m\Phi)$, on the other hand $\ker(\mathrm{d}_m\Phi) = T_m\mathcal{O}^\omega$. The symplectic vector space $V = T_m\mathcal{O}^\omega/T_m\mathcal{O}$ with the action of $H = G_m$ is called the symplectic slice representation at m.

Definition 7.6. Let H act on a symplectic vector space (V, ω_V) by linear symplectic transformations, and let $\Phi_V :\ V \to \mathfrak{h}^*$ be the unique moment map vanishing at 0 (cf. 4.5),

$$\Phi_V :\, V \to \mathfrak{h}^*,\quad \langle \Phi(v), \xi\rangle = \tfrac{1}{2}\omega(v, \xi.v)$$

One calls the symplectic quotient

$$E = (T^*G \times V)/\!\!/H$$

the *model* defined by V. Here H acts on $T^*G \times V$ by the diagonal action, where the action on T^*G is given by the cotangent lift of the right-action of H on G. We let $\Phi_E :\ E \to \mathfrak{g}^*$ be the moment map for the G-action on E inherited from the cotangent lift of the left-G-action on T^*G.

The orbit $\mathcal{O} = G/H$ is naturally embedded as an isotropic submanifold of E, namely as the zero section of $T^*G/\!\!/H = T^*(G/H)$. Its symplectic normal bundle in E is an associated bundle, $G \times_H V$.

Remark 7.7. The model can also be written as an associated bundle: Identify $T^*G = G \times \mathfrak{g}^*$ using left trivialization. The zero level set for the H-action consists of points (g, μ, v) such that $-\operatorname{pr}_{\mathfrak{h}^*}\mu + \Phi_V(v) = 0$. Thus, if we choose an H-equivariant complement to $\operatorname{ann}(\mathfrak{h})$ in $\mathfrak{g}^*$, identifying $\mathfrak{g}^* = \mathfrak{h}^* \oplus \operatorname{ann}(\mathfrak{h})$, we see that the zero level set consists of points $(g, \Phi_V(v) + \nu, v)$ with $\nu \in \operatorname{ann}(\mathfrak{h})$, and is therefore isomorphic to $G \times \operatorname{ann}(\mathfrak{h}) \times V$. Thus

$$E \cong G \times_H (\operatorname{ann}(\mathfrak{h}) \times V).$$

In this description the moment map Φ_E is given by

$$\Phi_E([\mathfrak{g}, \nu, v]) = g.(\nu + \Phi_V(v)).$$

Note however that this identification depends on the choice of splitting.

THEOREM 7.8. *Let (M, ω, Φ) be a Hamiltonian G-manifold, and $\mathcal{O} = G.m$ an orbit in the zero level set of Φ. There exists a G-equivariant symplectomorphism between neighborhoods of $\mathcal{O}$ in M and in the model E defined by the symplectic slice representation $V = T_m\mathcal{O}^\omega / T_m\mathcal{O}$ of $H = G_m$, intertwining the two moment maps.*

PROOF. This follows from (equivariant version of) the constant rank embedding theorem: The symplectic normal bundles of $\mathcal{O}$ in both spaces are $G \times_H V$. □

The symplectic slice theorem is extremely useful: For example we obtain a model for the singularities of $M/\!\!/G$ in case 0 is a singular value. Indeed, by reduction in stages we have

$$(T^*G \times V/\!\!/H)/\!\!/G = (T^*G \times V/\!\!/G)/\!\!/H = V/\!\!/H$$

which shows that the singularities are modeled by symplectic reductions of unitary representations. Since the moment map for a unitary representation is homogeneous, the zero level set $\Phi_V^{-1}(0)$ is a cone and hence the singularities are *conic singularities*. This discussion can be carried much further, see the paper Sjamaar-Lerman, "Stratified symplectic spaces and reduction", Ann. of Math. (2) 134 (1991), no. 2, 375–422.

PROPOSITION 7.9. *Let (M, ω, Φ) be a Hamiltonian G-space, $H \subset G$ a closed subgroup. The connected components of M^H and M_H are symplectic submanifolds of M. For every connected open subset $U \subset M_H$, the image $\Phi(U)$ is an open subset of an affine subspace $\mu + \mathrm{ann}(\mathfrak{h})^H \subset \mathfrak{g}^*$ for some $\mu \in (\mathfrak{g}^*)^H$.*

PROOF. For all $m \in M^H$, the tangent space $T_m(M^H)$ is equal to $(T_mM)^H$. But for any symplectic representation V of a compact Lie group H, the subspace V^H is symplectic. (Proof: Choose an H-invariant compatible complex structure. Then V^H is a complex, hence symplectic, subspace.) This shows that M^H and the open subset $M_H \subset M^H$ are symplectic.

The second part follows from the local model, or alternatively follows: Let $Z = Z_G(H)$ be the centralizer and $K = N_G(H)$ the normalizer of H in G, respectively. Thus $Z \subset K \subset G$ and $\mathfrak{z} = \mathfrak{k}^H = \mathfrak{g}^H$. Dually, identify $\mathfrak{z}^* = (\mathfrak{k}^*)^H = (\mathfrak{g}^*)^H$. By equivariance of the moment map, $\Phi(M_H) \subset \Phi(M^H) \subset (\mathfrak{g}^*)^H = \mathfrak{z}^*$. The action of $K \subset G$ preserves M_H. Its moment map $\Psi : M_H \to \mathfrak{k}^*$ is the restriction of Φ followed by projection $\mathfrak{g}^* \to \mathfrak{k}^*$, but since it takes values in $(\mathfrak{k}^*)^H = \mathfrak{z}^*$ it is actually just the restriction of Φ. Since $\mathrm{im}\, \mathrm{d}_m\Psi = \mathrm{ann}_{\mathfrak{k}^*}(\mathfrak{h}) = \mathrm{ann}(\mathfrak{h})^H$ is independent of $m \in U$, we conclude that $\Phi(U)$ is an open neighborhood of $\Phi(m)$ in $\Phi(m) + \mathrm{ann}(\mathfrak{h})^H$. □

CHAPTER 7

Hamiltonian torus actions

1. The Atiyah-Guillemin-Sternberg convexity theorem

1.1. Motivation. As a motivating example, which on first sight seems quite unrelated to symplectic geometry, consider the following problem about self-adjoint matrices. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be an n-tuple of real numbers, and let $\mathcal{O}(\lambda)$ be the set of all self-adjoint complex $n \times n$-matrices having eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\pi : \mathcal{O}(\lambda) \to \mathbb{R}^n$ be the projection to the diagonal.

THEOREM 1.1. *The image $\pi(\mathcal{O}(\lambda))$ is the convex hull*

$$\Delta := \mathrm{hull}\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}),\ \sigma \in \mathfrak{S}_n\}$$

where $\mathfrak{S}_n$ is the permutation group.

This and related results were proved by Schur and Horn, later greatly generalized by Kostant and Heckman.

The relation to symplectic geometry is as follows. First, instead of self-adjoint matrices we can equivalently consider skew-adjoint matrices, i.e. the Lie algebra $\mathfrak{g}$ of $G = \mathrm{U}(n)$. Since all matrices with given eigenvalues are conjugate, $\mathcal{O}(\lambda)$ is an orbit for the action of $\mathrm{U}(n)$. Using the inner product

$$(A, B) = -\operatorname{tr}(AB)$$

on $\mathfrak{g}$ we can also view it as a coadjoint orbit.

The projection π is just orthogonal projection onto the diagonal matrices, which are a maximal commutative subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$. Using the inner product to identify $\mathfrak{t} \cong \mathfrak{t}^*$ it becomes the moment map for the induced $T \subset G$ action.

For this reason the Schur-Horn theorem can be viewed as a convexity theorem for Hamiltonian torus actions on coadjoint orbits of $\mathrm{U}(n)$. Nothing is special about $\mathrm{U}(n)$, analogous results hold for arbitrary compact groups.

1.2. Local convexity. Our discussion of the convexity theorem will follow the exposition in Guillemin-Sternberg, "Symplectic Techniques in Physics". In order to understand how images of moment maps for Hamiltonian T-spaces look like, we first have to understand how they look like "locally". We will work with the local model E for the T-action near any orbit $\mathcal{O} = T.m$. (In that section we assumed $T.m \subset \Phi^{-1}(0)$, but this can be arranged by adding a constant to the moment map.) Letting $H = T_m$ be the stabilizer, we have the model

$$E = (T^*(T) \times V)/\!\!/H$$

where V is a unitary H-representation. Using the identification $T^*(T) = T \times \mathfrak{t}^*$, the moment map for the H-action on $T^*(T) \times V$ is $(t, \nu, v) \mapsto -\mathrm{pr}_{\mathfrak{h}^*}(\nu) + \Phi_V(v)$ and the moment map for the T-action on E is induced from the map $(t, \nu, v) \mapsto \mu + \nu$. The zero level set condition for the H-action reads $\mathrm{pr}_{\mathfrak{h}^*}(\nu) = \Phi_V(v)$. This shows:

LEMMA 1.2. *The image of the moment map Φ_E is of the form*

$$\Phi_E(E) = \mu + (\mathrm{pr}_{\mathfrak{h}^*})^{-1}(\Phi_V(V)).$$

To understand the shape of this set we need to describe the moment map images of unitary torus representations (here for the identity component of H acting on V).

Let T be a torus, $\Lambda \subset \mathfrak{t}$ the integral lattice. Every 1-dimensional unitary representation

$$T \to \mathrm{U}(1) \cong S^1 = \mathbb{R}/\mathbb{Z}$$

defines a map $\mathfrak{t} \to \mathbb{R}$, which restricts to a homomorphism of lattices,

$$\alpha : \Lambda \to \mathbb{Z}$$

called the *weight* of the representation. One calls

$$\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subset \mathfrak{t}^*$$

Given $\alpha \in \Lambda^*$ one defines a 1-dimensional representation $\mathbb{C}_\alpha$ by

$$\exp(\xi) \cdot z = e^{2\pi i \langle \alpha, \xi \rangle}\, z.$$

By Schur's Lemma, any unitary T-representation V splits into a sum of 1-dimensional representations, i.e. is isomorphic to a representation of the form

$$V = \oplus_{j=1}^n \mathbb{C}_{\alpha_j}$$

where α_j are called the weights of V. Given a *symplectic* vector space V with a symplectic T-representation, one chooses an invariant compatible complex structure I, which makes V into a unitary T-representation. The weights α_j for this representation are independent of the choice of I, since any two I's are deformation equivalent. They are called the weights of the symplectic T-representation.

LEMMA 1.3. *Let (V, ω_V, Φ_V) be a symplectic T-representation with moment map $\langle \Phi_V(v), \xi \rangle = \frac{1}{2}\omega(v, \xi\, v)$. The image of the moment map Φ_V is a convex, rational polyhedral cone $\Phi_V(V) = C_V$ spanned by the weight $\alpha_j \in \Lambda^*$ of the representation:*

$$C_V = \mathrm{cone}\{\alpha_1, \dots, \alpha_n\}$$

For any open subset $U \subset V$, the image $\Phi_V(U)$ is open in C_V; in particular any open neighborhood of the origin gets mapped onto an open neighborhood of the tip of the cone.

PROOF. It is convenient to write the moment map in a slightly different form. By the above discussion, we can choose an equivariant symplectomorphism $V \cong \mathbb{C}^n$ such that T acts on $\mathbb{C}^n$ by the homomorphism $T \to (S^1)^n$ determined by the weights. The moment map for the standard $(S^1)^n$-action on $\mathbb{C}^n$ is

$$\Phi_0(z_1, \ldots, z_n) = \pi(|z_1|^2, \ldots, |z_n|^2) = \pi \sum_j |z_j|^2 e_j$$

where e_j is the jth standard basis vector for $\mathbb{R}^n \cong (\mathbb{R}^n)^*$. Hence Φ_V is a composition of Φ_0 with the map $(\mathbb{R}^n)^* \to \mathfrak{t}^*$ dual to the tangent map. The latter map takes e_j to α_j. Thus

$$\Phi_V(z_1, \ldots, z_n) = \pi \sum_j |z_j|^2 \alpha_j.$$

The description of the image of Φ_V is now immediate. If $U \subset V$ is open, then $\Phi_0(U)$ is open in the positive orthant $\mathbb{R}^n_+$, i.e. is the intersection of $\mathbb{R}^n_+$ with an open set $U' \subset (\mathbb{R}^n)$. (This amounts to saying that the quotient topology on $\mathbb{R}^n_+ = \mathbb{C}^n/(S^1)^n$ coincides with the subset topology.) Since Φ is obtained from Φ_0 by composition with a linear map, it follows that Φ is open onto its image. $\square$

Following Sjamaar, we find it useful to make the following definition:

DEFINITION 1.4. Let (M, ω, Φ) be a Hamiltonian T-space, $m \in M$. Let $H = T_m$ be the stabilizer of m and H_0 its identity component. Let $C \subset \mathfrak{h}^*$ be the cone spanned by the weights for the H_0-action on $T_m M$. The affine cone

$$C_m = \Phi(m) + (\mathrm{pr}_{\mathfrak{h}^*})^{-1}(C)$$

is called the *local moment cone* at $m \in M$.

Thus C_m is exactly the moment map image for the local model at m. We have shown:

THEOREM 1.5 (Local convexity theorem). *Let (M, ω, Φ) be a Hamiltonian T-space, $m \in M$, $\mathcal{O} = T \cdot m$. Then for any sufficiently small T-invariant neighborhood U of $\mathcal{O}$ there is a neighborhood V of $\Phi(m) \in C_m$ such that*

$$\Phi(U) = C_m \cap V.$$

1.3. Global convexity. We now come to the key observation of Guillemin-Sternberg. Given $\xi \in \mathfrak{t}$ consider the corresponding component $\Phi^\xi = \langle \Phi, \xi \rangle$ of the moment map. A value $s \in \mathbb{R}$ is called a local minimum for Φ^ξ if there exists $m \in M$ with $\Phi^\xi(m) = s$ and $\Phi^\xi \geq s$ on some neighborhood.

LEMMA 1.6 (Guillemin-Sternberg). *Let (M, ω, Φ) be a compact connected Hamiltonian T-space. Then all fibers of Φ are connected. Moreover, the function Φ^ξ has a unique local minimum/maximum.*

We will prove this Lemma in the next section. For any subset $S \subset \mathfrak{t}^*$ and $\mu \in \mathfrak{t}^*$ let

$$\mathrm{cone}_\mu(S) = \{\mu + t(\nu - \mu) |\, \nu \in S\}$$

be the cone over S at μ.

THEOREM 1.7 (Guillemin-Sternberg, Atiyah). *Let* (M, ω, Φ) *be a compact, connected Hamiltonian* T*-space on which* T *acts effectively. The image* $\Delta = \Phi(M)$ *is a convex rational polytope of dimension* $\dim T$. *For all* $m \in M$, *one has*

$$C_m = \mathrm{cone}_{\Phi(m)}(\Delta). \tag{11}$$

PROOF. Since local convexity of a compact set implies global convexity it suffices to prove Equation (11). Since a neighborhood of the tip of the cone C_m gets mapped into Δ, it follows that $\mathrm{cone}_m(\Delta) \supset C_m$. To see the opposite inclusion, we define, for all $\xi \in \mathfrak{t}$, the affine linear functional $f_\xi = \langle \cdot, \xi \rangle - \langle \mu, \xi \rangle$ on $\mathfrak{t}^*$. We have to show that for al ξ,

$$f_\xi|_{C_m} \geq 0 \Rightarrow f_\xi|_\Delta \geq 0.$$

But $f_\xi \geq 0$ on C_m means, by the model, that $\langle \mu, \xi \rangle$ is a local minimum for Φ^ξ. By the Lemma, this has to be a global minimum, or equivalently $f_\xi \geq 0$ on Δ. This proves (11). □

We note that (11) was first observed by Sjamaar, who generalized it also to the non-abelian case. Since C_m is the moment map image for the local model, it shows that Φ is *open* as a map onto Δ.

We obtain the following description of the faces and the "fine structure" of Δ. Let $H \subset T$ be in the (finite) list of stabilizer groups, and M_H the points with stabilizer H. Recall again that M_H is an open subset of the symplectic submanifold M^H. Each connected component of M^H is a Hamiltonian T-space in its own right, with H acting trivially. Thus its moment map image is a convex polytope of dimension $\dim(T/H)$ inside an affine subspace $\mu + \mathrm{ann}(\mathfrak{h})$, with the corresponding component of M_H mapping to its interior. That is, the (open) faces of Δ correspond to orbit type strata, and in particular the vertices of Δ correspond to fixed points M^T. That is,

$$\Delta = \mathrm{hull}(\Phi(M^T))$$

is the convex hull of the fixed point set. Note however that some of the polytopes $\Phi(M^H)$ get mapped to the *interior* of Δ. Thus Δ gets subdivided into polyhedral subregions, consisting of regular values of Φ.

THEOREM 1.8. *Let* (M, ω, Φ) *be a Hamiltonian* T*-space, with* T *acting effectively, and* $\Delta \subset \mathfrak{t}^*$ *its moment polytope. For any closed face* Δ_i *of* Δ *of codimension* d_i, *the pre-image* $\Phi^{-1}(\Delta_i)$ *is symplectic, and is a connected component of the fixed point set for some* d_i*-dimensional stabilizer group* $H_i \subset T$ *where* $\mathfrak{h}_i$ *is the subspace orthogonal to* Δ_i. *In particular, the vertices of* Δ *correspond to fixed point manifolds.*

PROOF. We note that each $\Phi^{-1}(\Delta_i) \subset M$ is closed and connected, by connectedness of the fibers of Φ. Hence it is a connected component of some M^{H_i}, where $\mathrm{ann}(\mathfrak{h}_i)$ is parallel to Δ_i. □

In particular, Hamiltonian torus actions on compact symplectic manifolds are *never fixed point free*. (This shows immediately that the standard 2k-torus action on itself cannot be Hamiltonian.)

EXERCISE 1.9. Let (M, ω, Φ) be a compact, connected Hamiltonian T-space where T acts effectively. Let $M_* = M_{\{e\}}$ be the subset on which the action is free. Show that M_* is connected, and that its image $\Phi(M_*)$ is precisely the interior of the moment polytope $\Delta = \Phi(M)$.

1.4. Duistermaat-Heckman measure. Let us assume (with no loss of generality) that the T-action on (M, ω, Φ) is effective. The images of the fixed point manifolds for non-trivial stabilizer groups define a subdivison of the polytope Δ into chambers, given as the connected components of the set of regular values of Φ. By the Duistermaat-Heckman theorem, the volume function $\mu \mapsto \mathrm{Vol}(M_\mu)$ is a polynomial on each of these connected components.

The volume function is equivalent to the *Duistermaat-Heckman measure* ϱ, defined as the push-forward

$$\varrho := \Phi_* \left|\frac{\omega^n}{n!}\right|$$

of the Liouville measure under the moment map. Thus ϱ is the measure such that for every continuous function ϕ on $\mathfrak{t}^*$,

$$\int_{\mathfrak{t}^*} \phi\, \varrho = \int_M \Phi^* \phi \, \left|\frac{\omega^n}{n!}\right|.$$

Let ϱ_{Leb} be Lebesgue measure on $\mathfrak{t}^*$, normalized in such a way that $\mathfrak{t}^*/\Lambda^*$ (where Λ^* is the weight lattice) has volume 1.

EXERCISE 1.10. Let (M, ω, Φ) be a compact Hamiltonian T-space, with T acting effectively. Show that at regular values of Φ, ϱ is smooth with respect to the normalized Lebesgue measure ϱ_{Leb}, and

$$\varrho(\mu) = \mathrm{Vol}(M_\mu)\ \varrho_{Leb}$$

The proof of this fact is left as an exercise. Hint: Use the local model for reduction near a regular level set. The Duistermaat-Heckman theorem may thus be re-phrased by saying that ϱ is a *piecewise polynomial measure*.

Duistermaat-Heckman in their paper use this fact to derive a remarkable "exact integration formula", which we will in Section 3.

2. Some basic Morse-Bott theory

The proof of the fact that every component $f = \Phi^\xi$ of the moment map has a unique local minimum relies on the idea of viewing f as a Morse-Bott function. For any function $f \in C^\infty(M, \mathbb{R})$ on a manifold M, the set of critical points is the closed subset

$$\mathcal{C} = \{m |\, \mathrm{d}f(m) = 0\}.$$

For all $m \in \mathcal{C}$ there is a well-defined symmetric bilinear form on T_mM, called the Hessian

$$\mathrm{d}^2 f(m)(X_m, Y_m) = (L_X\, L_Y\, f)(m)$$

for all $X, Y \in \mathrm{Vect}(M)$. In local coordinates, the Hessian is simply given by the matrix of second derivatives of f.

The function f is called a Morse function if $\mathcal{C}$ is discrete and for all $m \in \mathcal{C}$ the Hessian is non-degenerate. More generally, f is called Morse-Bott if the connected components $\mathcal{C}^j$ of $\mathcal{C} = \{m |\, \mathrm{d}f(m) = 0\}$ are smooth manifolds, and for all $m \in \mathcal{C}^j$ we have

$$\ker(\mathrm{d}^2 f(m)) = T_m\mathcal{C}^j.$$

Given a Riemannian metric on M, consider the negative gradient flow of f, i.e. the flow F^t of the vector field $-\nabla(f) \in \mathrm{Vect}(M)$. For all connected components $\mathcal{C}^j$ we can consider the sets

$$W^j_+ = \{m \in M,\ \lim_{t\to\infty} F^t(m) \in \mathcal{C}^j\}$$

and

$$W^j_- = \{m \in M,\ \lim_{t\to\infty} F^t(m) \in \mathcal{C}^j\}.$$

If f is Morse-Bott then all $W^j_\pm$ are smooth manifolds, and one has natural finite decompositions

$$M = \cup_j W^j_- = \cup_j W^j_+$$

into unstable/stable manifolds. The dimension of W^j_- (resp. W^j_+) is equal to the dimension of $\mathcal{C}^j$ plus the dimension of the negative eigenspace of $\mathrm{Hess}(f)$, denoted $n^j_\pm$. Thus

$$n^j_\mp = \mathrm{codim}(W^j_\pm).$$

The number n^j_- is called the index of $\mathcal{C}^j$.

Proposition 2.1. *If none of the indices n^j_- is equal to 1, there exists a unique critical manifold of index 0, i.e. a unique local minimum of f. If moreover all $n^j_+ \neq 1$ then all level sets $f^{-1}(c)$ are connected.*

Proof. The condition $n^j_- \neq 1$ means that all W^+_j of positive index have codimension at least 2, so that their complement is connected. Hence there is a unique stable manifold W^+_j with $n^-_j = 0$. If in addition $n^j_+ \neq 1$, the set M_* obtained from M by removing all M^j_+ with $n^j_- > 0$ and all M^j_- with $n^j_+ > 0$ is open, dense and connected in M. Notice that M_* consists of all points which flow to the (unique) minimum of f for $t \to \infty$ and to the (unique) maximum of f for $t \to -\infty$. If $\min(f) < c < \max(f)$ then every trajectory of the gradient flow of a point in M_* intersects $f^{-1}(c)$ in a unique point. Therefore the map

$$(f^{-1}(c) \cap M_*) \times \mathbb{R} \to M_*,\ (m, t) \to F^t(m)$$

is a diffeomorphism, and in particular $f^{-1}(c) \cap M_*$ is connected. To prove the proposition it suffices to show that $f^{-1}(c) \cap M_*$ is dense in $f^{-1}(c)$. Let $m \in f^{-1}(c)$ and U a connected open neighborhood of m. Since c is neither maximum or minimum, $U \cap M^*$ meets both the sets where $f < c$ and $f > c$, and since it is connected it meets $f^{-1}(c)$. $\square$

Returning to the symplectic geometry context, we need to show:

THEOREM 2.2. *Let (M, ω, Φ) be a Hamiltonian G-space, $\xi \in \mathfrak{g}$. Then $f = \Phi^\xi$ is a Morse-Bott function. Moreover all critical manifolds $\mathbb{C}^j$ are symplectic submanifolds of M, and the indices n_-^j are all even.*

PROOF. Let $H \subset G$ be the closure of the 1-parameter subgroup generated by ξ. Then H is a torus. The critical set of f is given by the condition

$$0 = \mathrm{d}\langle \Phi, \xi \rangle(m) = \iota(\xi_M(m))\omega_m.$$

Since ω is non-degenerate, it is precisely the set of zeroes of the vector field ξ_M, or equivalently the fixed point set for the 1-parameter subgroup $\{\exp(t\xi) |\, t \in \mathbb{R}\} \subseteq G$. Let

$$H = \overline{\{\exp(t\xi) |\, t \in \mathbb{R}\}}.$$

then H is abelian and connected, hence is a torus, and $\mathbb{C}$ is just the set of fixed points for this torus action. Let $m \in \mathbb{C}$, and equip $T_m M = V$ with an H-invariant compatible complex structure. As a unitary representation, V is equivalent to $V = \oplus \mathbb{C}_{\alpha_j}$ where α_j are the weights for the action. By the equivariant Darboux-theorem, V serves as a model for the H-action near m. In particular the fixed point manifold $\mathbb{C} = M^H$ gets modeled by the space of fixed vectors V^H, which is a complex, hence also symplectic subspace. This shows that all $\mathbb{C}_j$ are symplectic manifolds. Moreover the moment map in this model is (a constant plus)

$$z \mapsto \pi \sum_j |z_j|^2 \alpha_j = \pi \sum_j (q_j^2 + p_j^2)\ \alpha_j,$$

in particular

$$f = \pi \sum_j |z_j|^2 \alpha_j = \pi \sum_j (q_j^2 + p_j^2)\ \langle \alpha_j, \xi \rangle.$$

From this it is evident that f is Morse-Bott and that all indices are even. $\square$

The fact that all indices are even has very strong implications in Morse theory: It implies that the so-called lacunary principle applies, and the Morse-Bott polynomial is equal to the Poincare polynomial. (I.e. the Morse inequalities are equalities – Morse functions for which this is the case are called *perfect*.) This gives a powerful tool to calculate the cohomology of Hamiltonian G-spaces: in particular for isolated fixed points, this gives

$$\dim H^k(M, \mathbb{Q}) = \#\{ \text{ critical points of index } k \ \};$$

in particular all cohomology sits in even degree if all indices are even.

COROLLARY 2.3. *Suppose M admits a Morse-Bott function f such that the minimum of f is an isolated point and all $n_-^j \neq 1$. Then M is simply connected.*

PROOF. Given any $m \in X$ and a loop $\gamma \in X$ based at m, one can always perturb γ so that it does not meet the stable manifolds of index > 0. Applying the gradient flow to γ contracts γ to the minimum. $\square$

Examples are coadjoint orbits of a compact Lie group (the fact that coadjoint orbits are compact submanifolds of a vector space allows one to show that for generic components of the moment map the minimum is isolated.) Thus coadjoint orbits are simply connected. (We remark that this is not true in general for conjugacy classes.). Let G/G_μ be a coadjoint orbit where G is compact, connected. View G_μ as the fiber over the identity coset. Given any two points in G_μ they can be joined by a path in G. The projection to G/G_μ is a closed path, hence can be contracted. Lifting the contraction to G produces a path in G_μ connecting the two points. Thus all stabilizer groups for the (co)-adjoint action are connected.

3. Localization formulas

Let (M, ω, Φ) be a compact Hamiltonian T-space. For simplicity we assume that the set M^T of fixed points is finite. (This is for example the case for the action of a maximal torus $T \subset G$ on a coadjoint orbit $\mathcal{O} = G\cdot\mu$.) Given $p \in M^T$ let $a_1(p), \ldots, a_n(p) \in \Lambda^* \subset \mathfrak{t}^*$ be the weights for the action on T_pM.

THEOREM 3.1 (Duistermaat-Heckman). *Let $\xi \in \mathfrak{t}^{\mathbb{C}}$ be such that $\langle a_j(p), \xi\rangle \neq 0$ for all p, j. Then one has the exact integration formula*

$$\int_M e^{\langle \Phi, \xi\rangle}\, \frac{\omega^n}{n!} = \sum_{p\in M^T} \frac{e^{\langle \Phi(p), \xi\rangle}}{\prod_j \langle a_j(p), \xi\rangle}.$$

One way of looking at this result is to say that the stationary phase approximation for the integral $\int_M e^{it\langle \Phi, \xi\rangle}\, \frac{\omega^n}{n!}$ is *exact*!

Our proof of the DH-formula will follow an argument of Berline-Vergne. Notice first that the integrand is just the top form degree part of

$$e^{\omega + \langle \Phi, \xi\rangle} = e^{\langle \Phi, \xi\rangle} \sum_{j=0}^{n} \frac{\omega^j}{j!} \in \Omega^*(M).$$

Consider the derivation

$$\mathrm{d}_\xi :\ \Omega^*(M) \to \Omega^*(M), \mathrm{d}_\xi := \mathrm{d} - \iota(\xi_M).$$

The differential form $\omega + \langle \Phi, \xi\rangle$ is d_ξ-closed, i.e. killed by d_ξ:

$$\mathrm{d}_\xi(\omega + \langle \Phi, \xi\rangle) = -\iota(\xi_M)\omega + \mathrm{d}\langle \Phi, \xi\rangle = 0.$$

Moreover $\alpha := e^{\omega + \langle \Phi, \xi\rangle}$ is d_ξ-closed as well. Berline-Vergne prove the following generalization of the DH-formula:

THEOREM 3.2. *Let M be a compact, oriented T-manifold with isolated fixed point set. Given $p \in M^T$ let $a_j(p)$ be the weights for the action on T_pM, defined with respect to some choice of T-invariant complex structure on T_pM. Suppose $\xi_M \neq 0$ on $M \backslash M^T$. Then for all forms $\alpha \in \Omega^*(M)$ such that $\mathrm{d}_\xi \alpha = 0$, one has the integration formula*

$$\int_M \alpha_{[\dim M]} = \sum_{p \in M^T} \frac{\alpha_{[0]}(p)}{\prod_j \langle a_j(p), \xi \rangle}.$$

In the proof we will use the useful notion of *real blow-ups*. Consider first the case of a real vector space V. Let

$$S(V) = V \backslash \{0\} / \mathbb{R}_{>0}$$

be its sphere, thought of as the space of rays based at 0. Define $\hat{V}$ as the subset of $V \times S(V)$,

$$\hat{V} := \{(v, x) \in V \times S(V) | \, v \text{ lies on the ray parametrized by } x\}.$$

Then $\hat{V}$ is a manifold with boundary. (In fact, if one introduces an inner product on V then $\hat{V} = S(V) \times \mathbb{R}_{\geq 0}$). There is a natural smooth map $\pi : \hat{V} \to V$ which is a diffeomorphism away from $S(V)$. If M is a manifold and $m \in M$, one can define its blow-up $\pi : \hat{M} \to M$ by using a coordinate chart based at m. Just as in the complex category, one shows that this is independent of the choice of chart (although this is actually not important for our purposes).

Suppose now that M is a T-space as above. Let $\pi : \hat{M} \to M$ be the manifold with boundary obtained by real blow-up at all the fixed points M^T. The T-action on M lifts to a T-action on $\hat{M}$ *with no fixed points*. In particular $\xi_{\hat{M}}$ has no zeroes. Choose an invariant Riemannian metric g on $\hat{M}$, and define

$$\theta := \frac{g(\xi_{\hat{M}}, \cdot)}{g(\xi_{\hat{M}}, \xi_{\hat{M}})} \in \Omega^1(\hat{M}).$$

Then θ satisfies $\iota(\xi_{\hat{M}})\theta = 1$ and $\mathrm{d}_\xi^2 \theta = L_{\xi_M} \theta = 0$. Therefore

$$\gamma := \frac{\theta}{\mathrm{d}_\xi \theta} = \frac{\theta}{\mathrm{d}\theta - 1} = -\theta \wedge \sum_j (\mathrm{d}\theta)^j$$

is a well-defined form satisfying $\mathrm{d}_\xi \gamma = 1$. The key idea of Berline-Vergne is to use this form for partial integration:

$$\begin{aligned}\int_M \alpha &= \int_{\hat{M}} \pi^* \alpha \\ &= \int_{\hat{M}} \pi^*\alpha \wedge \mathrm{d}_\xi \gamma \\ &= \int_{\hat{M}} \mathrm{d}_\xi (\pi^*\alpha \wedge \gamma) \\ &= \int_{\hat{M}} \mathrm{d}(\pi^*\alpha \wedge \gamma) \\ &= \sum_{p \in M^T} \int_{S(T_pM)} \pi^*\alpha \wedge \gamma \\ &= \sum_{p \in M^T} \alpha_{[0]}(p) \int_{S(T_pM)} \gamma\end{aligned}$$

Thus, to complete the proof we have to carry out the remaining integral over the sphere. We will do this by a trick, defining a d_ξ-closed form α where we can actually compute the integral by hand.

Consider the T-action on $T_pM = \sum_{j=1}^n \mathbb{C}_{a_j(p)}$ for a given $p \in M^T$. Introduce coordinates $r_j \geq 0, t_j \in [0,1]$ by $z_j = r_j \; e^{2\pi\sqrt{-1}t_j}$. Given $\epsilon > 0$ let $\chi \in C^\infty(\mathbb{R}_{\geq 0})$ be a cut-off function, with $\chi(r) = 1$ for $r \leq \epsilon$ and $\sigma = 0$ for $\epsilon \geq 2$. Define a form

$$\alpha = \prod_{j=1}^n (-\mathrm{d}_\xi(\chi(r_j)\,\mathrm{d}t_j)) = \prod_{j=1}^n \big(\langle a_j(p), \xi\rangle - \chi'(r_j)\mathrm{d}r_j \wedge \mathrm{d}t_j\big).$$

Note that this form is well-defined (even though the coordinates are not globally well-defined), compactly supported and d_ξ-closed. Its integral is equal to

$$\int_{T_pM} \alpha = \prod_{j=1}^n (-\chi'(r_j)\mathrm{d}r_j) = 1.$$

On the other hand $\alpha_{[0]} = \prod_{j=1}^n (\langle a_j(p), \xi\rangle)$.

Choosing ϵ sufficiently small, we can consider α as a form on M, vanishing at all the other fixed points. Applying the localization formula we find

$$1 = \int_M \alpha = \prod_{j=1}^n (\langle a_j(p), \xi\rangle) \int_{S(T_pM)} \gamma,$$

thus

$$\int_{S(T_pM)} \gamma = \frac{1}{\prod_{j=1}^n \langle a_j(p), \xi\rangle}.$$

Q.E.D.

The above discussion extends to non-isolated fixed points, in this case the product $\prod_{j=1}^{n}\langle a_j(p),\xi\rangle$ is replaced by the equivariant Euler class of the normal bundle of the fixed point manifold.

One often applies the Duistermaat-Heckman theorem in order to compute Liouville volumes of symplectic manifolds with Hamiltonian group action. Consider for example a Hamiltonian $S^1=\mathbb{R}/\mathbb{Z}$-action with isolated fixed points. Identify $\mathrm{Lie}(S^1)$, so that the integral lattice and its dual are just $\Lambda=\mathbb{Z}$, $\Lambda^*=\mathbb{Z}$. Let $H=\langle\Phi,\xi\rangle$ where ξ corresponds to $1\in\mathbb{R}$. By Duistermaat-Heckman,

$$\int_M e^{tH}\frac{\omega^n}{n!}=\frac{1}{t^n}\sum_{p\in M^{S^1}}\frac{e^{tH(p)}}{\prod_j a_j(p)}.$$

Notice by the way that the individual terms on the right hand side are singular for $t=0$. This implies very subtle relationships between the weight, for example one must have

$$\sum_{p\in M^{S^1}}\frac{H(p)^k}{\prod_j a_j(p)}=0$$

for all $k<n$. For the volume one reads off,

$$\mathrm{Vol}(M)=\frac{1}{n!}\sum_{p\in M^{S^1}}\frac{H(p)^n}{\prod_j a_j(p)}.$$

4. Frankel's theorem

As we have seen, Hamiltonian torus actions are very special in many respects: In particular they always have fixed points. It is a classical result of Frankel (long before moment maps were invented) that on Kähler manifolds the converse is true:

THEOREM 4.1. *Let M be a compact Kähler manifold, with Kähler form ω. Consider a symplectic S^1-action on M with at least one fixed point. Then the action is Hamiltonian.*

PROOF. Let $\dim M=2n$. We need one non-trivial result from complex geometry, which is a particular case of the hard Lefschetz theorem: Wedge product with ω^{n-1} induces an isomorphism in cohomology,

$$\wedge\,[\omega]^{n-1}:\ H^1(M)\cong H^{2n-1}(M).$$

Let $X\in\mathrm{Vect}(M)$ be the vector field corresponding to $1\in\mathbb{R}=\mathrm{Lie}(S^1)$. We need to show that $\iota_X\omega$ is exact. By hard Lefschetz, this is equivalent to showing that $\iota_X\omega^n$ is exact. Let $m\in M^{S^1}$ be a fixed point. In a neighborhood of m we can identify M as a T-space with T_mM. Let $\sigma\in\Omega^{2n}(T_mM)$ be an invariant form supported in an ϵ-ball around T_mM, normalized so that $\int_{T_mM}\sigma=\int_M\omega^n$. Choosing ϵ sufficiently small we can view σ as a form on M. Since σ and ω^n have the same integral, it follows that $\omega^n-\sigma=\mathrm{d}\beta$ for some invariant form $\beta\in\omega^{2n-1}M$.Then

$$\iota(X)(\sigma-\omega^n)=\iota(X)\mathrm{d}\beta=L_X\beta-\mathrm{d}\iota(X)\beta=-\mathrm{d}\iota(X)\beta,$$

showing that $\iota(X)(\sigma - \omega^n)$ is exact. We thus need to show that $\iota(X)\sigma$ is exact. This, however, follows from the Poincare lemma since it is supported in a ball around m, where one can just apply the homotopy operator. $\square$

5. Delzant spaces

DEFINITION 5.1. A Hamiltonian T-space (M, ω, Φ) with proper moment map Φ is called multiplicity-free if all reduced space M_μ are either empty or 0-dimensional. We call (M, ω, Φ) a *Delzant*-space if in addition M is connected, the moment map is proper, and the number of orbit type strata is finite.[1]

Thus, if T acts effectively, (M, ω, Φ) is Delzant if and only if $\dim M = 2\dim T$.

EXAMPLES 5.2.
(a) $M = \mathbb{C}^n$ with the standard action of $T = (S^1)^n$. The moment map image is the positive orthant $\mathbb{R}^n_+ \subset \mathbb{R}^n \cong \mathfrak{t}^*$. More abstractly, if V is a Hermitian vector space, the action of the maximal torus $T \subset U(V)$ on V os Delzant.
(b) $M = \mathbb{C}P(n)$ with the action of $T = (S^1)^{n+1}/S^1$ (quotient by diagonal subgroup) coming from the action of $(S^1)^{n+1}$ on $\mathbb{C}^{n+1}$. The moment map image is a simplex, given as the intersection of the positive orthant $\mathbb{R}^{n+1}_+$ with the hyperplane $\sum_{i=0}^n t_i = \pi$. More generally, if V is a Hermitian vector space, the action of the maximal torus $T \subset \mathrm{U}(V)$ on the projectivization $P(V)$ is Delzant.
(c) $M = T^*(T)$ with the cotangent lift of the left-action of T on itself. The moment map image is all of $\mathfrak{t}^*$. We will call this, from now on, the standard T-action on $T^*(T)$.
(d) Suppose (M, ω, Φ) is a Delzant T-space, and $H \subset T$ is a subgroup acting freely on the level set of $\mu \in \mathfrak{h}^*$. Then the H-reduced space $(M_\mu, \omega_\mu, \Phi_\mu)$ is Delzant. The moment map image $\Phi(M_\mu) \in \mathfrak{t}^*$ is the intersection of $\Phi(M)$ with the affine subspace $\mathrm{pr}_{\mathfrak{h}^*}^{-1}(\mu)$. We can view M_μ as a Delzant T/H-space, after choosing a moment map for the T/H-action; such a choice amounts to choosing a point in $\mathrm{pr}_{\mathfrak{h}^*}^{-1}(\mu)$.

The moment map images for Delzant spaces can be characterized as follows. Let $\Lambda \subset \mathfrak{t}$ be the integral lattice, i.e. the kernel of $\exp : \mathfrak{t} \to T$. Let $\Delta \subset \mathfrak{t}^*$ be a rational convex polyhedral set of dimension $d = \dim T$, with k boundary hyperplanes. That is, Δ is of the form

$$\Delta = \bigcap_{i=1}^k \mathcal{H}_{v_i,\lambda_i} \tag{12}$$

where $v_i \in \Lambda$ are primitive lattice vectors and $\lambda_i \in \mathbb{R}$, and

$$\mathcal{H}_{v_i,\lambda_i} = \{\mu \in \mathfrak{t}^* | \langle \mu, v_i \rangle \leq \lambda_i\}.$$

[1]The finiteness assumption is not very important, and is of course automatic if M is compact.

For any subset $I \subset \{1, \ldots, k\}$ let Δ_I be the set of all μ with $\langle \mu, v_i \rangle = \lambda_i$ for $i \in I$. We set $\Delta_\emptyset = \mathrm{int}(\Delta)$.

DEFINITION 5.3. The polyhedral set $\Delta \subset \mathfrak{t}^*$ is called *Delzant* if for all I with $\Delta_I \neq \emptyset$, the vectors v_i, $i \in I$ are linearly independent, and

$$\mathrm{span}_{\mathbb{Z}}\{v_i |\, i \in I\} = \Lambda \cap \mathrm{span}_{\mathbb{R}}\{v_i |\, i \in I\}.$$

REMARK 5.4. For compact polyhedral sets, (that is, polytopes) it is enough to check the Delzant condition at the vertices. The Delzant condition means in particular that each v_i has to be a *primitive* normal vector, i.e. is not of the form $v_i = a\, v_i'$ where $v_i' \in \Lambda$ and $a \in \mathbb{Z}_{>0}$.

EXAMPLE 5.5. Let $T = (S^1)^2$ and identify $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}^2$ and $\Lambda \cong \Lambda^* = \mathbb{Z}^2$. The polytope with vertices at $(0, 0), (0, 1), (1, 0)$ is Delzant. However, the polytope with vertices at $(0, 0), (0, 2), (1, 0)$ is not Delzant. Indeed, for the vertex at $(1, 0)$ the two primitive normal vectors are $v_1 = (0, -1)$ and $v_2 = (2, 1)$, and they do not span the lattice $\mathbb{Z}^2$.

The Delzant condition for $\Delta_I \neq \emptyset$ says that $\sum_{j \in I} s_j v_j \in \Lambda \Leftrightarrow s_j \in \mathbb{Z}$ for all $j \in I$, or equivalently,

$$\exp(\sum_{j \in I} s_j v_j) = 1 \Leftrightarrow s_j = 0 \bmod \mathbb{Z} \text{ for all } j \in I.$$

Thus if we define a homomorphism

$$\phi_\Delta :\ (S^1)^k \to T,\ [(s_1, \ldots, s_k)] \mapsto \exp(\sum_{i=1}^k s_i v_i)$$

and let

$$(S^1)^I = \{[(s_1, \ldots, s_k)] \in (S^1)^k |\ s_j = 0 \bmod \mathbb{Z} \ \text{ for } \ j \notin I\}$$

be the product of S^1-factors corresponding to indices $j \in I$, the Delzant condition is equivalent to saying that ϕ_Δ restricts to an inclusion $\phi_\Delta : (S^1)^I \hookrightarrow T$. The image $H_I = \phi_\Delta((S^1)^I) \subset T$ is obtained by exponentiating $\mathfrak{h}_I = \mathrm{span}_{\mathbb{R}}\{v_j |\ j \in I\}$; by definition it is the subspace perpendicular to $\Delta_I \subset \mathfrak{t}^*$.

THEOREM 5.6. *Let (M, ω, Φ) be a Delzant T-space with effective T-action. Then $\Delta = \Phi(M)$ is a Delzant polyhedron. For all open faces $F \subset \Delta$, the pre-image $\Phi^{-1}(F)$ is a connected component of the orbit type stratum $M_H \subset M$ for $H = \exp(\mathfrak{h}_F)$, where $\mathfrak{h}_F \subset \mathfrak{t}$ is the subspace perpendicular to F. In particular, all stabilizer groups are connected.*

PROOF. Let $\mu \in F$, $\mathcal{O} = T.m \in \Phi^{-1}(\mu)$ an orbit, and $H = T_m$ the stabilizer group. We had seen that the $\mathrm{cone}_\mu(\Delta)$ is equal to the local moment cone

$$C_m = \mu + (\mathrm{pr}_{\mathfrak{h}}^*)^{-1}(C),$$

where $C \subset \mathfrak{h}^*$ is the cone spanned by the weights $\alpha_1, \ldots, \alpha_k \in \mathfrak{h}^*$ for the H-action on the symplectic vector space $V = T_m(\mathcal{O})^\omega / T_m(\mathcal{O})$. By dimension count, $k = \dim_{\mathbb{C}} V =$

$\frac{1}{2}\dim M - \dim(T/H) = \dim H$. It follows that α_i are a basis of $\mathfrak{h}^*$. Since $\mathrm{ann}(\mathfrak{h}) \subset \mathfrak{t}^*$ is the maximal linear subspace inside the cone $(\mathrm{pr}_{\mathfrak{h}^*})^{-1}(C)$, it must coincide with the space parallel to F. That is, $\mathfrak{h} = \mathfrak{h}_F$.

The action of H on V must be effective since the T-action on E is effective. Thus H acts as a compact abelian subgroup of $\mathrm{U}(V)$ of dimension $\dim H = \dim_{\mathbb{C}} V$. So its identity component H_0 is a maximal torus. But it is a well-known fact from Lie group theory that maximal tori are maximal abelian, so $H = H_0$. In particular, we have shown that all points in $\Phi^{-1}(F)$ have the same stabilizer group.

It follows that the map $H \to (S^1)^k$ defined by the roots is an isomorphism. This means that $\alpha_1, \ldots, \alpha_k$ are a basis for the weight lattice weight lattice $(\Lambda \cap \mathfrak{h})^*$ in $\mathfrak{h}^*$. Equivalently, the dual basis $w_1, \ldots, w_k \in \mathfrak{h}$ are a basis for $\Lambda \cap \mathfrak{h}$. We have

$$C = \mathrm{cone}\{\alpha_1, \ldots, \alpha_n\} = \{\nu \in \mathfrak{h}^* |\, \langle \nu, w_i \rangle \geq 0\},$$

which identifies the $\{w_1, \ldots, w_k\}$ with $\{v_i |\; i \in I\}$. $\square$

Delzant gave an explicit recipe for constructing a Delzant space with moment polytope a given Delzant polyhedron. The following version of Delzant's construction is due to Eugene Lerman.

Let $(S^1)^k$ act on the cotangent bundle $T^*(T)$ via the composition of ϕ_Δ with the standard T-action on $T^*(T)$. In the left trivialization $T^*(T) = T \times \mathfrak{t}^*$, a moment map for the T-action is projection to $\mathfrak{t}^*$. Hence

$$\Psi_\Delta(t, \mu) = \sum_{j=1}^{k} \langle \mu, v_j \rangle e_j - \sum_j \lambda_j \, e_j$$

is a moment map for the action of $(S^1)^k$. Let $(S^1)^k$ act on $\mathbb{C}^k$ in the standard way, with moment map $\pi \sum_j |z_j|^2 \, e_j$.

Definition 5.7. For any polyhedron Δ let D_Δ be the symplectic quotient

$$D_\Delta = (T^*(T) \times \mathbb{C}^k) /\!\!/ (S^1)^k,$$

by the diagonal action, with T-action induced from the standard T-action on $T^*(T)$.

Theorem 5.8. *Suppose Δ is a Delzant polyhedron. Then the action of $(S^1)^k$ on the zero level set of $(T^*(T) \times \mathbb{C}^k)$ is free, and the quotient D_Δ is a Delzant-T-space. The moment map image of D_Δ is exactly Δ.*

Proof. Let $((t, \mu), z)$ in the zero level set. Thus

$$\langle \mu, v_i \rangle = \lambda_i - \pi |z_i|^2.$$

If $z_i \neq 0$ then the ith factor of $(S^1)^k$ acts freely at $((t, \mu), z)$. Thus we need only worry about the set I of indices i with $z_i = 0$. For these indices $\langle \mu, v_i \rangle = \lambda_i$. Let $(S^1)^I$ be the product of copies of S^1 corresponding to these indices. By the Delzant condition, ϕ_Δ restricts to an *embedding* $(S^1)^I \to T$. Since T acts freely on $T^*(T)$, so does $(S^1)^I$. This shows that the action is free, and D_Δ is a smooth symplectic manifold. To identify the image of the T-moment map note that, given $\mu \in \mathfrak{t}^*$, one can find t, z with $((t, \mu), z)$ is in the zero level set if and only if $\langle \mu, v_i \rangle \leq \lambda_i$. $\square$

Definition 5.9 (Lerman). Let Δ be a Delzant polyhedron, and (M, ω, Φ) a Hamiltonian T-space. The *cut space* defined by Δ is the symplectic quotient

$$M_\Delta = (M \times D_\Delta^-) /\!\!/ T$$

with T-action induced from the action on the first factor.

It is immediate that $T^*(T)_\Delta = D_\Delta$: In particular, $T^*(S^1)_{[0,\infty)} \cong \mathbb{C}$. We will now use these two facts to prove:

Theorem 5.10 (Delzant). *Every Delzant space (M, ω, Φ) is determined by its moment polyhedron $\Delta = \Phi(M)$, up to equivariant symplectomorphism intertwining the moment maps.*

Proof. Usually this is proved using a Čech theoretic argument. Below we sketch a more elementary (?) approach. The idea is to present M as a symplectic cut $\tilde{M}_\Delta$ of a connected, multiplicity free Hamiltonian T-space $\tilde{M}$ with *free* T-action. Since the action of T on $\tilde{M}$ is free, the map $\tilde{\Phi}$ is a Lagrangian fibration over its image. Thus we can introduce action-angle variables which identifies $\tilde{M}$ as an open subset of $T^*(T)$. Therefore, $M = \tilde{M}_\Delta = T^*(T)_\Delta = D_\Delta$.

We now indicate how to construct such a space $\tilde{M}$. Let $i_1 \in \{1, \ldots, k\}$ be an index such that $\Delta_{i_1} \neq 0$, and $S = \Phi^{-1}(\Delta_{i_1})$ the symplectic submanifold obtained as its preimage. It is a connected component of the fixed point set of H_{i_1}, and has codimension 2. Let $\nu_S = TS^\omega$ be its symplectic normal bundle. After choosing a compatible complex structure it can be viewed as a Hermitian line bundle. Let $Q \subset \nu_S$ be the unit circle bundle inside Q. It is a T-equivariant principal S^1-bundle, and $\nu_S = Q \times_{S^1} \mathbb{C}$. Let $\pi_Q : Q \to S$ be the projection map. Let $\alpha \in \Omega^1(Q)^T$ be a T-invariant connection 1-form, and consider the closed 2-form

$$\omega_{Q \times \mathbb{C}} := \pi_Q^* \omega_S + \omega_{\mathbb{C}} + \pi \mathrm{d}(|z|^2 \alpha).$$

It is easy to check that this 2-form is basic for the S^1-action, so it descends to a closed 2-form

$$\omega_{\nu_S} \in \Omega^2(\nu_S).$$

Furthermore, ω_{ν_S} is non-degenerate near $S = Q/S^1$. It follows that there exists an equivariant symplectomorphism between open neighborhoods of S in M and in ν_S. Now $\nu_S = (Q \times \mathbb{R})_{[0,\infty)}$ (cut with respect to the S^1-action), where $Q \times \mathbb{R}$ is equipped with the 2-form

$$\omega_{Q \times \mathbb{R}} = \pi_Q^* \omega_S + \omega_{\mathbb{C}} + \mathrm{d}(s\alpha).$$

We have a natural diffeomorphism between $Q \times \mathbb{R}_{>0}$ and $\nu_S \backslash S$, preserving 2-forms. We can thus glue $M \backslash S$ with a small neighborhood of Q in $Q \times \mathbb{R}_-$, to obtain a new connected multiplicity free Hamiltonian T-space (M_1, ω_1, Φ_1) with one orbit type stratum less. The original space is obtained from M_1 by cutting,

$$M = (M_1)_{\mathcal{H}_1}$$

where $\mathcal{H}_1$ is the affine half-space $\langle \mu, v_i \rangle \geq \lambda_i$. Continuing in this fashion, construct spaces $M_1, M_2, \ldots, M_n = \tilde{M}$ where n is the number of faces of Δ. We have

$$M = (M_1)_{\mathcal{H}_1} = (M_2)_{\mathcal{H}_1 \cap \mathcal{H}_2} = \ldots = (M_n)_{\Delta},$$

The final space $M_n = \tilde{M}$ no longer has 1-dimensional stabilizer groups, so the T-action is free as required. $\square$