

Notes on Moment Maps
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Preface

in preparation

The argument in the present notes is a more detailed version of [1, §2], except for (D_k) and the part of (A_k) beyond (A_k^*) ; see Remark 1.1 below.

Resources that the author found useful, but are not cited for specific statements, include [8].

1 Introduction and Overview

For a smooth manifold X , we denote by $\text{Diff}(X)$ the group of diffeomorphisms of X (with the product given by the composition of functions). For a symplectic manifold (X, ω) , we denote by $\text{Symp}(X, \omega)$ the group of symplectomorphisms of (X, ω) , i.e. diffeomorphisms ψ of X such that $\psi^*\omega = \omega$. A smooth action of a Lie group G on a smooth manifold X (resp. symplectic manifold (X, ω)) is a group homomorphism

$$\psi: G \longrightarrow \text{Diff}(X) \quad (\text{resp. } G \longrightarrow \text{Symp}(X, \omega)), \quad u \longrightarrow \psi_u, \quad (1.1)$$

such that the map

$$\Psi: G \times X \longrightarrow X, \quad \Psi(u, x) = \psi_u(x),$$

is smooth. For each $v \in T_{\mathbb{1}}G$,

$$\zeta_v \equiv d_{\mathbb{1}}\psi(v) \in \Gamma(X; TX), \quad \zeta_v(x) = d_{(\mathbb{1}, x)}\Psi(v, 0) \in T_x X \quad \forall x \in X, \quad (1.2)$$

is then a well-defined smooth vector field on X . For any action ψ as in (1.1), let

$$X^\psi \equiv \{x \in X : \psi_u(x) = x \quad \forall u \in G\}$$

denote its fixed locus. If in addition $x \in X$, let

$$G_x(\psi) \equiv \{u \in G : \psi_u(x) = x\} \quad (1.3)$$

denote the stabilizer of x in G .

An action ψ in (1.1) is called **effective** if the group homomorphism ψ is injective. It is called **irreducible** if the associated vector space homomorphism

$$d_{\mathbb{1}}\psi: T_{\mathbb{1}}G \longrightarrow \Gamma(X; TX), \quad v \longrightarrow \zeta_v,$$

is injective; otherwise, it is called **reducible**. An effective action is irreducible, but an irreducible action may have a discrete nontrivial kernel and thus not be effective.

Let G be a Lie group. A **moment map** for a smooth action of G on a symplectic manifold (X, ω) is a smooth map

$$\mu: X \longrightarrow T_{\mathbb{1}}^*G \quad \text{s.t.} \quad \begin{aligned} -d(\{\mu(\cdot)\})(v) &= \iota_{\zeta_v}\omega \equiv \omega(\zeta_v, \cdot) & \forall v \in T_{\mathbb{1}}G, \\ \mu(\psi_u(x)) &= \text{Ad}_{u^{-1}}^*(\mu(x)) & \forall x \in X, u \in G, \end{aligned} \quad (1.4)$$

where $\text{Ad}_{u^{-1}}^* : T_{\mathbb{1}}^*G \longrightarrow T_{\mathbb{1}}^*G$ is the dual of the adjoint action $\text{Ad}_{u^{-1}}$ of G on $T_{\mathbb{1}}G$; see [34, Sections 3.46]. On the left-hand side of the first equation in (1.4), $\{\mu(\cdot)\}(v)$ denotes the smooth function on X given by

$$\{\mu(\cdot)\}(v) : X \longrightarrow \mathbb{R}, \quad \{\{\mu(\cdot)\}(v)\}(x) = \{\mu(x)\}(v).$$

On the right-hand side of this equation, $\omega(\zeta_v, \cdot)$ denotes the 1-form on X given by

$$\omega(\zeta_v, \cdot) : TX \longrightarrow \mathbb{R}, \quad \{\omega(\zeta_v, \cdot)\}(w) = \omega(\zeta_v(x), w) \quad \forall x \in X, w \in T_x X.$$

If G is abelian, the second equation in (1.4) is equivalent to μ being G -invariant. A smooth G -action on (X, ω) that admits a smooth map $\mu : X \longrightarrow T_{\mathbb{1}}^*G$ satisfying the first condition in (1.4) (resp. admits a moment map) is called **weakly Hamiltonian** (resp. **Hamiltonian**). In such a case, μ is determined up to an additive constant (respectively, an additive constant fixed by the Ad^* -action of G on $T_{\mathbb{1}}^*G$). If G is connected, Exercise 3.13(a) implies that a smooth action of G on X that admits a smooth map $\mu : X \longrightarrow T_{\mathbb{1}}^*G$ satisfying the first condition in (1.4) is in fact an action on (X, ω) . By Proposition 3.25(3), such a map is G -invariant (and so is a moment map) if G is connected, abelian, and either $G \approx \mathbb{R}$ or either G or X is compact. If G is connected and X is compact or G is semisimple, a smooth map $\mu : X \longrightarrow T_{\mathbb{1}}^*G$ satisfying the first condition in (1.4) can be replaced by a moment map $\hat{\mu} : X \longrightarrow T_{\mathbb{1}}^*G$ for the G -action ψ on (X, ω) ; see Proposition 3.25(4) and Corollary 3.28, respectively. We will call a tuple (X, ω, ψ, μ) a **Hamiltonian G -manifold** if (X, ω) is a symplectic manifold, ψ is a smooth G -action on (X, ω) , and μ is a moment map for this action.

A **closed manifold** is a compact manifold without boundary. We call a closed subset Z of a smooth manifold X a **closed submanifold** if every topological component, i.e. a maximal connected subset, of Z is open in Z and is a smooth manifold without boundary smoothly embedded into X . In other words, every topological component of Z has an open neighborhood in X disjoint from the rest of Z and is a submanifold of X in the usual sense, but the dimensions of these submanifolds may not be the same. In such a case, the components of Z are also its path components; as usual, we denote the set of these components by $\pi_0(Z)$. We will call a Hamiltonian G -manifold (X, ω, ψ, μ) **closed** (resp. **connected**) if the manifold X is closed (resp. connected).

1.1 Convexity Theorem

The **convex hull** of a subset S of a vector space V is the subset

$$\text{CH}(S) \equiv \left\{ \sum_{i=1}^m r_i s_i : m \in \mathbb{Z}^+, s_1, \dots, s_m \in S, r_1, \dots, r_m \in \mathbb{R}^{\geq 0}, \sum_{i=1}^m r_i = 1 \right\} \subset V.$$

If $S \subset V$ is a finite subset of V , then $\text{CH}(S)$ is called a **polytope**. In such a case, we denote by $\text{Ver}(S)$ the set of **vertices** of $\text{CH}(S)$, i.e. the minimal subset of S so that $\text{CH}(\text{Ver}(S)) = \text{CH}(S)$. Since this subset of the polytope $P \equiv \text{CH}(S)$ is determined by P itself, we will also denote it by $\text{Ver}(P)$. The **dimension** of a polytope P is the dimension of the minimal affine subspace of V containing P . A (closed) **face** of a polytope P is the intersection P with the hyperplane $L^{-1}(c)$ for some nonzero linear functional $L : V \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$L(v) \geq c \quad \forall v \in P.$$

Such a face is the convex hull of $\text{Ver}(P) \cap L^{-1}(c)$ and thus is a polytope in itself. The interior P° of a polytope P is the complement of the proper faces of P in P . An **open face** of P is the interior of a face of P . An **edge** (resp. **facet**) of a polytope P is a face of P of dimension 1 (resp. codimension 1). We denote by $\text{Edg}(P)$ the set of edges of a polytope P and by $\text{Edg}_v(P) \subset \text{Edg}(P)$ for each $v \in \text{Ver}(P)$ the subset of the edges containing v . For $e \in \text{Edg}(P)$, we call $v_e \in V$ an **edge vector** for e if

$$P \cap \{v + tv_e : t \in \mathbb{R}\} = e \subset V$$

for a vertex $v \in \text{Ver}(e)$. A **full tuple of edge vectors** for a polytope P is an element $(v_e)_{e \in \text{Edg}(P)}$ of $V^{\text{Edg}(P)}$ so that v_e is an edge vector for each $e \in \text{Edg}_v(P)$.

Following [1], we call a smooth \mathbb{R}^k -action ψ on a smooth manifold X **almost periodic** if there exists a smooth action ψ' of a torus \mathbb{T} , i.e. a compact connected abelian Lie group, on X and a group homomorphism

$$\rho: \mathbb{R}^k \longrightarrow \mathbb{T} \quad \text{s.t.} \quad \psi = \psi' \circ \rho: \mathbb{R}^k \longrightarrow \text{Diff}(X). \quad (1.5)$$

If the image of ρ is dense in \mathbb{T} (which can be achieved by replacing \mathbb{T} with the closure of $\rho(\mathbb{R}^k)$), then $X^\psi = X^{\psi'}$. If in addition ψ preserves a symplectic form ω on X , then so does ψ' . This implies the existence of an \mathbb{R}^k -invariant Riemannian metric on X and thus of an \mathbb{R}^k -invariant ω -compatible almost complex structure on X in Theorem 1 below; see Exercises 3.7 and 3.14.

Theorem 1. *Suppose $k \in \mathbb{Z}^+$, (X, ω, ψ, μ) is a closed connected Hamiltonian \mathbb{R}^k -manifold, and the \mathbb{R}^k -action ψ is almost periodic.*

- (A_k) *The subset $\mu^{-1}(\alpha) \subset X$ is connected for every $\alpha \in T_0^* \mathbb{R}^k$.*
- (B_k) *The image $\mu(X) \subset T_0^* \mathbb{R}^k$ of X is a convex subset.*
- (C_k) *The ψ -fixed locus X^ψ is a closed symplectic submanifold of (X, ω) , $\mu|_Y$ is constant for each $Y \in \pi_0(X^\psi)$, and $\mu(X)$ is the convex hull of the finite subset $\mu(X^\psi) \subset T_0^* \mathbb{R}^k$ with at most $(\dim X)/2$ edges at each vertex.*
- (D_k) *The map $\mu: X \longrightarrow \mu(X)$ is open.*
- (E_k) *The components of a full tuple of edge vectors for the polytope $\mu(X)$ at any given vertex of $\mu(X)$ span $T_0^* \mathbb{R}^k$ if and only if the action ψ is irreducible.*
- (F_k) *If the action ψ is irreducible, then the subset $\text{Crit}(\mu)$ of points $x \in X$ so that $d_x \mu$ is not surjective is a finite union of (not necessarily disjoint) closed symplectic proper submanifolds of (X, ω) , and the image of each such submanifold under μ is contained in a hyperplane of $T_0^* \mathbb{R}^k$.*

The first claim in (C_k) is straightforward and holds for any smooth Lie group action on a symplectic manifold (X, ω) preserving a Riemannian metric on X ; see Remark 3.9 and the proof of Proposition 3.16(1). The second claim in (C_k) follows from the first and the observation that

$$X^\psi = \{x \in X : d_x \mu = 0\};$$

this identity is a consequence of (1.4) and Proposition 3.2(1). The first claim in (F_k) is a straightforward consequence of the equivariant splitting (3.14) of $TX|_Y$ for each $Y \in \pi_0(X^\psi)$.

The interesting parts of Theorem 1 are (A_k) , (B_k) , (D_k) , and the remaining claims in (C_k) and (F_k) . The fundamental reason behind these statements is the local form of the moment map μ provided by the first part of Corollary 3.40. This part of Corollary 3.40 is in a sense a Hamiltonian version of the Darboux Theorem. It ensures that (A_k) , (B_k) , and (D_k) hold locally and establishes (E_k) , the last claim in (F_k) , and Theorem 2(1) on page 7. We use the global, Morse-Bott theory statement of Proposition A.8 and the first, local statement of Proposition A.5 to obtain

(A_k^*) $\mu^{-1}(\alpha) \subset X$ is connected for every regular value $\alpha \in T_0^* \mathbb{R}^k$ of μ ,

via an induction of the dimension of the torus \mathbb{T} , as in [1, 23], and to deduce (D_k) from its local version. The remaining part of (A_k) follows from (A_k^*) , (D_k) , and (F_k) via Exercise A.10. Claim (B_k) of Theorem 1 is obvious for $k=1$, follows readily from (A_k) for $k \geq 2$, and leads to the last claim in (C_k) .

By the last claim in (C_k) , $\mu(X) \subset T_0^* \mathbb{R}^k$ is a polytope, called a **moment polytope** for the Hamiltonian action ψ on (X, ω) . It is well-defined up to translation. Since a torus \mathbb{T} is the quotient of a finite-dimensional vector space by a lattice, Theorem 1 immediately implies its statement with \mathbb{R}^k replaced by \mathbb{T} . In such a case, the moment polytope $\mu(X) \subset T_{\mathbb{T}}^* \mathbb{T}$ has additional properties; see Theorems 2 and 3 on pages 7 and 8, respectively. Figure 1 on page 16 shows moment polytopes for the two torus actions on $(\mathbb{C}P^2, \omega_{\text{FS};2})$ of Exercise 2.9.

Remark 1.1. The three parts of the statement of [1, Theorem 1] are (A_k) , (B_k) , and (C_k) in Theorem 1, while (F_k) is stated at the beginning of the proof of (A_k^*) and is justified at the end of Section 2 in [1]. The proof in [1] contains (at least) two gaps, both at the top of page 6.

(G1) It is claimed that (A_k^*) implies (A_k) *by continuity*. This is indeed the case for $k=1$; see Lemma A.9. However, this may not be the case for $k \geq 2$, even if the regular values are dense in the image (as is the case at the top of page 6). As an example, pinch the points $+1, -1$ of the unit circle $S^1 \subset \mathbb{R}^2$ to the origin (so that the preimage of $0 \in \mathbb{R}^2$ consists of two points, while the preimages of all other points contain at most one point). Replacing the circle with a 2-torus, we can ensure that the set of regular values is dense in the image. Statement (D_k) is the key property of μ needed for the *by continuity* claim in [1], but it does not even appear in [1], and neither does the local description of μ of Corollary 3.40 needed for this statement. Both play prominent roles in other approaches to the convexity theorem; see Section 1.4.

(G2) It is implicitly assumed that if $(c_1, \dots, c_k) \in \mathbb{R}^k$ is a regular value of an \mathbb{R}^k -valued smooth function (f_1, \dots, f_k) on a smooth manifold X , then $(c_1, \dots, c_{k-1}) \in \mathbb{R}^{k-1}$ is a regular value of (f_1, \dots, f_{k-1}) . This need not be the case, including in the setting on page 6 in [1].

Remark 1.2. The statements (B_k) and (C_k) in Theorem 1, with \mathbb{R}^k replaced by a torus \mathbb{T} , form [2, Theorem IV.4.3] and [23, Theorem 5.5.1]; [9, Theorem 27.1] includes (A_k) as well. The arguments in [2, 9, 23] generally follow [1], with [2] stating (A_k) as part of the proof and replicating the two gaps of Remark 1.1 in the middle of page 115 almost *verbatim*. In [23], only (A_k^*) is stated as part of the argument, making the issue (G1) in Remark 1.1 extraneous, while the gap (G2) is resolved. In order to deduce (B_k) from (A_k^*) , it is claimed in the last full sentence on page 239 in [23] that any two points in X (M in [23]) with the same value of a Hamiltonian ($A^{\text{tr}} \mu$ in [23]) can be approximated by two points in the preimage of a regular value of the Hamiltonian (the same regular value for both points). However, this is precisely what is needed to deduce (A_k) from (A_k^*) ,

as indicated by the proof of Lemma A.9. This property is implied by the Hamiltonian being an open map onto its image, i.e. (D_k) , as suggested by Exercise A.10, but neither the openness of the Hamiltonian nor its local description as in the first part of Corollary 3.40 is ever brought up in [23]. Thus, the attempt in [23] to bypass (G1) while establishing (B_k) contains fundamentally the same gap. In [9], the proof of (A_k) is relegated to Homework 21, which deals only with (A_k^*) , while resolving (G2) as in [23]; neither the openness of the Hamiltonian nor its local description is mentioned in [9] either. While (F_k) is also stated at the beginning of the proof of (A_k^*) in [23] with a note that it is established later in the proof, (F_k) is never addressed in [23]. The equivariant splitting (3.14) of $TX|_Y$ for each $Y \in \pi_0(X^\psi)$ needed to establish even the first claim in (F_k) does not appear anywhere in [23].

1.2 Kähler case

For a Lie group G and $v \in T_1 G$, let $e^v \in G$ be the exponential of v ; see [34, §3.30]. Define

$$(T_1 G)_\mathbb{Z} = \{v \in T_1 G : e^v = \{1\}\} \quad \text{and} \quad (T_1^* G)_\mathbb{Z} = \{\alpha \in T_1^* G : \alpha(v) \in \mathbb{Z} \ \forall v \in (T_1 G)_\mathbb{Z}\}.$$

If $G' \subset G$ is a Lie subgroup, then

$$(T_1 G')_\mathbb{Z} = (T_1 G)_\mathbb{Z} \cap T_1 G' \subset T_1 G$$

and the image of $(T_1^* G)_\mathbb{Z}$ under the restriction homomorphism $(T_1^* G)_\mathbb{Z} \rightarrow T_1^* G$ is contained in $(T_1^* G')_\mathbb{Z}$. If \mathbb{T} is a torus, then $(T_1 \mathbb{T})_\mathbb{Z} \subset T_1 \mathbb{T}$ and $(T_1^* \mathbb{T})_\mathbb{Z} \subset T_1^* \mathbb{T}$ are lattices, i.e. the homomorphisms

$$(T_1 \mathbb{T})_\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow T_1 \mathbb{T}, \quad v \otimes c \rightarrow cv, \quad \text{and} \quad (T_1^* \mathbb{T})_\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow T_1^* \mathbb{T}, \quad \alpha \otimes c \rightarrow c\alpha,$$

are isomorphisms of real vector spaces, and the map

$$T_1 \mathbb{T} / (T_1 \mathbb{T})_\mathbb{Z} \rightarrow \mathbb{T}, \quad [v] \rightarrow e^v,$$

is an isomorphism of Lie groups. We call $\alpha \in T_1^* \mathbb{T}$ **integral** if $\alpha \in (T_1^* \mathbb{T})_\mathbb{Z}$ and a line segment in $T_1^* \mathbb{T}$ **rational** if it is parallel to an integral element of $T_1^* \mathbb{T}$.

Exercise 1.3. Let \mathbb{T} be a torus.

- (a) Suppose that $V \subset T_1 \mathbb{T}$ is a linear subspace. Show that

$$e^V \equiv \{e^v : v \in V\} \subset \mathbb{T}$$

is a subtorus if and only if V is the \mathbb{R} -span of a finite subset of $(T_1 \mathbb{T})_\mathbb{Z}$.

- (b) Suppose $\mathbb{T}' \subset \mathbb{T}$ is a subtorus. Show that there exists a subtorus $\mathbb{T}'^c \subset \mathbb{T}$ so that the Lie group homomorphism

$$\mathbb{T}' \times \mathbb{T}'^c \rightarrow \mathbb{T}, \quad (u', u'^c) \rightarrow u' u'^c,$$

is an isomorphism. Conclude that the restriction homomorphism $(T_1^* \mathbb{T})_\mathbb{Z} \rightarrow (T_1^* \mathbb{T}')_\mathbb{Z}$ is surjective.

- (c) Suppose $S \subset (T_1^* \mathbb{T})_\mathbb{Z} - \{0\}$ and $\rho : \mathbb{R}^k \rightarrow \mathbb{T}$ is a homomorphism with dense image. Show that

$$\rho^* \alpha \equiv \alpha \circ d_0 \rho \neq 0 \in T_0^* \mathbb{R}^k \quad \text{and} \quad \rho^* \alpha \neq \rho^* \alpha' \quad \forall \alpha, \alpha' \in S, \alpha \neq \alpha'.$$

The complexification of a torus \mathbb{T} is the complex Lie group

$$\mathbb{T}_{\mathbb{C}} \equiv (T_{\mathbb{1}}\mathbb{T} \otimes_{\mathbb{R}} \mathbb{C}) / (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \otimes_{\mathbb{R}} \{1\}$$

with $T_{\mathbb{1}}\mathbb{T}_{\mathbb{C}} = T_{\mathbb{1}}\mathbb{T} \otimes_{\mathbb{R}} \mathbb{C}$. An almost complex structure on a smooth manifold X is an endomorphism J of the real vector bundle $TX \rightarrow X$ covering the identity on X so that $J^2 = -\text{Id}_{TX}$. A complexification of a smooth \mathbb{T} -action ψ on a symplectic manifold (X, ω) is a smooth $\mathbb{T}_{\mathbb{C}}$ -action on X ,

$$\begin{aligned} \psi_{\mathbb{C}}: \mathbb{T}_{\mathbb{C}} &\longrightarrow \text{Diff}(X), \quad u \longrightarrow \psi_{\mathbb{C};u}, \quad \text{s.t.} \\ d_{\mathbb{1}}\psi_{\mathbb{C}}(v+iv') &= d_{\mathbb{1}}\psi(v) + Jd_{\mathbb{1}}\psi(v') \in \Gamma(X; TX) \quad \forall v, v' \in T_{\mathbb{1}}\mathbb{T}, \end{aligned} \quad (1.6)$$

for some almost complex structure J on X compatible with ω and preserved by ψ , i.e.

$$\begin{aligned} \omega(w, Jw) &> 0 \quad \forall w \in TX, w \neq 0, \quad \omega(Jw, Jw') = \omega(w, w') \quad \forall w, w' \in TX, \quad \text{and} \\ \{d_x\psi_u\}^{-1} \circ J \circ d_x\psi_u &= J: T_x X \longrightarrow T_x X \quad \forall u \in \mathbb{T}, x \in X. \end{aligned}$$

If X is compact, the conditions (1.6) with $v=0$ determine \mathbb{R} -actions in the imaginary directions. However, these actions may not commute with ψ or each other and thus not give rise to a $\mathbb{T}_{\mathbb{C}}$ -action.

Theorem 2. *Suppose \mathbb{T} is a torus and (X, ω, ψ, μ) is a closed connected Hamiltonian \mathbb{T} -manifold.*

- (1) *All edges of the polytope $\mu(X) \subset T_{\mathbb{1}}^*\mathbb{T}$ are rational.*
- (2) *If $\psi_{\mathbb{C}}$ is a complexification of ψ , $x \in X$, and $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \subset X$ is the closure of the $\mathbb{T}_{\mathbb{C}}$ -orbit $\mathcal{O}_x(\psi_{\mathbb{C}}) \equiv \mathbb{T}_{\mathbb{C}}x$ of x , then*
 - (2a) $\text{Ver}(\mu(\{Y \in \pi_0(X^\psi): Y \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \neq \emptyset\})) = \mu(\{Y \in \pi_0(X^\psi): Y \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \neq \emptyset\});$
 - (2b) $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}) = \text{CH}(\mu(\{Y \in \pi_0(X^\psi): Y \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \neq \emptyset\}));$
 - (2c) *for every open face σ of the polytope $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})$, $\mu^{-1}(\sigma) \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ is a single $\mathbb{T}_{\mathbb{C}}$ -orbit;*
 - (2d) *the map $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}/\mathbb{T} \rightarrow \mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})$, $[x'] \rightarrow \mu(x')$, is a well-defined homeomorphism.*

If $\mathbb{T} \approx S^1$ or J is an integrable almost complex structure on X compatible with ω and preserved by a smooth \mathbb{T} -action ψ on (X, ω) , then (1.6) determines a complexification $\psi_{\mathbb{C}}$ of ψ ; see Exercise 4.1. In the latter case, Theorem 2(2) reduces to [1, Theorem 2], but the argument in [1] applies to the general case of Theorem 2(2). The crucial implication of (1.6) is that the action of the imaginary components (which correspond to the radial direction in \mathbb{C}^*) is given by the gradient flow of projections of μ to one-dimensional subspaces of $T_{\mathbb{1}}^*\mathbb{T}$; see (3.28).

Remark 1.4. As stated, [1, (3.6)] is wrong. For example, it fails if $C \subset N^u - N$ consists of a single point. However, [1, (3.6)] is used in the proof of Theorem 2(2) in [1] only to obtain the second statement in [1, (3.7)]. The latter is correct, as it follows from Proposition A.7(6), which corrects [1, (3.6)].

1.3 Symplectic toric manifolds

Let \mathbb{T} be a torus. A polytope $P \subset T_{\mathbb{1}}^*\mathbb{T}$ is Delzant if there exists a full tuple $(\alpha_e)_{e \in \text{Edg}(P)}$ of (integral) edge vectors for P such that for each vertex η of P the components α_e with $e \in \text{Edg}_{\eta}(P)$ form a \mathbb{Z} -basis for $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$; this property of P is typically called smoothness in the literature. In particular,

all edges of a Delzant polytope are rational. Furthermore, the number of edges containing any given vertex is the same as the dimension of \mathbb{T} ; this property of P is typically called **simplicity**. A symplectic toric \mathbb{T} -manifold is a connected Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) so that

$$\dim X = 2 \dim \mathbb{T}, \quad (1.7)$$

the action ψ on X is effective, and the moment map $\mu: X \rightarrow T_{\mathbb{1}}^*\mathbb{T}$ is proper. By Delzant's Theorem, Theorem 3 below, the map

$$(X, \omega, \psi, \mu) \longrightarrow \mu(X)$$

induces a bijection between the equivalence classes of compact symplectic toric \mathbb{T} -manifolds and Delzant polytopes in $T_{\mathbb{1}}^*\mathbb{T}$.

Theorem 3 ([13, Théorème 2.1, Section 3.2]). *Let \mathbb{T} be a torus.*

- (0) *If (X, ω, ψ, μ) is a compact symplectic toric \mathbb{T} -manifold, then the moment polytope $\mu(X) \subset T_{\mathbb{1}}^*\mathbb{T}$ is Delzant.*
- (1) *For every Delzant polytope $P \subset T_{\mathbb{1}}^*\mathbb{T}$, there exists a (necessarily compact) symplectic toric \mathbb{T} -manifold (X, ω, ψ, μ) with $\mu(X) = P$.*
- (2) *If (X, ω, ψ, μ) and $(X', \omega', \psi', \mu')$ are compact symplectic toric \mathbb{T} -manifolds with $\mu(X) = \mu(X')$, then there exists a \mathbb{T} -equivariant diffeomorphism*

$$\Phi: X \longrightarrow X' \quad \text{s.t.} \quad \omega = \Phi^*\omega', \quad \mu = \mu' \circ \Phi.$$

Claim 0 of this theorem is a consequence of the following description of the moment map $\mu: X \rightarrow T_{\mathbb{1}}^*\mathbb{T}$.

- (0⁺) If (X, ω, ψ, μ) is a compact symplectic toric \mathbb{T} -manifold, then there exist a subtorus $\mathbb{T}_F \subset \mathbb{T}$ for every face F of the polytope $\mu(X)$ and a full tuple $(\alpha_e)_{e \in \text{Edg}(P)}$ of integral edge vectors for $\mu(X)$ so that
 - (0⁺a) $\mathbb{T}_F = \{e^v: v \in T_{\mathbb{1}}\mathbb{T}, \alpha_e(v) = 0 \forall e \in \text{Edg}_\eta(F)\}$ for every face F of P and $\eta \in \text{Ver}(F)$;
 - (0⁺b) $\mathbb{T}_x(\psi) = \mathbb{T}_F$ for every face F of P and every $x \in \mu^{-1}(F^\circ)$;
 - (0⁺c) the restriction $\mu: \mu^{-1}(F^\circ) \rightarrow F^\circ$ is a principal \mathbb{T}/\mathbb{T}_F -bundle with ω -isotropic fibers, i.e. $\omega|_{T_{\mu^{-1}(\eta)}} = 0$ for every $\eta \in F^\circ$.

Similarly to Theorems 1(E_k) and 2(1), this statement is a consequence of the equivariant splitting (3.14) of $TX|_Y$ for each $Y \in \pi_0(X^\psi)$ and Corollary 3.40; we establish it at the end of Section 4.2. By (1.7) and (0⁺c), the dimension of $\mu(X)$ is the same as the dimension of \mathbb{T} and $\mathbb{T}_{\mu(X)} = \{\mathbb{1}\}$. Along with (0⁺a), the latter implies that for each vertex η of $\mu(X)$ the components α_e with $\eta \in e$ span $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ over \mathbb{Z} . Combining this with the last claim of Theorem 1(C_k) and (1.7) again, we conclude that these components form a \mathbb{Z} -basis for $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$.

Theorem 3(1) is readily obtained by applying the Hamiltonian symplectic cut of [21, Proposition 2.4], which is detailed in Section 5.2, to the Hamiltonian \mathbb{T} -manifold $(T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})$ of Exercise 2.14 with respect to the collection \mathcal{H} of half-spaces of $T_{\mathbb{1}}^*\mathbb{T}$ cutting out the polytope P . We establish Theorem 3(2) by describing a reverse of the Hamiltonian symplectic cut in Section 5.3; this implements the argument sketched in the proof of [24, Theorem 7.5.10].

1.4 Alternative approaches to Convexity Theorem

Other approaches to Theorem 1(C_k) on page 4, with \mathbb{R}^k replaced by a torus \mathbb{T} , have appeared in particular in [17, 18, 5, 6]. In contrast to [1, 2, 23, 9], they clearly emphasize the significance of the local form of the moment map μ as in Corollary 3.40 and make it possible to relax the compactness condition on X to the properness of μ . These approaches differ in how they pass from the local versions of (A_k), (B_k), and (D_k) implied by this local form to the global versions appearing in Theorem 1. The global, Morse-Bott theory statement of Proposition A.8 and the first, local statement of Proposition A.5 used in [1, 2, 23, 9] to inductively confirm (A_k^*) on page 5 is used in [17, §5] instead to deduce (B_k) from its local version and to establish (C_k); the remaining statements of Theorem 1 do not appear in [17]. An enlightening summary of the reasoning in [17] appears in [30].

The most succinct approach to passing from the local properties provided by the first part of Corollary 3.40 to the global statements of Theorem 1 is arguably presented in [6]. It is motivated by the following point set topology result from the 1920s.

Proposition 1.5 (Tietze-Nakajima Theorem, [33, Satz 1], [6, Theorem 1]). *A closed connected locally convex subset of \mathbb{R}^k is convex.*

Remark 1.6. Nakajima's paper [29] typically credited for Proposition 1.5 contains several statements in the same spirit, most concerning subsets of \mathbb{R}^2 and \mathbb{R}^3 , but not the actual statement of this proposition.

Let $f: X \rightarrow V$ be a continuous map between topological spaces. Such a map is fiber connected if $f^{-1}(v) \subset X$ is connected for every $v \in V$. If in addition V is a vector space, f is convex if for any $x_0, x_1 \in X$ there exists a path

$$\gamma: ([0, 1], 0, 1) \rightarrow (X, x_0, x_1)$$

from x_0 to x_1 in X such that the map $f \circ \gamma$ is fiber connected and $f(\gamma([0, 1]))$ is contained in the line segment from $f(x_0)$ to $f(x_1)$ in V . The conditions on γ mean that the path

$$f \circ \gamma: ([0, 1], 0, 1) \rightarrow (V, f(x_0), f(x_1))$$

traces the line segment from $f(x_0)$ to $f(x_1)$ in V without reversing the direction at any point in time.

Proposition 1.7 ([6, Theorem 15]). *Suppose X is a connected Hausdorff topological space, V is a finite-dimensional vector space, and $f: X \rightarrow V$ is a proper continuous map. If for every $x \in X$ there exists an open neighborhood $U \subset X$ of x such that $f|_U$ is convex and $f: U \rightarrow \Phi(U)$ is open, then f is a convex map and $f: X \rightarrow f(X)$ is an open map. In particular, f is fiber connected and $f(X) \subset V$ is convex.*

Taking $V = \mathbb{R}^k$, $X \subset \mathbb{R}^k$ to be a closed connected locally convex subset, and $f: X \rightarrow V$ to be the inclusion in this proposition, we recover Proposition 1.5; the closedness of X implies the properness of f . By Exercises 1.8-1.10 below, Corollary 3.40 implies that the moment map μ of Theorem 1 satisfies the local condition on f in Proposition 1.7.

Exercise 1.8. Suppose $f_i: X_i \rightarrow V_i$ for $i=1, 2$ are convex maps. Show that the map

$$f_1 \times f_2: X_1 \times X_2 \rightarrow V_1 \times V_2$$

is also convex.

Exercise 1.9. Suppose $k, m \in \mathbb{Z}^{\geq 0}$, $f: \mathbb{R}^k \rightarrow \mathbb{R}^m$ is a smooth function so that the differential $d_0 f$ is surjective, and \mathcal{U} is a neighborhood of 0 in \mathbb{R}^k . Show that f is open and convex on some neighborhood \mathcal{U}' of 0 in \mathcal{U} .

Exercise 1.10. Suppose $m \in \mathbb{Z}^{\geq 0}$, $S \subset \mathbb{R}^m$ is a finite subset,

$$f: \mathbb{C}^S \rightarrow \mathbb{R}^m, \quad f((w_\alpha)_{\alpha \in S}) = \sum_{\alpha \in S} |w_\alpha|^2 \alpha,$$

and \mathcal{U} is a neighborhood of 0 in \mathbb{C}^S . Show that the map

$$f: \mathbb{C}^S \rightarrow \mathcal{C}_0(S) \equiv \left\{ \sum_{\alpha \in S} t_\alpha \alpha : t_\alpha \in \mathbb{R}^{\geq 0} \forall \alpha \in S \right\}$$

is open and f is convex on some neighborhood \mathcal{U}' of 0 in \mathcal{U} .

Proposition 1.7 is a variation on the purely topological local-to-global theorems of [18, 19, 4, 5], which do not involve geometric input as in Propositions A.8 and A.5 used in [1, 17, 2, 23, 9]. The conditions on the continuous function f in [19, Theorem 3.10] and [5, Theorem 2.28], for example, are arguably more ad hoc, explicitly involving an assignment of a cone in the target vector space V of f to each point in the domain topological space X and thus fitting more closely with the output of Corollary 3.40; see Proposition 1.11 below.

Let V be a vector space. For $S \subset V$ and $v \in S$, define

$$\mathbb{R}^+(S-v) = \{r(v'-v) : v' \in S, r \in \mathbb{R}^+\}, \quad L_v(S) = \overline{\mathbb{R}^+(S-v)} \subset V.$$

A subset $S \subset V$ is **locally polyhedral** if for every $v \in S$ there exists a neighborhood $U \subset V$ of v such that

$$S \cap U = \{v+v' : v' \in L_v(S)\} \cap U.$$

A closed convex subset $S \subset V$ is a **(closed) cone with vertex** at $v \in V$ if $v+t(v'-v) \in S$ whenever $v' \in S$ and $t \in \mathbb{R}^+$. Such a subset is locally polyhedral. If $S \neq V$ is a cone and V is a finite-dimensional vector space, then S is contained in a **(closed) half-space**, i.e.

$$S \subset \{w \in V : L(w) \geq c\}$$

for some nonzero linear functional $L: V \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$.

Let $f: X \rightarrow V$ be a continuous map between topological spaces. Such a map is **locally fiber connected** if for every $x \in X$ and open neighborhood $U \subset X$ of x there exists an open neighborhood $U' \subset U$ such that $f|_{U'}$ is fiber connected. If in addition V is a finite-dimensional vector space, a tuple $(\mathcal{C}_x)_{x \in X}$ of closed convex cones in V based at 0 is called **local convexity data** if for every $y \in X$ and an open neighborhood $U \subset X$ of y there exists an open neighborhood $U_y \subset U$ such that $f|_{U_y}$ is fiber connected, $f(x) - f(y) \in \mathcal{C}_y$ for every $x \in U_y$, and

$$U_y \rightarrow \mathcal{C}_y, \quad x \rightarrow f(x) - f(y),$$

is an open map.

Proposition 1.11 ([18, Theorem 3.4],[19, Theorem 3.10]). *Suppose X is a connected Hausdorff topological space, V is a finite-dimensional vector space, and $f: X \rightarrow V$ is a proper locally fiber connected map. If f admits local convexity data $(\mathcal{C}_x)_{x \in X}$, then f is a fiber connected map, $f(X) \subset V$ is a closed convex locally polyhedral subset, $f: X \rightarrow f(X)$ is an open map, and $\mathcal{C}_x = L_{f(x)}(f(X))$ for every $x \in X$.*

For a topological space X , $A \subset X$, and $x \in A$, the connected component of A containing x is the maximal connected subset $A_x \subset A$ containing x . For a continuous map $f: X \rightarrow V$ between reasonable topological spaces, the Reeb quotient space,

$$X_f \equiv X / \sim, \quad x \sim x' \text{ if } f(x) = f(x') \in V, \quad (f^{-1}(f(x)))_x = (f^{-1}(f(x)))_{x'} \subset X,$$

is Hausdorff; see [14, Theorem 4.5]. A continuous map $f: X \rightarrow V$ is fiber connected if and only if the induced map

$$X_f \rightarrow V, \quad [x] \rightarrow f(x),$$

is injective. This perspective on fiber connectivity appearing in [12] provides the motivation for the proof of Proposition 1.11 in [19].

2 Preliminaries

2.1 Notation and terminology

For $k \in \mathbb{Z}^{\geq 0}$, let $[k] = \{1, 2, \dots, k\}$. If $Y \subset X$ is a smooth submanifold of a smooth manifold, let

$$\mathcal{N}_X Y \equiv TX|_Y / TY \rightarrow Y$$

denote the normal bundle of Y in X . If $H: X \rightarrow \mathbb{R}$ is a smooth function, we denote by

$$\text{Crit}(H) \equiv \{x \in X : d_x H = 0\}$$

its set of critical points. The gradient of H with respect to a Riemannian metric g on X is the vector field $\nabla^g H$ on M defined by

$$g(\nabla^g H|_x, w) = d_x H(w) \quad \forall x \in X, w \in T_x X. \quad (2.1)$$

Let X be a smooth manifold and G be a Lie group. For a map $\mu: X \rightarrow T_{\mathbb{1}}^* G$ and $v \in T_{\mathbb{1}} G$, define

$$\mu_v: X \rightarrow \mathbb{R}, \quad \mu_v(x) = \{\mu(x)\}(v). \quad (2.2)$$

A basis v_1, \dots, v_k for $T_{\mathbb{1}} G$ determines identifications

$$\mathbb{R}^k \rightarrow T_{\mathbb{1}} G, \quad (r_1, \dots, r_k) \rightarrow \sum_{i=1}^k r_i v_i, \quad \text{and} \quad T_{\mathbb{1}}^* G \rightarrow \mathbb{R}, \quad \alpha \rightarrow (\alpha(v_1), \dots, \alpha(v_k)). \quad (2.3)$$

The latter isomorphism identifies smooth (G -invariant) maps $\mu: X \rightarrow T_{\mathbb{1}}^* G$ with smooth (G -invariant) maps $H: X \rightarrow \mathbb{R}^k$ by

$$\mu \longleftrightarrow H \equiv (\mu_{v_1}, \dots, \mu_{v_k}). \quad (2.4)$$

A smooth (G -invariant) map $\mu: X \rightarrow T_{\mathbb{1}}^*G$ satisfying the first condition in (1.4) with respect to a smooth action ψ of G on a symplectic manifold (X, ω) corresponds via (2.4) to a smooth (G -invariant) map

$$H \equiv (H_1, \dots, H_k): X \rightarrow \mathbb{R}^k \quad \text{s.t.} \quad dH_i = -\iota_{\zeta_{v_i}} \omega \quad \forall i \in [k], \quad (2.5)$$

where $\zeta_{v_i} = d_{\mathbb{1}}\psi(v_i) \in \Gamma(X; TX)$.

We call such a smooth function a **weak Hamiltonian** for the action ψ on (X, ω) with respect to basis v_1, \dots, v_k for $T_{\mathbb{1}}G$. If H corresponds to a moment map via (2.4), then we call it a **Hamiltonian** for the action ψ on (X, ω) with respect to basis v_1, \dots, v_k for $T_{\mathbb{1}}G$. If G is a connected abelian Lie group, a weak Hamiltonian is a Hamiltonian if and only if it is G -invariant.

We denote by $e_1, \dots, e_k \in \mathbb{R}^k$ the standard orthonormal basis and by $\mathbb{R}_i \subset \mathbb{R}^k$ the \mathbb{R} -span of e_i . A smooth action ψ of \mathbb{R}^k on a smooth manifold X (resp. symplectic manifold (X, ω)) is equivalent to k commuting smooth \mathbb{R} -actions $\psi_i \equiv \psi|_{\mathbb{R}_i}$ on X (resp. (X, ω)). In such a case, we will call

$$\zeta_i \equiv d_{\mathbb{1}}\psi(e_i) = \left. \frac{d}{dt} \psi_{i;t} \right|_{t=0} \in \Gamma(X; TX), \quad i \in [k], \quad (2.6)$$

the **generating vector fields** of ψ and a **Hamiltonian** H as in (2.5) with respect to the basis e_1, \dots, e_k for $T_0\mathbb{R}^k$ simply a **Hamiltonian** for ψ . If $k=1$ or X is compact, such a Hamiltonian corresponds to k smooth functions $H_i: X \rightarrow \mathbb{R}$ satisfying the condition in (2.5) with $\zeta_{v_i} = \zeta_i$; see Proposition 3.25(3).

Suppose \mathbb{T} is a torus and $v_1, \dots, v_k \in T_{\mathbb{1}}\mathbb{T}$ is a \mathbb{Z} -basis for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \subset T_{\mathbb{1}}\mathbb{T}$ and thus an \mathbb{R} -basis for $T_{\mathbb{1}}\mathbb{T}$. The isomorphisms in (2.3) then identify $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}} \subset T_{\mathbb{1}}^*\mathbb{T}$ with $\mathbb{Z}^k \subset \mathbb{R}^k$. The first isomorphism in (2.3) also induces a Lie group identification of \mathbb{T} with the standard torus \mathbb{T}^k ,

$$\phi_{v_1 \dots v_k}: \mathbb{T}^k \equiv (\mathbb{R}/\mathbb{Z})^k = \mathbb{R}^k / \mathbb{Z}^k \rightarrow \mathbb{T}, \quad \phi_{v_1 \dots v_k}([r_1, \dots, r_k]) = \prod_{i=1}^k e^{r_i v_i}. \quad (2.7)$$

Exercise 2.1. Suppose \mathbb{T} is a torus and $v_1, \dots, v_k \in T_{\mathbb{1}}\mathbb{T}$ is \mathbb{Z} -basis for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \subset T_{\mathbb{1}}\mathbb{T}$. Let $\alpha_1, \dots, \alpha_k \in (T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ be the dual basis. For $u \in \mathbb{T}$, let $m_u: \mathbb{T} \rightarrow \mathbb{T}$ be the multiplication by u . Show that the diffeomorphism

$$\Phi_{v_1 \dots v_k}: \mathbb{R}^k \times \mathbb{T}^k \rightarrow T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}, \quad \Phi_{v_1 \dots v_k}((x_1, \dots, x_k), [r]) = \left(\sum_{i=1}^k x_i \alpha_i, \phi_{v_1 \dots v_k}([r]) \right), \quad (2.8)$$

is equivariant with respect to the identification $\phi_{v_1 \dots v_k}$ in (2.7) and satisfies

$$d\pi_2(\{d\Phi_{v_1 \dots v_k}\}(\partial_{x_i})) = 0 \quad \text{and} \quad \{\pi_1(\Phi_{v_1 \dots v_k}((x_1, \dots, x_k), [r]))\} \left(d_{\phi_{v_1 \dots v_k}([r])} m_{\phi_{v_1 \dots v_k}([-r])} (d\pi_2(\{d\Phi_{v_1 \dots v_k}\}(\partial_{r_i}))) \right) = x_i \quad \forall i \in [k], \quad (2.9)$$

where $\pi_1, \pi_2: T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T} \rightarrow T_{\mathbb{1}}^*\mathbb{T}, \mathbb{T}$ are the projections, ∂_{x_i} is the i -th coordinate vector field on \mathbb{R}^k , and ∂_{r_i} is the coordinate vector field on \mathbb{T}^k induced by the i -th coordinate vector field on \mathbb{R}^k .

We take the standard \mathbb{Z} -basis for $(T_{\mathbb{1}}\mathbb{T}^k)_{\mathbb{Z}}$ to be

$$2\pi i e_1, \dots, 2\pi i e_k \in T_{\mathbb{1}}\mathbb{T}^k \subset T_{(1, \dots, 1)}\mathbb{C}^k.$$

A smooth action ψ of \mathbb{T}^k on a smooth manifold X (resp. symplectic manifold (X, ω)) is equivalent to k commuting smooth S^1 -actions $\psi_i \equiv \psi|_{S^1_i}$ on X (resp. (X, ω)), where $S^1_i \subset \mathbb{T}^k = (S^1)^k$ is the i -th component subgroup S^1 . Similarly to the affine case, we then call

$$\zeta_i \equiv d_{\mathbb{1}}\psi(2\pi i e_i) = \left. \frac{d}{dt} \psi_{i; e^{2\pi i t}} \right|_{t=0} \in \Gamma(X; TX), \quad i \in [k],$$

the **generating vector fields** of ψ and a Hamiltonian H as in (2.5) with respect to the basis $2\pi i e_1, \dots, 2\pi i e_k$ for $T_{\mathbb{1}}\mathbb{T}^k$ simply a Hamiltonian for ψ . By Proposition 3.25(3), such a Hamiltonian corresponds to k smooth functions $H_i: X \rightarrow \mathbb{R}$ satisfying the condition in (2.5) with $\zeta_{v_i} = \zeta_i$.

Exercise 2.2. Suppose ψ is a smooth action of a Lie group G on a manifold X as in (1.1), ω is a symplectic form on X , $\mu: X \rightarrow T_{\mathbb{1}}^*G$ is a smooth map, and $\rho: G' \rightarrow G$ is a Lie group homomorphism. Let

$$\rho^* \equiv (d_{\mathbb{1}}\rho)^*: T_{\mathbb{1}}^*G \rightarrow T_{\mathbb{1}}^*G'$$

the homomorphism induced by ρ . Show that

- (a) $\psi \circ \rho$ is a smooth action of G' on X and the map $\rho^* \circ \mu: X \rightarrow T_{\mathbb{1}}^*G'$ is smooth;
- (b) if the action ψ preserves ω , so does the action $\psi \circ \rho$;
- (c) if the map μ satisfies the first (resp. second) condition in (1.4), then the map $\rho^* \circ \mu$ satisfies the first (resp. second) condition in (1.4) with (ψ, μ) replaced by $(\psi \circ \rho, \rho^* \circ \mu)$.

Exercise 2.3. Suppose ψ is a smooth \mathbb{R}^k -action on a manifold X as in (1.1), ω is a symplectic form on X , $H: X \rightarrow \mathbb{R}^k$ is a smooth map, and A is a real $k \times m$ -matrix (determining a linear map from \mathbb{R}^m to \mathbb{R}^k). Show that

- (a) $\psi \circ A$ is a smooth \mathbb{R}^m -action on X and the map $A^{\text{tr}} \circ H: X \rightarrow \mathbb{R}^m$ is smooth;
- (b) if the action ψ preserves ω , so does the action $\psi \circ A$;
- (c) if the map H is a weak Hamiltonian for the action ψ on (X, ω) (resp. \mathbb{R}^k -invariant), then the map $A^{\text{tr}} \circ H$ is a weak Hamiltonian for the action $\psi \circ A$ on (X, ω) (resp. \mathbb{R}^m -invariant).

Exercise 2.4. Suppose ψ is a smooth action of a Lie group G on a manifold (X, ω) as in (1.1), ω is a symplectic form on X , $\mu: X \rightarrow T_{\mathbb{1}}^*G$ is a smooth map, and $G' \subset G$ is a subgroup acting trivially on X . Let $q: G \rightarrow G/G'$ be the quotient projection. Show that

- (a) if $G' \subset G$ is a normal subgroup, then the action ψ of G determines a smooth action ψ' of the Lie group G/G' on X so that $\psi = \psi' \circ q$;
- (b) if the action ψ preserves ω and $G' \subset G$ is a normal subgroup, then the induced action ψ' of G/G' on X also preserves ω ;
- (c) if X is connected and the map μ satisfies the first condition in (1.4), then

$$\alpha|_{T_{\mathbb{1}}G'} = \alpha_0|_{T_{\mathbb{1}}G'} \quad \forall \alpha, \alpha_0 \in \mu(X);$$

- (d) if X is connected, the map μ satisfies the first condition in (1.4), $G' \subset G$ is a normal subgroup, and $\alpha_0 \in \mu(X)$, then the smooth map

$$\mu - \alpha_0: X \longrightarrow \{\alpha \in T_{\mathbb{1}}^*G: \alpha|_{T_{\mathbb{1}}G'} = 0\} = T_{\mathbb{1}}(G/G') \quad (2.10)$$

satisfies the first condition in (1.4) with (ψ, μ) replaced by $(\psi', \mu - \alpha_0)$;

- (e) if (X, ω, ψ, μ) is a Hamiltonian G -manifold, $G' \subset G$ is a normal subgroup, and

$$\alpha_0 \in (T_{\mathbb{1}}^*G)^G \equiv \{\alpha \in T_{\mathbb{1}}^*G: \text{Ad}_u^*(\alpha) = \alpha \forall u \in G\}$$

satisfies (2.10), then $(X, \omega, \psi', \mu - \alpha_0)$ is a Hamiltonian G/G' -manifold;

- (f) if G is abelian, (X, ω, ψ, μ) is a Hamiltonian G -manifold, and $\alpha_0 \in T_{\mathbb{1}}^*G$ satisfies (2.10), then $(X, \omega, \psi', \mu - \alpha_0)$ is a Hamiltonian G/G' -manifold.

Exercise 2.5. Suppose $\tilde{\psi}$ and ψ are smooth actions of a Lie group G on manifolds \tilde{X} and X , respectively, $\tilde{Y} \subset \tilde{X}$ is a smooth submanifold preserved by $\tilde{\psi}$, $p: \tilde{Y} \rightarrow X$ is a G -equivariant surjective submersion, $\tilde{\omega}$ and ω are symplectic forms on \tilde{X} and X , respectively, such that $p^*\omega = \tilde{\omega}|_{T\tilde{Y}}$, and

$$\tilde{\mu}: \tilde{X} \longrightarrow T_{\mathbb{1}}^*G \quad \text{and} \quad \mu: X \longrightarrow T_{\mathbb{1}}^*G$$

are maps such that $\mu \circ p = \tilde{\mu}|_{\tilde{Y}}$. Show that

- (a) if the action $\tilde{\psi}$ preserves $\tilde{\omega}$, then the action ψ also preserves ω ;
(b) if the map $\tilde{\mu}$ is smooth, then the map μ is also smooth;
(c) if the map $\tilde{\mu}$ is smooth and satisfies the first (resp. second) condition in (1.4) with (X, ω, ψ, μ) replaced by $(\tilde{X}, \tilde{\omega}, \tilde{\psi}, \tilde{\mu})$, then the map μ is also smooth and satisfies the first (resp. second) condition in (1.4) as stated.

2.2 Paradigmatic examples

Exercise 2.6. Let $k, n \in \mathbb{Z}^+$ with $k \leq n$ and

$$\omega_{\mathbb{C}^n} \equiv \sum_{i=1}^n dx_i \wedge dy_i \quad (2.11)$$

be the standard symplectic form on $\omega_{\mathbb{C}^n}$. Show that the action of S^1 on \mathbb{C}^n given by

$$e^{2\pi it} \cdot (z_1, \dots, z_n) = (z_1, \dots, z_{k-1}, e^{2\pi it} z_k, z_{k+1}, \dots, z_n)$$

is Hamiltonian with respect to $\omega_{\mathbb{C}^n}$ with a Hamiltonian

$$H: \mathbb{C}^n \longrightarrow \mathbb{R}, \quad H(z_1, \dots, z_n) = \pi |z_k|^2.$$

Exercise 2.7. With n and $\omega_{\mathbb{C}^n}$ as in Exercise 2.6, let ψ be the action of \mathbb{C}^n on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ by translations, i.e.

$$\psi_u: \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad \psi_u(z) = z + u, \quad \forall u \in \mathbb{C}^n.$$

Let e_1, \dots, e_n be the standard \mathbb{C} -basis for \mathbb{C}^n and $z_k = x_k + iy_k$ be the split of the k -th complex coordinate on \mathbb{C}^n into real ones. Show that

$$H \equiv (-y_1 + a_1, x_1 + b_1, \dots, -y_n + a_n, x_n + b_n) : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n} \quad (2.12)$$

is a weak Hamiltonian for the action ψ of \mathbb{C}^n on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ with respect to the \mathbb{R} -basis $e_1, ie_1, \dots, e_n, ie_n$ for $T_0\mathbb{C}^n = \mathbb{C}^n$ for any $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$ and that every weak Hamiltonian for the action ψ of \mathbb{C}^n on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ with respect to this basis is given by (2.12) for some $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$. Conclude that the action ψ of \mathbb{C}^n on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ is not Hamiltonian.

Exercise 2.8. Suppose (V, i) is a finite-dimensional complex vector space and Ω is a nondegenerate 2-form on V compatible with i , i.e.

$$\Omega(w, iw) > 0 \quad \forall w \in V - \{0\}, \quad \Omega(iw, iw') = \Omega(w, w') \quad \forall w, w' \in V.$$

Via the canonical identification $T_w V \approx V$ for each $w \in V$, i and Ω determine an almost complex structure J on V and a symplectic form ω compatible with J . Let $\psi : \mathbb{T} \longrightarrow \mathrm{GL}_{\mathbb{C}} V$ be a complex representation of a torus \mathbb{T} on V . Show that

(a) there exist a subset $S(Y) \subset (T_{\mathbb{1}}^* \mathbb{T})_{\mathbb{Z}}$ and a splitting

$$V = \bigoplus_{\alpha \in S} V_{\alpha} \quad \text{s.t.} \quad \psi_{e^v}((w_{\alpha})_{\alpha \in S}) = (e^{2\pi i \alpha(v)} w_{\alpha})_{\alpha \in S} \quad \forall v \in T_{\mathbb{1}} \mathbb{T}, (w_{\alpha})_{\alpha \in S} \in \bigoplus_{\alpha \in S} V_{\alpha};$$

(b) the action ψ is Hamiltonian with respect to ω with a moment map

$$\mu : V \longrightarrow T_{\mathbb{1}}^* \mathbb{T}, \quad \mu((w_{\alpha})_{\alpha \in S}) = \pi \sum_{\alpha \in S} |w_{\alpha}|^2 \alpha \quad \forall (w_{\alpha})_{\alpha \in S} \in \bigoplus_{\alpha \in S} V_{\alpha},$$

where $|\cdot|$ is the norm on V with respect to the metric $g(\cdot, \cdot) \equiv \Omega(\cdot, i\cdot)$.

Exercise 2.9. Let $n \in \mathbb{Z}^+$ and $q : \mathbb{C}^n - \{0\} \longrightarrow \mathbb{C}P^{n-1}$ be the usual quotient projection.

(a) Suppose $U \subset \mathbb{C}P^{n-1}$ is an open subset and $s : U \longrightarrow \mathbb{C}^n - \{0\}$ is a holomorphic section of q , i.e. $q \circ s = \mathrm{id}_U$. Show that the 2-form

$$\omega_{\mathrm{FS}; n-1}|_U \equiv \frac{i}{2\pi} \partial \bar{\partial} \ln |s|^2, \quad (2.13)$$

where $|\cdot|$ is the standard (round) norm on \mathbb{C}^n , is independent of the choice of s .

(b) By (a), (2.13) determines a global 2-form $\omega_{\mathrm{FS}; n-1}$ on $\mathbb{C}P^{n-1}$, called the Fubini-Study symplectic form. Show that this form is indeed symplectic,

$$q^* \omega_{\mathrm{FS}; n-1}|_{TS^{2n-1}} = \frac{1}{\pi} \omega_{\mathbb{C}^n}|_{TS^{2n-1}}, \quad \text{and} \quad \int_{\mathbb{C}P^1} \omega_{\mathrm{FS}; 1} = 1.$$

Hint. The restriction of q to the interior of the upper hemisphere $S_+^2 \subset S^2 = S^3 \cap (\mathbb{C} \times \mathbb{R})$ is a diffeomorphism onto the complement of a point in $\mathbb{C}P^1$.

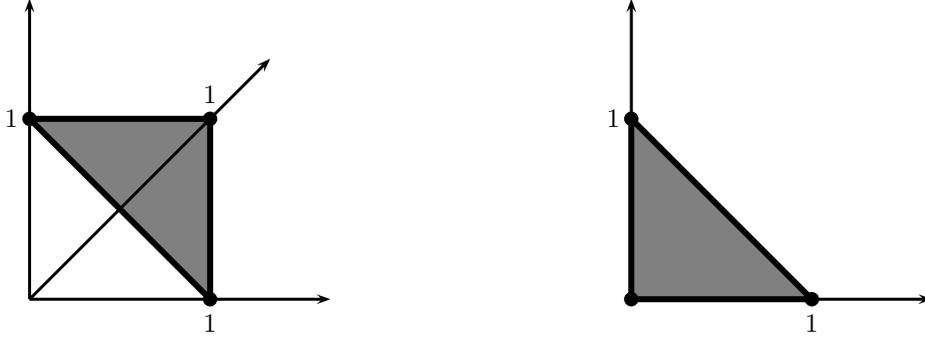


Figure 1: The images of $\mathbb{C}P^2$ under Hamiltonians for the torus actions of Exercise 2.9.

(c) Show that the actions of $\mathbb{T}^n \equiv (S^1)^n$ and $\mathbb{T}^{n-1} \equiv (S^1)^{n-1}$ on $\mathbb{C}P^{n-1}$ given by

$$\begin{aligned} (e^{2\pi i t_1}, \dots, e^{2\pi i t_n}) \cdot [z_1, \dots, z_n] &= [e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_n} z_n], \\ (e^{2\pi i t_1}, \dots, e^{2\pi i t_{n-1}}) \cdot [z_1, \dots, z_n] &= [e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_{n-1}} z_{n-1}, z_n] \end{aligned} \quad (2.14)$$

are Hamiltonian with respect to the symplectic form $\omega_{\text{FS};n-1}$. Determine the moment polytopes for these actions, in particular showing that in the $n=3$ they are as depicted in Figure 1.

Exercise 2.10. Let $k \in \mathbb{Z}^+$, $\mathbb{T}^k \equiv \mathbb{R}^k / \mathbb{Z}^k$ is the standard k -torus, and

$$x \equiv (x_1, \dots, x_k), y \equiv (y_1, \dots, y_k): \mathbb{R}^k \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

be the projections to the two components. Show that

(a) $\omega_k \equiv \sum_{i=1}^k dx_i \wedge dy_i$ is a well-defined symplectic form on $\mathbb{R}^k \times \mathbb{T}^k$;

(b) the action ψ_k of \mathbb{T}^k on $\mathbb{R}^k \times \mathbb{T}^k$ given by

$$\psi_{\mathbb{T};[r]}(x, [y]) = (x, [y+r]),$$

is well-defined, free, and smooth, preserves ω_k , and has

$$H_k: \mathbb{R}^k \times \mathbb{T}^k \longrightarrow \mathbb{R}^k, \quad H_k(x, [y]) = x,$$

as a Hamiltonian with respect to ω_k .

An automorphism $\phi: X \longrightarrow X$ of a set X is an involution if $\phi^2 = \text{id}_X$. If (X, ω) is a symplectic manifold, a smooth involution ϕ on X is called anti-symplectic if $\phi^* \omega = -\omega$. A Lagrangian submanifold of a symplectic manifold (X, ω) is a submanifold $Y \subset X$ such that

$$\dim Y = \frac{1}{2} \dim X \quad \text{and} \quad \omega|_{TY} = 0.$$

Exercise 2.11. Suppose (X, ω) and (X', ω') are symplectic manifolds of the same dimension and $f: X \rightarrow X'$ is a smooth map. Show that f is a symplectomorphism with respect to ω and ω' if and only if the graph of f ,

$$\text{Gr}(f) \equiv \{(x, f(x)) : x \in X\} \subset X \times X',$$

is a Lagrangian submanifold of $X \times X'$ with respect to the symplectic form $\pi_1^* \omega - \pi_2^* \omega'$, where

$$\pi_1, \pi_2: X \times X' \rightarrow X, X'$$

are the component projections.

Exercise 2.12. Let Y be a smooth manifold and λ_{T^*Y} be the 1-form on (the total space of) its cotangent bundle $\pi: T^*Y \rightarrow Y$ given by

$$\lambda_{T^*Y}|_{\theta}(w) = \{\pi^* \theta\}(w) \equiv \theta(\{d\theta\pi\}(w)) \quad \forall \theta \in T^*Y, w \in T_{\theta}(T^*Y).$$

Show that

- (a) $\{(df)^*\}^* \lambda_{T^*Y} = \lambda_{T^*Y'}$ for every diffeomorphism $f: Y \rightarrow Y'$ between smooth manifolds (thus $\{df\}^*: T^*Y' \rightarrow T^*Y$ is well-defined);
- (b) $\omega_{T^*Y} \equiv -d\lambda_{T^*Y}$ is a symplectic form on (the total space of) T^*Y and $\omega_{T^*\mathbb{R}^n} = \omega_{\mathbb{C}^n}$ under the natural identification of $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with \mathbb{C}^n ;
- (c) for every $y \in Y$, there is a canonical decomposition $T_y(T^*Y) = T_y Y \oplus T_y^* Y$ and

$$\omega_{T^*Y}|_y(v, w) = \begin{cases} 0, & \text{if } v, w \in T_y Y \text{ or } v, w \in T_y^* Y; \\ w(v), & v \in T_y Y \text{ and } w \in T_y^* Y. \end{cases}$$

Exercise 2.13. Let Y be a smooth manifold, λ_{T^*Y} be the canonical 1-form on the cotangent bundle $\pi: T^*Y \rightarrow Y$ of Y as in Exercise 2.12, and α be a 1-form on Y . Show that

- (a) $\alpha^* \lambda_{T^*Y} = \alpha$ and the map

$$\phi_{\alpha}: T^*Y \rightarrow T^*Y, \quad \phi_{\alpha}(\theta) = \alpha_{\pi(\theta)} - \theta,$$

is a smooth involution satisfying $\phi_{\alpha}^* \lambda_{T^*Y} = \pi^* \alpha - \lambda_{T^*Y}$;

- (b) the involution ϕ_{α} above is anti-symplectic with respect to ω_{T^*Y} if and only if $d\alpha = 0$.

Exercise 2.14. Let \mathbb{T} , v_1, \dots, v_k , $\Phi_{v_1 \dots v_k}$, and m_u for $u \in \mathbb{T}$ be as in Exercise 2.1, $(\mathbb{R}^k \times \mathbb{T}^k, \omega_k, \psi_k, H_k)$ be as in Exercise 2.10, and $\omega_{T^*\mathbb{T}}$ be as in Exercise 2.12. Define

$$\Phi_{\mathbb{T}}: T_{\mathbb{1}}^* \mathbb{T} \times \mathbb{T} \rightarrow T_{\mathbb{1}}^* \mathbb{T}, \quad \Phi_{\mathbb{T}}(\alpha, u) = \{d_u m_{u-1}\}^* \alpha, \quad \text{and} \quad \omega_{\mathbb{T}} = \{\Phi_{v_1 \dots v_k}^{-1}\}^* \omega_k.$$

Denote by $\psi_{\mathbb{T}}$ the action of \mathbb{T} on $T_{\mathbb{1}}^* \mathbb{T} \times \mathbb{T}$ by the multiplication on the second component and by $\mu_{\mathbb{T}}: T_{\mathbb{1}}^* \mathbb{T} \times \mathbb{T} \rightarrow T_{\mathbb{1}}^* \mathbb{T}$ the projection to the first component. Show that $\omega_{\mathbb{T}} = -\Phi_{\mathbb{T}}^* \omega_{T^*\mathbb{T}}$ and $(T_{\mathbb{1}}^* \mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})$ is a Hamiltonian \mathbb{T} -manifold which does not depend on the choice of a \mathbb{Z} -basis $v_1, \dots, v_k \in T_{\mathbb{1}} \mathbb{T}$ for the lattice $(T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}} \subset T_{\mathbb{1}} \mathbb{T}$ and is identified with $(\mathbb{R}^k \times \mathbb{T}^k, \omega_k, \psi_k, H_k)$ via $\Phi_{v_1 \dots v_k}$.

Exercise 2.15. (a) Suppose ϕ is an involution on a neighborhood of $0 \in \mathbb{R}^n$ with $\phi(0) = 0$. Let $\text{Jac}_0(\phi): \mathbb{R}^n \rightarrow \mathbb{R}^n$ be its Jacobian at 0 so that

$$\phi(x) = \{\text{Jac}_0(\phi)\}x + Q(x)$$

for some quadratic term $Q: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($Q(0) = 0$, $\text{Jac}_0(Q) = 0$) and all x in a neighborhood of $0 \in \mathbb{R}^n$. Show that there exist neighborhoods U and W of $0 \in \mathbb{R}^n$ so that

$$h: U \rightarrow W, \quad h(x) = x + \frac{1}{2}\{\text{Jac}_0(\phi)\}Q(x),$$

is a well-defined diffeomorphism satisfying $h \circ \phi = \{\text{Jac}_0(\phi)\}h$.

(b) Let X be a smooth manifold and $\phi: X \rightarrow X$ be a smooth involution. Show that every connected component of the fixed locus of ϕ ,

$$X^\phi \equiv \{x \in X: \phi(x) = x\},$$

is a smooth submanifold of X .

(c) Suppose in addition ω is a nondegenerate 2-form on X such that $\phi^*\omega = -\omega$. Show that $X^\phi \subset X$ is a Lagrangian submanifold of (X, ω) .

Exercise 2.16. Suppose (X, ω) is a symplectic manifold, $Y \subset X$ is a Lagrangian submanifold, J is an ω -compatible almost complex structure on X , and ω_{T^*Y} is the canonical symplectic form on T^*Y as in Exercise 2.12. Show that

(a) $J(TY) \subset TX|_Y$ is a subbundle complementary to TY ;

(b) the map $\Phi_{Y;J}: J(TY) \rightarrow T^*Y$, $\Phi_{Y;J}(w) = \omega(\cdot, w)$, is an isomorphism of vector bundles over Y ;

(c) $\Phi_{Y;J}^* \omega_{T^*Y}|_{T_y(J(TY))} = \omega|_{T_y X}$ under the canonical identification

$$T_y(J(TY)) = T_y Y \oplus J(T_y Y) = T_y X.$$

3 Group Actions on Manifolds

3.1 Group actions and flows of vector fields

This section collects basic facts about smooth actions of Lie groups, especially abelian ones, on smooth manifolds. For a group G and $u \in G$, let

$$L_u, R_u, \mathfrak{c}_u: G \rightarrow G, \quad L_u(u') = uu', \quad R_u(u') = u'u, \quad \mathfrak{c}_u(u') = uu'u^{-1}$$

denote the left multiplication, right multiplication, and conjugation, respectively, by u . If in addition G is a Lie group, $u' \in G$, and $v' \in T_{u'}G$, define

$$uw' = d_{u'}L_u(v') \in T_{uu'}G \quad \text{and} \quad v'u = d_{u'}R_u(v') \in T_{u'u}G.$$

If G is a Lie group, $u \in G$, and $v \in T_{\mathbb{1}}G$, let

$$\text{Ad}_u(v) = d_{\mathbb{1}}\mathfrak{c}_u(v) \in T_{\mathbb{1}}G. \tag{3.1}$$

By [34, §3.46/7],

$$\left. \frac{d}{dt} \text{Ad}_{e^{tv}}(v') \right|_{t=0} = [v, v'], \quad \text{Ad}_u([v, v']) = [\text{Ad}_u(v), \text{Ad}_u(v')] \quad \forall u \in G, v, v' \in T_{\mathbb{1}}G, \quad (3.2)$$

where $[v, v']$ is the Lie bracket on $T_{\mathbb{1}}G$ as in [34, §3.8]. If $h : [a, b] \rightarrow G$ is a smooth path, this implies that

$$\frac{d}{dt} \text{Ad}_{h^{-1}}(v') = -\text{Ad}_{h^{-1}}([h'h^{-1}, v']), \quad \text{where } h' = \frac{dh}{dt} : [a, b] \rightarrow TG. \quad (3.3)$$

Exercise 3.1. Suppose ψ is a smooth action of a Lie group G on a smooth manifold X as in (1.1). Show that

(a) the maps

$$d\psi : G \rightarrow \text{Diff}(TX), \quad u \rightarrow d\psi_u, \quad \text{and} \quad d\psi^* : G \rightarrow \text{Diff}(T^*X), \quad u \rightarrow d\psi_u^* \equiv \{d\psi_u^{-1}\}^*,$$

are smooth actions of G on TX and T^*X , respectively, lifting the G -action ψ on X and linear on the fibers of the vector bundles $TX, T^*X \rightarrow X$;

(b) the 1-form λ_{T^*X} of Exercise 2.12 is preserved by the action $d\psi^*$.

Proposition 3.2. *Let X be a smooth manifold.*

(1) *The flow of a complete vector field ζ on X determines a smooth \mathbb{R} -action ψ on X by*

$$\psi_0 = \text{id}_X, \quad \frac{d}{dt} \psi_t(x) = \zeta(\psi_t(x)) \quad \forall t \in \mathbb{R}, x \in X.$$

Conversely, a smooth \mathbb{R} -action ψ on X is the flow of the vector field ζ on X given by

$$\zeta(x) = \left. \frac{d}{dt} \psi_t(x) \right|_{t=0} \quad \forall x \in X. \quad (3.4)$$

In particular, $X^\psi = \{x \in X : \zeta(x) = 0\}$.

(2) *If ψ is a smooth \mathbb{R} -action on X with associated vector field ζ and $x \in X^\psi$, the linear \mathbb{R} -action $d_x\psi$ on T_xX is the flow of the vector field*

$$\nabla\zeta|_x : T_xX \rightarrow T_xX, \quad w \rightarrow \nabla_w\zeta, \quad (3.5)$$

on T_xX , where ∇ is any connection in the vector bundle $TX \rightarrow X$. If in addition J is a ψ -invariant endomorphism of this vector bundle, i.e.

$$\{d_{x'}\psi_t\}^{-1} \circ J \circ d_{x'}\psi_t = J : T_{x'}X \rightarrow T_{x'}X \quad \forall t \in \mathbb{R}, x' \in X, \quad (3.6)$$

then

$$\nabla_{Jw}\zeta = J\nabla_w\zeta \quad \forall w \in T_xX. \quad (3.7)$$

Proof. (1) If $\psi_t: X \rightarrow X$ is the time t flow of $\zeta \in \Gamma(X; TX)$ for each $t \in \mathbb{R}$, then

$$\psi_{s+t} = \psi_s \circ \psi_t: X \rightarrow X;$$

see [34, Theorem 1.48]. Thus, the map (1.1) is a group homomorphism and ψ is a smooth \mathbb{R} -action on X . Conversely, if ψ is a smooth \mathbb{R} -action on X and $\zeta \in \Gamma(X; TX)$ is given by (3.4), then

$$\psi_0 = \text{id}_X, \quad \frac{d}{dt} \psi_t(x) = \frac{d}{ds} \psi_{s+t}(x) \Big|_{s=0} = \frac{d}{ds} \psi_s(\psi_t(x)) \Big|_{s=0} \equiv \zeta(\psi_t(x));$$

the second equality above holds because (1.1) is a group homomorphism. Thus, ψ_t is the time t flow of ζ . The last claim in (1) follows immediately.

(2) Let $x \in X^\psi$ and $\gamma: (-\delta, \delta) \rightarrow X$ be a smooth curve such that $\gamma(0) = x$. Then,

$$\frac{d}{dt} d_x \psi_t(\gamma'(0)) \Big|_{t=0} = \frac{d}{dt} \frac{d}{ds} \psi_t(\gamma(s)) \Big|_{s,t=0} = \frac{D}{ds} \frac{d}{dt} \psi_t(\gamma(s)) \Big|_{s,t=0} = \frac{D}{ds} \zeta(\gamma(s)) \Big|_{s=0} = \nabla_{\gamma'(0)} \zeta,$$

where D/ds denotes the covariant derivative with respect to any torsion-free connection in TX ; the penultimate equality above holds by the second claim in (1) for the \mathbb{R} -action ψ on X . By second claim of (1) for the \mathbb{R} -action $d_x \psi$ on $T_x X$, $d_x \psi_t$ is thus the time t flow of the vector field $\nabla \zeta|_x$ on $T_x X$ given by (3.5) (which is independent of the choice of ∇ because $\zeta(x) = 0$).

If in addition J is a ψ -invariant endomorphism of the vector bundle $TX \rightarrow X$, then

$$d_{x'} \psi_t(Jw) = J d_{x'} \psi_t(w) \quad \forall w \in T_{x'} X, x' \in X.$$

Setting $x' = x$ above, differentiating the resulting equation at $t = 0$, and using the previous statement, we obtain (3.7). \square

Exercise 3.3. Suppose $k \in \mathbb{Z}^{\geq 0}$, ψ is a smooth \mathbb{R}^k -action on a smooth manifold X , and $x \in X$. Let $\mathbb{R}_x^k \subset \mathbb{R}^k$ be the largest linear subspace fixing x and $\mathbb{R}_x^c \subset \mathbb{R}^k$ be a complementary linear subspace. Show that

$$\frac{d}{dt} \psi_{tv}(x) \Big|_{t=0} \neq 0 \in T_x X \quad \forall v \in \mathbb{R}_x^c - \{0\}.$$

Exercise 3.4. Let ψ be a nontrivial smooth action of \mathbb{R}^k (resp. k -torus $\mathbb{T}^k \equiv (\mathbb{R}/\mathbb{Z})^k$) on a smooth manifold X as in (1.1). Show that there exist an irreducible smooth action ψ' of \mathbb{R}^m (resp. \mathbb{T}^m) on X for some $m \in \mathbb{Z}^+$ and a full-rank real (resp. integer) $m \times k$ -matrix A so that $\psi = \psi' \circ A$, i.e.

$$\psi_v = \psi'_{Av} \in \text{Diff}(X) \quad \forall v \in \mathbb{R}^k \quad (\text{resp. } v \in \mathbb{T}^k).$$

Furthermore, if the action ψ is almost periodic in the first case, then so is the action ψ' . *Hint.* Let G be a compact Lie group. For every $v \in T_1 G$, the closure of the one-parameter subgroup $\{e^{tv}: t \in \mathbb{R}\}$ is a torus.

Exercise 3.5. Suppose ψ is a smooth action of a Lie group G on a smooth manifold X as in (1.1). For each $v \in T_1 G$, let $\zeta_v \in \Gamma(X; TX)$ be the vector field on X generated by v as in (1.2). Show that

(a) if $u \in G$ and $v, v' \in T_{\mathbb{1}}G$, then

$$\zeta_{\text{Ad}_{u^{-1}}(v)} = \psi_u^* \zeta_v \equiv d\psi_u^{-1}(\zeta_v \circ \psi_u), \quad \zeta_{[v, v']} = \left. \frac{d}{dt} \zeta_{\text{Ad}_{e^{tv}}(v')} \right|_{t=0} = -\mathcal{L}_{\zeta_v} \zeta_{v'} = -[\zeta_v, \zeta_{v'}], \quad (3.8)$$

where \mathcal{L} is the Lie derivative;

(b) if $h: [a, b] \rightarrow G$ is a smooth path and $h' = \frac{dh}{dt}: [a, b] \rightarrow TG$, then

$$\frac{d\psi_h}{dt} = \zeta_{h'h^{-1}} \circ \psi_h. \quad (3.9)$$

Exercise 3.6. Let Y be a smooth manifold, λ_{T^*Y} and ω_{T^*Y} be the canonical 1-form and the canonical symplectic form, respectively, on T^*Y as in Exercise 2.12, and ψ be a smooth action of a Lie group G on T^*Y .

(a) For each $v \in T_{\mathbb{1}}G$, let $\zeta_v \in \Gamma(T^*Y; T(T^*Y))$ be the vector field on T^*Y generated by v as in (1.2) with $X = T^*Y$. Show that the smooth map

$$\mu: T^*Y \rightarrow T_{\mathbb{1}}^*G \quad \text{defined by} \quad \{\mu(x)\}(v) = -\lambda_{T^*Y}(\zeta_v(x)) \quad \forall v \in T_{\mathbb{1}}G,$$

satisfies the second condition in (1.4).

(b) Suppose in addition the action ψ preserves λ_{T^*Y} , i.e. $\psi_u^* \lambda_{T^*Y} = \lambda_{T^*Y}$ for every $u \in G$. Show that the map μ above is a moment map for the action ψ on (T^*Y, ω_{T^*Y}) .

3.2 Fixed loci of group actions

Exercise 3.7. Let ψ be a smooth action of a compact Lie group on a smooth manifold X as in (1.1). Show that there exists a ψ -invariant Riemannian metric on X , i.e. a Riemannian metric g on X such that

$$g(d_x \psi_u(w), d_x \psi_u(w')) = g(w, w') \quad \forall u \in G, x \in X, w, w' \in T_x X.$$

Suppose $Y \subset X$ is a closed submanifold of a smooth manifold. A tubular neighborhood identification for Y in X is a diffeomorphism $\Phi: \mathcal{U} \rightarrow U$ from an open neighborhood of Y in a subbundle $TY^c \subset TX|_Y$ complementary to TY to an open neighborhood of $Y \subset X$ such that

$$\Phi(y) = y, \quad d_y \Phi = \text{id}: T_y \mathcal{U} = T_y(TY^c) = T_y Y \oplus TY^c|_y \rightarrow T_y Y \oplus TY^c|_y = T_y X = T_y U \quad \forall y \in Y. \quad (3.10)$$

Proposition 3.8. Let ψ be a smooth action of a compact Lie group G on a smooth manifold X as in (1.1).

(1) The fixed locus $X^\psi \subset X$ of ψ is a closed submanifold with

$$T(X^\psi) = (TX)^{d\psi}. \quad (3.11)$$

(2) If $Y \subset X$ is a closed submanifold preserved by ψ and $TY^c \subset TX|_Y$ is a subbundle complementary to TY and preserved by ψ , then there exists a tubular neighborhood identification $\Phi: \mathcal{U} \rightarrow U$ for Y in X with $\mathcal{U} \subset TY^c$ which is G -equivariant with respect to the actions ψ on X and $d\psi$ on TX .

Proof. Let g be a Riemannian metric preserved by the group action ψ , as provided by Exercise 3.7. Its Levi-Civita connection ∇ is also preserved by G . If $w \in TX$, $\gamma_w: (a, b) \rightarrow X$ with $a < 0 < b$ is the geodesic with respect to ∇ of g with $\gamma_w'(0) = w$, and $u \in G$, then

$$\psi_u \circ \gamma_w: (a, b) \rightarrow X$$

is the geodesic with respect to ∇ with $(\psi_u \circ \gamma_w)'(0) = \{d_{\gamma_w(0)}\psi_u\}(w)$, i.e. $\psi_u \circ \gamma_w = \gamma_{\{d_{\gamma_w(0)}\psi_u\}(w)}$. Thus, the exponential map

$$\exp: \mathcal{W} \rightarrow X, \quad \exp(w) = \gamma_w(1) \quad \forall w \in \mathcal{W} \subset TX,$$

with respect to ∇ satisfies

$$\{d\psi_u\}(\mathcal{W}) = \mathcal{W} \quad \text{and} \quad \exp(\{d\psi_u\}(w)) = \psi_u(\exp(w)) \quad \forall u \in G, w \in \mathcal{W}, \quad (3.12)$$

i.e. it is G -equivariant with respect to the actions ψ on X and $d\psi$ on TX .

(2) Since

$$\{d_x \exp\}(w) = w \in T_x X \quad \forall x \in X, w \in W, \quad (3.13)$$

for each $y \in Y$ the restriction of \exp to a neighborhood of y in $TY^c \cap \mathcal{W}$ is a diffeomorphism onto an open neighborhood of y in X by the Inverse Function Theorem. Since $Y \subset X$ is closed, it follows that there exists a neighborhood \mathcal{U}' of Y in $TY^c \cap \mathcal{W}$ so that

$$\exp: \mathcal{U}' \rightarrow \Psi(\mathcal{U}')$$

is a diffeomorphism onto an open subset of X . This map satisfies both conditions in (3.10) by the definition of the exponential map. Since G is compact,

$$\mathcal{U} \equiv \bigcap_{u \in G} d\psi_u(\mathcal{U}') \subset \mathcal{U}' \subset TY^c \cap \mathcal{W}$$

is a neighborhood of $Y \subset TY^c$ preserved by the G -action. By (3.12), the restriction

$$\Phi \equiv \exp|_{\mathcal{U}}: \mathcal{U} \rightarrow U \equiv \exp(\mathcal{U}) \subset X$$

is a G -equivariant diffeomorphism from an open neighborhood of Y in TY^c to an open neighborhood of Y in X with the required properties. This establishes (2).

(1) It is immediate that $X^\psi \subset X$ is a closed subset. For each $y \in X^\psi$, let $\Phi_y: \mathcal{U}_y \rightarrow U_y$ be a G -equivariant tubular neighborhood identification as in (2) with $Y = \{y\}$ and $\mathcal{U}_y \subset T_y X$. By the G -equivariance of Φ_y ,

$$\Phi_y: (T_y X)^{d\psi} \cap \mathcal{U}_y \rightarrow X^\psi \cap U_y$$

is a homeomorphism for every $y \in X^\psi$. Thus, each topological component of X^ψ is a submanifold of X ; see [34, 1.33(b)]. By the G -equivariance of Φ_y , (3.11) holds as well. \square

Remark 3.9. The conclusion of Proposition 3.8(1) also holds if $\pi_0(G)$ is finite and X admits a G -invariant metric (but G is not necessarily compact). The first paragraph in the proof of Proposition 3.8 still applies. For (1) in this proof, $\mathcal{U}_y \subset T_y X$ can be taken to be any neighborhood of $0 \in T_y X$ on which the map \exp is injective.

Corollary 3.10. *Let ψ be an irreducible almost periodic action of \mathbb{R}^k on a smooth manifold X as in (1.1). The subspace $\text{Crit}(\psi)$ of points of X with stabilizers containing a one-dimensional linear subspace of \mathbb{R}^k is a countable union of (not necessarily disjoint) closed proper submanifolds of X .*

Proof. Let $\rho: \mathbb{R}^k \rightarrow \mathbb{T}$ and ψ' be as in (1.5). For each one-dimensional linear subspace $L \subset \mathbb{R}^k$, the closure $\mathbb{T}_L \subset \mathbb{T}$ of $\rho(L)$ in \mathbb{T} is a torus. By (1.5), the fixed locus $X^{\mathbb{T}_L} \subset X$ of the smooth action $\psi'|_{\mathbb{T}_L}$ on X is the same as the fixed locus X^L of the smooth action $\psi|_L$. By Proposition 3.8(1), X^L is thus a closed submanifold of X (possibly empty). Since the action ψ is irreducible, $\psi|_L$ is a nontrivial action and $X^L \neq X$. Thus, $\text{Crit}(\psi)$ is the union of the closed submanifolds $X^{\mathbb{T}'} \neq X$ taken over the subcollection

$$\mathcal{A} \equiv \{\mathbb{T}_L : L \in \mathbb{R}\mathbb{P}^{k-1}\}$$

of subtori of \mathbb{T} . Since the subtori of \mathbb{T} are generated by finite sets of vectors in $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$, the collection \mathcal{A} is (at most) countable. \square

For a torus \mathbb{T} and $\alpha \in (T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$, let

$$\mathbb{T}_{\alpha} = \{e^v : v \in T_{\mathbb{1}}\mathbb{T}, \alpha(v) \in \mathbb{Z}\}.$$

If $\alpha \neq 0$, $\mathbb{T}_{\alpha} \subset \mathbb{T}$ is a codimension 1 closed subgroup. If α is primitive, i.e. $\alpha \neq k\alpha'$ for any $\alpha' \in (T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ and $k \in \mathbb{Z}$ with $k \geq 2$, then $\mathbb{T}_{\alpha} \subset \mathbb{T}$ is a codimension 1 subtorus. For a subset $S \subset (T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ and a closed subgroup $G \subset \mathbb{T}$, let

$$\mathbb{T}_S = \bigcap_{\alpha \in S} \mathbb{T}_{\alpha} \quad \text{and} \quad S_G = \{\alpha \in S : G \subset \mathbb{T}_{\alpha}\}.$$

Thus, $\mathbb{T}_S \subset \mathbb{T}$ is a closed subgroup of codimension at most $|S|$, $S_G \subset S$ is the maximal subset so that $G \subset \mathbb{T}_{S_G}$, and $S_{\mathbb{T}_S} = S$.

Proposition 3.11. *Let ψ be a smooth action of a torus \mathbb{T} on a smooth manifold X as in (1.1) and $Y \subset X^{\psi}$ be a topological component of the ψ -fixed locus.*

(1) *There exist a subset $S(Y) \subset (T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}} - \{0\}$ and a splitting*

$$TX|_Y = TY \oplus \bigoplus_{\alpha \in S(Y)} \mathcal{N}_X^{\alpha} Y \longrightarrow Y \tag{3.14}$$

of $TX|_Y$ into a direct sum of vector bundles preserved by $d\psi$ so that the bundles $\mathcal{N}_X^{\alpha} Y$ are nonzero and complex with

$$d\psi_{e^v}(w) = e^{2\pi i \alpha(v)} w \quad \forall v \in T_{\mathbb{1}}\mathbb{T}, w \in \mathcal{N}_X^{\alpha} Y, \alpha \in S(Y). \tag{3.15}$$

In particular, $2|S(Y)| \leq \text{codim}_X Y$. If X is connected and the action ψ is irreducible (resp. effective), then the \mathbb{R} -span (resp. \mathbb{Z} -span) of $S(Y)$ is $T_{\mathbb{1}}^\mathbb{T}$ (resp. $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$). If $TX|_Y$ is a complex vector bundle and $d\psi$ preserves its complex structure J , then the complex structure on each subbundle $\mathcal{N}_X^{\alpha} Y \subset TX|_Y$ can be taken to be the restriction of J .*

(2) *If $G \subset \mathbb{T}$ is a closed subgroup, $Z \subset X^G$ is a topological component of the G -action $\psi|_G$ on X , and $Y \subset Z$, then*

$$TZ|_Y = TY \oplus \bigoplus_{\alpha \in S(Y)_G} \mathcal{N}_X^{\alpha} Y \subset TX|_Y \longrightarrow Y$$

and Z is a topological component of the fixed locus $X^{\mathbb{T}_{S(Y)_G}}$ of the $\mathbb{T}_{S(Y)_G}$ -action $\psi|_{\mathbb{T}_{S(Y)_G}}$ on X .

Proof. (1) For each $y \in Y$, $d_y\psi$ is a real representation of \mathbb{T} on T_yX . Every such representation splits as a direct sum of a trivial real representation and of one-dimensional complex representations with the action on each factor given by (3.15) for some $\alpha \in (T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ nonzero. By (3.11), the trivial representation summand is T_yY . Since $d_y\psi$ depends smoothly on y , the weights α are independent of $y \in Y$ and the corresponding component representations vary smoothly with $y \in Y$. Thus, the latter form vector subbundles $\mathcal{N}_X^\alpha Y \subset TX|_Y$ as in (3.14) with complex structures. If $TX|_Y$ is a complex vector bundle and $d\psi$ preserves its complex structure, then $d_y\psi$ is a complex representation of \mathbb{T} on T_yY and the same reasoning applies.

If $S(Y)$ does not span $T_{\mathbb{1}}^*\mathbb{T}$ over \mathbb{R} (resp. $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ over \mathbb{Z}), there exists $v \in T_{\mathbb{1}}\mathbb{T} - (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ such that $\alpha(v) = 0$ (resp. $\alpha(v) \in \mathbb{Z}$) for all $\alpha \in S(Y)$. Let G be the closure of the subgroup $\{e^{tv} : t \in \mathbb{R}\}$ (resp. the subgroup generated by v) in \mathbb{T} . This subgroup acts trivially on $TX|_Y$. By Proposition 3.8(1), this implies that the connected component of the G -fixed locus X^G containing Y is a connected component of X , i.e. G acts trivially on X (and so the action ψ is not effective) if X is connected. If $\alpha(v) = 0$ for all $\alpha \in S(Y)$ and X is connected, then

$$\zeta_v \equiv d_{\mathbb{1}}\psi(v) = 0 \in \Gamma(X; TX),$$

i.e. the action ψ is reducible.

(2) By Proposition 3.8(1) applied to $\psi|_G$, ψ , and $\psi|_{\mathbb{T}S(Y)_G}$,

$$TZ|_Y = \{w \in TX|_Y : d\psi_u(w) = w \ \forall u \in G\} = TY \oplus \bigoplus_{\alpha \in S(Y)_G} \mathcal{N}_X^\alpha Y = TX^{\mathbb{T}S(Y)_G}|_Y.$$

This establishes both claims. \square

Corollary 3.12. *Let ψ be an irreducible almost periodic action of \mathbb{R}^k on a smooth manifold X as in (1.1). For each $L \in \mathbb{R}\mathbb{P}^{k-1}$, let $X^L \subset X$ be the fixed locus of the action $\psi|_L$. If X is compact, then the set*

$$\tilde{\pi}_0^*(\text{Crit}(\psi)) \equiv \{Z \in \pi_0(X^L) : L \in \mathbb{R}\mathbb{P}^{k-1}, Z \cap X^\psi \neq \emptyset\}$$

is finite.

Proof. Let $\rho : \mathbb{R}^k \rightarrow \mathbb{T}$ and ψ' be as in (1.5). We can assume that the image of ρ is dense in \mathbb{T} and so $X^\psi = X^{\psi'}$. For each $L \in \mathbb{R}\mathbb{P}^{k-1}$, let $\mathbb{T}_L \subset \mathbb{T}$ be as in the proof of Corollary 3.10. For each subtorus $\mathbb{T}' \subset \mathbb{T}$, let $X^{\mathbb{T}'} \subset X$ be the fixed locus of the action $\psi'|_{\mathbb{T}'}$. In particular, $X^L = X^{\mathbb{T}_L}$. By Proposition 3.11(2), every element Z of $\tilde{\pi}_0^*(\text{Crit}(\psi))$ intersecting a topological component Y of $X^\psi = X^{\psi'}$ is thus the unique topological component of $X^{\mathbb{T}S} \subset X$ for some $S \subset S(Y)$ intersecting Y . The number of subsets of $S \subset S(Y)$ is finite for each $Y \in \pi_0(X^\psi)$. Since X is compact, $\pi_0(X^\psi)$ is finite as well. \square

3.3 Group actions on symplectic manifolds

This section provides an analogue of Proposition 3.8 in the symplectic setting. In particular, Proposition 3.16(2) is an equivariant version of the Symplectic Tubular Neighborhood Theorem.

Exercise 3.13. Suppose X is a smooth manifold, ω is a closed 2-form on X , and ψ is a smooth action of a Lie group G on X as in (1.1). For each $v \in T_{\mathbb{1}}G$, let $\zeta_v \in \Gamma(X; TX)$ be the vector field on X generated by v as in (1.2)..

(a) Suppose G is connected. Show that ψ preserves ω if and only if $d(\iota_{\zeta_v}\omega) = 0$ for all $v \in T_{\mathbb{1}}G$.

(b) Suppose ψ preserves ω . Show that

$$d(\omega(\zeta_v, \zeta_{v'})) = -\iota_{[\zeta_v, \zeta_{v'}]}\omega \quad v, v' \in T_{\mathbb{1}}G, \quad (3.16)$$

$$\omega([\zeta_v, \zeta_{v'}], \zeta_{v''}) + \omega([\zeta_{v'}, \zeta_{v''}], \zeta_v) + \omega([\zeta_{v''}, \zeta_v], \zeta_{v'}) = 0 \quad \forall v, v', v'' \in T_{\mathbb{1}}G. \quad (3.17)$$

Hint: use Cartan's formula for the Lie derivative.

Exercise 3.14. Let ψ be a smooth action of a compact Lie group on a symplectic manifold (X, ω) as in (1.1). Show that there exists a ψ -invariant ω -compatible almost complex structure on X , i.e. an ω -compatible almost complex structure J on X such that

$$\{d_x\psi_u\}^{-1} \circ J \circ d_x\psi_u = J: T_xX \longrightarrow T_xX \quad \forall u \in G, x \in X.$$

Hint: a Riemannian metric g and a nondegenerate 2-form ω on X determine an ω -compatible almost complex structure $J_{g,\omega}$ on X ; see the proof of Proposition 2.3 in [37].

Exercise 3.15. Suppose ψ is a smooth action of a Lie group on a symplectic manifold (X, ω) as in (1.1), J is a ψ -invariant ω -compatible almost complex structure on X , and $Y \subset X$ is a Lagrangian submanifold. Show that the isomorphism

$$\Phi_{Y;J}: J(TY) \longrightarrow T^*Y, \quad \Phi_{Y;J}(w) = \omega(\cdot, w),$$

of real vector bundles over Y is G -equivariant with respect to the actions $d\psi$ and $d\psi^*$ of Exercise 3.5.

Proposition 3.16. *Let ψ be a smooth action of a compact Lie group G on a symplectic manifold (X, ω) as in (1.1).*

- (1) *The fixed locus $X^\psi \subset X$ of ψ is a closed symplectic submanifold with $T(X^\psi) = (TX)^{d\psi}$.*
- (2) *Suppose $Y \subset X$ is a closed submanifold preserved by ψ , $TY^c \subset TX|_Y$ is a subbundle complementary to TY and preserved by $d\psi$, and $\tilde{\omega}$ is a G -invariant closed 2-form on a neighborhood of Y in TY^c preserved by $d\psi$. If*

$$\tilde{\omega}|_{T_y(TY^c)} = \omega|_{T_yX} \quad \forall y \in Y, \quad (3.18)$$

there exists a G -equivariant tubular neighborhood identification $\Phi: \mathcal{U} \longrightarrow U$ for Y in X such that $\mathcal{U} \subset TY^c$ and $\Phi^\omega = \tilde{\omega}|_{\mathcal{U}}$.*

Proof. (1) Let J be a G -invariant ω -compatible almost complex structure on X , as provided by Exercise 3.14. By Proposition 3.8(1), $X^\psi \subset X$ is a closed submanifold with $T(X^\psi) = (TX)^{d\psi}$. Since J is G -invariant, $J(TX^\psi) \subset TX^\psi$ by (3.11). Since J is ω -compatible, $\omega(v, Jv) > 0$ for all $v \in TX$ nonzero. Thus, $\omega|_{TX^\psi}$ is nondegenerate.

(2) Let $\Phi: \mathcal{U} \longrightarrow U$ be a G -equivariant tubular neighborhood identification for Y in X with $\mathcal{U} \subset TY^c$, as provided by Proposition 3.8(2). In particular, $\Phi^*\omega$ is a symplectic form on \mathcal{U} . By (3.18) and (3.10),

$$(\Phi^*\omega)|_{T_y(TY^c)} = \tilde{\omega}|_{T_y(TY^c)} \quad \forall y \in Y. \quad (3.19)$$

Since Φ is G -equivariant, the 2-form $\Phi^*\omega$ is G -invariant. Since the subset of \mathcal{U} on which $\tilde{\omega}$ is nondegenerate contains Y by (3.18) and is open and preserved by G , we can assume that the 2-form $\tilde{\omega}$ is nondegenerate on \mathcal{U} (by replacing \mathcal{U} by its subset on which $\tilde{\omega}$ is nondegenerate) .

Let $m_\tau: TY^c \rightarrow TY^c$ be the scalar multiplication by τ as in (B.1) and $\zeta_{TY^c} \in \Gamma(TY^c; T(TY^c))$ be the canonical vertical vector field as in (B.5). Define a 1-form α on \mathcal{U} by

$$\alpha = \int_0^1 m_\tau^*(\iota_{\tau^{-1}\zeta_{TY^c}}(\Phi^*\omega - \tilde{\omega}))d\tau.$$

By Exercise B.1 and (3.19),

$$d\alpha = \Phi^*\omega - \tilde{\omega}, \quad d\alpha|_{T(TY^c)|_Y} = 0, \quad \alpha|_{T(TY^c)|_Y} = 0, \quad \nabla\alpha|_{T(TY^c)|_Y} = 0, \quad (3.20)$$

where ∇ is any connection in $T^*(TY^c)$.

By the second statement in (3.20) and the compactness of $[0, 1]$,

$$\omega_t \equiv \tilde{\omega} + t d\alpha$$

is a symplectic form on a neighborhood \mathcal{U}' of $Y \subset \mathcal{U}$ for every $t \in [0, 1]$. For each $t \in [0, 1]$, define

$$\xi_t \in \Gamma(\mathcal{U}'; T\mathcal{U}') \quad \text{by} \quad \iota_{\xi_t}\omega_t = -\alpha. \quad (3.21)$$

By the third statement in (3.20), $\xi_t|_Y = 0$. Since $[0, 1]$ is compact, it follows that there exists a neighborhood \mathcal{U}'' of Y in \mathcal{U}' so that the flow of ξ_t ,

$$\psi_t: \mathcal{U}'' \rightarrow \mathcal{U}', \quad \psi_0(w) = w \quad \forall w \in \mathcal{U}'', \quad \frac{d}{dt}\psi_t = \xi_t \circ \psi_t,$$

is well-defined for every $t \in [0, 1]$. By the first, third, and fourth statements in (3.20),

$$\tilde{\omega} = \psi_t^*\omega_t, \quad \psi_t(w) = w \quad \forall w \in \mathcal{U}'', \quad \text{and} \quad d\psi_t|_{T(TY^c)|_Y} = \text{id}_{T(TY^c)|_Y} \quad \forall t \in [0, 1]; \quad (3.22)$$

see Exercise 3.17 below for the first identity.

Since G is compact, the set

$$\mathcal{U}''' \equiv \bigcap_{u \in G} d\psi_u(\mathcal{U}'') \subset \mathcal{U}' \subset TY^c$$

is a neighborhood of $Y \subset TY^c$ preserved by the G -action so that $\psi_1: \mathcal{U}''' \rightarrow \mathcal{U}$ is a well-defined diffeomorphism onto an open subset of \mathcal{U} . Since the 2-forms $\Phi^*\omega$ and $\tilde{\omega}$ are G -invariant, so are the 1-form α and the vector fields ξ_t . Thus, the smooth map ψ_1 is G -equivariant, as is the diffeomorphism

$$\Phi \circ \psi_1: \mathcal{U}''' \rightarrow \Phi(\psi_1(\mathcal{U}''')) \subset U \subset X.$$

By the second and third statements in (3.22), this diffeomorphism is a tubular neighborhood identification for Y in X (because Φ is). By the first statement in (3.22), $\{\Phi \circ \psi_1\}^*\omega = \tilde{\omega}|_{\mathcal{U}'''}$. \square

Exercise 3.17. Suppose X is a smooth manifold, $(\omega_t)_{t \in [0,1]}$ is a smooth family of symplectic forms on X , $(\xi_t)_{t \in [0,1]}$ is the smooth family of vector fields on X defined by

$$d(\iota_{\xi_t} \omega_t) = -\frac{d}{dt} \omega_t,$$

and $\psi_t: \mathcal{U} \rightarrow X$ is a flow of $(\xi_t)_{t \in [0,1]}$ on an open subset of X , i.e.

$$\psi_0(x) = x, \quad \frac{d}{dt}(\psi_t(x)) = \xi_t(\psi_t(x)) \quad \forall x \in \mathcal{U}, t \in [0,1].$$

Show that $\psi_t^* \omega_t = \omega_0|_{\mathcal{U}}$ for all $t \in [0,1]$. *Hint:* differentiate both sides and use Cartan's formula for the Lie derivative.

Example 3.18 (Symplectic Tubular Neighborhood Theorem). Suppose ψ is a smooth action of a compact Lie group G on a symplectic manifold (X, ω) and Y is a symplectic submanifold of (X, ω) preserved by ψ . The restriction ω_Y^\perp of ω to the ω -symplectic complement

$$TY^\omega \equiv \{w \in TX|_Y : \omega(w, w') = 0 \ \forall w' \in TY\} \rightarrow Y$$

of TY in $TX|_Y$ is then a G -invariant nondegenerate fiberwise 2-form. Let $\zeta_{TY^\omega} \in \Gamma(TY^\omega; T(TY^\omega))$ be the canonical vertical vector field as in (B.5). A G -invariant connection ∇ in the real vector bundle $\pi: TY^\omega \rightarrow Y$ extends ω_Y^\perp to a G -invariant 2-form ω_∇^\perp on (the total space of) TY^ω ; see Exercise B.5. By (B.19), the G -invariant closed 2-form

$$\tilde{\omega}_\nabla \equiv \pi^* \omega + \frac{1}{2} d(\iota_{\zeta_{TY^\omega}} \omega_\nabla^\perp)$$

satisfies (3.18) with $\tilde{\omega} = \tilde{\omega}_\nabla$. If in addition $Y \subset X$ is a closed subspace, by Proposition 3.16(2) there then exists a G -equivariant tubular neighborhood identification $\Phi: \mathcal{U} \rightarrow U$ for Y in X such that $\mathcal{U} \subset TY^\omega$ and $\Phi^* \omega = \tilde{\omega}_\nabla|_{\mathcal{U}}$. If $Y \equiv \{x\}$ is a one-point set, $TY^\omega = T_x X$ and $\tilde{\omega}_\nabla = \pi^* \omega_x$. This yields Corollary 3.19 below.

Corollary 3.19 (Darboux Theorem). *Suppose ψ is a smooth action of a compact Lie group G on a symplectic manifold (X, ω) and $x \in X^\psi$. There exist a G -invariant tubular neighborhood \mathcal{U} of 0 in $T_x X$ and a G -equivariant diffeomorphism $\Phi: \mathcal{U} \rightarrow U$ onto a neighborhood U of x in X such that*

$$\Phi(0) = x, \quad d_0 \Phi = \text{id}: T_0(T_x X) = T_x X \rightarrow T_x X, \quad \text{and} \quad \Phi^* \omega = \pi^* \omega_x|_{\mathcal{U}},$$

where $\pi: T_x X \rightarrow \{x\}$ is the projection.

Corollary 3.20 (Lagrangian Tubular Neighborhood Theorem). *Suppose ψ is a closed action of a compact Lie group G on a symplectic manifold (X, ω) , $Y \subset X$ is a compact Lagrangian submanifold preserved by ψ , and ω_{T^*Y} is the canonical symplectic form on T^*Y as in Exercise 2.12. There exists a G -equivariant diffeomorphism $\Phi: \mathcal{U} \rightarrow U$ from an open neighborhood of Y in T^*Y onto an open neighborhood of Y in X so that*

$$\Phi(y) = y \quad \forall y \in Y \quad \text{and} \quad \Phi^* \omega = \omega_{T^*Y}|_{\mathcal{U}}. \quad (3.23)$$

Proof. By Exercise 3.14, there exists a ψ -invariant ω -compatible almost complex structure J on X . By Exercise 3.15, the isomorphism

$$\Phi_{Y;J}: J(TY) \longrightarrow T^*Y, \quad \Phi_{Y;J}(w) = \omega(\cdot, w),$$

of real vector bundles over Y is G -equivariant with respect to the actions $d\psi$ and $d\psi^*$ of Exercise 3.5. Along with the latter exercise, this implies that the closed 2-form $\Phi_{Y;J}^*\omega_{T^*X}$ on (the total space of) $J(TY)$ is G -invariant. By Exercise 2.16(c), this form satisfies (3.18). By Proposition 3.16(2), there thus exists a G -equivariant tubular neighborhood identification $\Phi: \mathcal{U} \longrightarrow U$ for Y in X such that $\mathcal{U} \subset J(TY)$ and $\Phi^*\omega = \Phi_{Y;J}^*\omega_{T^*Y}|_{\mathcal{U}}$. The map

$$\Phi \circ \Phi_{Y;J}^{-1}: \Phi_{Y;J}(\mathcal{U}) \longrightarrow U$$

is then a G -equivariant diffeomorphism satisfying (3.23) with Φ replaced by $\Phi \circ \Phi_{Y;J}^{-1}$. \square

Exercise 3.21 (Moser's Stability). Let X be a closed manifold. Suppose $p \in \mathbb{Z}^{\geq 0}$ and $(\omega_t)_{t \in [0,1]}$ is a smooth family of cohomologous closed smooth p -forms on X , i.e. $[\omega_t] = [\omega_0] \in H_{\text{deR}}^p(X)$ for every $t \in [0, 1]$.

(a) Show that there exists a smooth family $(\eta_t)_{t \in [0,1]}$ of $(p-1)$ -forms on X such that

$$\omega_t - \omega_0 = d\eta_t \quad \forall t \in [0, 1].$$

(b) With η_t as in (a), suppose also that $(\zeta_t)_{t \in [0,1]}$ is a smooth family of vector fields on X satisfying

$$\iota_{\zeta_t} \omega_t = -\frac{d}{dt} \eta_t \quad \forall t \in [0, 1]$$

and $\psi_t: X \longrightarrow X$ for $t \in [0, 1]$ is its flow. Show that $\psi_t^* \omega_t = \omega_0$ for all $t \in [0, 1]$.

(c) Suppose ω_t is a symplectic form on X for every $t \in [0, 1]$. Show that there exists a diffeomorphism $\psi: X \longrightarrow X$ such that $\psi^* \omega_1 = \omega_0$.

(d) Suppose X is connected and oriented and Ω_0, Ω_1 are volume forms on X . Show that there exists a diffeomorphism $\psi: X \longrightarrow X$ such that $\psi^* \Omega_1 = \Omega_0$ if and only if $\int_X \Omega_0 = \int_X \Omega_1$.

The assumptions that X is compact and the symplectic forms ω_t are cohomologous necessary for the conclusion of Exercise 3.21(c). For example, \mathbb{C}^n with $n \geq 2$ admits a symplectic structure ω so that (\mathbb{C}^n, ω) is not symplectomorphic to $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$; see [16, 0.4.A'_2]. A smooth family $(\omega_t)_{t \in \mathbb{R}}$ of symplectic forms on a closed 8-dimensional smooth manifold \tilde{Y} is constructed in [22] so that all forms ω_k with $k \in \mathbb{Z}$ are cohomologous and the symplectic manifolds (\tilde{Y}, ω_k) and (\tilde{Y}, ω_ℓ) with $k, \ell \in \mathbb{Z}$ are symplectomorphic if and only if $|k| = |\ell|$; see Theorem 2.1 in [22]. In the case of Exercise 3.21(d), the equality of the integrals implies that

$$\Omega_t \equiv (1-t)\Omega_0 + t\Omega_1, \quad t \in [0, 1],$$

is a smooth family of cohomologous volume forms.

Remark 3.22. The analogue of Proposition 3.16(2) in [23] is Lemma 3.2.1, which unnecessarily requires Y (Q in [23]) to be compact. As a consequence, the Symplectic and Lagrangian Tubular Neighborhood Theorems, i.e. Example 3.18 and Corollary 3.20 above, are restricted to compact submanifolds in [23]; see Theorems 3.4.10 and 3.4.13 in [23]. Even if one is interested only in compact symplectic manifolds, the Symplectic Tubular Neighborhood Theorem without the compactness restriction is needed for the proof of Theorem 6. The latter is a key step in the proof of Delzant's Theorem, Theorem 3, following the modern efficient approach sketched in [21, 24]; see page 68.

3.4 Weakly Hamiltonian vs. Hamiltonian group actions

Exercise 3.23. Let X be a closed oriented manifold.

(1) Suppose $\zeta \in \Gamma(X; TX)$ and Ω is a (top) form on X . Show that

$$\int_X \mathcal{L}_\zeta \Omega = 0.$$

(2) Suppose the dimension of X is $2n$, ω is a symplectic form on X , and ψ is a smooth action of a Lie group G on (X, ω) . For each $v \in T_{\mathbb{1}}G$, let $\zeta_v \in \Gamma(X; TX)$ be the vector field on X generated by v as in (1.2). If $v_1 \in T_{\mathbb{1}}G$ and there exists a smooth map $\mu_{v_1}: X \rightarrow \mathbb{R}$ satisfying the first condition in (1.4) with $v = v_1$, then

$$\int_X \omega(\zeta_{v_1}, \zeta_{v_2}) \omega^n = 0 \quad \forall v_2 \in T_{\mathbb{1}}G.$$

Hint: use Cartan's formula for the Lie derivative.

Exercise 3.24. Suppose (X, ω) is a symplectic manifold, ψ is a smooth action of a connected Lie group G on X as in (1.1), $\mu: X \rightarrow T_{\mathbb{1}}^*G$ is a smooth map satisfying the first condition in (1.4), $v \in T_{\mathbb{1}}G$, and $\zeta_v \in \Gamma(X; TX)$ is the vector field on X generated by v as in (1.2). Let $h: [a, b] \rightarrow G$ be a smooth path. Show that

$$\frac{d}{dt} \mu_v \circ \psi_h = -\omega(\zeta_{\text{Ad}_{h^{-1}}(v)}, \zeta_{\text{Ad}_{h^{-1}}(h'h^{-1})}), \quad \text{where } h' = \frac{dh}{dt}: [a, b] \rightarrow TG.$$

Hint: apply the first condition in (1.4) and (3.9), then the first equation in (3.8) twice, and finally the ψ -invariance of ω .

Proposition 3.25. Suppose (X, ω) is a symplectic manifold, ψ is a smooth action of a connected Lie group G on X as in (1.1), and $\mu: X \rightarrow T_{\mathbb{1}}^*G$ is a smooth map satisfying the first condition in (1.4). For each $v \in T_{\mathbb{1}}G$, let $\zeta_v \in \Gamma(X; TX)$ be the vector field on X generated by v as in (1.2).

(1) For all $v, v' \in T_{\mathbb{1}}G$, the function

$$\mu_{[v, v']} + \omega(\zeta_v, \zeta_{v'}): X \rightarrow \mathbb{R}$$

is constant on each connected component of X .

(2) The map μ satisfies the second condition in (1.4) if and only if

$$\mu_{[v, v']} = -\omega(\zeta_v, \zeta_{v'}): X \rightarrow \mathbb{R} \quad \forall v, v' \in T_{\mathbb{1}}G. \quad (3.24)$$

(3) If G is abelian and either $G \approx \mathbb{R}$, or G is compact, or X is compact, then the map μ satisfies the second condition in (1.4).

(4) If X is closed, connected, and of dimension $2n$, then the smooth map

$$\widehat{\mu}: X \longrightarrow T_{\mathbb{1}}^*G, \quad \widehat{\mu}(x) = \mu(x) - \int_X \mu \omega^n / \int_X \omega^n,$$

is a moment map for the G -action ψ on (X, ω) .

Proof. (1) By the first condition in (1.4) and the second identity in (3.8),

$$d\mu_{[v, v']} = -\omega(\zeta_{[v, v']}, \cdot) = \omega([\zeta_v, \zeta_{v'}], \cdot).$$

By (3.16),

$$d(\omega(\zeta_v, \zeta_{v'})) = -\omega([\zeta_v, \zeta_{v'}], \cdot).$$

The two statements give the claim.

(2) Let $v \in T_{\mathbb{1}}G$, $h: [0, b] \longrightarrow G$ be a smooth path with $h(0) = \mathbb{1}$, and $h': [0, b] \longrightarrow TG$ be its derivative as in (3.3). Thus,

$$\begin{aligned} \mu_v \circ \psi_{h(0)} &= \mu_{\text{Ad}_{h(0)^{-1}}v}, & \frac{d}{dt} \mu_v \circ \psi_h &= -\omega(\zeta_{\text{Ad}_{h^{-1}}(v)}, \zeta_{\text{Ad}_{h^{-1}}(h'h^{-1})}), \\ & & \frac{d}{dt} \mu_{\text{Ad}_{h^{-1}}(v)} &= -\mu_{\text{Ad}_{h^{-1}}([h'h^{-1}, v])} = -\mu_{[\text{Ad}_{h^{-1}}(h'h^{-1}), \text{Ad}_{h^{-1}}(v)]}; \end{aligned} \quad (3.25)$$

the equality in the top equation on the right-hand side above holds by Exercise 3.24, while the two equalities in the bottom equation follow from (3.3) and the second identity in (3.2). Since G is connected, the second condition in (1.4) is thus equivalent to the condition that

$$\mu_{[\text{Ad}_{h^{-1}}(v), \text{Ad}_{h^{-1}}(h'h^{-1})]} = -\omega(\zeta_{\text{Ad}_{h^{-1}}(v)}, \zeta_{\text{Ad}_{h^{-1}}(h'h^{-1})})$$

for every $v \in T_{\mathbb{1}}G$ and every smooth path $h: [0, b] \longrightarrow G$ with $h(0) = \mathbb{1}$. Since Ad_u is an automorphism of $T_{\mathbb{1}}G$, the last condition is equivalent to (3.24).

(3) We can assume that X is connected. Let $v, v' \in T_{\mathbb{1}}G$ and $h(t) = e^{tv'}$ for $t \in \mathbb{R}$. Since G is abelian,

$$\text{Ad}_{h(t)^{-1}}(v) = v \quad \text{and} \quad \text{Ad}_{h^{-1}}(h'h^{-1}) = v'.$$

Along with (3.25), this implies that

$$-\frac{d}{dt} \mu_v \circ \psi_h = -\frac{d}{dt} (\mu_v \circ \psi_h - \mu_{\text{Ad}_{h^{-1}}(v)}) = \mu_{[v, v']} + \omega(\zeta_v, \zeta_{v'}); \quad (3.26)$$

the two sides above are a priori functions on $X \times \mathbb{R}$, but the right-hand side above is explicitly independent of the \mathbb{R} -input and is constant in the X -input by (1). If the right-hand side in (3.26) vanishes (as is automatically the case if $G \approx \mathbb{R}$), then μ satisfies the second condition in (1.4) by (2). If the constant on the right-hand side of (3.26) is not zero, then the function $\mu_v \circ \psi_h$ is unbounded on \mathbb{R} (with the X -input fixed), which implies that G and X are not compact.

(4) Since $\widehat{\mu}$ differs from μ by a constant, $\widehat{\mu}$ satisfies the first condition in (1.4) with μ replaced by $\widehat{\mu}$. Let $v, v' \in T_{\mathbb{1}}G$. By (1),

$$\widehat{\mu}_{[v, v']} + \omega(\zeta_v, \zeta_{v'}) : X \longrightarrow \mathbb{R}$$

is a constant function on X . By the definition of $\widehat{\mu}$ and Exercise 3.23(2),

$$\int_X \widehat{\mu}_{[v, v']} \omega^n = 0 \quad \text{and} \quad \int_X \omega(\zeta_v, \zeta_{v'}) \omega^n = 0,$$

respectively. Thus,

$$\widehat{\mu}_{[v, v']} = -\omega(\zeta_v, \zeta_{v'}) : X \longrightarrow \mathbb{R} \quad \forall v, v' \in T_{\mathbb{1}}G.$$

Along with (2), this implies that $\widehat{\mu}$ satisfies the second condition in (1.4) with μ replaced by $\widehat{\mu}$. \square

Let G be a Lie group and V be a vector space (over \mathbb{R}). For $k \in \mathbb{Z}^{\geq 0}$, define

$$\begin{aligned} d_k^G : \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^k(T_{\mathbb{1}}G); V) &\longrightarrow \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^{k+1}(T_{\mathbb{1}}G); V), \\ \{d_k^G f\}(v_0, v_1, \dots, v_k) &= \sum_{0 \leq i < j \leq k} (-1)^{i+j} f([v_i, v_j], v_0, v_1, \dots, \widehat{v}_i, \dots, \widehat{v}_j, \dots, v_k), \end{aligned}$$

where $\widehat{v}_i, \widehat{v}_j$ denote skipped inputs. By the Jacobi identity, $d_k^G \circ d_{k-1}^G = 0$ for every $k \in \mathbb{Z}^+$. Thus, the k -th cohomology group of $T_{\mathbb{1}}G$ with coefficients in V ,

$$H^k(T_{\mathbb{1}}G; V) \equiv \frac{\ker(d_k^G : \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^k(T_{\mathbb{1}}G); V) \longrightarrow \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^{k+1}(T_{\mathbb{1}}G); V))}{\text{Im}(d_{k-1}^G : \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^{k-1}(T_{\mathbb{1}}G); V) \longrightarrow \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^k(T_{\mathbb{1}}G); V))}$$

is well-defined. Let $H^k(T_{\mathbb{1}}G) = H^k(T_{\mathbb{1}}G; \mathbb{R})$. In particular,

$$H^1(T_{\mathbb{1}}G; V) \equiv \ker(d_1^G : \text{Hom}_{\mathbb{R}}(T_{\mathbb{1}}G; V) \longrightarrow \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^2(T_{\mathbb{1}}G); V)).$$

If the identity component of G is abelian, then $d_k^G = 0$ for all $k \in \mathbb{Z}^{\geq 0}$ and so

$$H^k(T_{\mathbb{1}}G; V) = \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^k(T_{\mathbb{1}}G); V).$$

Exercise 3.26. Suppose G is a Lie group and $G' \subset G$ is a normal subgroup. Let

$$\mathcal{K}_{G'; G}^1 = \{\alpha \in T_{\mathbb{1}}^*G' : \alpha([v, v']) = 0 \ \forall v \in T_{\mathbb{1}}G, v' \in T_{\mathbb{1}}G'\} \subset H^1(T_{\mathbb{1}}G').$$

Show that

(a) the short exact sequence

$$0 \longrightarrow T_{\mathbb{1}}^*(G/G') = \{\alpha \in T_{\mathbb{1}}^*G : \alpha|_{T_{\mathbb{1}}G'} = 0\} \longrightarrow T_{\mathbb{1}}^*G \longrightarrow T_{\mathbb{1}}G \longrightarrow 0$$

of Lie algebras induces an exact sequence

$$0 \longrightarrow H^1(T_{\mathbb{1}}(G/G')) \longrightarrow H^1(T_{\mathbb{1}}G) \longrightarrow \mathcal{K}_{G'; G}^1 \xrightarrow{\delta_1} H^2(T_{\mathbb{1}}(G/G')) \longrightarrow H^2(T_{\mathbb{1}}G);$$

(b) if the identity component of G is abelian, then $\delta_1 = 0$ above.

Exercise 3.27. Let ψ be a smooth action of a Lie group G on a symplectic manifold (X, ω) .

- (a) For each $v \in T_{\mathbb{1}}G$, let $\zeta_v \in \Gamma(X; TX)$ be the vector field on X generated by v as in (1.2). By Exercise 3.13(a), the homomorphism

$$\psi_\omega : T_{\mathbb{1}}G \longrightarrow H_{\text{deR}}^1(X), \quad \psi_\omega(v) = [\iota_{\zeta_v} \omega],$$

is well-defined. Show that ψ_ω lies in the kernel of

$$d_1^G : \text{Hom}_{\mathbb{R}}(T_{\mathbb{1}}G; H_{\text{deR}}^1(X)) \longrightarrow \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^2(T_{\mathbb{1}}G); H_{\text{deR}}^1(X)).$$

- (b) Suppose in addition X is connected and $\mu : X \longrightarrow T_{\mathbb{1}}^*G$ is a smooth map satisfying the first condition in (1.4). For $v, v' \in T_{\mathbb{1}}G$, let $\psi_{\omega; \mu}(v, v') \in \mathbb{R}$ be the constant value of the function in Proposition 3.25(1). Show that

$$\begin{aligned} \psi_{\omega; \mu} &\in \ker(d_2^G : \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^2(T_{\mathbb{1}}G); \mathbb{R}) \longrightarrow \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^3(T_{\mathbb{1}}G); \mathbb{R})) \\ \text{and} \quad \psi_{\omega; \mu + \alpha_0} - \psi_{\omega; \mu} &= -d_1^G \alpha_0 \quad \forall \alpha_0 \in T_{\mathbb{1}}^*G. \end{aligned}$$

Hint: use (3.16) for (a), the second identity in (3.8) and (3.17) for (b).

By Exercise 3.27(a), a smooth action ψ on a symplectic manifold (X, ω) determines an element

$$\psi_\omega^1 \equiv [\psi_\omega] \in H^1(T_{\mathbb{1}}G; H_{\text{deR}}^1(X)).$$

This action is weakly Hamiltonian if and only if $\psi_\omega^1 = 0$. By Exercise 3.27(b), a weakly Hamiltonian action ψ on a connected symplectic manifold (X, ω) determines an element

$$\psi_\omega^2 \equiv [\psi_{\omega; \mu}] \in H^2(T_{\mathbb{1}}G).$$

By Proposition 3.25(2), this action on (X, ω) is Hamiltonian if and only if $\psi_\omega^2 = 0$.

Corollary 3.28. *Suppose (X, ω) is a symplectic manifold and G is a semisimple Lie group G . Every smooth G -action on (X, ω) is Hamiltonian.*

Proof. We can assume that X is connected. By [10, Theorem 21.1], $H^1(T_{\mathbb{1}}G), H^2(T_{\mathbb{1}}G) = \{0\}$. By Exercise 3.27(a), every smooth G -action on (X, ω) is thus weakly Hamiltonian. By Exercise 3.27(b) and Proposition 3.25(2), every weakly Hamiltonian G -action on (X, ω) is similarly Hamiltonian. \square

Example 3.29. With $n \in \mathbb{Z}^+$, let ψ be the action of \mathbb{C}^n on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ as in Exercise 2.7. Let $e_1, ie_1, \dots, e_n, ie_n$ be the standard \mathbb{R} -basis for $T_0\mathbb{C}^n = \mathbb{C}^n$ as before. By Exercise 2.7, this action is weakly Hamiltonian, but not Hamiltonian, with

$$\psi_\omega^2 \in H^2(T_{\mathbb{1}}\mathbb{C}^n) = \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}^2(\mathbb{C}^n); \mathbb{R})$$

given by

$$\psi_\omega^2(e_j \wedge e_k), \psi_\omega^2(ie_j \wedge ie_k) = 0, \quad \psi_\omega^2(e_j \wedge ie_k) = \delta_{jk}. \quad \forall j, k \in [n].$$

Exercise 3.30. Suppose ψ is a smooth action of a Lie group G on a nonempty symplectic manifold (X, ω) as in (1.1), and $G' \subset G$ is a normal subgroup acting trivially on X . Let ψ' be the induced action of G/G' as in Exercise 2.4(d) and

$$\delta_1 : \mathcal{K}_{G'; G}^1 \longrightarrow H^2(T_{\mathbb{1}}(G/G'))$$

be as in Exercise 3.26. Show that

- (a) $\psi_\omega^1|_{T_{\mathbb{1}}G'} = 0$ and thus the G/G' -action ψ' on (X, ω) is weakly Hamiltonian if and only if the G -action ψ is;
- (b) if (X, ω, ψ, μ) is a Hamiltonian G -manifold and $\alpha_0 \in T_{\mathbb{1}}^*G$ satisfies (2.10), then $\alpha_0|_{T_{\mathbb{1}}G'} \in \mathcal{K}_{G';G}^1$;
- (c) if (X, ω, ψ, μ) is a connected Hamiltonian G -manifold and $\alpha_0 \in T_{\mathbb{1}}^*G$ satisfies (2.10), then

$$\psi_\omega'^2 = \delta_1(\alpha_0|_{T_{\mathbb{1}}G'}) \in H^2(T_{\mathbb{1}}(G/G')).$$

Hint: first show (b) for $\alpha_0 \in \mu(X)$ via Proposition 3.25(2).

Remark 3.31. Let ψ be a smooth action of a Lie group G on a symplectic manifold (X, ω) and $\mu: X \rightarrow T_{\mathbb{1}}^*G$ be a smooth map satisfying the first condition in (1.4). The condition (3.24) is then equivalent to the vector space homomorphism

$$T_{\mathbb{1}}G \rightarrow C^\infty(X; \mathbb{R}), \quad v \rightarrow \mu_v,$$

being a Lie algebra or anti Lie algebra homomorphism, depending on the exact conventions, with respect to the Poisson bracket determined by ω on $C^\infty(X; \mathbb{R})$; see [23, p99]. Some authors use (3.24) instead of the second condition in (1.4) in the definition of moment map. By Proposition 3.25(2), (3.24) is equivalent to the second condition in (1.4) restricted to the identity component of G and completely ignores the remaining components of G . Thus, the second condition in (1.4) is more natural for the notion of moment map than (3.24).

3.5 Hamiltonian group actions

We next obtain structural results for Hamiltonian group actions and their moment maps, in particular Proposition A.5 and Corollary 3.40.

Exercise 3.32. Suppose G is a Lie group, (X, ω, ψ, μ) is a Hamiltonian G -manifold, and $x \in X$. For each $v \in T_{\mathbb{1}}G$, let $\zeta_v \in \Gamma(X; TX)$ be as in (1.2). Show that

$$\begin{aligned} \ker d_x \mu &= \{\zeta_v(x) : v \in T_{\mathbb{1}}G\}^\omega \equiv \{w \in T_x X : \omega(w, \zeta_v(x)) = 0 \ \forall v \in T_{\mathbb{1}}G\}, \\ \text{Im } d_x \mu &= \text{Ann}(\{v \in T_{\mathbb{1}}G : \zeta_v(x) = 0\}) \equiv \{\alpha \in T_{\mathbb{1}}^*G : \alpha(v) = 0 \ \forall v \in T_{\mathbb{1}}G \text{ s.t. } \zeta_v(x) = 0\}. \end{aligned} \quad (3.27)$$

Conclude that

- (a) the G -orbit $Gx \subset X$ of x is open if and only if $d_x \mu$ is injective;
- (b) the stabilizer $\text{Stab}_x(\psi) \subset G$ of x is discrete if and only if $d_x \mu$ is surjective.

Exercise 3.33. Suppose \mathbb{T} is a torus and (X, ω, ψ, μ) is a Hamiltonian \mathbb{T} -manifold so that (1.7) holds, the action ψ is free, and the fibers of μ are connected.

- (a) Show that $\mu(X) \subset T_{\mathbb{1}}^*\mathbb{T}$ is an open subset, $\mu: X \rightarrow \mu(X)$ is a principal \mathbb{T} -bundle, and the fibers of μ are Lagrangian submanifolds of (X, ω) , i.e.

$$\dim \mu^{-1}(\alpha) = n \quad \text{and} \quad \omega|_{T\mu^{-1}(\alpha)} = 0 \quad \forall \alpha \in \mu(X).$$

- (b) Let η be a 1-form on $\mu(X)$. Show that the vector field ζ_η on X defined by $\iota_{\zeta_\eta} \omega = \mu^* \eta$ is μ -vertical, i.e.

$$d\mu(\zeta_\eta) = 0 \in \Gamma(\mu(X); T\mu(X)).$$

- (c) Let $(T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})$ be the Hamiltonian \mathbb{T} -manifold of Exercise 2.14, with $k=n$. Suppose $s: \mu(X) \rightarrow X$ is a (smooth) Lagrangian section of μ , i.e. $\mu \circ s = \text{id}_{\mu(X)}$ and $s^*\omega = 0$. Show that the map

$$\Phi: \mu(X) \times \mathbb{T} \rightarrow X, \quad \Phi(\alpha, u) = \psi_u(s(\alpha)),$$

is a \mathbb{T} -equivariant diffeomorphism such that $\Phi^*\omega = \omega_{\mathbb{T}}|_{\mu(X) \times \mathbb{T}}$ and $\mu \circ \Phi = \mu_{\mathbb{T}}|_{\mu(X) \times \mathbb{T}}$. *Hint:* choose a \mathbb{Z} -basis $v_1, \dots, v_n \in T_{\mathbb{1}}\mathbb{T}$ for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \subset T_{\mathbb{1}}\mathbb{T}$ and replace μ by the corresponding Hamiltonian $H: X \rightarrow \mathbb{R}^n$ and $(T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, H_{\mathbb{T}})$ by the Hamiltonian \mathbb{T} -manifold $(\mathbb{R}^n \times \mathbb{T}^n, \omega_n, \psi_n, H_n)$ as in Exercise 2.10.

Exercise 3.34. Suppose G is a positive-dimensional Lie group, (X, ω) is a compact positive-dimensional symplectic manifold, and ψ is a smooth G -action on (X, ω) .

- (a) Suppose ψ is a Hamiltonian action. Show that the ψ -fixed locus X^ψ contains at least 2 points.
(b) Give an example of a compact positive-dimensional symplectic manifold (X, ω) and an action ψ on (X, ω) so that $X^\psi = \emptyset$.

Exercise 3.35. Suppose \mathbb{T} is a torus, (X, ω, ψ, μ) is a compact Hamiltonian \mathbb{T} -manifold, $x \in X$, and $Z \in \pi_0(X^{\mathbb{T}^\psi(x)})$ is the topological component of the $\psi|_{\mathbb{T}^\psi(x)}$ -fixed locus containing x . For each $Y \in \pi_0(X^\psi)$, let $S(Y) \subset T_{\mathbb{1}}^*\mathbb{T}$ be as in Proposition 3.11(1). Show that

- (a) $Z \cap X^\psi \neq \emptyset$;
(b) if $Y \in \pi_0(X^\psi)$ and $Y \subset Z$, then $\mathbb{T}_x(\psi) = \mathbb{T}_S$ for some $S \subset S(Y)$.

Hint: use Exercise 3.34(a) and Proposition 3.11(2).

Exercise 3.36. Suppose G is a Lie group, (X, ω) is a symplectic manifold, ψ is a smooth G -action on (X, ω) , J is an ω -compatible almost complex structure on X , and $x \in X$. For each $v \in T_{\mathbb{1}}G$, let $\zeta_v \in \Gamma(X; TX)$ be as in (1.2).

- (a) Let $\mu: X \rightarrow T_{\mathbb{1}}^*G$ be a smooth map satisfying the first condition in (1.4) and $g(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$ be the Riemannian metric on X determined by ω and J . Show that

$$\nabla^g \mu_v = -J\zeta_v \in \Gamma(X; TX) \quad \forall v \in T_{\mathbb{1}}G. \quad (3.28)$$

- (b) Let $\mu: X \rightarrow T_{\mathbb{1}}^*G$ be a G -invariant smooth map satisfying the first condition in (1.4). Show that

$$\{\zeta_v(x): v \in T_{\mathbb{1}}G\} \cap \{J\zeta_v(x): v \in T_{\mathbb{1}}G\} = \{0\} \subset T_x X.$$

Suppose in addition that $\alpha \equiv \mu(x) \in T_{\mathbb{1}}^*G$ is a regular value of μ and thus $\mu^{-1}(\alpha) \subset X$ is a smooth submanifold. Show that

$$T_x X = T_x(\mu^{-1}(\alpha)) \oplus \{J\zeta_v(x): v \in T_{\mathbb{1}}G\}. \quad (3.29)$$

- (c) Give an example of a positive-dimensional symplectic manifold (X, ω) and an action ψ on (X, ω) so that

$$\{\zeta_v(x): v \in T_{\mathbb{1}}G\} = \{J\zeta_v(x): v \in T_{\mathbb{1}}G\} = T_x X.$$

Exercise 3.37. Suppose (X, ω) is a symplectic manifold, ψ is a smooth action of \mathbb{R}^k on (X, ω) , $X' \subset X$ is an ω -symplectic manifold preserved ϕ so that the inclusion $i: X' \rightarrow X$ is a homotopy equivalence, and $\mu': X' \rightarrow T_0^*\mathbb{R}^k$ is a moment map for the restriction of the action ψ to X' . Show that μ' extends to a moment map $\mu: X \rightarrow T_0^*\mathbb{R}^k$ for ψ . *Hint:* first show this for $k=1$.

Proposition 3.38. Suppose $k \in \mathbb{Z}^+$, (X, ω, ψ, μ) is a Hamiltonian \mathbb{R}^k -manifold, ψ' is a smooth action of a torus \mathbb{T} on X , $\rho: \mathbb{R}^k \rightarrow \mathbb{T}$ is a homomorphism with dense image so that $\psi = \psi' \circ \rho$, $Y \subset X$ is a topological component of $X^\psi = X^{\psi'}$, and J is a ψ -invariant (or equivalently ψ' -invariant) ω -compatible almost complex structure on X . Let $S(Y) \subset (T_1^*\mathbb{T})_{\mathbb{Z}}$ and $\mathcal{N}_X^\alpha Y \subset TX|_Y$ for each $\alpha \in S(Y)$ be as in Proposition 3.11(1) with ψ replaced by ψ' so that the complex structure on $\mathcal{N}_X^\alpha Y$ is induced by J . For every $y \in Y$, there exists a \mathbb{T} -equivariant tubular neighborhood identification $\Phi_y: \mathcal{U}_y \rightarrow U_y$ for y in X such that

$$\begin{aligned} \Phi_y^* \omega = \omega_y|_{\mathcal{U}_y} \quad \text{and} \quad \mu(\Phi_y(w_0, (w_\alpha)_{\alpha \in S(Y)})) &= \mu(Y) + \pi \sum_{\alpha \in S(Y)} |w_\alpha|^2 \rho^* \alpha \\ \forall (w_0, (w_\alpha)_{\alpha \in S(Y)}) \in \mathcal{U}_y \subset T_y X = T_y Y \oplus \bigoplus_{\alpha \in S(Y)} \mathcal{N}_X^\alpha Y|_y, \end{aligned} \quad (3.30)$$

where $|\cdot|$ is the norm on TX with respect to the metric $g(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$. If in addition X is closed and connected, then

$$\mu(X) \subset \mathcal{C}_{\mu(Y)}(\rho^* S(Y)) \equiv \left\{ \mu(Y) + \sum_{\alpha \in S(Y)} t_\alpha \rho^* \alpha : t_\alpha \in \mathbb{R}^{\geq 0} \forall \alpha \in S(Y) \right\}. \quad (3.31)$$

Proof. By Corollary 3.19 with ψ replaced by ψ' , there exists a \mathbb{T} -equivariant (or equivalently \mathbb{R}^k -equivariant) tubular neighborhood identification $\Phi_y: \mathcal{U}_y \rightarrow U_y$ for y in X satisfying the first condition in (3.30). By Proposition 3.11(1), the complex vector space $(T_y X, J_y)$ splits as

$$T_y X|_Y = T_y Y \oplus \bigoplus_{\alpha \in S(Y)} \mathcal{N}_X^\alpha Y|_y$$

with the \mathbb{T} -action $d_y \psi'$ given by (3.15) with ψ replaced by ψ' . By Example 2.8, a moment map for this action with respect to ω_y is

$$\mu': T_y X \rightarrow T_1^* \mathbb{T}, \quad \mu'(w_0, (w_\alpha)_{\alpha \in S(Y)}) = \pi \sum_{\alpha \in S(Y)} |w_\alpha|^2 \alpha \quad \forall (w_0, (w_\alpha)_{\alpha \in S(Y)}) \in T_y Y \oplus \bigoplus_{\alpha \in S(Y)} \mathcal{N}_X^\alpha Y|_y.$$

Since a moment map is unique up to an additive constant on each connected component of the domain, it follows that

$$\Phi_y^* \mu = \mu(y) + \rho^* \circ \mu'|_{\mathcal{U}_y}: \mathcal{U}_y \rightarrow T_0^* \mathbb{R}^k.$$

This establishes the second condition in (3.30).

Suppose in addition that X is closed and connected and $\eta_0 \in T_0^* \mathbb{R}^k - \mathcal{C}_{\mu(Y)}(\rho^* S(Y))$. Thus, $\mathcal{C}_{\mu(Y)}(\rho^* S(Y))$ is contained in a (closed) half-space in $T_0^* \mathbb{R}^k$ and there exists $v \in T_0^* \mathbb{R}^k$ so that

$$\eta_0(v) < \inf \{ \eta(v) : \eta \in \mathcal{C}_{\mu(Y)}(\rho^* S(Y)) \} = \{ \mu(Y) \}(v) \equiv \mu_v(Y). \quad (3.32)$$

By the second equality in (3.30), this implies that $\{\rho^*\alpha\}(v) \geq 0$ for all $\alpha \in S(Y)$. Thus,

$$\mu_v(Y) = \inf\{\mu_v(x) : x \in U_y\}.$$

Combining this with Proposition A.5(2), we conclude that

$$\mu_v(Y) = \inf\{\mu_v(x) \equiv \{\mu(x)\}(v) : x \in X\} \quad \forall y \in Y.$$

Along with (3.32), this implies that $\eta_0 \notin \mu(X)$ and establishes (3.31). \square

Corollary 3.39. *Suppose $k \in \mathbb{Z}^+$, (X, ω, ψ, μ) is a compact Hamiltonian \mathbb{R}^k -manifold, the \mathbb{R}^k -action ψ is almost periodic, and the ψ -fixed locus $X^\psi \subset X$ is discrete. Then,*

- (1) $H_{\text{odd}}(X; \mathbb{Z})$ is trivial and $H_{\text{even}}(X; \mathbb{Z})$ is a free \mathbb{Z} -module of rank $|X^\psi|$;
- (2) if $\rho: \mathbb{R}^k \rightarrow \mathbb{T}$, $S(y) \subset (T_0^*\mathbb{T})_{\mathbb{Z}}$ for each $y \in X^\psi$, and $\mathcal{N}_X^\alpha\{y\} \subset T_y X$ for each $\alpha \in S(y)$ are as in Proposition 3.38, $v \in T_0\mathbb{R}^k$ is generic, $S_v(y) = \{\alpha \in S(y) : \alpha(\rho(v)) < 0\}$, and $p \in \mathbb{Z}^{\geq 0}$, then

$$\text{rk}_{\mathbb{Z}} H_{2p}(X; \mathbb{Z}) = |X_{v;2p}^\psi|, \quad \text{where } X_{v;2p}^\psi = \left\{ y \in X^\psi : \sum_{\alpha \in S_v(y)} \dim_{\mathbb{R}} \mathcal{N}_X^\alpha\{y\} = 2p \right\}; \quad (3.33)$$

- (3) if in addition the linear spans of $S_v(y)$ and $S_{-v}(y)$ in $T_0^*\mathbb{R}^k$ are disjoint (except at 0) for every $y \in X^\psi$, $\mathbb{R}_{y;v}^- \subset \mathbb{R}^k = T_0\mathbb{R}^k$ is the annihilator of $S_v(y)$, and $X_{y;v}^-(\psi) \subset X$ is the topological component of the $\psi|_{\mathbb{R}_{y;v}^-}$ -fixed locus containing $y \in X^\psi$, then $H_{2p}(X; \mathbb{Z})$ is freely generated by the fundamental homology classes of the submanifolds $X_{y;v}^-(\psi)$ with $y \in X_{v;2p}^\psi$.

Proof. Let ρ , $S(y)$, and $\mathcal{N}_X^\alpha\{y\} \subset T_y X$ be as in Proposition 3.38. For each $\alpha \in \mathcal{N}_X^\alpha\{y\}$, let $n_{y\alpha} \in \mathbb{Z}^+$ be half the (real) dimension of $\mathcal{N}_X^\alpha\{y\}$. For a generic choice of $v \in \mathbb{R}^k$, $\rho(\mathbb{R}v) \subset \mathbb{T}$ is dense and thus $X^{\mathbb{R}v} = X^\psi$. We apply Proposition 3.38 to the restriction of the action ψ to $\mathbb{R}v \subset \mathbb{R}^k$ and its Hamiltonian

$$\mu_v: X \rightarrow \mathbb{R}, \quad \mu_v(x) = \{\mu(x)\}(x).$$

For every $y \in X^\psi$, there thus exists a coordinate chart

$$(w_\alpha)_{\alpha \in S(y)}: U_y \rightarrow \prod_{\alpha \in S(y)} \mathbb{C}^{n_{y\alpha}} \quad \text{s.t.} \quad \mu_v = \mu_v(y) + \pi \sum_{\alpha \in S(y)} \alpha(\rho(v)) |w_\alpha|^2: U_y \rightarrow \mathbb{R}. \quad (3.34)$$

By Exercise 1.3(c), $\alpha(\rho(v)) \neq 0$ for all $\alpha \in S(y)$. By the first condition in (1.4) and Proposition 3.2(1),

$$\text{Crit}(\mu_v) \equiv \{y \in X : d_y \mu_v = 0\} = X^\psi.$$

Thus, μ_v is a Morse function, i.e. a Morse-Bott function with only isolated critical points. Along with [26, Theorem 3.5] and (3.34), this implies that X has the homotopy type of a CW complex with only even-dimensional cells and the number of cells of dimension $2p$ with $p \in \mathbb{Z}^{\geq 0}$ given by the right-hand side in (3.33). This gives (1) and (2).

We now assume that the linear spans of $S_v(y)$ and $S_{-v}(y)$ in $T_0^*\mathbb{R}^k$ are disjoint (except at 0) for every $y \in X^\psi$. Let $\psi_{v;t}: X \rightarrow X$ be the negative gradient flow of μ_v , as in Exercise A.6, with

respect to a ψ -invariant Riemannian metric g and $X_y^-(\mu_v) \subset X$ for each $y \in X^\psi$ be the μ_v -unstable manifold of y , as in (A.5). By (A.6) and the last equality in (3.30),

$$T_y(X_y^-(\mu_v)) = E_y^-(\mu_v) = \bigoplus_{\alpha \in S_v(y)} \mathcal{N}_X^\alpha\{y\} = T_y(X_{y;v}^-(\psi)).$$

Since the flow $\psi_{v;t}$ commutes with the action ψ , it follows that $X_y^-(\mu_v) \subset X_{y;v}^-(\psi)$ is an open subset and $X_{y;v}^-(\psi) - X_y^-(\mu_v)$ is the union of all other unstable manifolds of $\mu_v|_{X_{y;v}^-(\psi)}$. Since the indices of all critical points are even, $X_y^-(\mu_v)$ is a $2p$ -pseudocycle in the sense of [36, Theorem 1.1] for every $y \in X_{v;2p}^\psi$ and the boundary operator in the Witten chain complex of [32, Section 3] vanishes. By [32, Theorem 3.1], $H_{2p}(X; \mathbb{Z})$ is thus freely generated by the homology classes of these pseudocycles. The same applies to every compact connected submanifold $X_{y;v}^-(\psi) \subset X$. In particular, the fundamental homology class of $X_{y;v}^-(\psi)$ and thus its image in X are the homology classes in $X_{y;v}^-(\psi)$ and in X , respectively, determined by the pseudocycle $X_y^-(\mu_v) \subset X_{y;v}^-(\psi)$, X for every $y \in X^\psi$. This gives (3). \square

Suppose ψ is a smooth action of \mathbb{R}^k on a smooth manifold X . For $y \in X$, let $\mathbb{R}_y^k \subset \mathbb{R}^k$ be the largest linear subspace preserving y and $\mathbb{R}_y^c \subset \mathbb{R}^k$ be a complementary linear subspace. The decomposition $\mathbb{R}^k = \mathbb{R}_y^k \oplus \mathbb{R}_y^c$ induces decompositions

$$T_0\mathbb{R}^k = T_0\mathbb{R}_y^k \times T_0\mathbb{R}_y^c \quad \text{and} \quad \mu \equiv (\mu_y, \mu_y^c): X \longrightarrow T_0^*\mathbb{R}_y^k \times T_0^*\mathbb{R}_y^c = T_0^*\mathbb{R}^k, \quad (3.35)$$

for any map $\mu: X \longrightarrow T_0^*\mathbb{R}^k$. If μ is a moment map for the action ψ on X with respect to a symplectic form ω , then μ_y and μ_y^c are moment maps for the \mathbb{R}_y^k -action $\psi_y \equiv \psi|_{\mathbb{R}_y^k}$ and \mathbb{R}_y^c -action $\psi_y^c \equiv \psi|_{\mathbb{R}_y^c}$, respectively, on X with respect to ω . If the action ψ is almost periodic, then so are the actions ψ_y and ψ_y^c . By Proposition 3.16(1), $(X^{\psi_y}, \omega|_{X^{\psi_y}}, \psi_y^c, \mu_y^c)$ is then a Hamiltonian \mathbb{R}_y^c -manifold. By Exercises 3.32(b) and 3.3 with (X, ω, ψ, μ) replaced by $(X^{\psi_y}, \omega|_{X^{\psi_y}}, \psi_y^c, \mu_y^c)$, the differential

$$d_y\mu_y^c: T_yX^{\psi_y} \longrightarrow T_0^*\mathbb{R}_y^c \quad (3.36)$$

is surjective in this case.

Corollary 3.40. *Suppose $k \in \mathbb{Z}^+$, (X, ω, ψ, μ) is a Hamiltonian \mathbb{R}^k -manifold, the \mathbb{R}^k -action ψ is almost periodic, $y \in X$, and $\mathbb{R}_y^k, \mathbb{R}_y^c \subset \mathbb{R}^k$, μ_y, μ_y^c , and ψ_y are as above. There exist a finite subset $S(y) \subset T_0^*\mathbb{R}_y^c$, neighborhoods $\mathcal{U}_{y;1}$ of y in X^{ψ_y} , $\mathcal{U}_{y;2}$ of 0 in $\mathbb{C}^{S(y)}$, and U_y of y in X , and a diffeomorphism $\Phi_y: \mathcal{U}_{y;1} \times \mathcal{U}_{y;2} \longrightarrow U_y$ such that*

$$\begin{aligned} |S(y)| &\leq (\dim X)/2 - \dim \mathbb{R}_y^c, \\ \Phi_y(y', 0) &= y', \quad \mu_y^c(\Phi_y(y', w)) = \mu_y^c(y'), \\ \mu_y(\Phi_y(y', (w_\alpha)_{\alpha \in S(y)})) &= \mu_y(y) + \pi \sum_{\alpha \in S(y)} |w_\alpha|^2 \alpha \quad \forall y' \in \mathcal{U}_{y;1}, w \equiv (w_\alpha)_{\alpha \in S(y)} \in \mathcal{U}_{y;2} \subset \mathbb{C}^{S(y)}. \end{aligned} \quad (3.37)$$

In particular, the map μ is locally convex. If in addition X is closed and connected, then there exist a cone $\mathcal{C}_y(\psi) \subset T_0^\mathbb{R}^k$ with vertex at $\mu(y)$ and a neighborhood $U_y \subset X$ of y so that $\mu(X) \subset \mathcal{C}_y(\psi)$ and the restriction $\mu: U_y \longrightarrow \mathcal{C}_y(\psi)$ is an open map.*

Proof. Let $Y \in \pi_0(X^{\psi_y})$ be the connected component of ψ_y -fixed locus containing y and ψ_y^c be as above. With $S(Y)$ and ρ as in Proposition 3.38 with (ψ, μ) replaced by (ψ_y, μ_y) , let

$$S(y) = \rho^* S(Y) \subset T_{\mathbb{1}}^* \mathbb{R}_y^k.$$

By Proposition 3.16(1), $Y \subset X$ is a symplectic submanifold with $\mu_y(Y) = \mu_y(y)$. Since the actions ψ_y and ψ_y^c commute, X^{ψ_y} is preserved by ψ_y^c . Since the differential (3.36) is surjective, the second equations in (3.27) and in (3.28) applied to $(Y, \omega|_Y, \psi_y^c|_Y, \mu_y^c|_Y)$ imply that

$$\dim Y \geq 2 \dim \mathbb{R}_y^c \implies 2|S(y)| \leq \dim T_y X - \dim T_y Y \leq \dim X - 2 \dim \mathbb{R}_y^c.$$

This establishes (3.37). The remainder of the first claim of the corollary follows from the first statement of Proposition 3.38 with (ψ, μ) replaced by (ψ_y, μ_y) and Exercise 3.41 below with $k = \dim Y$, $\ell = \text{codim } Y$, $m = \mathbb{R}_y^c$, and $f = \mu_y^c$. Along with Exercises 1.8-1.10, this claim implies the convexity claim.

Suppose in addition that X is closed and connected. By the second statement of Proposition 3.38,

$$\mu(X) \subset \mathcal{C}_y(\psi) \equiv \mathcal{C}_{\mu(y)}(S(y)) \times T_0^* \mathbb{R}_y^c \subset T_0^* \mathbb{R}^k.$$

By the first claim of the corollary and Exercises 1.8-1.10, there exists a neighborhood $U_y \subset X$ of y so that the restriction $\mu: U_y \rightarrow \mathcal{C}_y(\psi)$ is an open map. \square

Exercise 3.41. Suppose $k, \ell, m \in \mathbb{Z}^{\geq 0}$ and $f: \mathbb{R}^k \times \mathbb{R}^\ell \rightarrow \mathbb{R}^m$ is a smooth function so that the restriction of the differential $d_{(0,0)} f$ to $\mathbb{R}^k \times \{0\}$ is surjective. Show that there exist neighborhoods \mathcal{U}_1 of $0 \in \mathbb{R}^k$ and \mathcal{U}_2 of $0 \in \mathbb{R}^\ell$ and a smooth map

$$\phi: \mathcal{U}_1 \times \mathcal{U}_2 \rightarrow \mathbb{R}^k \quad \text{s.t.} \quad \phi(x, 0) = x, \quad f(\phi(x, w), w) = f(x, 0) \quad \forall x \in \mathcal{U}_1, w \in \mathcal{U}_2,$$

and for each $w \in \mathcal{U}_2$ the map $\mathcal{U}_1 \rightarrow \mathbb{R}^k$, $x \rightarrow \phi(x, w)$, is a diffeomorphism onto an open subset of \mathbb{R}^k . *Hint:* assume that the restriction of $d_{(0,0)} f$ to $\mathbb{R}^m \times \{0\} \times \{0\} \subset \mathbb{R}^k \times \{0\}$ is surjective; show that there exist neighborhoods \mathcal{U}_1 of $0 \in \mathbb{R}^k$ and \mathcal{U}_2 of $0 \in \mathbb{R}^\ell$ so that for each $w \in \mathcal{U}_2$ the map

$$\Phi_w: \mathcal{U}_1 \rightarrow \mathbb{R}^k, \quad \Phi_w(x_1, x_2) = (f((x_1, x_2), w), x_2) \quad \forall (x_1, x_2) \in \mathcal{U}_1 \subset \mathbb{R}^m \times \mathbb{R}^{k-m},$$

is a diffeomorphism onto an open subset of \mathbb{R}^k .

Proposition 3.42. Suppose \mathbb{T} is a torus and (X, ω, ψ, μ) is a Hamiltonian \mathbb{T} -manifold so that (1.7) holds, the action ψ is free, the fibers of μ are connected, and $\mu(X) \subset T_{\mathbb{1}}^* \mathbb{T}$ is contractible. Let $(T_{\mathbb{1}}^* \mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})$ be the Hamiltonian \mathbb{T} -manifold of Exercise 2.14. Then $\mu(X) \subset T_{\mathbb{1}}^* \mathbb{T}$ is an open subset and there exists a \mathbb{T} -equivariant diffeomorphism

$$\Phi: \mu(X) \times \mathbb{T} \rightarrow X \quad \text{s.t.} \quad \Phi^* \omega = \omega_{\mathbb{T}}|_{\mu(X) \times \mathbb{T}}, \quad \mu \circ \Phi = \mu_{\mathbb{T}}|_{\mu(X) \times \mathbb{T}}.$$

Proof. The subset $\mu(X) \subset T_{\mathbb{1}}^* \mathbb{T}$ is open by Exercise 3.32(b). By Exercise 3.33(c), it remains to show that μ admits a Lagrangian section $\mu(X) \rightarrow X$. Since $\mu(X) \subset T_{\mathbb{1}}^* \mathbb{T}$ is contractible, μ admits a section $s: \mu(X) \rightarrow X$ and $s^* \omega = d\eta$ for some 1-form η on $\mu(X)$. Let $\zeta_\eta \in \Gamma(X; TX)$ be the μ -vertical vector field of Exercise 3.33(b). Since the fibers of μ are compact and the vector field ζ_η is vertical, the flow of $-\zeta_\eta$,

$$\psi_t: X \rightarrow X, \quad \psi_0 = \text{id}_X, \quad \frac{d}{dt} \psi_t = -\zeta_\eta \circ \psi_t,$$

is defined for all $t \in \mathbb{R}$, $\psi_t \circ s: \mu(X) \rightarrow X$ is a section of μ for every $t \in \mathbb{R}$, and

$$\frac{d}{dt} \psi_t^* \omega = \psi_t^* (\mathcal{L}_{-\zeta_\eta} \omega) = \psi_t^* (\mathcal{L}_{-\zeta_\eta} \omega + \iota_{-\zeta_\eta} (d\omega)) = \psi_t^* (d(\mu^*(-\eta)) + 0) = -d(\mu^*\eta) = -\mu^* s^* \omega,$$

where \mathcal{L} is the Lie derivative; the second equality above holds by Cartan's formula for the Lie derivative. Thus,

$$s^* \psi_1^* \omega = s^* (\omega - \mu^* s^* \omega) = 0,$$

i.e. $\psi_1 \circ s: \mu(X) \rightarrow X$ is a Lagrangian section of μ . \square

4 Properties of Moment Polytopes

4.1 Complexified Hamiltonian group actions

This section establishes Theorem 2(2). The key steps in the proof are Lemma 4.2, which describes the behavior of the moment map μ on $\mathcal{O}_x(\psi_{\mathbb{C}})$, and Proposition 4.8, which concerns the images of $\mathcal{O}_x(\psi_{\mathbb{C}})$ and $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} - \mathcal{O}_x(\psi_{\mathbb{C}})$ under μ .

Exercise 4.1. Suppose ψ is a smooth action of a torus \mathbb{T} on a compact almost complex manifold (M, J) , i.e. ψ preserves J . Show that (1.6) determines a complexification $\psi_{\mathbb{C}}$ of ψ if either \mathbb{T} is one-dimensional or J is integrable. *Hint:* J is preserved by ψ if and only if $\mathcal{L}_{\zeta_v} J = 0$ for every $v \in T_{\mathbb{1}}\mathbb{T}$, where \mathcal{L} is the Lie derivative and $\zeta_v \in \Gamma(X; TX)$ is as in (1.2); J is integrable if and only $\mathcal{L}_{J\xi} J = J(\mathcal{L}_\xi J)$ for every $\xi \in \Gamma(X; TX)$.

Lemma 4.2. *Suppose \mathbb{T} is a torus, (X, ω, ψ, μ) is a Hamiltonian \mathbb{T} -manifold, and $\psi_{\mathbb{C}}$ is a complexification of ψ with respect to a \mathbb{T} -invariant ω -compatible almost complex structure J as in (1.6). For each $v \in T_{\mathbb{1}}\mathbb{T}$, let $\zeta_v \in \Gamma(X; TX)$ and $\mu_v \in C^\infty(X)$ be as in (1.2) and (2.2), respectively. Then,*

$$\frac{d}{dt} \mu_v(\psi_{\mathbb{C};[itv']}(x)) = -g(\zeta_v, \zeta_{v'})|_{\psi_{\mathbb{C};[itv']}(x)} \quad \forall v, v' \in T_{\mathbb{1}}\mathbb{T}, x \in X, \quad (4.1)$$

where $g(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$ is the Riemannian metric determined by ω and J . Furthermore,

$$\mu(\psi_{\mathbb{C};[iv]}(x)) \neq \mu(x) \in T_{\mathbb{1}}\mathbb{T}^* \quad \forall v \in T_{\mathbb{1}}\mathbb{T}, x \in X \text{ s.t. } \zeta_v(x) \neq 0. \quad (4.2)$$

Proof. By (1.4), (1.6), and Proposition 3.2(1),

$$\frac{d}{dt} \mu_v(\psi_{\mathbb{C};[itv']}(x)) \equiv d_{\psi_{\mathbb{C};[itv']}(x)} \mu_v \left(\frac{d}{dt} \psi_{\mathbb{C};[itv']}(x) \right) = -\omega \left(\zeta_v(\psi_{\mathbb{C};[itv']}(x)), (J\zeta_{v'}(\psi_{\mathbb{C};[itv']}(x))) \right).$$

This gives (4.1). Along with Proposition 3.2(1) again, this implies that

$$\mathbb{R} \rightarrow \mathbb{R}, \quad t \rightarrow \mu_v(\psi_{\mathbb{C};[itv]}(x)),$$

is a strictly decreasing function unless $\zeta_v(x) = 0$. This gives the second claim of the lemma. \square

Corollary 4.3. *Suppose \mathbb{T} , (X, ω, μ, ψ) , and $\psi_{\mathbb{C}}$ are as in Lemma 4.2. If $\alpha \in T_{\mathbb{1}}^*\mathbb{T}$ is a regular value of μ , then $\mu^{-1}(\alpha) \subset X$ is a smooth submanifold and the map*

$$\Psi: \mathbb{T}_{\mathbb{1}} \times \mu^{-1}(\alpha) \rightarrow X, \quad \Psi(u, x) = \psi_{\mathbb{C};u}(x),$$

is a diffeomorphism onto an open subset of X .

Proof. By the regularity assumption, $\mu^{-1}(\alpha) \subset X$ is a smooth submanifold. By (3.29) and Exercise 3.32(b), the homomorphism

$$T_{\mathbb{1}}\mathbb{T}_i \oplus T_x(\mu^{-1}(\alpha)) \longrightarrow T_x X, \quad (iv, w) \longrightarrow J\zeta_v(x) + w, \quad (4.3)$$

is an isomorphism for every $x \in \mu^{-1}(\alpha)$. Along with \mathbb{T}_i being abelian, (3.8), and (1.6), this implies that Ψ is an open map. By (4.2) and the injectivity of (4.3) on $T_{\mathbb{1}}\mathbb{T}_i$, Ψ is injective. \square

Exercise 4.4. Suppose \mathbb{T} , (X, ω, μ, ψ) , and $\psi_{\mathbb{C}}$ are as in Lemma 4.2, $v \in T_{\mathbb{1}}\mathbb{T}$, $x \in X$, and ζ_v and μ_v are also as in Lemma 4.2. Show that the function

$$\mathbb{R} \longrightarrow \mathbb{R}, \quad t \longrightarrow \mu_v(\psi_{\mathbb{C};itv}(x)), \quad (4.4)$$

is strictly decreasing if $\zeta_v(x) \neq 0$ and is constant otherwise. If in addition,

$$x_{\infty}(v) = \lim_{t \rightarrow \infty} \psi_{\mathbb{C};[itv]}(x) \in X, \quad (4.5)$$

i.e. the limit above also exists, show that

$$\zeta_v(x_{\infty}(v)) = 0, \quad d_{x_{\infty}(v)}\mu_v = 0, \quad (\psi_{\mathbb{C};u}(x))_{\infty}(v) = \psi_{\mathbb{C};u}(x_{\infty}(v)) \quad \forall u \in \mathbb{T}_{\mathbb{C}}. \quad (4.6)$$

Exercise 4.5. With the assumptions as in Exercise 4.4, suppose also that X is compact. Show that the limit in (4.5) exists.

Suppose ψ is a smooth action of a torus \mathbb{T} on a smooth manifold X . For $x \in X$ and $v \in T_{\mathbb{1}}\mathbb{T}$, let $\mathbb{T}_{x;v}(\psi) \subset \mathbb{T}$ be the closed subgroup spanned by $\text{Stab}_x(\psi)$ and the closure of $\{e^{tv} : t \in \mathbb{R}\}$ in \mathbb{T} .

Corollary 4.6. *Suppose \mathbb{T} , (X, ω, μ, ψ) , and $\psi_{\mathbb{C}}$ are as in Lemma 4.2, $v \in T_{\mathbb{1}}\mathbb{T}$, $x \in X$, μ_v is also as in Lemma 4.2, and the limit in (4.5) exists. There exists a topological component $Z_{x;v}$ of the $\mathbb{T}_{x;v}(\psi)$ -fixed locus $X^{\mathbb{T}_{x;v}(\psi)} \subset X$ so that*

$$x'_{\infty}(v) \in Z_{x;v} \quad \text{and} \quad \lim_{t \rightarrow \infty} \mu_v(\psi_{\mathbb{C};itv}(x')) = \mu_v(Z_{x;v}) = \inf_{\mathcal{O}_x(\psi_{\mathbb{C}})} \mu_v \quad \forall x' \in \mathcal{O}_x(\psi_{\mathbb{C}}); \quad (4.7)$$

in particular, the limit in (4.5) with x replaced by x' also exists. If in addition X is compact, then

$$\inf_{\mathcal{O}_x(\psi_{\mathbb{C}})} \mu_v = \inf_{\substack{Y \in \pi_0(X^{\psi}) \\ Y \cap \mathcal{O}_x(\psi_{\mathbb{C}}) \neq \emptyset}} \mu_v(Y). \quad (4.8)$$

Proof. Let $x' \in \mathcal{O}_x(\psi_{\mathbb{C}})$. By the continuity of the action ψ , $x'_{\infty}(v)$ is fixed by $\text{Stab}_{x'}(\psi) = \text{Stab}_x(\psi)$. By the first equation in (4.6) with x replaced by x' , $x'_{\infty}(v)$ is also fixed by the closure of $\{e^{tv} : t \in \mathbb{R}\}$ in \mathbb{T} . Thus, $x'_{\infty}(v)$ lies in $X^{\mathbb{T}_{x;v}(\psi)}$, which is a closed symplectic submanifold of (X, ω) by Proposition 3.16(1). Let $Z_{x;v} \subset X^{\mathbb{T}_{x;v}(\psi)}$ be the component containing $x_{\infty}(v)$. Since $\mathbb{T}_{\mathbb{C}}$ is connected, the last equation in (4.6) implies that $x'_{\infty}(v) \in Z_{x;v}$ as well.

Since $d\mu_v$ vanishes on $X^{\mathbb{T}_{x;v}(\psi)}$, μ_v is constant on $Z_{x;v}$. Along with (4.5) with x replaced by x' and the function (4.4) with x replaced by x' being non-increasing, this yields the second equation in (4.7). If $v \in T_{\mathbb{1}}\mathbb{T}$ is generic, $\mathbb{T}_{x;v}(\psi) = \mathbb{T}$ and so $Z_{x;v} \subset X^{\psi}$. Thus,

$$\inf_{\mathcal{O}_x(\psi_{\mathbb{C}})} \mu_v \leq \inf_{\substack{Y \in \pi_0(X^{\psi}) \\ Y \cap \mathcal{O}_x(\psi_{\mathbb{C}}) \neq \emptyset}} \mu_v(Y) \leq \mu_v(Z_{x;v}) = \inf_{\mathcal{O}_x(\psi_{\mathbb{C}})} \mu_v \implies \inf_{\mathcal{O}_x(\psi_{\mathbb{C}})} \mu_v = \inf_{\substack{Y \in \pi_0(X^{\psi}) \\ Y \cap \mathcal{O}_x(\psi_{\mathbb{C}}) \neq \emptyset}} \mu_v(Y).$$

If X is compact, then so is $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$. The continuity of μ in both inputs then implies that the last equality holds for all $v \in T_{\mathbb{1}}\mathbb{T}$. \square

Corollary 4.7. *Suppose \mathbb{T} , (X, ω, ψ, μ) , $\psi_{\mathbb{C}}$, and μ_v for each $v \in T_{\mathbb{1}}\mathbb{T}$ are as in Lemma 4.2 and $x \in X$. Then,*

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} - \mathcal{O}_x(\psi_{\mathbb{C}}) = \{x' \in \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} : \exists v \in T_{\mathbb{1}}\mathbb{T} \text{ s.t. } d_x \mu_v \neq 0, \mu_v(x') = \inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}} \mu_v\}. \quad (4.9)$$

Proof. Let $T_{\mathbb{1}}^c\mathbb{T} \subset T_{\mathbb{1}}\mathbb{T}$ be a complement of

$$T_{\mathbb{1}}\text{Stab}_x(\psi) = \{v \in T_{\mathbb{1}}\mathbb{T} : d_x \mu_v = 0\} \quad (4.10)$$

and $S(T_{\mathbb{1}}^c\mathbb{T}) \subset T_{\mathbb{1}}^c\mathbb{T}$ be the unit sphere in $T_{\mathbb{1}}^c\mathbb{T}$ with respect to some metric. In particular,

$$\mathcal{O}_x(\psi_{\mathbb{C}}) = \{\psi_{\mathbb{C};[v+iv']}(x) : v, v' \in T_{\mathbb{1}}^c\mathbb{T}\}.$$

Since the group $\mathbb{T}_{\mathbb{C}}$ is abelian, $\xi_v(x') \neq 0$ and $d_{x'} \mu_v \neq 0$ for all $v \in T_{\mathbb{1}}^c\mathbb{T}$ and $x' \in \mathcal{O}_x(\psi_{\mathbb{C}})$. Along with the first condition in (1.4), this implies that the left-hand set in (4.9) contains the right-hand set.

By (4.1), the continuous function

$$S(T_{\mathbb{1}}^c\mathbb{T}) \times \mathbb{R} \longrightarrow \mathbb{R}, \quad h(v, t) = \mu_v(\psi_{\mathbb{C};itv}(x)),$$

is non-increasing in t . By (4.7),

$$\lim_{t \rightarrow \infty} h(v, t) = \inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}} \mu_v \quad \forall v \in S(T_{\mathbb{1}}^c\mathbb{T}). \quad (4.11)$$

Suppose $x' \in \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} - \mathcal{O}_x(\psi_{\mathbb{C}})$ is the limit of a sequence $\psi_{\mathbb{C};it_k v_k}(\psi_{u_k}(x))$ with $u_k \in \mathbb{T}$ converging to some u , $v_k \in S(T_{\mathbb{1}}^c\mathbb{T})$ converging to some v , and $t_k \in \mathbb{R}$ converging to ∞ . Since μ is a continuous \mathbb{T} -invariant function on $T_{\mathbb{1}}\mathbb{T} \times X$ and the functions $h(v_k, t)$ are non-increasing in t ,

$$\mu_v(x') = \lim_{k \rightarrow \infty} \mu_{v_k}(\psi_{u_k}(\psi_{\mathbb{C};it_k v_k}(x))) = \lim_{k \rightarrow \infty} h(v_k, t_k) \leq \lim_{t \rightarrow \infty} h(v, t).$$

Along with (4.11), this implies that $\mu_v(x') = \inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}} \mu_v$. □

For a polytope $P \subset T_{\mathbb{1}}^*\mathbb{T}$, we denote by $\partial P \subset P$ the union of the proper faces of P . Let $\text{Int } P = P - \partial P$. With \mathbb{T} , (X, ω, ψ, μ) , and $\psi_{\mathbb{C}}$ as in Theorem 2(2) and $x \in X$, let

$$P_x(\psi_{\mathbb{C}}) = \text{CH}(\mu(\{Y \in \pi_0(X^\psi) : Y \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \neq \emptyset\})).$$

Proposition 4.8. *Suppose \mathbb{T} , (X, ω, ψ, μ) , and $\psi_{\mathbb{C}}$ are as in Theorem 2(2) and $x \in X$. Then,*

$$\mu(\mathcal{O}_x(\psi_{\mathbb{C}})) = \text{Int } P_x(\psi_{\mathbb{C}}), \quad \mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} - \mathcal{O}_x(\psi_{\mathbb{C}})) = \partial P_x(\psi_{\mathbb{C}}), \quad (4.12)$$

and the map $\mathcal{O}_x(\psi_{\mathbb{C}})/\mathbb{T} \rightarrow T_{\mathbb{1}}^*\mathbb{T}$ induced by μ is injective.

Proof. For each $v \in T_{\mathbb{1}}\mathbb{T}$, the map

$$L_v : T_{\mathbb{1}}^*\mathbb{T} \longrightarrow \mathbb{R}, \quad L_v(\alpha) = \alpha(v), \quad (4.13)$$

is a linear functional and $\mu_v \equiv L_v \circ \mu : X \rightarrow \mathbb{R}$. Let $\mathbb{T}_x^c \subset \mathbb{T}$ be a subtorus complementary to the identity component $(\text{Stab}_x(\psi))_0$ of $\text{Stab}_x(\psi)$ and $\iota : \mathbb{T}_x^c \rightarrow \mathbb{T}$ be the inclusion so that

$$\mu_x^c \equiv \iota^* \circ \mu : X \rightarrow T_{\mathbb{1}}^*\mathbb{T}_x^c$$

is a moment map for the restriction of the \mathbb{T} -action ψ on \mathbb{T}_x^c . In particular,

$$\begin{aligned}\mathcal{O}_x(\psi_{\mathbb{C}}) &\equiv \mathbb{T}_{\mathbb{C}}x = (\mathbb{T}_x^c)_{\mathbb{C}}x, \quad \{Y \in \pi_0(X^\psi) : Y \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \neq \emptyset\} = \{Y \in \pi_0(X^{\mathbb{T}_x^c}) : Y \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \neq \emptyset\}, \\ \mathcal{O}_x(\psi_{\mathbb{C}})/\mathbb{T} &= \mathcal{O}_x(\psi_{\mathbb{C}})/\mathbb{T}_x^c, \\ \mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}) &\subset (T_{\mathbb{1}}^*\mathbb{T})_{\mu;x} \equiv \{\alpha \in T_{\mathbb{1}}^*\mathbb{T} : L_v(\alpha) = L_v(\mu(x)) \ \forall v \in T_{\mathbb{1}}\text{Stab}_x(\psi)\}.\end{aligned}$$

Since $\iota^* : (T_{\mathbb{1}}^*\mathbb{T})_{\mu;x} \rightarrow T_{\mathbb{1}}^*\mathbb{T}_x^c$ is a homeomorphism sending line segments to line segments, it suffices to establish the claims with (ψ, μ) replaced by $(\psi|_{\mathbb{T}_x^c}, \mu_x^c)$. We can thus assume that $(\text{Stab}_x(\psi))_0 = \{\mathbb{1}\}$, as is done below.

For $v \in T_{\mathbb{1}}\mathbb{T}$, let $\zeta_v \in \Gamma(X; TX)$ be as in (1.2). Since

$$(\text{Stab}_{x'}(\psi))_0 = (\text{Stab}_x(\psi))_0 = \{\mathbb{1}\} \quad \forall x' \in \mathcal{O}_x(\psi_{\mathbb{C}}),$$

$\zeta_v(x') \neq 0$ for all $v \in T_{\mathbb{1}}\mathbb{T} - \{0\}$ and $x' \in \mathcal{O}_x(\psi_{\mathbb{C}})$. By (4.1) and Exercise 4.9 below, the map

$$T_{\mathbb{1}}\mathbb{T} \rightarrow T_{\mathbb{1}}^*\mathbb{T}, \quad v \rightarrow \mu(\psi_{\mathbb{C};[iv]}(x)),$$

is thus an immersion. By (4.2), this map is injective and thus a diffeomorphism onto an open subset of $T_{\mathbb{1}}^*\mathbb{T}$. Since μ is \mathbb{T} -invariant, this open subset is $\mu(\mathcal{O}_x(\psi_{\mathbb{C}}))$. By (4.8), $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}) \subset P_x(\psi_{\mathbb{C}})$. Thus, the polytope $P_x(\psi_{\mathbb{C}}) \subset T_{\mathbb{1}}^*\mathbb{T}$ is of full dimension, $\mu(\mathcal{O}_x) \subset \text{Int } P_x(\psi_{\mathbb{C}})$, and the last claim of the proposition holds.

For $v \in T_{\mathbb{1}}\mathbb{T} - \{0\}$, the level sets of L_v are hyperplanes. Thus, the restriction of $\mu_v \equiv L_v \circ \mu$ to $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ achieves its minimum along the preimage of a proper face of $P_x(\psi_{\mathbb{C}})$ under μ and not at any point of $\mu^{-1}(\text{Int } P_x(\psi_{\mathbb{C}}))$; the former preimage contains $Y \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ for at least one element $Y \in \pi_0(X^\psi)$ with $Y \cap \mathcal{O}_x(\psi_{\mathbb{C}}) \neq \emptyset$. From the compactness of $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ and (4.9), we then conclude that

$$\begin{aligned}\overline{\mu(\mathcal{O}_x(\psi_{\mathbb{C}}))} - \mu(\mathcal{O}_x(\psi_{\mathbb{C}})) &\subset \mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} - \mathcal{O}_x(\psi_{\mathbb{C}})) \\ &= \{\mu(x') : x' \in \overline{\mathcal{O}_x(\psi_{\mathbb{C}})}, \exists v \in T_{\mathbb{1}}\mathbb{T} - \{0\} \text{ s.t. } \mu_v(x') = \inf_{\mathcal{O}_x(\psi_{\mathbb{C}})} \mu_v\} \subset \partial P_x(\psi_{\mathbb{C}}).\end{aligned}$$

Thus, $\mu(\mathcal{O}_x(\psi_{\mathbb{C}})) \supset \text{Int } P_x(\psi_{\mathbb{C}})$, which establishes the first equality in (4.12). The second equality in (4.12) then follows from the compactness of $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$. \square

Exercise 4.9. Suppose V is a vector space with an inner-product product $\langle \cdot, \cdot \rangle$ and $v_1, \dots, v_k \in V$. Show that the matrix $(\langle v_i, v_j \rangle)_{i,j \in [k]}$ is nondegenerate if and only if the vectors v_1, \dots, v_k are linearly independent.

Proof of Theorem 2(2). The two equations in (4.12) give (2b), as well as (2c) with $\sigma = \text{Int } P_x(\psi_{\mathbb{C}})$. Suppose σ is the interior of a proper face F of $P_x(\psi_{\mathbb{C}})$. Choose $v \in T_{\mathbb{1}}\mathbb{T}$ and $c \in \mathbb{R}$ be so that

$$F = P_x(\psi_{\mathbb{C}}) \cap L_v^{-1}(c),$$

with L_v as in (4.13). Let $Z_{x;v} \subset X$ be as in (4.7). Thus,

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \cap \mu^{-1}(F) \subset \text{Crit}(\mu_v), \quad \mu_v|_{\mu^{-1}(F)} = c,$$

and $Z_{x;v}$ is a topological component of $\text{Crit}(\mu_v)$ with $\mu_v|_{Z_{x;v}} = c$. By (1.6) and (3.28), $\psi_{\mathbb{C};itv}$ is the negative gradient flow $\psi_{\mu_v;t}$ of μ_v with respect to the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, with J as in (1.6). By the first statement in (4.7),

$$\mathcal{O}_x(\psi_{\mathbb{C}}) \subset X_{Z_{x;v}}^+(\mu_v) \equiv \{x' \in X : \lim_{t \rightarrow \infty} \psi_{\mathbb{C};[itv]}(x') \in Z_{x;v}\}.$$

By the first statement of Proposition A.5, $\mu_v : X \rightarrow \mathbb{R}$ is a Morse-Bott function. Proposition A.7(6) and the last equation in (4.6) thus imply that

$$\begin{aligned} \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \cap \mu^{-1}(F) &\equiv \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \cap \mu_v^{-1}(c) = \overline{\{x'_\infty(v) : x' \in \mathcal{O}_x(\psi_{\mathbb{C}})\}} = \overline{\mathcal{O}_{x_\infty(v)}(\psi_{\mathbb{C}})}, \\ P_{x_\infty(v)}(\psi_{\mathbb{C}}) &\equiv \text{CH}(\mu(\{Y \in \pi_0(X^\psi) : Y \cap \overline{\mathcal{O}_{x_\infty(v)}(\psi_{\mathbb{C}})} \neq \emptyset\})) \\ &= \text{CH}(\mu(\{Y \in \pi_0(X^\psi) : Y \cap \overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \neq \emptyset\})) \cap F = F. \end{aligned}$$

From (4.12) with x replaced by $x_\infty(v)$, we then conclude that

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \cap \mu^{-1}(\sigma) = \overline{\mathcal{O}_{x_\infty(v)}(\psi_{\mathbb{C}})} \cap \mu^{-1}(\sigma) = \mathcal{O}_{x_\infty(v)}(\psi_{\mathbb{C}}).$$

This establishes (2c). Since $\mathcal{O}_y(\psi_{\mathbb{C}}) = \{y\}$ for any $y \in X^\psi$, (2a) follows from (2c).

Since the moment map μ is \mathbb{T} -invariant, the map

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}/\mathbb{T} \rightarrow \mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}), \quad [x'] \rightarrow \mu(x'), \quad (4.14)$$

is well-defined. It is surjective by definition. Its domain is compact, while the target is Hausdorff. By (2c), for every open face σ of the polytope $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})$ there exists $x_\sigma \in \mathcal{O}_x(\psi_{\mathbb{C}})$ so that

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \cap \mu^{-1}(\sigma) = \mathcal{O}_{x_\sigma}(\psi_{\mathbb{C}}).$$

By the last statement of Proposition 4.8 with x replaced by x_σ , the restriction of the map (4.14) to $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})} \cap \mu^{-1}(\sigma)/\mathbb{T}$ is thus injective for every open face σ of $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})$. It follows that the entire map (4.14) is injective as well and thus a homeomorphism. \square

Exercise 4.10. Show that

- (a) the S^1 -action ψ on $\mathbb{C}P^2$ given by

$$S^1 \times \mathbb{C}P^2 \rightarrow \mathbb{C}P^2, \quad u \cdot [z_0, z_1, z_2] = [z_0, u^2 z_1, u^3 z_2],$$

is effective and Hamiltonian with respect to the symplectic form $\omega_{\text{FS};2}$ of Exercise 2.9;

- (b) the closure $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ of the \mathbb{C}^* -orbit $\mathcal{O}_x(\psi_{\mathbb{C}})$ of x under the complexification of ψ with respect to the standard complex structure on $\mathbb{C}P^2$ is a rational cubic curve (and in particular is not smooth) for any point $x \equiv [z_0, z_1, z_2]$ in $\mathbb{C}P^2$ with $z_0, z_1, z_2 \neq 0$.

4.2 Proofs of Theorems 1, 2(1), and 3(0⁺)

The last statement of Corollary 3.40 establishes (D_k). As already noted in Section 1, (F_k) is a straightforward consequence of the equivariant splitting (3.14) of $TX|_Y$ for each $Y \in \pi_0(X^\psi)$ and is deduced first below. We then establish the main part of the proof of Theorem 1, (A_k^{*}) on page 5, and wrap up this section with the remaining statements of Theorem 1, Theorem 2(1), and (0⁺) on page 8.

Proof of (F_k) . For each $L \in \mathbb{R}P^{k-1}$, let $X^L \subset X$ be the fixed locus of the action $\psi|_L$. By the second equality in (3.27) and Proposition 3.2(1),

$$\text{Crit}(\mu) = \{x \in X^L : L \in \mathbb{R}P^{k-1}\}.$$

By Proposition 3.16(1), $X^L \subset X$ is a compact symplectic submanifold. Every topological component $Z \subset X^L$ is preserved ψ . The restriction of the ψ -action to such a component is Hamiltonian. Thus,

$$Z \cap X^\psi = Z^\psi \neq \emptyset \quad \forall Z \in \pi_0(X^L), L \in \mathbb{R}P^{k-1}$$

by Exercise 3.34(a). Along with Corollary 3.12, this implies that $\text{Crit}(\mu)$ is a *finite* union of the topological components Z of the symplectic submanifolds $X^L \subset X$ with $L \in \mathbb{R}P^{k-1}$. By the second equality in (3.27) and Proposition 3.2(1), the smooth map

$$\mu_v : X \longrightarrow T_0^*\mathbb{R}^k, \quad \mu_v(x) = \{\mu(x)\}(v), \quad (4.15)$$

is constant along each topological component $Z \subset X^L$ for every $v \in L$, i.e. for any $v \in L$ there exists $c_v \in \mathbb{R}$ so that

$$\mu(Z) \subset \{\alpha \in T_0^*\mathbb{R}^k : \alpha(v) = c_v\};$$

the right-hand side above is a *hyperplane* in $T_0^*\mathbb{R}^k$ if $v \neq 0$. □

Lemma 4.11. *Suppose $k \in \mathbb{Z}^{\geq 0}$, (X, ω) is a symplectic manifold, $\psi_1, \dots, \psi_{k+1}$ are \mathbb{R} -actions on (M, ω) with Hamiltonians*

$$H_1, \dots, H_{k+1} : X \longrightarrow \mathbb{R},$$

respectively, the \mathbb{R} -action ψ_{k+1} is almost periodic, and its Hamiltonian H_{k+1} is ψ_i -invariant for every $i \in [k]$. If $c \in \mathbb{R}^k$ is a regular value of the map

$$H \equiv (H_1, \dots, H_k) : X \longrightarrow \mathbb{R}^k,$$

then the submanifolds $Z \equiv H^{-1}(c)$ and $\text{Crit}(H_{k+1})$ of X are transverse, $Z \cap \text{Crit}(H_{k+1})$ is an open subset of $\text{Crit}(H_{k+1}|_Z)$, and

$$T_x(Z \cap \text{Crit}(H_{k+1})) = T_x Z \cap T_x \text{Crit}(H_{k+1}) = E_x^0(H_{k+1}|_Z), \quad n_x^\pm(H_{k+1}|_Z) = n_x^\pm(H_{k+1}) \in 2\mathbb{Z}^{\geq 0}$$

for all $x \in Z \cap \text{Crit}(H_{k+1})$.

Proof. By the first statement of Proposition A.5, $\text{Crit}(H_{k+1}) \subset X$ is a closed symplectic submanifold and $H_{k+1} : X \longrightarrow \mathbb{R}$ is a Morse-Bott function. Since $c \in \mathbb{R}^k$ is a regular value of H , $Z \equiv H^{-1}(c)$ is a submanifold of X . In light of Exercise A.4 and the first statement of Proposition A.5, it remains to prove that the submanifolds $\text{Crit}(H_{k+1}), Z \subset X$ are transverse. Let $Y \in \pi_0(\text{Crit}(H_{k+1}))$.

Let $\zeta_1, \dots, \zeta_k \in \Gamma(X; TX)$ be the vector fields generating the \mathbb{R} -actions ψ_1, \dots, ψ_k and thus satisfying the middle equation in (2.5) with $\zeta_{v_i} = \zeta_i$. Since $H_{k+1} \circ \psi_i = H_{k+1}$ for each $i \in [k]$, ψ_i preserves $\text{Crit}(H_{k+1})$ and thus Y . Therefore,

$$\zeta_i|_Y \in \Gamma(Y; TY) \subset \Gamma(Y; TX|_Y) \quad \forall i \in [k].$$

If $x \in Z$, then $d_x H_1, \dots, d_x H_k$ vanish on $T_x Z$. Since $d_x H$ is surjective, it follows that

$$d_x H \equiv (d_x H_1, \dots, d_x H_k) : T_x X / T_x Z \longrightarrow \mathbb{R}^k \quad (4.16)$$

is a well-defined isomorphism. The middle equation in (2.5) with $\zeta_{v_i} = \zeta_i$ then implies that the tangent vectors $\zeta_1(x), \dots, \zeta_k(x) \in T_x X$ are linearly independent.

Suppose $x \in Y \cap Z$ and $(r_1, \dots, r_k) \in \mathbb{R}^k - \{0\}$. Since

$$r \cdot \zeta(x) \equiv r_1 \zeta_1(x) + \dots + r_k \zeta_k(x) \in T_x Y - \{0\}$$

and $\omega|_{T_x Y}$ is a nondegenerate, there exists $w \in T_x Y$ so that

$$\sum_{i=1}^k r_i d_x H_i(w) \equiv -\omega(r \cdot \zeta(x), w) \neq 0.$$

Thus, the restrictions of $d_x H_1, \dots, d_x H_k$ to $T_x Y$ are linearly independent. Since (4.16) is a well-defined isomorphism, it follows that $T_x X = T_x Y \oplus T_x Z$, i.e. the submanifolds $Y, Z \subset X$ are transverse at x . \square

Corollary 4.12. *Suppose $k \in \mathbb{Z}^{\geq 0}$, (X, ω) is a symplectic manifold, $\tilde{\psi}$ is an almost periodic \mathbb{R}^{k+1} -action on (X, ω) with Hamiltonian*

$$\tilde{H} \equiv (H, H_{k+1}): X \longrightarrow \mathbb{R}^k \times \mathbb{R} = \mathbb{R}^{k+1}.$$

If $c \in \mathbb{R}^k$ is a regular value of H , then the restriction of H_{k+1} to the submanifold $Z \equiv H^{-1}(c)$ of X is a Morse-Bott function with $n_x^\pm(H) \in 2\mathbb{Z}^{\geq 0}$ for every $x \in \text{Crit}(H_{k+1}|_Z)$.

Proof. Let $H = (H_1, \dots, H_k)$ and $x \in \text{Crit}(H_{k+1}|_Z)$. Suppose $x \in \text{Crit}(H_{k+1}|_Z)$. Since the map (4.16) is a well-defined isomorphism,

$$d_x H_{k+1} = r_1 H_1 + \dots + r_k H_k: T_x X \longrightarrow \mathbb{R}$$

for some $r \equiv (r_1, \dots, r_k) \in \mathbb{R}^k$. The map

$$H_{k+1;r} \equiv H_{k+1} - (r_1 H_1 + \dots + r_k H_k): X \longrightarrow \mathbb{R}$$

is then a Hamiltonian for an almost periodic \mathbb{R} -action on (X, ω) so that $x \in Z \cap \text{Crit}(H_{k+1;r})$. This Hamiltonian is preserved by the restriction of the action $\tilde{\psi}$ to $\mathbb{R}^k \times \{0\}$. Since $H_{k+1;r} - H_{k+1}$ restricts to the constant $r \cdot c$ on Z ,

$$\text{Crit}(H_{k+1;r}|_Z) = \text{Crit}(H_{k+1}|_Z), \quad E_x^0(H_{k+1;r}|_Z) = E_x^0(H_{k+1}|_Z), \quad n_x^\pm(H_{k+1;r}|_Z) = n_x^\pm(H_{k+1}|_Z).$$

By Lemma 4.11, the closed submanifold $Z \cap \text{Crit}(H_{k+1;r})$ of Z is thus an open subset of $\text{Crit}(H_{k+1})$,

$$T_x(Z \cap \text{Crit}(H_{k+1;r})) = E_x^0(H_{k+1}|_Z), \quad \text{and} \quad n_x^\pm(H_{k+1}|_Z) \in 2\mathbb{Z}^{\geq 0}. \quad (4.17)$$

We conclude that $\text{Crit}(H_{k+1}|_Z)$ is a finite union of submanifolds $Z \cap \text{Crit}(H_{k+1;r})$ of Z with $r \in \mathbb{R}^k$, each of which is a union of the topological components of $\text{Crit}(H_{k+1}|_Z)$ and satisfies (4.17) for all $x \in Z \cap \text{Crit}(H_{k+1;r})$. \square

Proof of (A_k^{*}) on page 5. The claim is trivially true for $k = 0$. We assume that it is true for some $k \in \mathbb{Z}^{\geq 0}$ and show that it also holds with k replaced by $k+1$. Let

$$\tilde{H} \equiv (H, H_{k+1}): X \longrightarrow \mathbb{R}^k \times \mathbb{R} = \mathbb{R}^{k+1}$$

be a Hamiltonian for an almost periodic \mathbb{R}^{k+1} -action $\tilde{\psi}$ on (X, ω) and $\tilde{c} \equiv (c, c_{k+1}) \in \mathbb{R}^k \times \mathbb{R}$.

Suppose first that $c \in \mathbb{R}^k$ is a regular value of H and thus $Z \subset H^{-1}(c)$ is a closed submanifold of X . It is connected by the inductive assumption. By Corollary 4.12, $H_{k+1}|_Z$ is a Morse-Bott function with $n_x^\pm(H_{k+1}|_Z) \in 2\mathbb{Z}^{\geq 0}$. By Proposition A.8,

$$\tilde{H}^{-1}(\tilde{c}) \equiv \{H_{k+1}|_Z\}^{-1}(c_{k+1}) \subset X$$

is thus connected.

Let $\mathbb{R}_H^k \subset \mathbb{R}^k$ and $\mathbb{R}_{\tilde{H}}^{k+1} \subset \mathbb{R}^{k+1}$ be the subsets of regular values of H and \tilde{H} , respectively. In particular,

$$\mathbb{R}_{\tilde{H}}^{k+1} = \{\tilde{c} \in \mathbb{R}^{k+1} : d_x H_1, \dots, d_x H_{k+1} \in T_x^* X \text{ are lin. independent } \forall x \in \tilde{H}^{-1}(\tilde{c})\}.$$

Since the subset $\tilde{H}^{-1}(\tilde{c}) \subset X$ is compact for every $\tilde{c} \in \mathbb{R}^{k+1}$, the subset $\mathbb{R}_{\tilde{H}}^{k+1} \subset \mathbb{R}^{k+1}$ is open. The function

$$\mathbb{R}_{\tilde{H}}^{k+1} \longrightarrow \mathbb{Z}^{\geq 0}, \quad \tilde{c} \longrightarrow |\pi_0(\tilde{H}^{-1}(\tilde{c}))|, \quad (4.18)$$

is constant on the connected components of $\mathbb{R}_{\tilde{H}}^{k+1}$ and takes value 0 or 1 on $\mathbb{R}_{\tilde{H}}^{k+1} \cap (\mathbb{R}_H^k \times \mathbb{R})$. Since the subset $\mathbb{R}_H^k \subset \mathbb{R}^k$ is dense by Sard's Theorem and each connected component of $\mathbb{R}_{\tilde{H}}^{k+1}$ is open in \mathbb{R}^{k+1} , the function (4.18) takes value 0 or 1 on each connected component of $\mathbb{R}_{\tilde{H}}^{k+1}$, i.e. $\tilde{H}^{-1}(\tilde{c}) \subset X$ is connected for every $\tilde{c} \in \mathbb{R}_{\tilde{H}}^{k+1}$. \square

Proof of (A_k). By Exercises 3.4 and 2.3, we can assume that the action ψ is irreducible. Let $(T_0^* \mathbb{R}^k)_\mu \subset T_0^* \mathbb{R}^k$ be the subset of regular values of μ . By (F_k), $\mu^{-1}((T_0^* \mathbb{R}^k)_\mu) \subset X$ is the complement of a finite union of submanifolds of positive codimensions. Thus, the subset

$$P^* \equiv \mu(\mu^{-1}((T_0^* \mathbb{R}^k)_\mu)) \subset \mu(X) \equiv P$$

is dense in P . By (D_k), the map $\mu: X \longrightarrow P$ is open. By (A_k^{*}), $\mu^{-1}(\alpha) \subset X$ is connected for every $\alpha \in P^*$. Thus, (A_k) now follows from Exercise A.10. \square

Proof of (B_k). This claim is trivially true for $k = 0$. Suppose $k \in \mathbb{Z}^+$ and $H: X \longrightarrow \mathbb{R}^k$ is a Hamiltonian for an almost periodic \mathbb{R}^k -action $\tilde{\psi}$ on (X, ω) . For a $k \times (k-1)$ real matrix A , the \mathbb{R}^{k-1} -action $\psi_A \equiv \psi \circ A$ is then also almost periodic with Hamiltonian

$$H_A \equiv A^{\text{tr}} \circ H: X \longrightarrow \mathbb{R}^{k-1}.$$

Suppose $x_0, x_1 \in X$. Let A be a $k \times (k-1)$ injective real matrix A so that

$$H(x_1) - H(x_0) \in \ker A^{\text{tr}}.$$

Thus, $x_1 \in H_A^{-1}(H_A(x_0))$ and

$$H(x_1) \in H(H_A^{-1}(H_A(x_0))) \subset \{H(x_0) + c : c \in \ker A^{\text{tr}}\}.$$

Since $\ker A^{\text{tr}}$ is a line and $H_A^{-1}(H_A(x_0)) \subset X$ is connected by (A_k),

$$H(H_A^{-1}(H_A(x_0))) \subset H(X)$$

contains the line segment between $H(x_0)$ and $H(x_1)$. Thus, $H(X) \subset \mathbb{R}^k$ is convex. \square

Proof of (C_k). Let $\rho : \mathbb{R}^k \rightarrow \mathbb{T}$ and ψ' be as in (1.5). We can assume that the image of ρ is dense in \mathbb{T} and so $X^\psi = X^{\psi'}$. The first claim then follows from Proposition 3.16(1). By Proposition 3.2(1) and the first equation in (1.4), $d\mu$ vanishes along X^ψ ; this implies the second claim. By (B_k), $\mu(X) \subset \text{CH}(\mu(X^\psi))$.

Suppose $\eta_0 \in T_0^* \mathbb{R}^k - \text{CH}(\mu(X^\psi))$. Thus, there exists $v \in T_0 \mathbb{R}^k$ so that

$$\eta_0(v) < \min\{\eta(v) : \eta \in \mu(X^\psi)\} = \min\{\eta(v) : \eta \in \text{CH}(\mu(X^\psi))\}. \quad (4.19)$$

Let $y \in X$ be a minimum of the smooth function μ_v as in (4.15). Thus, $d_y \mu_v = 0$, the vector field ζ_v as in (1.2) vanishes at y , and y lies in the fixed locus $X^{\mathbb{R}v}$ of the restriction of the ψ -action to $\mathbb{R}v \subset \mathbb{R}^k$. For a generic choice of $v \in \mathbb{R}^k$ satisfying (4.19), $\rho(\mathbb{R}v) \subset \mathbb{T}$ is dense and thus $X^{\mathbb{R}v} = X^\psi$. It follows that

$$\eta_0(v) < \min\{\eta(v) : \eta \in \mu(X^\psi)\} = \min\{\mu_v(x) : x \in X\} = \min\{\eta(v) : \eta \in \mu(X)\},$$

i.e. $\eta_0 \notin \mu(X)$.

Thus, $\mu(X) = \text{CH}(\mu(X^\psi))$. The vertices of this polytope are of the form $\mu(Y)$ with $Y \in \pi_0(X^\psi)$. By (3.37), the number of edges at any such vertex $\mu(Y)$ is at most $|S(Y)|$. Since the real rank of each subbundle $\mathcal{N}_X^\alpha Y \subset TX|_Y$ is at least 2, $|S(Y)| \leq (\dim X)/2$. \square

Proof of (E_k). Suppose $Y \in \pi_0(X^\psi)$. Let $\rho, J, S(Y), \mathcal{N}_X^\alpha Y$ for each $\alpha \in S(Y)$, and $\mathcal{C}_{\mu(Y)}(\rho^* S(Y))$ be as in Proposition 3.38. If $\rho^* S(Y)$ does not span $T_0^* \mathbb{R}^k$ over \mathbb{R} , there exists

$$v \in T_0 \mathbb{R}^k - \{0\} \quad \text{s.t.} \quad \{\rho^* \alpha\}(v) = 0 \quad \forall \alpha \in S(Y).$$

The subgroup $\mathbb{R}v \subset \mathbb{R}^k$ then acts trivially on $TX|_Y$. By Proposition 3.8(1), this implies that the connected component of the $\mathbb{R}v$ -fixed locus $X^{\mathbb{R}v}$ containing Y is a connected component of X , i.e. $\mathbb{R}v$ acts trivially on X (and so the action ψ is reducible), since X is connected. If the action ψ is reducible, then $\mathbb{R}v \subset \mathbb{R}^k$ acts trivially on X and thus on $TX|_Y$ for some $v \in T_0 \mathbb{R}^k$ nonzero and thus $\{\rho^* \alpha\}(v) = 0$ for every $\alpha \in S(Y)$, i.e. $\rho^* S(Y)$ does not span $T_0^* \mathbb{R}^k$ over \mathbb{R} .

Thus, $\rho^* S(Y)$ spans $T_0^* \mathbb{R}^k$ over \mathbb{R} if and only if the action ψ is irreducible. Suppose $\mu(Y) \in \text{Ver}(\mu(X^\psi))$ is a vertex of the polytope $\mu(X) = \text{CH}(\mu(X^\psi))$. By Proposition 3.38, the edges of $\mu(X)$ at $\mu(Y)$ are the edges of the cone $\mathcal{C}_{\mu(Y)}(\rho^* S(Y))$. A subset $S_\mu(Y)$ of $\rho^* S(Y)$ thus forms a collection of edge vectors of the polytope $\mu(X)$ at the vertex $\mu(Y)$, while the elements of $\rho^* S(Y) - S_\mu(Y)$ lie in the span of $S_\mu(Y)$. We conclude that $S_\mu(Y)$ spans $T_0^* \mathbb{R}^k$ over \mathbb{R} if and only if the action ψ is irreducible. \square

Proof of Theorem 2(1). Suppose $Y \in \pi_0(X^\psi)$. Let $\rho, J, S(Y)$, and $\mathcal{N}_X^\alpha Y$ for each $\alpha \in S(Y)$ be as in Proposition 3.38 with $(\mathbb{R}^k, 0)$ replaced by $(\mathbb{T}, \mathbb{1})$ and thus $\rho = \text{id}$. As above, a subset $S_\mu(Y)$ of $\rho^*S(Y) = S(Y)$ forms a collection of edge vectors of the polytope $\mu(X)$ at $\mu(Y)$. Since $S(Y) \subset (T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$, all edges of the polytope $\mu(X)$ at $\mu(Y)$ are rational. \square

Proof of (0⁺) on page 8. Let $\rho, J, S(Y)$ for each $Y \in \pi_0(X^\psi)$, $\mathcal{N}_X^\alpha Y$ for each $\alpha \in S(Y)$, and $\mathcal{C}_{\mu(Y)}(\rho^*S(Y))$ be as in Proposition 3.38 with $(\mathbb{R}^k, 0)$ replaced by $(\mathbb{T}, \mathbb{1})$ and thus $\rho = \text{id}$. By (3.37) and (1.7), $|S(Y)|$ is at most the dimension of \mathbb{T} . Since the action ψ is effective, Proposition 3.11(1) then implies that $S(Y)$ is a \mathbb{Z} -basis for $T_{\mathbb{1}}^*\mathbb{T}$ for every $Y \in \pi_0(X^\psi)$; this remains the case if some of the elements of $S(Y)$ are negated. In particular, the polytope $\mu(X) \subset T_{\mathbb{1}}^*\mathbb{T}$ is of full dimension. Furthermore, for every $Y \in \pi_0(X^\psi)$, the cone $\mathcal{C}_{\mu(Y)}(S(Y))$ contains no lines, $\mu(Y) \in \text{Ver}(\mu(X))$, and for every $S \subset S(Y)$ the μ -image of the topological component X_Y^S of the $\psi|_{\mathbb{T}_S}$ -fixed locus X^S containing Y lies in the cone $\mathcal{C}_{\mu(Y)}(S)$ of dimension $|S|$ and contains a neighborhood of the vertex $\mu(Y)$ of this cone.

By Proposition 3.16(1), $(X_Y^S, \omega|_{X_Y^S}, \psi|_{X_Y^S}, \mu|_{X_Y^S})$ is a closed connected Hamiltonian \mathbb{T} -manifold for every $Y \in \pi_0(X^\psi)$ and $S \subset S(Y)$. Thus,

$$\mu(X_Y^S) = \text{CH}(\mu((X_Y^S)^\psi)) = \text{CH}(\mu(X^\psi \cap X_Y^S))$$

by Theorem 1(C_k). Since $\mu(X^\psi \cap X_Y^S) \subset \text{Ver}(\mu(X))$, it follows that $\mu(X_Y^S)$ is the face $F_{\mu(Y);S}(\mu(X))$ of the polytope $\mu(X)$ containing the edges

$$e_{\mu(Y);\alpha} \equiv \mu(X) \cap \{\mu(Y) + t_\alpha \alpha : t_\alpha \in \mathbb{R}^{\geq 0}\}$$

with $\alpha \in S$. Since $S(Y)$ is a \mathbb{Z} -basis for $T_{\mathbb{1}}^*\mathbb{T}$ for every $Y \in \pi_0(X^\psi)$, Exercise 3.35 implies that for each $x \in X$ there exist $Y_x \in \pi_0(X^\psi)$ and $S_x \subset S(Y_x)$ so that the ψ -stabilizer $\mathbb{T}_x(\psi) \subset \mathbb{T}$ of x is the subtorus $\mathbb{T}_{S_x} \subset \mathbb{T}$ and $x \in X_{Y_x}^{S_x}$. It follows that

$$\mu^{-1}(F_{\mu(Y);S}^\circ(\mu(X))) = \{x \in X_{Y_x}^{\mathbb{T}_x} : \mathbb{T}_x(\psi) = \mathbb{T}_S\} \quad \forall Y \in \pi_0(X^\psi), S \subset S(Y), \quad (4.20)$$

where $F_{\mu(Y);S}^\circ(\mu(X)) \subset F_{\mu(Y);S}(\mu(X))$ is the interior.

Suppose $e \in \text{Edg}(\mu(X))$ is an edge of the polytope $\mu(X)$ and thus $e = e_{\mu(Y);\alpha}$ for some $Y \in \pi_0(X^\psi)$ and $\alpha \in S(Y)$. We then set $\alpha_e = \alpha$. If $Y' \in \pi_0(X^\psi)$ is such that $\mu(Y')$ is the vertex of $e_{\mu(Y);\alpha}$ other than $\mu(Y)$, then $-\alpha \in S(Y')$ and $e = e_{\mu(Y');-\alpha}$. Thus, $(\alpha_e)_{e \in \text{Edg}(\mu(X))}$ is a full tuple of integral edge vectors for the polytope $\mu(X)$ such that for each vertex η of $\mu(X)$ the components α_e with $e \in \text{Edg}_\eta(\mu(X))$ form a \mathbb{Z} -basis for $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ and

$$\mathbb{T}_S = \bigcap_{\alpha \in S} \mathbb{T}_{\alpha_{e_{\mu(Y);\alpha}}} \quad \forall Y \in \pi_0(X^\psi), S \subset S(Y). \quad (4.21)$$

Suppose $F \subset \mu(X)$ is a face of the polytope $\mu(X)$ and thus $F = F_{\mu(Y);S}(\mu(X))$ for some $Y \in \pi_0(X^\psi)$ and $S \subset S(Y)$. We then set $\mathbb{T}_F = \mathbb{T}_S$. Thus, $\mathbb{T}_F \subset \mathbb{T}$ is a subtorus. By (4.20), (0⁺b) holds. This implies that \mathbb{T}_F is independent of the choice of $\mu(Y) \in F$. By (4.21), (0⁺a) thus holds for all $\eta \in F$.

Let $Y \in \pi_0(X^\psi)$, $S \subset S(Y)$, and $F = F_{\mu(Y);S}(\mu(X))$. Since

$$\dim Y + \sum_{\alpha \in S(Y)} \text{rk } \mathcal{N}_X^\alpha Y = \dim X, \quad \text{rk } \mathcal{N}_X^\alpha Y \geq 2 \quad \forall \alpha \in S(Y), \quad \text{and} \quad |S(Y)| = \dim \mathbb{T} = (\dim X)/2,$$

we conclude that $\dim Y = 0$ and $\text{rk } \mathcal{N}_X^\alpha Y = 2$ for every $\alpha \in S(Y)$. Along with Proposition 3.11(2), this implies that

$$\dim X_Y^{\mathbb{T}^S} = 2|S| = 2 \dim F = 2 \dim(\mathbb{T}/\mathbb{T}_F). \quad (4.22)$$

Since $\alpha(v) = 0$ for all $\alpha \in S$ and $v \in T_1 \mathbb{T}_S$, the affine map

$$\Phi_{Y;S}: \left\{ \mu(Y) + \sum_{\alpha \in S} t_\alpha \alpha : t_\alpha \in \mathbb{R} \right\} \longrightarrow T_1^*(\mathbb{T}/\mathbb{T}_S), \quad \{\Phi_{Y;S}(\eta)\}(v) = \eta(v) - \{\mu(Y)\}(v),$$

is a well-defined surjection and thus affine isomorphism by the last equality in (4.22). By (4.20), the torus \mathbb{T}/\mathbb{T}_F acts freely on $\mu^{-1}(F^\circ) \subset X_Y^{\mathbb{T}^S}$ via ψ ; we denote this action by $\psi_{Y;S}$. By Theorem 1(A_k), the fibers of the restriction

$$\mu: \mu^{-1}(F^\circ) \longrightarrow F^\circ$$

are connected. Since $F^\circ \subset F$ is open, $\mu^{-1}(F^\circ) \subset X_Y^{\mathbb{T}^S}$ is a symplectic submanifold. Thus, (0⁺c) follows from Exercise 3.33(a) with \mathbb{T} and (X, ω, ψ, μ) replaced by \mathbb{T}/\mathbb{T}_S and

$$(\mu^{-1}(F^\circ), \omega|_{\mu^{-1}(F^\circ)}, \psi_{Y;S}, \Phi_{Y;S} \circ \mu|_{\mu^{-1}(F^\circ)}),$$

respectively. □

5 Symplectic Quotient and Cut Constructions

5.1 Symplectic quotient

For a Lie group G , let

$$(T_1^*G)^G \equiv \{ \alpha \in T_1^*G : \text{Ad}_g^*(\alpha) = \alpha \ \forall g \in G \}$$

be the fixed locus of the dual of the adjoint action of G on T_1^*G . If ψ is a G -action on a space X , $\mu: X \longrightarrow T_1^*G$ is a map satisfying the second condition in (1.4), and $\alpha \in (T_1^*G)^G$, then ψ restricts to a G -action on $\mu^{-1}(\alpha) \subset X$. If G is abelian, then $(T_1^*G)^G = T_1^*G$.

Theorem 4 ([25, Theorems 3,4]). *Suppose G is a compact Lie group, $(\tilde{X}, \tilde{\omega}, \tilde{\psi}, \tilde{\mu})$ is a Hamiltonian G -manifold, and $\alpha \in (T_1^*G)^G$ are such that G acts freely on $\tilde{\mu}^{-1}(\alpha)$. Then,*

(SQ0) $\alpha \in T_1^*G$ is a regular value of $\tilde{\mu}$;

(SQ1) there is a unique smooth structure on $X_\alpha \equiv \tilde{\mu}^{-1}(\alpha)/G$ so that the quotient projection

$$p_\alpha: \tilde{\mu}^{-1}(\alpha) \longrightarrow X_\alpha$$

is a principal G -bundle;

(SQ2) there exists a unique 2-form ω_α on X_α so that $p_\alpha^* \omega_\alpha = \tilde{\omega}|_{T\tilde{\mu}^{-1}(\alpha)}$;

(SQ3) the 2-form ω_α is symplectic.

Suppose in addition G' is another Lie group, and $(\tilde{X}, \tilde{\omega}, \tilde{\psi}', \tilde{\mu}')$ is a Hamiltonian G' -manifold. If the actions $\tilde{\psi}$ and $\tilde{\psi}'$ commute, $\tilde{\mu}'$ is ψ -invariant, and $\tilde{\mu}$ is ψ' -invariant, then

(SQ4) $\tilde{\psi}'$ and $\tilde{\mu}'$ descend to a G' -action ψ'_α on X_α and a smooth map $\mu'_\alpha: X_\alpha \longrightarrow T_1^*G'$, respectively, so that $(X_\alpha, \omega_\alpha, \psi'_\alpha, \mu'_\alpha)$ is a Hamiltonian G' -manifold;

(SQ5) if $\alpha' \in (T_{\mathbb{1}}^*G')^{G'}$, G' acts freely on $\mu'^{-1}(\alpha')$ if and only if $G \times G'$ acts freely on $(\tilde{\mu}, \tilde{\mu}')^{-1}(\alpha, \alpha')$;

(SQ6) if $\alpha' \in (T_{\mathbb{1}}^*G')^{G'}$ and $G \times G'$ acts freely on $(\tilde{\mu}, \tilde{\mu}')^{-1}(\alpha, \alpha')$, then

$$p_{(\alpha, \alpha')} = p_{\alpha'} \circ p_{\alpha} \Big|_{\tilde{\mu}^{-1}(\alpha) \cap \tilde{\mu}'^{-1}(\alpha')} : (\tilde{\mu}, \tilde{\mu}')^{-1}(\alpha, \alpha') \longrightarrow X_{(\alpha, \alpha')} = (X_{\alpha})_{\alpha'}$$

and $\omega_{(\alpha, \alpha')} = (\omega_{\alpha})_{\alpha'}$.

The symplectic manifold $(X_{\alpha}, \omega_{\alpha})$ of Theorem 4 is called the **symplectic quotient** of $(\tilde{X}, \tilde{\omega}, \tilde{\psi}, \tilde{\mu})$ at α . We will similarly call the Hamiltonian G' -manifold $(X_{\alpha}, \omega_{\alpha}, \psi'_{\alpha}, \mu'_{\alpha})$ of this theorem the **Hamiltonian quotient** of $(\tilde{X}, \tilde{\omega}, (\tilde{\psi}, \tilde{\psi}'), (\tilde{\mu}, \tilde{\mu}'))$ at α .

Proof of Theorem 4. Exercise 3.32(b) establishes (SQ0). Since G is compact, the quotient projection p_{α} is a closed map. By [28, Lemma 37.1], the quotient space X_{α} is thus Hausdorff. By (SQ0) and the Implicit Function Theorem, $\tilde{\mu}^{-1}(\alpha) \subset \tilde{X}$ is a smooth submanifold on which the compact Lie G acts smoothly and freely. By the Slice Theorem (equivariant version of the Tubular Neighborhood Theorem), for every $x \in \tilde{\mu}^{-1}(\alpha)$ there thus exists a submanifold $S_x \subset \tilde{X}$ so that $x \in S_x$ and the map

$$S_x \times G \longrightarrow \tilde{X}, \quad (x', u) \longrightarrow \psi_u(x'),$$

is a diffeomorphism onto an open neighborhood $\tilde{U}_x \subset \tilde{X}$ of x preserved by G . This submanifold is then transverse to the orbits Gx' of G and thus to $\tilde{\mu}^{-1}(\alpha)$. The restriction

$$p_{\alpha} : \tilde{\mu}^{-1}(\alpha) \cap S_x \longrightarrow p_{\alpha}(\tilde{\mu}^{-1}(\alpha) \cap \tilde{U}_x) \subset X_{\alpha}$$

of the quotient map is a homeomorphism onto an open subset of X_{α} and induces a smooth structure on $p_{\alpha}(\tilde{\mu}^{-1}(\alpha) \cap \tilde{U}_x)$ so that the restriction

$$p_{\alpha} : \tilde{\mu}^{-1}(\alpha) \cap \tilde{U}_x \longrightarrow p_{\alpha}(\tilde{\mu}^{-1}(\alpha) \cap \tilde{U}_x)$$

is a (trivial) principle G -bundle. Since there is at most one smooth structure on $p_{\alpha}(\tilde{\mu}^{-1}(\alpha) \cap \tilde{U}_x)$ so that the latter restriction is a submersion, the smooth structures on open subset of X_{α} obtained in this way overlap smoothly. This establishes (SQ1).

By (SQ1), for every $x \in \tilde{\mu}^{-1}(\alpha)$ the map

$$d_x p_{\alpha} : T_x \tilde{\mu}^{-1}(\alpha) / T_x(Gx) \longrightarrow T_{p_{\alpha}(x)} X_{\alpha}$$

is a well-defined isomorphism. For each $v \in T_{\mathbb{1}}G$, let $\zeta_v \in \Gamma(\tilde{X}; T\tilde{X})$ be as in (1.2) with (X, ψ) replaced by $(\tilde{X}, \tilde{\psi})$. Thus,

$$\begin{aligned} T_x(Gx) &= \{\zeta_v(x) : v \in T_{\mathbb{1}}G\} \quad \forall x \in \tilde{\mu}^{-1}(\alpha) \quad \text{and} \\ -(\iota_{\zeta_v(x)} \tilde{\omega}) \Big|_{T_x \tilde{\mu}^{-1}(\alpha)} &= d_x(\{\tilde{\mu}(\cdot)\}(v)) \Big|_{T_x \tilde{\mu}^{-1}(\alpha)} = d_x(\alpha(v)) \Big|_{T_x \tilde{\mu}^{-1}(\alpha)} = 0 \quad \forall v \in T_{\mathbb{1}}G, x \in \tilde{\mu}^{-1}(\alpha). \end{aligned}$$

It follows that there is a unique alternating 2-tensor $\omega_{\alpha}|_{p_{\alpha}(x)}$ on $T_{p_{\alpha}(x)} X_{\alpha}$ for each $x \in \tilde{\mu}^{-1}(\alpha)$ so that

$$\omega_{\alpha}|_{p_{\alpha}(x)}(d_x p_{\alpha}(w), d_x p_{\alpha}(w')) = \tilde{\omega}|_x(w, w') \quad \forall w, w' \in T_x \tilde{\mu}^{-1}(\alpha),$$

i.e. $p_\alpha^*(\omega_\alpha|_{p_\alpha(x)}) = \tilde{\omega}|_{T_x\tilde{\mu}^{-1}(\alpha)}$. Since ω is G -invariant, $\omega_\alpha|_{p_\alpha(x)}$ does not depend on the choice of x in $p_\alpha^{-1}(p_\alpha(x))$, i.e. ω_α is a well-defined 2-form on X_α . Since p_α is a submersion and the 2-form $\tilde{\omega}|_{T\tilde{\mu}^{-1}(\alpha)}$ is smooth and closed, so is the 2-form ω_α . Since α is a regular value of $\tilde{\mu}$ and $\tilde{\omega}$ is nondegenerate on \tilde{X} , the first statement of Exercise 3.32 with (ω, μ) replaced by $(\tilde{\omega}, \tilde{\mu})$ implies that

$$(T_x\tilde{\mu}^{-1}(\alpha))^{\tilde{\omega}} = (\ker d_x\tilde{\mu})^{\tilde{\omega}} = T_x(Gx).$$

Thus, ω_α is nondegenerate.

Suppose G' is another Lie group and $(\tilde{X}, \tilde{\omega}, \tilde{\psi}', \tilde{\mu}')$ is a Hamiltonian G' -manifold such that the actions $\tilde{\psi}$ and $\tilde{\psi}'$ commute, $\tilde{\mu}'$ is ψ -invariant, and $\tilde{\mu}$ is ψ' -invariant. Since $\tilde{\mu}$ is ψ' -invariant, ψ' preserves $\tilde{\mu}^{-1}(\alpha) \subset \tilde{X}$. Since the actions $\tilde{\psi}$ and $\tilde{\psi}'$ commute and $\tilde{\mu}'$ is ψ -invariant, the restriction of $\tilde{\psi}'$ and $\tilde{\mu}'$ to $\tilde{\mu}^{-1}(\alpha)$ descend to a G' -action ψ'_α on X_α and a smooth map $\mu'_\alpha: X_\alpha \rightarrow T_1G'$. By Exercise 2.5 with (ψ, ψ) replaced by $(\tilde{\psi}', \psi')$, $(X_\alpha, \omega_\alpha, \psi'_\alpha, \mu'_\alpha)$ is thus a Hamiltonian G' -manifold. Since G acts freely on $\tilde{\mu}^{-1}(\alpha)$, (SQ5) follows from

$$\mu_\alpha^{-1}(\alpha') = (\tilde{\mu}, \tilde{\mu}')^{-1}(\alpha, \alpha')/G.$$

Along with (SQ1) and (SQ2), this identity also implies (SQ6). \square

Exercise 5.1. Suppose G , $(\tilde{X}, \tilde{\omega}, \tilde{\psi}, \tilde{\mu})$, α , $(X_\alpha, \omega_\alpha)$, and p_α are as in Theorem 4 and $\tilde{X}' \subset \tilde{X}$ is an $\tilde{\omega}$ -symplectic submanifold preserved by the G -action $\tilde{\psi}$.

- (a) Show that the symplectic quotient $(X'_\alpha, \omega'_\alpha)$ of $(\tilde{X}', \tilde{\omega}|_{\tilde{X}'}, \tilde{\psi}|_{\tilde{X}'}, \tilde{\mu}|_{\tilde{X}'})$ at α is a symplectic submanifold of $(X_\alpha, \omega_\alpha)$ and the bundle homomorphisms

$$\mathcal{N}_{\tilde{X}}\tilde{X}'|_{\tilde{\mu}^{-1}(\alpha)\cap\tilde{X}'} \longleftarrow \mathcal{N}_{\tilde{\mu}^{-1}(\alpha)}(\tilde{\mu}^{-1}(\alpha)\cap\tilde{X}') \xrightarrow{dp_\alpha} \{p_\alpha|_{\tilde{\mu}^{-1}(\alpha)\cap\tilde{X}'}\}^* \mathcal{N}_{X_\alpha}X'_\alpha \quad (5.1)$$

over $\tilde{\mu}^{-1}(\alpha)\cap\tilde{X}'$ induced by the inclusions and the quotient projection are isomorphisms.

- (b) Suppose that G' , $\tilde{\psi}'$, $\tilde{\mu}'$, and $(X_\alpha, \omega_\alpha, \psi'_\alpha, \mu'_\alpha)$ are also as in Theorem 4 and the submanifold $\tilde{X}' \subset \tilde{X}$ is preserved by the G' -action $\tilde{\psi}'$. Show that the submanifold $X'_\alpha \subset X_\alpha$ is preserved by the G' -action ψ' , $(X'_\alpha, \omega'_\alpha, \psi'_\alpha|_{X'_\alpha}, \mu'_\alpha|_{X'_\alpha})$ is the Hamiltonian quotient of

$$(\tilde{X}', \tilde{\omega}|_{\tilde{X}'}, (\tilde{\psi}, \tilde{\psi}')|_{\tilde{X}'}, (\tilde{\mu}, \tilde{\mu}')|_{\tilde{X}'})$$

at α , and the bundle isomorphisms in (5.1) are G' -equivariant.

Exercise 5.2. Suppose G , $(\tilde{X}, \tilde{\omega}, \tilde{\psi}, \tilde{\mu})$, α , $(X_\alpha, \omega_\alpha)$, and p_α are as in Theorem 4 and \tilde{J} is a ψ -invariant almost complex structure on \tilde{X} compatible with $\tilde{\omega}$. Show that

- (a) the restriction of the differential

$$d_x p_\alpha: T_x\tilde{\mu}^{-1}(\alpha)\cap\tilde{J}(T_x\tilde{\mu}^{-1}(\alpha)) \longrightarrow T_{p_\alpha(x)}X_\alpha$$

is an isomorphism for every $x \in \tilde{\mu}^{-1}(\alpha)$ and thus induces an almost complex structure J_α on X_α compatible with ω_α ;

- (b) if G' , $\tilde{\psi}'$, $\tilde{\mu}'$, and $(X_\alpha, \omega_\alpha, \psi'_\alpha, \mu'_\alpha)$ are also as in Theorem 4 and the almost complex structure \tilde{J} on \tilde{X} is ψ' -invariant as well, then the almost complex structure J_α is ψ'_α -invariant;

- (c) if $\tilde{X}' \subset \tilde{X}$ is an almost complex submanifold preserved by the G -action $\tilde{\psi}$ and $(X'_\alpha, \omega'_\alpha)$ is the symplectic quotient of $(\tilde{X}', \tilde{\omega}|_{\tilde{X}'}, \tilde{\psi}|_{\tilde{X}'}, \tilde{\mu}|_{\tilde{X}'})$ at α , then X'_α is an almost complex submanifold of X_α and the composite isomorphism from the left-hand side in (5.1) to the right-hand side is \mathbb{C} -linear with respect to the complex structures induced by \tilde{J} and J_α .

Example 5.3. Let $n \in \mathbb{Z}^+$. By Exercise 2.6,

$$H: \mathbb{C}^n \longrightarrow \mathbb{R}, \quad H(z_1, \dots, z_n) = \pi \sum_{k=1}^n |z_k|^2,$$

is a Hamiltonian for the standard action ψ of S^1 on \mathbb{C}^n ,

$$\psi_u: \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad \psi_u(z) = uz, \quad \forall u \in S^1 \subset \mathbb{C}.$$

For each $r \in \mathbb{R}^+$, the group S^1 acts freely on $H^{-1}(\pi r^2)$, the sphere of radius r centered at the origin. The associated quotient of Theorem 4 is the complex projective space $\mathbb{C}P^{n-1}$ with a symplectic form $\omega_{\mathbb{C}P^{n-1};r}$. By Exercise 2.9(b),

$$\omega_{\mathbb{C}P^{n-1};r} = \pi r^2 \omega_{\text{FS};n-1}.$$

Exercise 5.4. Let $n \in \mathbb{Z}^+$ and $q: \mathbb{C}^n - \{0\} \longrightarrow \mathbb{C}P^{n-1}$ be the usual quotient projection. The \mathbb{C}^* -action on \mathbb{C}^n by the coordinate multiplication restricts to a \mathbb{C}^* -action on $\mathbb{C}^n - \{0\}$ and S^1 -actions on \mathbb{C}^n and the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. Show that

- (a) the quotient topologies on $\mathbb{C}P^{n-1}$ given by $(\mathbb{C}^n - \{0\})/\mathbb{C}^*$ and S^{2n-1}/S^1 are the same (i.e. the map $S^{2n-1}/S^1 \longrightarrow (\mathbb{C}^n - \{0\})/\mathbb{C}^*$ induced by inclusions is a homeomorphism);
- (b) $\mathbb{C}P^{n-1}$ is a compact topological $2(n-1)$ -manifold that admits a complex structure so that the quotient projections

$$q: \mathbb{C}^n - \{0\} \longrightarrow \mathbb{C}P^{n-1} = (\mathbb{C}^n - \{0\})/\mathbb{C}^* \quad \text{and} \quad p: S^{2n-1} \longrightarrow \mathbb{C}P^{n-1} = S^{2n-1}/S^1$$

are a holomorphic submersion and a smooth submersion, respectively.

- (c) the above complex structure is compatible with the Fubini-Study symplectic form $\omega_{\text{FS};n-1}$ of Exercise 2.9(b).

5.2 Hamiltonian symplectic cut

Let S be any set. By a finite subcollection $\mathcal{H} \subset S$, we mean an element of a finite symmetric product of S , i.e. the quotient of the Cartesian product S^k of k copies of S for some $k \in \mathbb{Z}^{\geq 0}$ by the action of the k -th symmetric group \mathbb{S}_k permuting the components of the elements of S^k . For a finite subcollection $\mathcal{H} \subset S$ with a representative $\tilde{\mathcal{H}} \in S^k$, we set $|\mathcal{H}| = k$. In such a case, we write $v \in \mathcal{H}$ and $\mathcal{H}' \subset \mathcal{H}$ if $v \in S$ is a component of $\tilde{\mathcal{H}}$ and \mathcal{H}' is a finite subcollection of S obtained by dropping some of the components of $\tilde{\mathcal{H}}$, respectively. In the latter case, $\mathcal{H} - \mathcal{H}'$ denotes the finite subcollection of S obtained from the dropped components of $\tilde{\mathcal{H}}$. We say $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ is a partition of \mathcal{H} if $\mathcal{H}_1 \subset \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{H} - \mathcal{H}_1$. We call finite subcollections $\mathcal{H}_1, \mathcal{H}_2 \subset S$ disjoint if the sets of the components of their representatives are disjoint subsets of S . In particular, \mathcal{H}' and $\mathcal{H} - \mathcal{H}'$ above may not be disjoint if $\mathcal{H} \subset S$ is not a subset in the usual sense. Informally, a

finite subcollection $\mathcal{H} \subset S$ is a ‘‘subset’’ which may contain several copies of an element of S . For the purposes of Sections 5.2 and 5.3, it would be sufficient to restrict to subsets in the usual sense. However, subcollections fit better with some of the considerations of Chapter 6, as certain relevant operations on subsets can turn them into subcollections that are no longer subsets.

Suppose \mathbb{T} is a torus. For $v \equiv (v, c) \in T_{\mathbb{1}}^* \mathbb{T} \times \mathbb{R}$, let

$$c_v = c, \quad \mathcal{H}_v \equiv \{\alpha \in T_{\mathbb{1}}^* \mathbb{T} : \alpha(v) \geq c\}, \quad \text{and} \quad \partial \mathcal{H}_v \equiv \{\alpha \in T_{\mathbb{1}}^* \mathbb{T} : \alpha(v) = c\};$$

the subspaces $\mathcal{H}_v, \partial \mathcal{H}_v \subset T_{\mathbb{1}}^* \mathbb{T}$ are a (closed) half-space and an affine hyperplane, respectively, if $v \neq 0$. For a subcollection $\mathcal{H} \subset T_{\mathbb{1}}^* \mathbb{T} \times \mathbb{R}$ and $\mathcal{H}' \subset \mathcal{H}$, let

$$\langle \mathcal{H} \rangle = \bigcap_{v \in \mathcal{H}} \mathcal{H}_v \subset T_{\mathbb{1}}^* \mathbb{T}, \quad \langle \mathcal{H} \rangle^\partial = \bigcap_{v \in \mathcal{H}} \partial \mathcal{H}_v \subset \langle \mathcal{H} \rangle, \quad \langle \mathcal{H}' \rangle_{\mathcal{H}}^\partial = \langle \mathcal{H}' \rangle^\partial \cap \langle \mathcal{H} \rangle - \bigcap_{\mathcal{H}' \subsetneq \mathcal{H}'' \subset \mathcal{H}} \langle \mathcal{H}'' \rangle^\partial.$$

In particular, $\langle \emptyset \rangle = \langle \emptyset \rangle^\partial = \langle \emptyset \rangle_0^\partial = T_{\mathbb{1}}^* \mathbb{T}$, $\langle \mathcal{H}' \rangle_{\mathcal{H}}^\partial \subset \langle \mathcal{H}' \rangle^\partial$ is an open subset,

$$\langle \mathcal{H}' \rangle \cap \langle \mathcal{H} - \mathcal{H}' \rangle = \langle \mathcal{H}' \rangle, \quad \langle \mathcal{H}' \rangle^\partial \cap \langle \mathcal{H} - \mathcal{H}' \rangle = \langle \mathcal{H}' \rangle^\partial \cap \langle \mathcal{H} \rangle, \quad \text{and} \quad \langle \mathcal{H}' \rangle^\partial \cap \langle \emptyset \rangle_{\mathcal{H} - \mathcal{H}'}^\partial = \langle \mathcal{H}' \rangle_{\mathcal{H}}^\partial.$$

We call a subcollection $\mathcal{H} \subset (T_{\mathbb{1}} \mathbb{T} - \{0\}) \times \mathbb{R}$ minimal if it is a subset of $(T_{\mathbb{1}} \mathbb{T} - \{0\}) \times \mathbb{R}$ in the usual sense and $\partial \mathcal{H}_v \cap \langle \mathcal{H} \rangle \neq \emptyset$ for every $v \in \mathcal{H}$. Every subcollection $\mathcal{H} \subset (T_{\mathbb{1}} \mathbb{T} - \{0\}) \times \mathbb{R}$ with $\langle \mathcal{H} \rangle \neq \emptyset$ contains a unique minimal subcollection \mathcal{H}' with $\langle \mathcal{H}' \rangle = \langle \mathcal{H} \rangle$.

If $\mathcal{H} \subset (T_{\mathbb{1}} \mathbb{T} - \{0\}) \times \mathbb{R}$ is a finite subcollection, $\langle \mathcal{H} \rangle$ is a polyhedron by definition. A polytope is easily seen to be a compact polyhedron. The converse, which is not needed for our purposes, follows from the Minkowski-Weyl Theorem [11, Theorem 3.13], which states that a polyhedron is a finitely generated cone on a polytope.

If $\mathcal{H} \subset (T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection, define

$$L_{\mathcal{H}} : \mathbb{R}^{\mathcal{H}} \longrightarrow T_{\mathbb{1}} \mathbb{T}, \quad L_{\mathcal{H}}((r_{v,c})_{(v,c) \in \mathcal{H}}) = \sum_{(v,c) \in \mathcal{H}} r_{v,c} v, \quad (5.2)$$

$$\Phi_{\mathcal{H}} : \mathbb{T}^{\mathcal{H}} \equiv \mathbb{R}^{\mathcal{H}} / \mathbb{Z}^{\mathcal{H}} \longrightarrow \mathbb{T}, \quad \Phi_{\mathcal{H}}([\mathbf{r}]) = e^{L_{\mathcal{H}}(\mathbf{r})}, \quad \mathbb{T}_{\mathcal{H}} = \text{Im } \Phi_{\mathcal{H}}.$$

In particular, $\mathbb{T}_{\mathcal{H}} \subset \mathbb{T}$ is a subtorus. We call a finite subcollection $\mathcal{H} \subset (T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ Delzant (resp. regular) if $\Phi_{\mathcal{H}'}$ is injective for every subcollection $\mathcal{H}' \subset \mathcal{H}$ such that $\langle \mathcal{H}' \rangle^\partial \cap \langle \mathcal{H} \rangle \neq \emptyset$ (resp. if $L_{\mathcal{H}}$ is surjective). The former implies in particular that $v \in (T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}}$ is primitive for every element $(v, c) \in \mathcal{H}$ such that $\partial \mathcal{H}_{v,c} \cap \langle \mathcal{H} \rangle \neq \emptyset$ and $\mathbb{T}_{\mathcal{H}'}$ is a subtorus of dimension $|\mathcal{H}'|$ for every subset $\mathcal{H}' \subset \mathcal{H}$ such that $\langle \mathcal{H}' \rangle^\partial \cap \langle \mathcal{H} \rangle \neq \emptyset$. However, the disjoint union of any Delzant subcollection \mathcal{H} of $(T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ and a pair $(0, c)$ with $c \in \mathbb{R}^\pm$ is still a Delzant subcollection of $(T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$.

Exercise 5.5. Let \mathbb{T} be a torus and $\mathcal{H} \subset (T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a finite subcollection.

- Suppose $\alpha_0 \in \langle \mathcal{H} \rangle^\partial$. Show that $\langle \mathcal{H} \rangle^\partial \subset T_{\mathbb{1}}^* \mathbb{T}$ is the preimage of $\alpha_0|_{T_{\mathbb{1}} \mathbb{T}, \mathcal{H}}$ under the restriction homomorphism $T_{\mathbb{1}}^* \mathbb{T} \longrightarrow T_{\mathbb{1}}^* \mathbb{T}_{\mathcal{H}}$.
- Suppose $\mathbb{T}' \subset \mathbb{T}$ is a subtorus of \mathbb{T} so that $\mathcal{H} \subset (T_{\mathbb{1}} \mathbb{T}')_{\mathbb{Z}} \times \mathbb{R}$. Show that the images of

$$\langle \mathcal{H} \rangle, \langle \mathcal{H} \rangle^\partial, \quad \langle \mathcal{H}' \rangle^\partial \cap \langle \mathcal{H} \rangle, \langle \mathcal{H}' \rangle_{\mathcal{H}}^\partial \subset T_{\mathbb{1}}^* \mathbb{T} \quad \text{with} \quad \mathcal{H}' \subset \mathcal{H}$$

under the restriction homomorphism $T_{\mathbb{1}}^* \mathbb{T} \longrightarrow T_{\mathbb{1}}^* \mathbb{T}'$ are the corresponding subsets of $T_{\mathbb{1}}^* \mathbb{T}'$. Conclude that \mathcal{H} is Delzant with respect to \mathbb{T} if and only if \mathcal{H} is Delzant with respect to \mathbb{T}' .

Exercise 5.6. Let \mathbb{T} be a torus and $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a finite subcollection.

- (a) Suppose $\mathcal{H}' \subset \mathcal{H}$ and $\langle \mathcal{H}' \rangle^{\partial} \cap \langle \mathcal{H} \rangle \neq \emptyset$. Show that there exists a subcollection $\mathcal{H}_{\bullet} \subset \mathcal{H}$ such that

$$\mathcal{H}' \subset \mathcal{H}_{\bullet}, \quad \langle \mathcal{H}_{\bullet} \rangle^{\partial} \cap \langle \mathcal{H} \rangle \neq \emptyset \subset T_1^* \mathbb{T}, \quad \text{and} \quad \text{Im } L_{\mathcal{H}_{\bullet}} = \text{Im } L_{\mathcal{H}} \subset T_1 \mathbb{T}.$$

- (b) Suppose \mathcal{H} is Delzant with $\langle \mathcal{H} \rangle \neq \emptyset$. Show that $\ker \Phi_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}}$ is a subtorus of codimension equal to the dimension of $\text{Im } L_{\mathcal{H}} \subset T_1 \mathbb{T}$.

Exercise 5.7. Suppose \mathbb{T} is a torus and $\mathcal{H} \subset \mathcal{H}^{\dagger} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ are finite subcollections. Show that

- (a) if \mathcal{H} is regular, then the sequence

$$0 \longrightarrow \ker \Phi_{\mathcal{H}} \longrightarrow \ker \Phi_{\mathcal{H}^{\dagger}} \longrightarrow \mathbb{T}^{\mathcal{H}^{\dagger} - \mathcal{H}} \longrightarrow 0$$

of abelian Lie groups, where the second homomorphism is the restriction of the projection $\mathbb{T}^{\mathcal{H}^{\dagger}} \longrightarrow \mathbb{T}^{\mathcal{H}^{\dagger} - \mathcal{H}}$, is exact;

- (b) if \mathcal{H}^{\dagger} is Delzant and $\mathcal{H}' \subset \mathcal{H}$ is a finite subcollection such that $\langle \mathcal{H}' \rangle^{\partial} \cap \langle \mathcal{H} \rangle = \langle \mathcal{H}' \rangle^{\partial} \cap \langle \mathcal{H}^{\dagger} \rangle$, then

$$\alpha(v) > c \quad \forall (v, c) \in \mathcal{H}^{\dagger} - \mathcal{H}, \quad \alpha \in \langle \mathcal{H}' \rangle^{\partial} \cap \langle \mathcal{H} \rangle.$$

Exercise 5.8. Suppose \mathbb{T} is a torus, $\mathcal{H}_0 \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection, $\alpha_0 \in \langle \mathcal{H}_0 \rangle^{\partial}$, and $q: \mathbb{T} \longrightarrow \mathbb{T}/\mathbb{T}_{\mathcal{H}_0}$ is the quotient projection. Define

$$q_{\alpha_0}: (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R} \longrightarrow (T_1(\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}))_{\mathbb{Z}} \times \mathbb{R}, \quad q_{\alpha_0}(v, c) = (d_1 q(v), c - \alpha_0(v)).$$

If $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection such that $\mathcal{H}_0 \subset \mathcal{H}$, let $\mathcal{H}/\mathcal{H}_0 = q_{\alpha_0}(\mathcal{H} - \mathcal{H}_0)$ and

$$\phi_{\mathcal{H}; \mathcal{H}_0} \equiv (\phi_{\mathcal{H}; \mathcal{H}_0; v})_{v \in \mathcal{H}/\mathcal{H}_0}: \mathbb{T}^{\mathcal{H}} \longrightarrow \mathbb{T}^{\mathcal{H}/\mathcal{H}_0}, \quad \phi_{\mathcal{H}; \mathcal{H}_0; q_{\alpha_0}(v)}((u_{v'})_{v' \in \mathcal{H}}) = u_v \quad \forall v \in \mathcal{H} - \mathcal{H}_0.$$

Show that

- (a) if $\mathcal{H}_0 \subset \mathcal{H}' \subset \mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ are finite subcollections, then $\langle \mathcal{H}'/\mathcal{H}_0 \rangle^{\partial} \cap \langle \mathcal{H}/\mathcal{H}_0 \rangle \neq \emptyset$ if and only if $\langle \mathcal{H}' \rangle^{\partial} \cap \langle \mathcal{H} \rangle \neq \emptyset$;

- (b) if $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection such that $\mathcal{H}_0 \subset \mathcal{H}$, then the diagram

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & \ker \Phi_{\mathcal{H}} & \longrightarrow & \mathbb{T}^{\mathcal{H}} & \xrightarrow{\Phi_{\mathcal{H}}} & \mathbb{T} & \longrightarrow & \{1\} \\ & & \downarrow & & \downarrow \phi_{\mathcal{H}; \mathcal{H}_0} & & \downarrow q & & \\ \{1\} & \longrightarrow & \ker \Phi_{\mathcal{H}/\mathcal{H}_0} & \longrightarrow & \mathbb{T}^{\mathcal{H}/\mathcal{H}_0} & \xrightarrow{\Phi_{\mathcal{H}/\mathcal{H}_0}} & \mathbb{T}/\mathbb{T}_{\mathcal{H}_0} & \longrightarrow & \{1\} \end{array}$$

of exact rows commutes and the left vertical homomorphism is surjective;

- (c) if the homomorphism $\Phi_{\mathcal{H}_0}$ is injective and $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection such that $\mathcal{H}_0 \subset \mathcal{H}$, then the homomorphism $\phi_{\mathcal{H}; \mathcal{H}_0}$ restricts to an isomorphism from $\ker \Phi_{\mathcal{H}}$ to $\ker \Phi_{\mathcal{H}/\mathcal{H}_0}$;

- (d) if $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant (resp. regular) subcollection such that $\mathcal{H}_0 \subset \mathcal{H}$, then the subcollection $\mathcal{H}/\mathcal{H}_0 \subset (T_1(\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}))_{\mathbb{Z}} \times \mathbb{R}$ is also Delzant (resp. regular).

Exercise 5.9. Suppose \mathbb{T} is a torus and $P \subset T_1^*\mathbb{T}$ is a polytope. Show that P is Delzant if and only if $P = \langle \mathcal{H} \rangle$ for some regular minimal Delzant subcollection $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$.

Exercise 5.10. Suppose \mathbb{T} is a torus, $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is finite subcollection, and (X, ω, ψ, μ) is a Hamiltonian \mathbb{T} -manifold. Show that

- (a) if $\langle \mathcal{H} \rangle^\partial \neq \emptyset$, then $\mu^{-1}(\langle \mathcal{H} \rangle^\partial) \subset X$ is a fiber of a moment map for the restriction of the action ψ to $\mathbb{T}_{\mathcal{H}} \subset \mathbb{T}$;
- (b) if $\mathbb{T}_{\mathcal{H}}$ acts freely on $\mu^{-1}(\langle \mathcal{H} \rangle^\partial)$, then $\mu^{-1}(\langle \mathcal{H} \rangle^\partial) \subset X$ is a closed submanifold of codimension equal to the dimension of $\mathbb{T}_{\mathcal{H}}$;
- (c) if $\mu(X) \subset \langle \mathcal{H} \rangle^\partial$, then $\mathbb{T}_{\mathcal{H}}$ acts trivially on X ;
- (d) if $\mu(X) \subset \langle \mathcal{H} \rangle^\partial$, $x \in X$, and $\text{Stab}_x(\psi) = \mathbb{T}_{\mathcal{H}}$, then the differential $d_x\mu: T_x X \rightarrow T_{\mu(x)}\langle \mathcal{H} \rangle^\partial$ is surjective.

Definition 5.11. Let \mathbb{T} be a torus and $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a finite subcollection. A Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) is

- \mathcal{H} -cuttable if for every subcollection $\mathcal{H}' \subset \mathcal{H}$ such that $\mu^{-1}(\langle \mathcal{H}' \rangle^\partial) \neq \emptyset$ the Lie group homomorphism $\Phi_{\mathcal{H}'}$ as in (5.2) is injective and the subtorus $\mathbb{T}_{\mathcal{H}'} \subset \mathbb{T}$ acts freely on $\mu^{-1}(\langle \mathcal{H}' \rangle^\partial)$;
- \mathcal{H} -cut if $\mu(X) \subset \langle \mathcal{H} \rangle$ and for every $\mathcal{H}' \subset \mathcal{H}$ the subspace $Y_{\mathcal{H}'} \equiv \mu^{-1}(\langle \mathcal{H}' \rangle^\partial)$ is a union of topological components of the fixed locus $X^{\mathbb{T}_{\mathcal{H}'}} \subset X$ of $\psi|_{\mathbb{T}_{\mathcal{H}'}}$ and there is a \mathbb{T} -equivariant splitting

$$TX|_{Y_{\mathcal{H}'}} = TY_{\mathcal{H}'} \oplus \bigoplus_{v \in \mathcal{H}'} \mathcal{N}_X^v Y_{\mathcal{H}'} \rightarrow Y_{\mathcal{H}'} \quad (5.3)$$

of $TX|_{Y_{\mathcal{H}'}}$, with a ψ -invariant complex structure J compatible with ω so that

$$\begin{aligned} \text{rk}_{\mathbb{C}} \mathcal{N}_X^v Y_{\mathcal{H}'} &= 1 \quad \forall v \in \mathcal{H}' \quad \text{and} \\ d\psi_{\Phi_{\mathcal{H}'}}((r_v)_{v \in \mathcal{H}'}) &= e^{2\pi i r_v} w \quad \forall (r_v)_{v \in \mathcal{H}'} \in \mathbb{R}^{\mathcal{H}'}, w \in \mathcal{N}_X^{v'} Y_{\mathcal{H}'}, v' \in \mathcal{H}'. \end{aligned} \quad (5.4)$$

Suppose \mathbb{T} is a torus, $\mathcal{H}' \subset \mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ are finite subcollections, and (X, ω, ψ, μ) is a Hamiltonian \mathbb{T} -manifold such that $\mu^{-1}(\langle \mathcal{H}' \rangle^\partial) \neq \emptyset$. If (X, ω, ψ, μ) is an \mathcal{H} -cuttable, then $\mu^{-1}(\langle \mathcal{H}' \rangle^\partial) \subset X$ is a closed submanifold of codimension $|\mathcal{H}'|$ by Exercise 5.10(b) and thus

$$\mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^\partial) = \mu^{-1}(\langle \mathcal{H}' \rangle^\partial \cap \langle \mathcal{H} \rangle) - \bigcap_{\mathcal{H}'' \subset \mathcal{H}' \subset \mathcal{H}} \mu^{-1}(\langle \mathcal{H}'' \rangle^\partial)$$

is an open subset. If (X, ω, ψ, μ) is \mathcal{H} -cut, then $\mu^{-1}(\langle \mathcal{H}' \rangle^\partial) \subset X$ is a closed ω -symplectic submanifold of codimension $2|\mathcal{H}'|$ and the Lie group homomorphism $\Phi_{\mathcal{H}'}$ as in (5.2) is injective by Proposition 3.16(1), (5.3), and (5.4). Thus, $\mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^\partial) \subset \mu^{-1}(\langle \mathcal{H}' \rangle^\partial)$ is again an open subset; it is dense in this case, since $\mu(X) \subset \langle \mathcal{H} \rangle$. If in addition $\mathcal{H}'_1 \subset \mathcal{H}$ and $\mathcal{H}'_2 \subset \mathcal{H} - \mathcal{H}'_1$, then the restriction of μ to the symplectic submanifold $\mu^{-1}(\langle \mathcal{H}'_2 \rangle^\partial) \subset X$ is transverse to $\langle \mathcal{H}'_1 \rangle^\partial \subset T_1^*\mathbb{T}$ by (5.3) and (5.4).

Exercise 5.12. Suppose (X, ω, ψ, μ) is a compact symplectic toric \mathbb{T} -manifold, $\mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant subcollection so that $\mu(X) = \langle \mathcal{H} \rangle$, and $\mathcal{H}' \subset \mathcal{H}$. Show that $\mathbb{T}_{\mathcal{H}'} \subset \mathbb{T}$ is the subtorus $\mathbb{T}_{\langle \mathcal{H}' \rangle \cap \langle \mathcal{H} \rangle} \subset \mathbb{T}$ as in (0⁺) on page 8, $Y_{\mathcal{H}'} \equiv \mu^{-1}(\langle \mathcal{H}' \rangle^{\partial})$ is a connected component of the fixed locus $X^{\mathbb{T}_{\mathcal{H}'}} \subset X$ of $\psi|_{\mathbb{T}_{\mathcal{H}'}}$, and $TX|_{Y_{\mathcal{H}'}}$ splits as in (5.3) and (5.4). Conclude that the Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) is \mathcal{H} -cut.

Exercise 5.13. Suppose \mathbb{T} is a torus, $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ is a partition of a finite subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$, and (X, ω, ψ, μ) is an \mathcal{H} -cuttable Hamiltonian \mathbb{T} -manifold. Show that

- (a) (X, ω, ψ, μ) is \mathcal{H}_1 -cuttable;
- (b) for all $\mathcal{H}'_1 \subset \mathcal{H}_1$ and $\mathcal{H}'_2 \subset \mathcal{H}_2$ such that $\mu^{-1}(\langle \mathcal{H}'_2 \rangle^{\partial}) \neq \emptyset$ the Lie group homomorphism $\Phi_{\mathcal{H}'_2}$ as in (5.2) is injective and the subtorus $\mathbb{T}_{\mathcal{H}'_2} \subset \mathbb{T}$ acts freely on $\mu^{-1}(\langle \mathcal{H}'_1 \sqcup \mathcal{H}'_2 \rangle^{\partial}) / \mathbb{T}_{\mathcal{H}'_1}$.

Theorem 5 ([21, Proposition 2.4]). *Suppose \mathbb{T} is a torus, $\mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection, and (X, ω, ψ, μ) is an \mathcal{H} -cuttable Hamiltonian \mathbb{T} -manifold. There exists a unique \mathcal{H} -cut Hamiltonian \mathbb{T} -manifold*

$$(X, \omega, \psi, \mu)_{\mathcal{H}} \equiv (X_{\mathcal{H}}, \omega_{\mathcal{H}}, \psi_{\mathcal{H}}, \mu_{\mathcal{H}}) \quad (5.5)$$

so that

(HC₁) $X_{\mathcal{H}} = \mu^{-1}(\langle \mathcal{H} \rangle) / \sim$ with $x \sim x'$ if there exist $\mathcal{H}' \subset \mathcal{H}$ and $u \in \mathbb{T}_{\mathcal{H}'}$ so that $\mu(x) \in \langle \mathcal{H}' \rangle^{\partial}$ and $x' = \psi_u(x)$;

(HC₂) the quotient projection $p_{\mathcal{H}}: \mu^{-1}(\langle \mathcal{H} \rangle) \rightarrow X_{\mathcal{H}}$ is \mathbb{T} -equivariant and $\mu|_{\mu^{-1}(\langle \mathcal{H} \rangle)} = \mu_{\mathcal{H}} \circ p_{\mathcal{H}}$;

(HC₃) for every $\mathcal{H}' \subset \mathcal{H}$, $p_{\mathcal{H}}: \mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial}) \rightarrow \mu_{\mathcal{H}}^{-1}(\langle \mathcal{H}' \rangle^{\partial})$ is a submersion (onto a dense open subset) and

$$\{p_{\mathcal{H}}|_{\mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial})}\}^* \omega_{\mathcal{H}} = \omega|_{T(\mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial}))}. \quad (5.6)$$

For any partition $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$, $(X, \omega, \psi, \mu)_{\mathcal{H}_1}$ is an \mathcal{H}_2 -cuttable Hamiltonian \mathbb{T} -manifold and

$$(X, \omega, \psi, \mu)_{\mathcal{H}} = ((X, \omega, \psi, \mu)_{\mathcal{H}_1})_{\mathcal{H}_2}. \quad (5.7)$$

Proof. Let $\mathbf{c} = (c_v)_{v \in \mathcal{H}} \in \mathbb{R}^{\mathcal{H}}$ and $\pi_1, \pi_2: X \times \mathbb{C}^{\mathcal{H}} \rightarrow X, \mathbb{C}^{\mathcal{H}}$ be the component projections. Denote by $\omega_{\mathcal{H}}$ the standard symplectic form on $\mathbb{C}^{\mathcal{H}}$, analogously to (2.11). The 2-form

$$\tilde{\omega} \equiv \pi_1^* \omega \oplus \pi_2^* \omega_{\mathcal{H}}$$

on $X \times \mathbb{C}^{\mathcal{H}}$ is then symplectic. Let $\tilde{\psi}$ be the $\mathbb{T}^{\mathcal{H}}$ -action on $X \times \mathbb{C}^{\mathcal{H}}$ given by

$$\tilde{\psi}_{[(r_v)_{v \in \mathcal{H}}]}(x, (z_v)_{v \in \mathcal{H}}) = (\psi_{\Phi_{\mathcal{H}}([(r_v)_{v \in \mathcal{H}}])}(x), (e^{-2\pi i r_v} z_v)_{v \in \mathcal{H}}). \quad (5.8)$$

This action commutes with the \mathbb{T} -action ψ on the first component and preserves its moment map

$$\mu \circ \pi_1: X \times \mathbb{C}^{\mathcal{H}} \rightarrow T_{\mathbb{1}}^* \mathbb{T} \quad (5.9)$$

with respect to $\tilde{\omega}$. By Exercises 2.2 and 2.6, the smooth function

$$\tilde{H}_{\mu}: X \times \mathbb{C}^{\mathcal{H}} \rightarrow T_{\mathbb{1}}^* \mathbb{T}^{\mathcal{H}} = \mathbb{R}^{\mathcal{H}}, \quad \tilde{H}_{\mu}(x, (z_{v,c})_{(v,c) \in \mathcal{H}}) = (\mu_v(x) - \pi |z_{v,c}|^2)_{(v,c) \in \mathcal{H}}, \quad (5.10)$$

is a Hamiltonian for the action $\tilde{\psi}$ with respect to $\tilde{\omega}$. It is preserved by the \mathbb{T} -action ψ on the first component.

Suppose $(x, (z_v)_{v \in \mathcal{H}}) \in \tilde{H}_\mu^{-1}(\mathbf{c})$, $u \in \mathbb{T}^{\mathcal{H}}$, and $\tilde{\psi}_u(x, (z_v)_{v \in \mathcal{H}}) = (x, (z_v)_{v \in \mathcal{H}})$. Let

$$\mathcal{H}' = \{v \in \mathcal{H} : z_v = 0\}.$$

From (5.10) and (5.8), we then obtain

$$\mu(x) \in \langle \mathcal{H}' \rangle_{\mathcal{H}}^\partial \subset T_1^* \mathbb{T} \quad \text{and} \quad u \in \mathbb{T}^{\mathcal{H}'} \subset \mathbb{T}^{\mathcal{H}}. \quad (5.11)$$

Since (X, ω, ψ, μ) is \mathcal{H} -cuttable, it follows that $u = \mathbb{1}$. Thus, $\mathbb{T}^{\mathcal{H}}$ acts freely on $\tilde{H}_\mu^{-1}(\mathbf{c})$ via (5.8). Let

$$(X_{\mathcal{H}}, \omega_{\mathcal{H}}, \psi_{\mathcal{H}}, \mu_{\mathcal{H}}) \equiv (X_\alpha, \omega_\alpha, \psi_\alpha, \mu_\alpha)$$

be the associated quotient Hamiltonian \mathbb{T} -manifold of Theorem 4 and

$$p: \tilde{H}_\mu^{-1}(\mathbf{c}) \longrightarrow X_{\mathcal{H}} \equiv \tilde{H}_\mu^{-1}(\mathbf{c}) / \mathbb{T}^{\mathcal{H}}$$

be the quotient projection.

The map

$$p_{\mathcal{H}}: \mu^{-1}(\langle \mathcal{H} \rangle) \longrightarrow X_{\mathcal{H}} \equiv \tilde{H}_\mu^{-1}(\mathbf{c}) / \mathbb{T}^{\mathcal{H}}, \quad p_{\mathcal{H}}(x) = [x, (\sqrt{(\mu_v(x) - c_v) / \pi})_{(v, c) \in \mathcal{H}})],$$

is well-defined, continuous, surjective, and \mathbb{T} -equivariant and satisfies the last condition in (HC₂). In particular, $\mu_{\mathcal{H}}(X_{\mathcal{H}}) \subset \langle \mathcal{H} \rangle$. By the first statement in (5.11), $p_{\mathcal{H}}$ induces an injective map from the quotient $\mu^{-1}(\langle \mathcal{H} \rangle) / \sim$ in (HC₁) to $X_{\mathcal{H}}$. Since the map

$$\tilde{p}_{\mathcal{H}}: \mu^{-1}(\langle \mathcal{H} \rangle) \longrightarrow \tilde{H}_\mu^{-1}(\mathbf{c}), \quad \tilde{p}_{\mathcal{H}}(x) = (x, (\sqrt{(\mu_v(x) - c_v) / \pi})_{(v, c) \in \mathcal{H}})), \quad (5.12)$$

is closed and the group $\mathbb{T}^{\mathcal{H}}$ is compact, $p_{\mathcal{H}}$ is a closed map and thus so is the induced map from $\mu^{-1}(\langle \mathcal{H} \rangle)$. This confirms (HC₁) and (HC₂).

Let J be a ψ -invariant almost complex structure on X compatible with ω , $J_{\mathcal{H}}$ be the standard complex structure on $\mathbb{C}^{\mathcal{H}}$, and $\tilde{\psi}'$ be the action of $\mathbb{T}_{\mathcal{H}}$ on $X \times \mathbb{C}^{\mathcal{H}}$ given by

$$\begin{aligned} \tilde{\psi}'_{\Phi_{\mathcal{H}}([(r_v)_{v \in \mathcal{H}}])}(x, (z_v)_{v \in \mathcal{H}}) &= \tilde{\psi}'_{[(-r_v)_{v \in \mathcal{H}}]}(\psi_{\Phi_{\mathcal{H}}([(r_v)_{v \in \mathcal{H}}])}(x), (z_v)_{v \in \mathcal{H}}) \\ &= (x, (e^{2\pi i r_v} z_v)_{v \in \mathcal{H}}). \end{aligned} \quad (5.13)$$

This action commutes with the $\mathbb{T}^{\mathcal{H}}$ -action $\tilde{\psi}$ and thus induces a $\mathbb{T}_{\mathcal{H}}$ -action on $X_{\mathcal{H}}$. By the middle expression in (5.13), the latter is the restriction of the \mathbb{T} -action $\psi_{\mathcal{H}}$ to $\mathbb{T}_{\mathcal{H}}$. The almost complex structure $\tilde{J} \equiv J \oplus J_{\mathcal{H}}$ on $X \times \mathbb{C}^{\mathcal{H}}$ is ψ -, $\tilde{\psi}$ -, and $\tilde{\psi}'$ -invariant and compatible with $\tilde{\omega}$. By Exercise 5.2, \tilde{J} thus descends to a $\psi_{\mathcal{H}}$ -invariant almost complex structure $J_{\mathcal{H}}$ on $X_{\mathcal{H}}$ which is compatible with $\omega_{\mathcal{H}}$.

Let $\mathcal{H}' \subset \mathcal{H}$. By the first statement in (5.11),

$$\begin{aligned} \tilde{Y}_{\mathcal{H}'} &\equiv (\mu^{-1}(\langle \mathcal{H}' \rangle^\partial) \times \mathbb{C}^{\mathcal{H}}) \cap \tilde{H}_\mu^{-1}(\mathbf{c}) = \{\tilde{H}_\mu|_{X \times \mathbb{C}^{\mathcal{H} - \mathcal{H}'}}\}^{-1}(\mathbf{c}) \\ &\subset X \times \mathbb{C}^{\mathcal{H} - \mathcal{H}'} = (X \times \mathbb{C}^{\mathcal{H}})^{\tilde{\psi}'|_{\mathbb{T}_{\mathcal{H}'}}}. \end{aligned} \quad (5.14)$$

Since the moment map $\mu_{\mathcal{H}} : X_{\mathcal{H}} \rightarrow T_{\mathbb{T}}^* \mathbb{T}$ is induced by (5.9),

$$Y_{\mathcal{H}'} \equiv \mu_{\mathcal{H}}^{-1}(\langle \mathcal{H}' \rangle^{\partial}) = p(\tilde{Y}_{\mathcal{H}'}) = p_{\mathcal{H}}(\mu^{-1}(\langle \mathcal{H}' \rangle^{\partial})) \subset X_{\mathcal{H}}^{\mathbb{T}^{\mathcal{H}'}} \subset X_{\mathcal{H}};$$

the first inclusion above follows from the last equality in (5.14). The $\tilde{\omega}$ -symplectic submanifold $X \times \mathbb{C}^{\mathcal{H}-\mathcal{H}'} \subset X \times \mathbb{C}^{\mathcal{H}}$ is preserved by the $\mathbb{T}^{\mathcal{H}}$ -action $\tilde{\psi}$, the \mathbb{T} -action ψ , and the $\mathbb{T}_{\mathcal{H}}$ -action $\tilde{\psi}'$. By Exercise 5.1, $Y_{\mathcal{H}'} \subset X_{\mathcal{H}}$ is thus an $\omega_{\mathcal{H}}$ -symplectic submanifold preserved by the \mathbb{T} -action $\psi_{\mathcal{H}}$. By (5.13), the natural splitting

$$\mathcal{N}_{X \times \mathbb{C}^{\mathcal{H}}}(X \times \mathbb{C}^{\mathcal{H}-\mathcal{H}'}) = \bigoplus_{v \in \mathcal{H}'} (X \times \mathbb{C}^{\{v\}})$$

is \mathbb{T} -, $\mathbb{T}^{\mathcal{H}}$ -, and $\mathbb{T}_{\mathcal{H}}$ -equivariant with respect to the actions $d\psi$, $d\tilde{\psi}$, and $d\tilde{\psi}'$ on the left-hand side and the actions

$$\begin{aligned} u \cdot (x, z_v) &= (\psi_u(x), z_v), \quad [(r_{v'})_{v' \in \mathcal{H}}] \cdot (x, z_v) = (\psi_{\Phi_{\mathcal{H}}((r_{v'})_{v' \in \mathcal{H}})}(x), e^{-2\pi i r_v} z_v), \\ \text{and} \quad \Phi_{\mathcal{H}}([(r_{v'})_{v' \in \mathcal{H}}]) \cdot (x, z_v) &= (x, e^{2\pi i r_v} z_v) \end{aligned}$$

on the summand $X \times \mathbb{C}^{\{v\}}$ on the right-hand side. By Exercises 5.1 and 5.2, $TX_{\mathcal{H}}|_{Y_{\mathcal{H}'}}$ thus splits \mathbb{T} -equivariantly as in (5.3) and (5.4). It follows that $Y_{\mathcal{H}'} \subset X_{\mathcal{H}}$ is a union of topological components of the fixed locus $X_{\mathcal{H}}^{\mathbb{T}^{\mathcal{H}'}}$ of the restriction of the \mathbb{T} -action $\psi_{\mathcal{H}}$ to $\mathbb{T}^{\mathcal{H}'} \subset \mathbb{T}_{\mathcal{H}}$. Thus, (5.5) is an \mathcal{H} -cut Hamiltonian \mathbb{T} -manifold.

The restriction of the map $\tilde{p}_{\mathcal{H}}$ in (5.12) to $\mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial})$ is a smooth embedding; its image is $\tilde{Y}_{\mathcal{H}'} \cap (X \times (\mathbb{R}^+)^{\mathcal{H}-\mathcal{H}'})$. Thus,

$$\{\tilde{p}_{\mathcal{H}}|_{\mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial})}\}^* (\tilde{\omega}|_{T\tilde{Y}_{\mathcal{H}'}}) = \omega|_{T(\mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial}))}. \quad (5.15)$$

Since $p^*(\omega_{\mathcal{H}}|_{TY_{\mathcal{H}'}}) = \tilde{\omega}|_{T\tilde{Y}_{\mathcal{H}'}}$ by Theorem 4(SQ2) and Exercise 5.1, (5.6) follows from (5.15). The map

$$\begin{aligned} \tilde{P}_{\mathcal{H}; \mathcal{H}'} : \mathbb{T}^{\mathcal{H}-\mathcal{H}'} \times \mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial}) &\longrightarrow \tilde{Y}_{\mathcal{H}'} \cap (X \times (\mathbb{C}^*)^{\mathcal{H}-\mathcal{H}'}), \\ \tilde{P}_{\mathcal{H}; \mathcal{H}'}(u, x) &= \tilde{\psi}_u(\tilde{p}_{\mathcal{H}}(x)), \end{aligned}$$

is a diffeomorphism. Since the map $p : \tilde{Y}_{\mathcal{H}'} \rightarrow Y_{\mathcal{H}'}$ is a principal $\mathbb{T}^{\mathcal{H}}$ -bundle (and thus a submersion) by Theorem 4(HC₂),

$$\mu_{\mathcal{H}}^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial}) = \tilde{Y}_{\mathcal{H}'} \cap (X \times (\mathbb{C}^*)^{\mathcal{H}-\mathcal{H}'}),$$

and dp vanishes on $d\tilde{P}_{\mathcal{H}; \mathcal{H}'}(T\mathbb{T}^{\mathcal{H}-\mathcal{H}'})$, the composition

$$p_{\mathcal{H}} = p \circ \tilde{p}_{\mathcal{H}} : \mu^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial}) \longrightarrow \mu_{\mathcal{H}}^{-1}(\langle \mathcal{H}' \rangle_{\mathcal{H}}^{\partial}) \subset \mu_{\mathcal{H}}^{-1}(\langle \mathcal{H}' \rangle^{\partial})$$

is a submersion. This confirms (HC₃). The conditions (HC₁)-(HC₃) ensure the uniqueness of \mathcal{H} -cut Hamiltonian \mathbb{T} -manifold satisfying these properties.

Suppose $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$. By Exercise 5.13, the Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) is \mathcal{H}_1 -cuttable. Let

$$(X, \omega, \psi, \mu)_{\mathcal{H}_1} \equiv (X_{\mathcal{H}_1}, \omega_{\mathcal{H}_1}, \psi_{\mathcal{H}_1}, \mu_{\mathcal{H}_1}) \quad (5.16)$$

be the corresponding \mathcal{H}_1 -cut Hamiltonian \mathbb{T} -manifold as in (5.5). If $\mathcal{H}'_2 \subset \mathcal{H}_2$,

$$\mu_{\mathcal{H}_1}^{-1}(\langle \mathcal{H}'_2 \rangle^\partial) = \bigcup_{\mathcal{H}'_1 \subset \mathcal{H}_1} \bigcup_{\mathcal{H}'_2 \subset \mathcal{H}_2} \mu_{\mathcal{H}_1}^{-1}(\langle \mathcal{H}'_1 \sqcup \mathcal{H}'_2 \rangle_{\mathcal{H}}^\partial) = \bigcup_{\mathcal{H}'_1 \subset \mathcal{H}_1} \bigcup_{\mathcal{H}'_2 \subset \mathcal{H}_2} \mu^{-1}(\langle \mathcal{H}'_1 \sqcup \mathcal{H}'_2 \rangle_{\mathcal{H}}^\partial) / \mathbb{T}_{\mathcal{H}'_1};$$

the last equality holds by (HC₁) with \mathcal{H} replaced by \mathcal{H}_1 . If $\mu_{\mathcal{H}_1}^{-1}(\langle \mathcal{H}'_2 \rangle^\partial) \neq \emptyset$, Exercise 5.13 thus implies that the Lie group homomorphism $\Phi_{\mathcal{H}'_2}$ as in (5.2) is injective and $\mathbb{T}_{\mathcal{H}'_2}$ acts freely on $\mu_{\mathcal{H}_1}^{-1}(\langle \mathcal{H}'_2 \rangle^\partial)$. We conclude that the Hamiltonian \mathbb{T} -manifold (5.16) is \mathcal{H}_2 -cuttable.

By (HC₁) and (HC₂) with \mathcal{H} replaced by \mathcal{H}_1 and \mathcal{H}_2 ,

$$(X_{\mathcal{H}_1})_{\mathcal{H}_2} = \mu_{\mathcal{H}_1}^{-1}(\langle \mathcal{H}_2 \rangle) / \sim_{\mathcal{H}_2} = p_{\mathcal{H}_1}(\mu^{-1}(\langle \mathcal{H}_1 \rangle) \cap \mu^{-1}(\langle \mathcal{H}_2 \rangle)) / \sim_{\mathcal{H}_2} = (\mu^{-1}(\langle \mathcal{H} \rangle) / \sim_{\mathcal{H}_1}) / \sim_{\mathcal{H}_2},$$

with $x, x' \in \mu^{-1}(\langle \mathcal{H} \rangle)$ being equivalent in the double quotient if there exist $\mathcal{H}'_1 \subset \mathcal{H}_1$, $\mathcal{H}'_2 \subset \mathcal{H}_2$, and u in the subgroup generated by $\mathbb{T}_{\mathcal{H}'_1}, \mathbb{T}_{\mathcal{H}'_2} \subset \mathbb{T}$ such that

$$\mu(x) \in \langle \mathcal{H}'_1 \rangle^\partial \cap \langle \mathcal{H}'_2 \rangle^\partial = \langle \mathcal{H}'_1 \sqcup \mathcal{H}'_2 \rangle^\partial \quad \text{and} \quad x' = \psi_u(x).$$

By definition, the subgroup generated by $\mathbb{T}_{\mathcal{H}'_1}, \mathbb{T}_{\mathcal{H}'_2} \subset \mathbb{T}$ is $\mathbb{T}_{\mathcal{H}'_1 \sqcup \mathcal{H}'_2}$. Thus,

$$p_{\mathcal{H}_2} \circ p_{\mathcal{H}_1} = p_{\mathcal{H}} : X \longrightarrow (X_{\mathcal{H}_1})_{\mathcal{H}_2} = X_{\mathcal{H}}, \quad (\mu_{\mathcal{H}_1})_{\mathcal{H}_2} = \mu_{\mathcal{H}} : X_{\mathcal{H}} \longrightarrow T_1^* \mathbb{T},$$

and the \mathbb{T} -actions $(\psi_{\mathcal{H}_1})_{\mathcal{H}_2}$ and $\psi_{\mathcal{H}}$ on $X_{\mathcal{H}}$ are the same (as they are induced by the same \mathbb{T} -action ψ on X). Since $\langle \emptyset \rangle_{\mathcal{H}}^\partial = \langle \emptyset \rangle_{\mathcal{H}_1}^\partial \cap \langle \emptyset \rangle_{\mathcal{H}_2}^\partial$ is an open subset of $T_1^* \mathbb{T}$, (5.6) gives

$$\begin{aligned} \{p_{\mathcal{H}}|_{\mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}}^\partial)}\}^*(\omega_{\mathcal{H}_1})_{\mathcal{H}_2} &= \{p_{\mathcal{H}_1}|_{\mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}}^\partial)}\}^* \{p_{\mathcal{H}_2}|_{\mu_{\mathcal{H}_1}^{-1}(\langle \emptyset \rangle_{\mathcal{H}_2}^\partial)}\}^*(\omega_{\mathcal{H}_1})_{\mathcal{H}_2} \\ &= \{p_{\mathcal{H}_1}|_{\mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}}^\partial)}\}^* \omega_{\mathcal{H}_1} = \omega|_{\mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}}^\partial)} = \{p_{\mathcal{H}}|_{\mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}}^\partial)}\}^* \omega_{\mathcal{H}}. \end{aligned}$$

Since $p_{\mathcal{H}}$ is a submersion on $\mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}}^\partial)$, it follows that $(\omega_{\mathcal{H}_1})_{\mathcal{H}_2} = \omega$ on the dense open subset $\mu_{\mathcal{H}}^{-1}(\langle \emptyset \rangle_{\mathcal{H}}^\partial) \subset X_{\mathcal{H}}$ and thus everywhere on $X_{\mathcal{H}}$. This establishes (5.7). \square

Exercise 5.14. Suppose \mathbb{T} is a torus, $\mathcal{H} \subset (T_1 \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection, and (X, ω, ψ, μ) is an \mathcal{H} -cuttable Hamiltonian \mathbb{T} -manifold.

(a) Suppose $\mu(X) \subset \langle \mathcal{H} \rangle$. Show that the projection

$$p_{\mathcal{H}} : \mu^{-1}(\langle \mathcal{H} \rangle) = X \longrightarrow X_{\mathcal{H}}$$

of Theorem 5 identifies (X, ω, ψ, μ) with $(X, \omega, \psi, \mu)_{\mathcal{H}}$.

(b) Conclude that $(X, \omega, \psi, \mu)_{\mathcal{H}}$ does not depend on the choice of finite subcollection $\mathcal{H} \subset (T_1 \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ with $\langle \mathcal{H} \rangle$ fixed so that (X, ω, ψ, μ) is \mathcal{H} -cuttable.

Exercise 5.15. Suppose \mathbb{T} is a torus, $\mathcal{H} \subset (T_1 \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection, and $\lambda \in \mathbb{R}^+$.

(a) Show that the finite subcollection

$$\lambda \mathcal{H} \equiv \{(v, \lambda c) : (v, c) \in \mathcal{H}\} \subset (T_1 \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$$

is Delzant (resp. minimal, regular) if and only if \mathcal{H} is. Show also that

$$\langle \lambda \mathcal{H} \rangle = \lambda \langle \mathcal{H} \rangle \equiv \{\lambda \alpha : \alpha \in \langle \mathcal{H} \rangle\} \subset T_1^* \mathbb{T}.$$

- (b) Suppose (X, ω, ψ, μ) is an \mathcal{H} -cuttable Hamiltonian \mathbb{T} -manifold. Show that $(X, \lambda\omega, \psi, \lambda\mu)$ is a $\lambda\mathcal{H}$ -cuttable Hamiltonian \mathbb{T} -manifold and that the map

$$\tilde{H}_\mu^{-1}(\mathbf{c}) \longrightarrow \tilde{H}_{\lambda\mu}^{-1}(\lambda\mathbf{c}) \subset X \times \mathbb{C}^{\lambda\mathcal{H}}, \quad (x, (z_{v,c})_{(v,c) \in \mathcal{H}}) \longrightarrow (x, (\sqrt{\lambda}z_{v,c})_{(v,\lambda c) \in \lambda\mathcal{H}}),$$

descends to an identification of $(X_{\mathcal{H}}, \lambda\omega_{\mathcal{H}}, \psi_{\mathcal{H}}, \lambda\mu_{\mathcal{H}})$ with $(X, \lambda\omega, \psi, \lambda\mu)_{\lambda\mathcal{H}}$.

Exercise 5.16. Suppose \mathbb{T} is a torus, $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection, and $\alpha_0 \in T_1^*\mathbb{T}$.

- (a) Show that the finite subcollection

$$\mathcal{H} + \alpha_0 \equiv \{(v, c + \alpha_0(v)) : (v, c) \in \mathcal{H}\} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$$

is Delzant (resp. minimal, regular) if and only if \mathcal{H} is. Show also that

$$\langle \mathcal{H} + \alpha_0 \rangle = \langle \mathcal{H} \rangle + \alpha_0 \equiv \{\alpha + \alpha_0 : \alpha \in \langle \mathcal{H} \rangle\} \subset T_1^*\mathbb{T}.$$

- (b) Suppose (X, ω, ψ, μ) is an \mathcal{H} -cuttable Hamiltonian \mathbb{T} -manifold. Let

$$\mathbf{c} + \alpha_0 = (c_v + \alpha_0(v))_{v \in \mathcal{H}} \in \mathbb{R}^{\mathcal{H}}.$$

Show that $(X, \omega, \psi, \mu + \alpha_0)$ is an $(\mathcal{H} + \alpha_0)$ -cuttable Hamiltonian \mathbb{T} -manifold and that the map

$$\tilde{H}_\mu^{-1}(\mathbf{c}) \longrightarrow \tilde{H}_{\mu + \alpha_0}^{-1}(\mathbf{c} + \alpha_0) \subset X \times \mathbb{C}^{\mathcal{H} + \alpha_0}, \quad (x, (z_{v,c})_{(v,c) \in \mathcal{H}}) \longrightarrow (x, (z_{v,c})_{(v,c + \alpha_0(v)) \in \mathcal{H} + \alpha_0}),$$

descends to an identification of $(X_{\mathcal{H}}, \omega_{\mathcal{H}}, \psi_{\mathcal{H}}, \mu_{\mathcal{H}} + \alpha_0)$ with $(X, \omega, \psi, \mu + \alpha_0)_{\mathcal{H} + \alpha_0}$.

Exercise 5.17. Suppose \mathbb{T} is a torus, $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection, and $\Theta \in \mathrm{GL}_{\mathbb{Z}}((T_1\mathbb{T})_{\mathbb{Z}})$. Let $\Theta^* \in \mathrm{GL}_{\mathbb{Z}}((T_1^*\mathbb{T})_{\mathbb{Z}})$ be the dual of Θ and ψ_{Θ} be the \mathbb{T} -action on X obtained by composing ψ with the automorphism of \mathbb{T} induced by Θ .

- (a) Show that the finite subcollection

$$\Theta^{-1}\mathcal{H} \equiv \{(\Theta^{-1}(v), c) : (v, c) \in \mathcal{H}\} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$$

is Delzant (resp. minimal, regular) if and only if \mathcal{H} is. Show also that

$$\langle \Theta^{-1}\mathcal{H} \rangle = \Theta^*(\langle \mathcal{H} \rangle) \equiv \{\Theta^*(\alpha) : \alpha \in \langle \mathcal{H} \rangle\} \subset T_1^*\mathbb{T}.$$

- (b) Suppose (X, ω, ψ, μ) is an \mathcal{H} -cuttable Hamiltonian \mathbb{T} -manifold. Show that $(X, \omega, \psi_{\Theta}, \Theta^* \circ \mu)$ is a $\Theta^{-1}\mathcal{H}$ -cuttable Hamiltonian \mathbb{T} -manifold and that the map

$$\tilde{H}_\mu^{-1}(\mathbf{c}) \longrightarrow \tilde{H}_{\Theta^* \circ \mu}^{-1}(\mathbf{c}) \subset X \times \mathbb{C}^{\Theta^{-1}\mathcal{H}}, \quad (x, (z_{v,c})_{(v,c) \in \mathcal{H}}) \longrightarrow (x, (z_{v,c})_{(\Theta^{-1}(v), c) \in \Theta^{-1}\mathcal{H}}),$$

descends to an identification of $(X_{\mathcal{H}}, \omega_{\mathcal{H}}, (\psi_{\mathcal{H}})_{\Theta}, \Theta^* \circ \mu_{\mathcal{H}})$ with $(X, \omega, \psi_{\Theta}, \Theta^* \circ \mu)_{\Theta^{-1}\mathcal{H}}$.

We will call two finite subcollections of $(T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ **equivalent** if they differ by a composition of rescalings, translations, and automorphisms of $(T_1\mathbb{T})_{\mathbb{Z}}$ as in Exercises 5.15(a), 5.16(a), and 5.17(a), respectively. A regular finite subcollection of $(T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ has at least $\dim \mathbb{T}$ elements. The next exercise describes the equivalence classes of regular minimal Delzant subsets \mathcal{H} of $(T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ of small cardinality relative to the dimension. The subsets $\langle \mathcal{H} \rangle \subset T_1^*\mathbb{T}$ determined by representatives of these classes when \mathbb{T} is two-dimensional are shown in Figures 2 and 3.

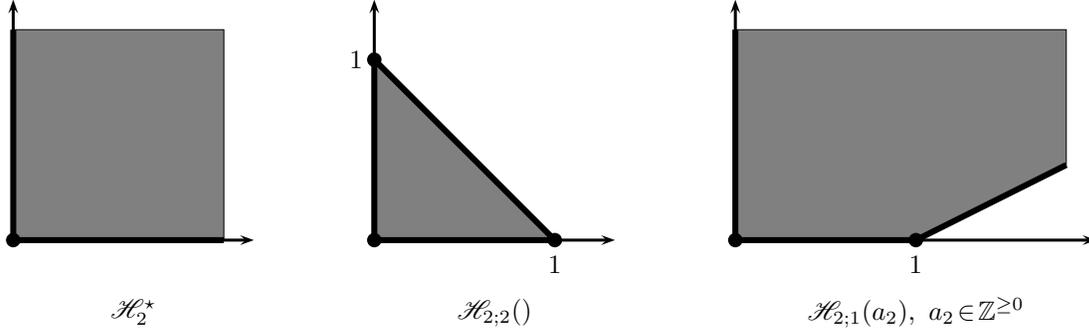


Figure 2: Representatives for the equivalence classes of regular minimal Delzant subsets \mathcal{H} of $(T_{\mathbb{1}}\mathbb{T}^2)_{\mathbb{Z}} \times \mathbb{R}$ with $|\mathcal{H}|=2, 3$ as in Exercise 5.18.

Exercise 5.18. Suppose $n \in \mathbb{Z}^+$ and $\mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}} \times \mathbb{R}$ is a regular minimal Delzant subset. Let e_1, \dots, e_n be the standard basis for $T_{\mathbb{1}}\mathbb{T}^n = \mathbb{R}^n$. Show that

- (a) if $|\mathcal{H}|=n$, then \mathcal{H} is equivalent to $\mathcal{H}_n^* \equiv \{(e_i, 0) : i \in [n]\}$;
- (b) if $|\mathcal{H}|=n+1$, then \mathcal{H} is equivalent to

$$\mathcal{H}_{n;k}(a_{k+1}, \dots, a_n) \equiv \mathcal{H}_n^* \sqcup \{(-e_1 - \dots - e_k + a_{k+1}e_{k+1} + \dots + a_n e_n, -1)\}$$

for some $k \in [n]$ and $a_{k+1}, \dots, a_n \in \mathbb{Z}^{\geq 0}$;

- (c) if $n=2$ and $|\mathcal{H}|=4$, then \mathcal{H} is equivalent to

$$\mathcal{H}_{2;1}(a; b, c) \equiv \mathcal{H}_{2;1}(a) \sqcup \{(-be_1 + (ab-1)e_2, -b-c)\}$$

for some $a, b \in \mathbb{Z}^{\geq 0}$ and $c \in \mathbb{R}^+$;

- (d) the equivalence class of $\mathcal{H}_{n;k}(a_{k+1}, \dots, a_n)$ in (b) is invariant under the permutations of a_{k+1}, \dots, a_n , the equivalence classes of $\mathcal{H}_{2;1}(a; b, c)$ and $\mathcal{H}_{2;1}(b; a, 1/c)$ in (c) are the same, and all other pairs of the subsets above represent distinct equivalence classes.

5.3 Hamiltonian symplectic uncut

In this section, we show that the Hamiltonian symplectic cut construction is reversible and complete the proof of Theorem 3. This formalizes the argument sketched in the proof of [24, Theorem 7.5.10].

Exercise 5.19. Suppose \mathbb{T} is a torus, $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ is a partition of a finite subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$, (X, ω, ψ, μ) is an \mathcal{H}_2 -cuttable Hamiltonian \mathbb{T} -manifold so that $(X, \omega, \psi, \mu)_{\mathcal{H}_2}$ is an \mathcal{H} -cut Hamiltonian \mathbb{T} -manifold, and $Y \subset X_{\mathcal{H}_2}$ is a topological component of $\mu_{\mathcal{H}_2}^{-1}(\langle \mathcal{H}_1 \rangle^{\partial})$. Show that

- (a) there exists a (unique) topological component $\tilde{Y} \subset X$ of $\mu^{-1}(\langle \mathcal{H}_1 \rangle^{\partial})$ which contains $p_{\mathcal{H}_2}^{-1}(Y)$;
- (b) \tilde{Y} is a topological component of the fixed locus $X^{\mathbb{T}_{\mathcal{H}_1}} \subset X$ of $\psi|_{\mathbb{T}_{\mathcal{H}_1}}$ with its normal bundle admitting a splitting as in (5.3) and (5.4) with $(Y_{\mathcal{H}_1}, \mathcal{H}_1)$ replaced by $(\tilde{Y}, \mathcal{H}_1)$.

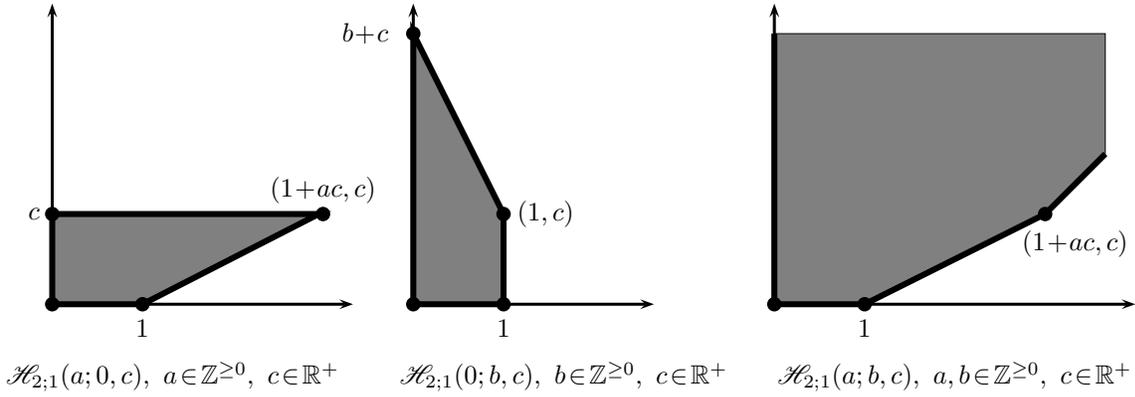


Figure 3: Representatives for the equivalence classes of regular minimal Delzant subsets \mathcal{H} of $(T_1\mathbb{T}^2)_{\mathbb{Z}} \times \mathbb{R}$ with $|\mathcal{H}| = 4$ as in Exercise 5.18; the subsets $\mathcal{H}_{2;1}(a; b, c)$ and $\mathcal{H}_{2;1}(b; a, 1/c)$ are equivalent.

Corollary 5.20. *Suppose \mathbb{T} is a torus, $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ is a partition of a finite subcollection of $(T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$, and (X, ω, ψ, μ) is an \mathcal{H}_2 -cuttable Hamiltonian \mathbb{T} -manifold. If $(X, \omega, \psi, \mu)_{\mathcal{H}_2}$ is an \mathcal{H} -cut Hamiltonian \mathbb{T} -manifold, then there exists an open \mathbb{T} -invariant subset $X' \subset X$ so that*

$$(X', \omega|_{X'}, \psi|_{X'}, \mu|_{X'})_{\mathcal{H}_2} = (X, \omega, \psi, \mu)_{\mathcal{H}_2} \quad (5.17)$$

and the Hamiltonian \mathbb{T} -manifold $(X', \omega|_{X'}, \psi|_{X'}, \mu|_{X'})$ is \mathcal{H}_1 -cut. If $X_{\mathcal{H}_2}$ is connected and/or the restriction

$$\mu: \mu^{-1}(\langle \mathcal{H}_1' \rangle_{\mathcal{H}_1}^{\partial}) \longrightarrow \langle \mathcal{H}_1' \rangle_{\mathcal{H}_1}^{\partial} \cap \mu(X) \quad (5.18)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}_1'}$ -bundle for every $\mathcal{H}_1' \subset \mathcal{H}_1$, then $(X', \omega', \psi', \mu')$ can be chosen so that X' is also connected and/or the restriction

$$\mu: \mu^{-1}(\langle \mathcal{H}_1' \rangle_{\mathcal{H}_1}^{\partial}) \cap X' \longrightarrow \langle \mathcal{H}_1' \rangle_{\mathcal{H}_1}^{\partial} \cap \mu(X') \quad (5.19)$$

is also a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}_1'}$ -bundle for every $\mathcal{H}_1' \subset \mathcal{H}_1$, respectively.

Proof. Suppose $\mathcal{H}_1' \subset \mathcal{H}_1$ and $Y \subset X_{\mathcal{H}_2}$ is a topological component of $\mu^{-1}(\langle \mathcal{H}_1' \rangle_{\mathcal{H}_1}^{\partial})$. It is thus contained in a topological component $Y_{\mathcal{H}_1''} \subset X_{\mathcal{H}_2}$ of $\mu^{-1}(\langle \mathcal{H}_1'' \rangle_{\mathcal{H}_1}^{\partial})$ for every $\mathcal{H}_1'' \subset \mathcal{H}_1'$. By Exercise 5.19(a), there exists a (unique) topological component $\tilde{Y}_{\mathcal{H}_1''} \subset X$ of $\mu^{-1}(\langle \mathcal{H}_1'' \rangle_{\mathcal{H}_1}^{\partial})$ which contains $p_{\mathcal{H}_2}^{-1}(Y_{\mathcal{H}_1''})$. In particular, $\tilde{Y}_{\mathcal{H}_1''}$ is disjoint from the closed subsets $\mu^{-1}(\langle \mathcal{H}_1'' \rangle_{\mathcal{H}_1}^{\partial}) - \tilde{Y}_{\mathcal{H}_1''}$ of X with $\mathcal{H}_1'' \subset \mathcal{H}_1'$.

By Exercise 5.19(b) and the first part of Proposition 3.38 with \mathbb{T} replaced by $\mathbb{T}_{\mathcal{H}_1'}$, there thus exists a \mathbb{T} -invariant neighborhood \tilde{U}_Y of $\tilde{Y}_{\mathcal{H}_1'}$ in X so that

$$\mu(\tilde{U}_Y) \subset \langle \mathcal{H}_1' \rangle \quad \text{and} \quad \tilde{U}_Y \cap \mu^{-1}(\langle \mathcal{H}_1'' \rangle_{\mathcal{H}_1}^{\partial}) \subset \tilde{Y}_{\mathcal{H}_1''} \quad \forall \mathcal{H}_1'' \subset \mathcal{H}_1'. \quad (5.20)$$

The \mathbb{T} -invariant neighborhood

$$\tilde{U}_Y' \equiv \tilde{U}_Y \cap \mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}_1 - \mathcal{H}_1'}^{\partial}) \subset \mu^{-1}(\langle \mathcal{H}_1' \rangle) \cap \mu^{-1}(\langle \mathcal{H}_1 - \mathcal{H}_1' \rangle) = \mu^{-1}(\langle \mathcal{H}_1 \rangle)$$

of $\tilde{Y}_{\mathcal{H}'_1} \cap \mu^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial)$ in X is then disjoint from $\mu^{-1}(\langle \mathcal{H}''_1 \rangle^\partial)$ for every subset $\mathcal{H}''_1 \subset \mathcal{H}'_1$ not contained in \mathcal{H}'_1 . Thus,

$$X' \equiv \bigcup_{\mathcal{H}'_1 \subset \mathcal{H}_1} \bigcup_{Y \in \pi_0(\mu_{\mathcal{H}'_2}^{-1}(\langle \mathcal{H}'_1 \rangle^\partial))} \tilde{U}'_Y \subset \mu^{-1}(\langle \mathcal{H}_1 \rangle) \quad (5.21)$$

is a \mathbb{T} -invariant neighborhood of

$$\bigcup_{\mathcal{H}'_1 \subset \mathcal{H}_1} \bigcup_{Y \in \pi_0(\mu_{\mathcal{H}'_2}^{-1}(\langle \mathcal{H}'_1 \rangle^\partial))} (\tilde{Y}_{\mathcal{H}'_1} \cap \mu^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial)) \supset \bigcup_{\mathcal{H}'_1 \subset \mathcal{H}_1} \bigcup_{Y \in \pi_0(\mu_{\mathcal{H}'_2}^{-1}(\langle \mathcal{H}'_1 \rangle^\partial))} p_{\mathcal{H}'_2}^{-1}(Y) = p_{\mathcal{H}'_2}^{-1}(X_{\mathcal{H}'_2}) = \mu^{-1}(\langle \mathcal{H}'_2 \rangle)$$

in X . By Theorem 5(HC₁) with \mathcal{H} replaced by \mathcal{H}'_2 , the above inclusion implies (5.17).

By (5.21), $\mu(X') \subset \langle \mathcal{H}_1 \rangle$. Let $\mathcal{H}''_1 \subset \mathcal{H}'_1$ be such that $\mu^{-1}(\langle \mathcal{H}''_1 \rangle^\partial) \neq \emptyset$. Since $\tilde{U}'_Y \subset \tilde{U}_Y$ with $Y \in \pi_0(\mu_{\mathcal{H}'_2}^{-1}(\langle \mathcal{H}'_1 \rangle^\partial))$ is disjoint from $\mu^{-1}(\langle \mathcal{H}''_1 \rangle^\partial)$ whenever $\mathcal{H}''_1 \not\subset \mathcal{H}'_1$, the second statement in (5.20) implies that

$$X' \cap \mu^{-1}(\langle \mathcal{H}''_1 \rangle^\partial) \subset \bigcup_{\mathcal{H}''_1 \subset \mathcal{H}'_1 \subset \mathcal{H}_1} \bigcup_{Y \in \pi_0(\mu_{\mathcal{H}'_2}^{-1}(\langle \mathcal{H}'_1 \rangle^\partial))} \tilde{Y}_{\mathcal{H}''_1} = \bigcup_{Y \in \pi_0(\mu_{\mathcal{H}'_2}^{-1}(\langle \mathcal{H}'_1 \rangle^\partial))} \tilde{Y}_{\mathcal{H}'_1}. \quad (5.22)$$

Since $(X, \omega, \psi, \mu)_{\mathcal{H}'_2}$ is \mathcal{H} -cut, it follows that the Lie group homomorphism $\Phi_{\mathcal{H}'_1}$ as in (5.2) is injective. By (5.22), $X' \cap \mu^{-1}(\langle \mathcal{H}''_1 \rangle^\partial) \subset X$ is the disjoint union of the open subspaces $X' \cap \tilde{Y}_{\mathcal{H}''_1}$ of $\tilde{Y}_{\mathcal{H}'_1}$ with $Y \in \pi_0(\mu_{\mathcal{H}'_2}^{-1}(\langle \mathcal{H}'_1 \rangle^\partial))$. By Exercise 5.19(b), $X' \cap \mu^{-1}(\langle \mathcal{H}''_1 \rangle^\partial)$ is thus a union of topological components of the fixed locus $X'^{\mathbb{T}_{\mathcal{H}'_1}}$ of the restriction of the \mathbb{T} -action ψ to $\mathbb{T}_{\mathcal{H}'_1} \subset \mathbb{T}$ and $X' \subset X$ with its normal bundles admitting a splitting as in (5.3) and (5.4) with $(Y_{\mathcal{H}'_1}, \mathcal{H}'_1)$ replaced by $(X' \cap \mu^{-1}(\langle \mathcal{H}''_1 \rangle^\partial), \mathcal{H}''_1)$. Thus, $(X', \omega|_{X'}, \psi|_{X'}, \mu|_{X'})$ is \mathcal{H}'_1 -cut.

Since $\mu_{\mathcal{H}'_2}(X_{\mathcal{H}'_2}) \subset \langle \mathcal{H}'_2 \rangle$, the subspaces $\tilde{Y}_{\mathcal{H}'_1} \cap \mu^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial) \subset X$ above intersect $\mu^{-1}(\langle \mathcal{H}'_2 \rangle)$ and thus so do their neighborhoods $\tilde{U}'_Y \subset X' \subset X$. If $X_{\mathcal{H}'_2}$ is connected, then so is $\mu^{-1}(\langle \mathcal{H}'_2 \rangle) \subset X$ by Theorem 5(HC₁) with \mathcal{H} replaced by \mathcal{H}'_2 . It follows that $X' \subset X$ is then connected. If the restriction (5.18) is a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle for some $\mathcal{H}'_1 \subset \mathcal{H}_1$, then \mathbb{T} acts transitively on the fibers of μ over $\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \mu(X)$. Since $X' \subset X$ is a \mathbb{T} -invariant subset, (5.19) is the restriction of principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle (5.18) to $\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \mu(X') \subset \langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \mu(X)$ and thus is still a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle. \square

Theorem 6. *Suppose \mathbb{T} is a torus, $\mathcal{H} = \mathcal{H}'_1 \sqcup \mathcal{H}'_2$ is a partition of a finite subcollection of $(T_1 \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$, and (X, ω, ψ, μ) is an \mathcal{H}'_2 -cut Hamiltonian \mathbb{T} -manifold. Then,*

$$(X, \omega, \psi, \mu) = (X', \omega', \psi', \mu')_{\mathcal{H}'_2} \quad (5.23)$$

for some \mathcal{H}'_2 -cuttable Hamiltonian \mathbb{T} -manifold $(X', \omega', \psi', \mu')$. If

- (a) X is connected and/or
- (b) (X, ω, ψ, μ) is \mathcal{H} -cut and the restriction

$$\mu: \mu^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial) \longrightarrow \langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \mu(X) \quad (5.24)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle for every $\mathcal{H}'_1 \subset \mathcal{H}$,

then $(X', \omega', \psi', \mu')$ can be chosen so that

(a') X' is connected and/or

(b') $(X', \omega', \psi', \mu')$ is \mathcal{H}_1 -cut and the restriction

$$\mu' : \mu'^{-1}(\langle \mathcal{H}_1' \rangle_{\mathcal{H}_1}^\partial) \longrightarrow \langle \mathcal{H}_1' \rangle_{\mathcal{H}_1}^\partial \cap \mu'(X') \quad (5.25)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}_1'}$ -bundle for every $\mathcal{H}_1' \subset \mathcal{H}_1$,

respectively.

Proof. By (5.7) and Corollary 5.20, it is sufficient to establish this proposition with \mathcal{H}_2 consisting of a single element $v \equiv (v, c)$ of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ with $v \neq 0$. We assume that the closed codimension 2 symplectic submanifold

$$Y \equiv \mu^{-1}(\langle \mathcal{H}_2 \rangle^\partial) \equiv \{x \in X : \mu_v(x) = c\}$$

of (X, ω) is nonempty; otherwise, we can take $(X', \omega', \psi', \mu') = (X, \omega, \psi, \mu)$. Since (X, ω, ψ, μ) is \mathcal{H}_2 -cut, the Lie group homomorphism $\Phi_{\mathcal{H}_2}$ as in (5.2) is then injective. We establish the proposition by removing Y and continuing the radial directions in the normal bundle of Y in X into the negative values without them coming together at 0. Since (X, ω, ψ, μ) is \mathcal{H}_2 -cut,

$$X - Y = \{x \in X : \mu_v(x) > c\} = \mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}_2}^\partial). \quad (5.26)$$

Let J be a \mathbb{T}^n -invariant almost complex structure on X compatible with ω and $g(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$ be the associated \mathbb{T}^n -invariant metric compatible with J .

Since (X, ω, ψ, μ) is \mathcal{H}_2 -cut, Y is a union of topological components of the fixed locus $X^{\mathbb{T}^v}$ of the restriction of the \mathbb{T} -action ψ to the circle $\mathbb{T}_v \subset \mathbb{T}$ generated by $v \in (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and

$$\pi : TY^\omega \equiv \{w \in TX|_Y : \omega(w, w') = 0 \ \forall w' \in TY\} \longrightarrow Y$$

is a complex line bundle complementary to TY . It is preserved by the \mathbb{T}^n -action $d\psi$ and

$$d\psi_{e^{tv}}(w) = e^{2\pi it} w \quad \forall t \in \mathbb{R}, w \in TY^\omega \quad (5.27)$$

by (5.4). Let

$$\zeta_v \in \Gamma(TY^\omega; T(TY^\omega)), \quad \zeta_v(w) = \frac{d}{dt} d\psi_{e^{tv}}(w) \Big|_{t=0} = 2\pi i w,$$

be the (vertical) vector field on TY^ω generating the S^1 -action (5.27). Since this action preserves the unit circle bundle of TY^ω ,

$$\pi : S(TY^\omega) \equiv \{w \in TY^\omega : g(w, w) = 1\} \longrightarrow Y, \quad (5.28)$$

the vector field $\zeta_v|_{S(TY^\omega)}$ is tangent to $S(TY^\omega)$. The maps

$$\begin{aligned} \tilde{\iota} : S(TY^\omega) \times \mathbb{R}^+ &\longrightarrow S(TY^\omega) \times \mathbb{C}, & \tilde{\iota}(w, t) &= (w, \sqrt{2t}), \\ p : S(TY^\omega) \times \mathbb{C} &\longrightarrow TY^\omega, & p(w, z) &= zw, \end{aligned}$$

are smooth. The map p descends to a \mathbb{T} -equivariant diffeomorphism from the quotient $S(TY^\omega) \times_{S^1} \mathbb{C}$ of $S(TY^\omega) \times \mathbb{C}$ by the S^1 -action

$$S^1 \times (S(TY^\omega) \times \mathbb{C}) \longrightarrow S(TY^\omega) \times \mathbb{C}, \quad u \cdot (w, z) = (uw, u^{-1}z), \quad (5.29)$$

to TY^ω . The generating vector field for this action is $\tilde{\zeta}_v \equiv (\zeta_v, -2\pi\partial_\theta)$. The map

$$\iota \equiv p \circ \tilde{\iota}: S(TY^\omega) \times \mathbb{R}^+ \longrightarrow TY^\omega$$

is a \mathbb{T} -equivariant diffeomorphism onto $TY^\omega - Y$.

Let λ be a \mathbb{T} -invariant connection 1-form on the principal S^1 -bundle $S(TY^\omega) \longrightarrow Y$, i.e. λ is a 1-form on $S(TY^\omega)$ so that

$$\lambda(\zeta_v) = 2\pi \quad \text{and} \quad \iota_{\zeta_v}(\mathrm{d}\lambda) = 0. \quad (5.30)$$

In light of the first condition above, the second condition is equivalent to λ being S^1 -invariant. Let $\tilde{\omega}$ and $\tilde{\omega}'$ be the \mathbb{T} -invariant closed 2-forms on $S(TY^\omega) \times \mathbb{C}$ and $S(TY^\omega) \times \mathbb{R}$, respectively, given by

$$\tilde{\omega} = \pi^*\omega + \omega_{\mathbb{C}} + \frac{1}{2}\mathrm{d}(|z|^2\lambda) \quad \text{and} \quad \tilde{\omega}' = \pi^*\omega + \mathrm{d}(t\lambda),$$

where $\omega_{\mathbb{C}}$ is the standard symplectic form on \mathbb{C} as in Example 2.6, z is the standard coordinate on \mathbb{C} , and t is the standard coordinate on \mathbb{R} . Since

$$\tilde{\omega}'_{(w,0)} = \omega_{\pi(w)} + \mathrm{d}_0 t \wedge \lambda_w \quad \forall w \in S(TY^\omega), \quad (5.31)$$

the 2-form $\tilde{\omega}'$ is nondegenerate (and thus symplectic) on some neighborhood $\mathcal{U}' \subset S(TY^\omega) \times \mathbb{R}$ of $S(TY^\omega) \times \{0\}$.

By (5.30) and the last coordinate of the map $\tilde{\iota}$ taking only real values,

$$\iota_{\zeta_v} \tilde{\omega}' = -2\pi \mathrm{d}t \quad \text{and} \quad \tilde{\omega}'|_{S(TY^\omega) \times \mathbb{R}^+} = \tilde{\iota}^* \tilde{\omega}, \quad (5.32)$$

respectively. With (r, θ) denoting the standard radius-angle coordinates on \mathbb{C} so that $\omega_{\mathbb{C}} = r \mathrm{d}r \wedge \mathrm{d}\theta$,

$$(\iota_{\tilde{\zeta}_v} \tilde{\omega})_{(w, r e^{i\theta})} = 0 - 2\pi \iota_{\partial_\theta} \omega_{\mathbb{C}} + \iota_{\zeta_v} (r \mathrm{d}r \wedge \lambda + \frac{1}{2} r^2 \mathrm{d}\lambda) = 2\pi r \mathrm{d}r - 2\pi r \mathrm{d}r + 0 = 0; \quad (5.33)$$

the middle equality above follows again from (5.30). Since the S^1 -action (5.29) preserves the 2-form $\tilde{\omega}$, (5.33) implies that there is a unique 2-form ω_{TY^ω} on (the total space of) TY^ω so that $p^* \omega_{TY^\omega} = \tilde{\omega}$. This form is \mathbb{T} -equivariant and closed and satisfies

$$\tilde{\omega}'|_{S(TY^\omega) \times \mathbb{R}^+} = \iota^* \omega_{TY^\omega}; \quad (5.34)$$

see the second equation in (5.32).

Let λ_S be the 1-form on $L_S = TY^\omega$ determined by λ as in Exercise B.8(a). Thus,

$$\omega_{TY^\omega} = \pi^* \omega + \frac{1}{2} \mathrm{d}\lambda_S.$$

Along with Exercise B.8(b), this implies that the closed 2-form ω_{TY^ω} on TY^ω satisfies (3.18) with $TY^c = TY^\omega$. By Proposition 3.16(2), there thus exists a \mathbb{T} -equivariant tubular neighborhood identification $\Phi: \mathcal{U} \rightarrow U$ for Y in X such that $\mathcal{U} \subset TY^\omega$ and $\Phi^*\omega = \omega_{TY^\omega}|_{\mathcal{U}}$. Along with (5.34), the last identity gives

$$\tilde{\omega}'|_{\iota^{-1}(\mathcal{U})} = \iota^*\Phi^*\omega. \quad (5.35)$$

Let $\mathcal{U}'' \subset \mathcal{U}' \subset S(TY^\omega) \times \mathbb{R}$ be a \mathbb{T} -invariant tubular neighborhood of $S(TY^\omega) \times \{0\}$ so that

$$\mathcal{U}^+ \equiv \mathcal{U}'' \cap (S(TY^\omega) \times \mathbb{R}^+) \subset \iota^{-1}(\mathcal{U}).$$

Since the diffeomorphism $\Phi \circ \iota: \mathcal{U}^+ \rightarrow \Phi(\iota(\mathcal{U}^+))$ is \mathbb{T} -equivariant and satisfies (5.35) with $\iota^{-1}(\mathcal{U})$ replaced by \mathcal{U}^+ ,

$$\mu \circ \Phi \circ \iota: \mathcal{U}^+ \rightarrow T_{\mathbb{1}}^*\mathbb{T} \quad (5.36)$$

is a moment map for the \mathbb{T} -action on $(\mathcal{U}^+, \tilde{\omega}'|_{\mathcal{U}^+})$. By Exercise 3.37, it extends to a moment map $\tilde{\mu}': \mathcal{U}'' \rightarrow T_{\mathbb{1}}^*\mathbb{T}$ for the \mathbb{T} -action on $(\mathcal{U}'', \tilde{\omega}'|_{\mathcal{U}''})$. Since

$$\lim_{t \rightarrow 0^+} \iota(w, t) = \pi(w) \in Y \subset TY^\omega, \quad \lim_{t \rightarrow 0^+} \mu(\Phi(\iota(w, t))) = \mu(\pi(w)) \quad \forall w \in S(TY^\omega),$$

the first equation in (5.32) implies that

$$\tilde{\mu}'_v(w, t) = 2\pi t + c, \quad \tilde{\mu}'^{-1}(\langle \mathcal{H} \rangle^\partial) = S(TY^\omega) \times \{0\}, \quad (5.37)$$

and $\tilde{\mu}' = \mu \circ \pi$ on $S(TY^\omega) \times \{0\} = S(TY^\omega)$.

We define

$$\begin{aligned} X' &= ((X - Y) \sqcup \mathcal{U}'') / \sim, \quad \mathcal{U}^+ \ni (w, t) \sim \Phi(\iota(w, t)) \in X - Y, \\ \omega'_{[x]} &= \begin{cases} \omega_x, & \text{if } x \in X - Y; \\ \tilde{\omega}'_x, & \text{if } x \in \mathcal{U}''; \end{cases} \quad \mu'([x]) = \begin{cases} \mu(x), & \text{if } x \in X - Y; \\ \tilde{\mu}'(x), & \text{if } x \in \mathcal{U}'' \end{cases} \end{aligned} \quad (5.38)$$

Suppose $x \in X - Y - \Phi(\iota(\mathcal{U}^+))$ and $x' \in \mathcal{U}'' - \mathcal{U}^+$. If x' does not lie in the closure $\text{Cl}_{\mathcal{U}''}\mathcal{U}^+$ of \mathcal{U}^+ in \mathcal{U}'' , then the images of $X - Y$ and $\mathcal{U}'' - \text{Cl}_{\mathcal{U}''}\mathcal{U}^+$ in X' under the quotient projection

$$q: (X - Y) \sqcup \mathcal{U}'' \rightarrow X'$$

are disjoint open subsets around $[x]$ and $[x']$, respectively. If

$$x' \in \text{Cl}_{\mathcal{U}''}\mathcal{U}^+ - \mathcal{U}^+ = Y \times \{0\}$$

and $U, U' \subset X$ are disjoint open neighborhoods of x and Y , respectively, then

$$q(U), q((\mathcal{U}'' \cap (S(TY^\omega) \times \mathbb{R}^{\leq 0})) \cup \iota^{-1}(\Phi^{-1}(U')))) \subset X'$$

are disjoint open subsets around $[x]$ and $[x']$, respectively. Since the restrictions of q to the Hausdorff spaces $X - Y$ and \mathcal{U}'' are homeomorphisms onto open subsets of X' , it follows that X' is a Hausdorff space and a smooth manifold. By (5.35) and the assumption on \mathcal{U}' below (5.31), ω' is a well-defined symplectic form on X' . Since the smooth map $\tilde{\mu}'$ is an extension of the map (5.36),

$\mu': X' \longrightarrow T_{\mathbb{1}}^*\mathbb{T}$ is a well-defined smooth map.

Since the identification of the spaces $X-Y$ and \mathcal{U}'' over $\mathcal{U}^+ \subset \mathcal{U}''$ in (5.38) is \mathbb{T} -equivariant, the \mathbb{T} -action ψ on $X-Y \subset X$ and the \mathbb{T} -action $d\psi$ on $\mathcal{U}'' \subset S(TY^\omega) \times \mathbb{R}$ induce a smooth \mathbb{T} -action ψ' on X' which preserves ω' . Since $\mu|_{X-Y}$ and $\tilde{\mu}'$ are moment maps for the \mathbb{T} -actions on $(X-Y, \omega|_{X-Y})$ and $(\mathcal{U}'', \tilde{\omega}'|_{\mathcal{U}''})$, μ' is a moment map for the \mathbb{T} -action ψ' on (X', ω') . By (5.26) and (5.37),

$$\mu'^{-1}(\langle \mathcal{H}_2 \rangle^\partial) = q(S(TY^\omega) \times \{0\}) \subset X'.$$

Since the restriction of q to \mathcal{U}'' is a diffeomorphism onto $q(\mathcal{U}'')$ and \mathbb{T}_v acts freely on $S(TY^\omega) \times \{0\}$, \mathbb{T}_v acts freely on $\mu'^{-1}(\langle \mathcal{H}_2 \rangle^\partial)$ as well. Thus, $(X', \omega', \psi', \mu')$ is an \mathcal{H}_2 -cuttable Hamiltonian \mathbb{T} -manifold with

$$\begin{aligned} \mu'^{-1}(\langle \emptyset \rangle_{\mathcal{H}_2}^\partial) &= q(X-Y) \approx Y, & \mu'^{-1}(\langle \mathcal{H}_2 \rangle^\partial) &= q(S(TY^\omega) \times \{0\}) \subset X', \\ X &= \mu'^{-1}(\langle \mathcal{H}_2 \rangle^\partial) / \sim, & x &\sim \psi'_u(x) \quad \forall x \in \mu'^{-1}(\langle \mathcal{H}_2 \rangle^\partial), u \in \mathbb{T}_v, & Y &= \mu'^{-1}(\langle \mathcal{H}_2 \rangle^\partial) / \mathbb{T}_v, \\ \{q|_{X-Y}\}^* p_{\mathcal{H}_2}^* \omega &= \omega|_{X-Y}, & \{q|_{S(TY^\omega) \times \{0\}}\}^* \{p_{\mathcal{H}_2}|_{\mu'^{-1}(\langle \mathcal{H}_2 \rangle^\partial)}\}^* (\omega|_{TY}) &= \tilde{\omega}'|_{S(TY^\omega) \times \{0\}}, \end{aligned}$$

where $p_{\mathcal{H}_2}: \mu'^{-1}(\langle \mathcal{H}_2 \rangle^\partial) \longrightarrow X$ is the quotient projection; the two identities on the first line above follow from (5.26) and the second statement in (5.37). Furthermore, the map $p_{\mathcal{H}_2}$ is \mathbb{T} -equivariant and the compositions

$$p_{\mathcal{H}_2} \circ q: X-Y \longrightarrow \mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}_2}^\partial) \quad \text{and} \quad p_{\mathcal{H}_2} \circ q: S(TY^\omega) \times \{0\} \longrightarrow \mu^{-1}(\langle \mathcal{H}_2 \rangle^\partial)$$

are submersions. By the uniqueness statement of Theorem 5, (5.23) thus holds.

Every topological component of the tubular neighborhood $q(\mathcal{U}'') \subset X'$ of $q(Y \times \{0\})$ intersects $q(X-Y)$. If X is connected, then so is $X-Y$ (because $Y \subset X$ is a submanifold of codimension 2). It then follows that X' is also connected. If (X, ω, ψ, μ) is \mathcal{H} -cut, then $(X', \omega', \psi', \mu')$ is \mathcal{H}_1 -cut by Corollary 5.20 if \mathcal{U}'' is sufficiently small.

Suppose both conditions in (b) hold and $\mathcal{H}'_1 \subset \mathcal{H}_1 = \mathcal{H} - \{v\}$. Let $\mathcal{H}'_1 v = \mathcal{H}'_1 \sqcup \{v\}$. By the above, $\mu'^{-1}(\langle \mathcal{H}'_1 \rangle^\partial) \subset X'$ is an ω' -symplectic submanifold consisting of components of the fixed locus $X'^{\mathbb{T}_{\mathcal{H}'_1}}$ of the restriction of the action ψ' to the subtorus $\mathbb{T}_{\mathcal{H}'_1} \subset \mathbb{T}$. Since the restriction of the quotient projection q above to \mathcal{U}'' is a \mathbb{T} -equivariant diffeomorphism onto the open subset $q(\mathcal{U}'') \subset X'$ and $\tilde{\mu}' = \mu' \circ q$ on \mathcal{U}'' , it follows that $\tilde{\mu}'^{-1}(\langle \mathcal{H}'_1 \rangle^\partial) \subset \mathcal{U}''$ is an $\tilde{\omega}'$ -symplectic submanifold consisting of components of $\mathcal{U}''^{\mathbb{T}_{\mathcal{H}'_1}}$. By (5.24) with $\mathcal{H}' = \mathcal{H}'_1 v$, the restriction

$$\mu: \mu^{-1}(\langle \mathcal{H}'_1 v \rangle_{\mathcal{H}}^\partial) = \mu^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}_1}^\partial) \cap Y \longrightarrow \langle \mathcal{H}'_1 v \rangle_{\mathcal{H}}^\partial \cap \mu(X) = \langle \mathcal{H}'_1 \rangle_{\mathcal{H}_1}^\partial \cap \mu(Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1 v}$ -bundle. Since \mathbb{T}_v acts freely on the fibers of the circle bundle (5.28), it follows that the restriction

$$\tilde{\mu}' = \mu \circ \pi: S(TY^\omega)|_{\mu^{-1}(\langle \mathcal{H}'_1 v \rangle_{\mathcal{H}}^\partial)} \times \{0\} = \tilde{\mu}'^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}_1}^\partial) \cap (S(TY^\omega) \times \{0\}) \longrightarrow \langle \mathcal{H}'_1 \rangle_{\mathcal{H}_1}^\partial \cap \mu(Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle. By Exercise 5.10(d) with \mathcal{H} replaced by \mathcal{H}'_1 and the submanifold $\langle \mathcal{H}'_1 v \rangle^\partial \subset \langle \mathcal{H}'_1 \rangle^\partial$ being closed, the moment map

$$\tilde{\mu}': \tilde{\mu}'^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}_1}^\partial) \longrightarrow \langle \mathcal{H}'_1 \rangle^\partial$$

is then a submersion and thus also a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle over its image if \mathcal{U}'' is sufficiently small. Since $q|_{\mathcal{U}''}$ is a \mathbb{T} -equivariant diffeomorphism onto $q(\mathcal{U}'') \subset X'$ and $\tilde{\mu}' = \mu' \circ q$ on \mathcal{U}'' , it follows that

$$\mu' : \mu'^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial) \cap q(\mathcal{U}'') \longrightarrow \langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \tilde{\mu}'(\mathcal{U}'') = \langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \mu'(q(\mathcal{U}'')).$$

By (5.24) with $\mathcal{H}' = \mathcal{H}'_1$ and (5.26), the restriction

$$\mu : \mu^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial) \cap (X - Y) = \mu^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial) \cap (X - Y) \longrightarrow \langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \mu(X - Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle. Since the restriction of q to $X - Y$ is a \mathbb{T} -equivariant diffeomorphism onto the open subset $q(X - Y) \subset X'$ and $\mu = \mu' \circ q$ on $X - Y$, it follows that the restriction

$$\mu' : \mu'^{-1}(\langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial) \cap q(X - Y) \longrightarrow \langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \mu(X - Y) = \langle \mathcal{H}'_1 \rangle_{\mathcal{H}'_1}^\partial \cap \mu'(q(X - Y))$$

is also a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle. Since $X' = q(\mathcal{U}'') \cup q(X - Y)$, we conclude that the restriction (5.25) is a principal $\mathbb{T}/\mathbb{T}_{\mathcal{H}'_1}$ -bundle. \square

Proof of Theorem 3. Since (0) in the statement of this theorem follows from (0⁺) on page 8, which was established in Section 4.2, it remains to establish (1) and (2). Suppose $P \subset T_{\mathbb{1}}^*\mathbb{T}$ is a Delzant polytope and $\mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant subcollection so that $\mu(X) = \langle \mathcal{H} \rangle$, and $\mathcal{H}' \subset \mathcal{H}$. Let $(T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})$ be the Hamiltonian \mathbb{T} -manifold of Exercise 2.14, with $k = n$. By Theorem 5, the Hamiltonian \mathbb{T} -manifold

$$(X, \omega, \psi, \mu) \equiv (T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})_{\mathcal{H}}$$

as in (5.5) is then a closed connected Hamiltonian \mathbb{T} -manifold so that (1.7) holds, the \mathbb{T} -action ψ is effective (it is free on $\mu^{-1}(\langle \emptyset \rangle_{\mathcal{H}}^\partial)$, and $\mu(X) = P$. This gives (1).

Suppose (X, ω, ψ, μ) is a (necessarily compact) symplectic toric \mathbb{T} -manifold with $\mu(X) = P$. In particular, X is connected. By Exercise 5.12 and (0⁺c) on page 8, (X, ω, ψ, μ) also satisfies (b) in the statement of Theorem 6. By Theorem 6 with $\mathcal{H}_1 = \emptyset$ and $\mathcal{H}_2 = \mathcal{H}$,

$$(X, \omega, \psi, \mu) = (X', \omega', \psi', \mu')_{\mathcal{H}} \tag{5.39}$$

for some \mathcal{H} -cuttable Hamiltonian \mathbb{T} -manifold $(X', \omega', \psi', \mu')$ so that X' is connected and

$$\mu' : X' = \mu'^{-1}(\langle \emptyset \rangle_{\emptyset}^\partial) \longrightarrow \langle \emptyset \rangle_{\emptyset}^\partial \cap \mu'(X') = \mu'(X')$$

is a principal \mathbb{T} -bundle. By Exercise 3.32(b) and (5.39), $\mu'(X') \subset T_{\mathbb{1}}^*\mathbb{T}$ is an open neighborhood of the polytope P . By replacing X' with the preimage of a contractible neighborhood of P in $\mu'(X')$, we can assume $\mu'(X')$ is contractible. By Proposition 3.42, $(X', \omega', \psi', \mu')$ is then isomorphic to $(U, \omega_{\mathbb{T}}|_U, \psi_{\mathbb{T}}|_U, \mu_{\mathbb{T}}|_U)$ for an open neighborhood $U \subset T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}$ of $\mu_{\mathbb{T}}^{-1}(P)$. Along with (5.39) and Theorem 5, this implies that

$$(X, \omega, \psi, \mu) \approx (U, \omega_{\mathbb{T}}|_U, \psi_{\mathbb{T}}|_U, \mu_{\mathbb{T}}|_U)_{\mathcal{H}} = (T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}}).$$

This gives (2). \square

6 Symplectic Toric Manifolds

In this chapter, we describe a construction of symplectic toric manifolds along the lines of [13, Section 3.2] and use it to obtain key properties of these manifolds. Example 2.9 is a special case of this construction. The structure of this chapter is motivated by [31, Chapter 2], which efficiently summarizes these properties from a more concrete perspective. We fix a torus \mathbb{T} of dimension n and continue with the notation and terminology introduced at the beginning of Section 5.2.

6.1 Symplectic quotient construction

In this section, we use the symplectic reduction of Theorem 4 to construct a symplectic toric \mathbb{T} -manifold (X, ω, ψ, μ) with $\mu(X) = \langle \mathcal{H} \rangle$ for any regular Delzant subcollection $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$. By Exercise 5.9, every Delzant polytope P equals $\langle \mathcal{H} \rangle$ for some regular Delzant subcollection $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$. Thus, the construction of this section provides another proof of Theorem 3(1).

Suppose $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection. Let

$$\mathcal{K}_{\mathcal{H}} \equiv \ker \Phi_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}} \equiv \mathbb{R}^{\mathcal{H}} / \mathbb{Z}^{\mathcal{H}}$$

be the kernel of the homomorphism $\Phi_{\mathcal{H}}$ in (5.2) and

$$\iota_{\mathcal{H}}^* : \mathbb{R}^{\mathcal{H}} = T_1^*\mathbb{T}^{\mathcal{H}} \longrightarrow T_1^*\mathcal{K}_{\mathcal{H}}$$

be the composition of the standard identification of $\mathbb{R}^{\mathcal{H}}$ with $T_1^*\mathbb{T}^{\mathcal{H}}$ and of the homomorphism induced by the inclusion $\mathcal{K}_{\mathcal{H}} \longrightarrow \mathbb{T}^{\mathcal{H}}$. Thus, the sequence

$$0 \longrightarrow T_1^*\mathbb{T} \xrightarrow{L_{\mathcal{H}}^*} T_1^*\mathbb{T}^{\mathcal{H}} = \mathbb{R}^{\mathcal{H}} \xrightarrow{\iota_{\mathcal{H}}^*} T_1^*\mathcal{K}_{\mathcal{H}} \longrightarrow 0 \quad (6.1)$$

of vector spaces is exact. Since $L_{\mathcal{H}}^*(\alpha) = (\alpha(v))_{(v,c) \in \mathcal{H}}$ for any $\alpha \in T_1^*\mathbb{T}$,

$$\begin{aligned} L_{\mathcal{H}}^*(\langle \mathcal{H}^1 \rangle) &= \{(s_v)_{v \in \mathcal{H}} \in \ker \iota_{\mathcal{H}}^* : s_v \geq c_v \ \forall v \in \mathcal{H}^1\} \quad \text{and} \\ L_{\mathcal{H}}^*(\langle \mathcal{H}^1 \rangle^\partial) &= \{(s_v)_{v \in \mathcal{H}} \in \ker \iota_{\mathcal{H}}^* : s_v = c_v \ \forall v \in \mathcal{H}^1\} \end{aligned} \quad (6.2)$$

for any $\mathcal{H}^1 \subset \mathcal{H}$. If $\mathcal{H}^1 \subset \mathcal{H}$ and $\iota_{\mathcal{H}; \mathcal{H}^1} : \mathbb{T}^{\mathcal{H}^1} \longrightarrow \mathbb{T}^{\mathcal{H}}$ is the induced inclusion homomorphism, then

$$\iota_{\mathcal{H}; \mathcal{H}^1}^* : \mathbb{R}^{\mathcal{H}} = \mathbb{R}^{\mathcal{H}^1} \times \mathbb{R}^{\mathcal{H} - \mathcal{H}^1} \longrightarrow \mathbb{R}^{\mathcal{H}^1}$$

is the projection to the $\mathbb{R}^{\mathcal{H}^1}$ -factor and

$$L_{\mathcal{H}^1}^* = \iota_{\mathcal{H}; \mathcal{H}^1}^* \circ L_{\mathcal{H}}^* : T_1^*\mathbb{T} \longrightarrow T_1^*\mathbb{T}^{\mathcal{H}^1} = \mathbb{R}^{\mathcal{H}^1}. \quad (6.3)$$

Denote by $\omega_{\mathcal{H}}$ the standard symplectic form on $\mathbb{C}^{\mathcal{H}}$ as in Example 2.6. By this example, the map

$$H_{\mathcal{H}} : \mathbb{C}^{\mathcal{H}} \longrightarrow \mathbb{R}^{\mathcal{H}}, \quad H_{\mathcal{H}}((z_v)_{v \in \mathcal{H}}) = (\pi|z_v|^2 + c_v)_{v \in \mathcal{H}}, \quad (6.4)$$

is a Hamiltonian with respect to $\omega_{\mathcal{H}}$ for the standard action $\psi_{\mathcal{H}}$ of $\mathbb{T}^{\mathcal{H}}$ on $\mathbb{C}^{\mathcal{H}}$,

$$\psi_{\mathcal{H}; [(r_v)_{v \in \mathcal{H}}]}((z_v)_{v \in \mathcal{H}}) = (e^{2\pi i r_v} z_v)_{v \in \mathcal{H}}. \quad (6.5)$$

Thus,

$$\mu_{\mathcal{H}} \equiv \iota_{\mathcal{H}}^* \circ H_{\mathcal{H}} : \mathbb{C}^{\mathcal{H}} \longrightarrow T_{\mathbb{1}}^* \mathcal{K}_{\mathcal{H}}$$

is a moment map with respect to $\omega_{\mathcal{H}}$ for the restriction of the action $\psi_{\mathcal{H}}$ to $\mathcal{K}_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}}$. By the exactness of (6.1) and by (6.2),

$$\mu_{\mathcal{H}}^{-1}(0) \cap \mathbb{C}^{\mathcal{H} - \mathcal{H}'} = H_{\mathcal{H}}^{-1}(L_{\mathcal{H}}^*(\langle \mathcal{H} \rangle \cap \langle \mathcal{H}' \rangle^{\partial})) = H_{\mathcal{H}}^{-1}(L_{\mathcal{H}}^*(\langle \mathcal{H}' \rangle^{\partial})) \quad \forall \mathcal{H}' \subset \mathcal{H}. \quad (6.6)$$

Exercise 6.1. Let $\mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a finite subcollection.

- (a) Show that the restriction $H_{\mathcal{H}} : \mu_{\mathcal{H}}^{-1}(0) \longrightarrow \ker \iota_{\mathcal{H}}^*$ is a proper map.
- (b) Let $K \subset T_{\mathbb{1}}^* \mathcal{K}_{\mathcal{H}}$ be a bounded subset (with respect to some norm on $\mathcal{K}_{\mathcal{H}}$) such that

$$\iota_{\mathcal{H}}^{*-1}(K) \cap (\mathbb{R}^{\geq 0})^{\mathcal{H}} \neq \emptyset \subset \mathbb{R}^{\mathcal{H}}.$$

Show that $\iota_{\mathcal{H}}^{*-1}(K) \cap (\mathbb{R}^{\geq 0})^{\mathcal{H}}$ is bounded if and only if $(\ker \iota_{\mathcal{H}}^*) \cap (\mathbb{R}^{\geq 0})^{\mathcal{H}} = \{0\}$.

Exercise 6.2. Let $\mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a finite subcollection and

$$\alpha_{\mathcal{H}} = -\iota_{\mathcal{H}}^*((c_v)_{v \in \mathcal{H}}) \in T_{\mathbb{1}}^* \mathcal{K}_{\mathcal{H}}. \quad (6.7)$$

Show that the following conditions on \mathcal{H} are equivalent:

- (a) $0 \in \mathbb{C}^{\mathcal{H}}$ is a regular value of the map $H_{\mathcal{H}}$ in (6.4),
- (b) $0 \in \mathbb{C}^{\mathcal{H}}$ is a regular value of $H_{\mathcal{H}}|_{\mathbb{C}^{\mathcal{H} - \mathcal{H}'}}$ for every $\mathcal{H}' \subset \mathcal{H}$,
- (c) $|\mathcal{H}'| \leq \dim L_{\mathcal{H}}(\mathbb{R}^{\mathcal{H}'})$ for every $\mathcal{H}' \subset \mathcal{H}$ such that $\alpha_{\mathcal{H}} \in \iota_{\mathcal{H}}^*((\mathbb{R}^{\geq 0})^{\mathcal{H} - \mathcal{H}'})$.

Exercise 6.3. Let $\mathcal{H}' \subset \mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be finite subcollections.

- (a) Show that the subgroup $\mathcal{K}_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}}$ acts freely on $\mu_{\mathcal{H}}^{-1}(0) \cap \mathbb{C}^{\mathcal{H} - \mathcal{H}'}$ if and only if the homomorphism $\Phi_{\mathcal{H}''}$ as in (5.2) is injective whenever $\mathcal{H}' \subset \mathcal{H}'' \subset \mathcal{H}$ and $\langle \mathcal{H}'' \rangle^{\partial} \cap \langle \mathcal{H}' \rangle \neq \emptyset$. *Hint:* proceed similarly to the paragraph containing (5.11).
- (b) Suppose the homomorphism $\Phi_{\mathcal{H}''}$ is injective whenever $\mathcal{H}' \subset \mathcal{H}'' \subset \mathcal{H}$ and $\langle \mathcal{H}'' \rangle^{\partial} \cap \langle \mathcal{H}' \rangle \neq \emptyset$. Show that $\mu_{\mathcal{H}}^{-1}(0) \cap \mathbb{C}^{\mathcal{H} - \mathcal{H}'}$ is a smooth submanifold of $\mathbb{C}^{\mathcal{H} - \mathcal{H}'}$ and $(\mu_{\mathcal{H}}^{-1}(0) \cap \mathbb{C}^{\mathcal{H} - \mathcal{H}'}) / \mathcal{K}_{\mathcal{H}}$ is a smooth manifold.

Exercise 6.4. Suppose $\mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection. Let $[\![\mathcal{H}]\!] \approx \mathbb{R}^{\mathcal{H}}$ be the collection of all subcollections $\mathcal{H}^{\dagger} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ that differ from \mathcal{H} only in the second input of each element of \mathcal{H} (i.e. there is a bijection $\Psi_{\mathcal{H}^{\dagger}, \mathcal{H}}$ between the elements of \mathcal{H} and of each $\mathcal{H}^{\dagger} \in [\![\mathcal{H}]\!]$ so that $\Psi_{\mathcal{H}^{\dagger}, \mathcal{H}}(v, c) = (v, c^{\dagger})$ for some $c^{\dagger} \in \mathbb{R}$). Show that

- (a) the subset $[\![\mathcal{H}]\!]^* \subset [\![\mathcal{H}]\!]$ of the subcollections \mathcal{H}^{\dagger} such that $0 \in \mathbb{C}^{\mathcal{H}}$ is a regular value of the map $H_{\mathcal{H}^{\dagger}}$ in (6.4) is open;
- (b) every topological component of $[\![\mathcal{H}]\!]^*$ is preserved under the multiplication by \mathbb{R}^+ (i.e. simultaneous multiplication of the second input of each element of $\mathcal{H}^{\dagger} \in [\![\mathcal{H}]\!]^*$);
- (c) the collections $\{\Psi_{\mathcal{H}^{\dagger}, \mathcal{H}}^{-1}(\mathcal{H}') : \mathcal{H}' \subset \mathcal{H}^{\dagger}, \langle \mathcal{H}' \rangle^{\partial} \cap \langle \mathcal{H}^{\dagger} \rangle \neq \emptyset\}$ with $\mathcal{H}^{\dagger} \in [\![\mathcal{H}]\!]^*$ are constant functions on the topological components of $[\![\mathcal{H}]\!]^*$;

(d) if the subcollection \mathcal{H} is Delzant, then $\mathcal{H} \in [\mathcal{H}]^*$ and every element \mathcal{H}^\dagger of the topological component $[\mathcal{H}]^* \subset [\mathcal{H}]^*$ containing \mathcal{H} is also a Delzant subcollection.

Lemma 6.5. *The subspace $\mu_{\mathcal{H}}^{-1}(0) \subset \mathbb{C}^{\mathcal{H}}$ is preserved by the $\mathbb{T}^{\mathcal{H}}$ -action (6.5) and is path-connected. If $\langle \emptyset \rangle_{\mathcal{H}}^\partial \neq \emptyset$, i.e. $\langle \mathcal{H} \rangle$ is full-dimensional, then $\text{Stab}_z(\psi_{\mathcal{H}}) = \{\mathbb{1}\}$ for some $z \in \mu_{\mathcal{H}}^{-1}(0)$.*

Proof. Since the Hamiltonian $H_{\mathcal{H}}$ is $\mathbb{T}^{\mathcal{H}}$ -invariant, the $\mathbb{T}^{\mathcal{H}}$ -action (6.5) preserves $\mu_{\mathcal{H}}^{-1}(0)$. By (6.2),

$$(\text{Im } H_{\mathcal{H}}) \cap (\ker \iota_{\mathcal{H}}^*) = L_{\mathcal{H}}^*(\langle \mathcal{H} \rangle) \subset \mathbb{R}^{\mathcal{H}}. \quad (6.8)$$

Since the subspace $\langle \mathcal{H} \rangle \subset T_{\mathbb{1}}^* \mathbb{T}$ is path-connected and the fibers of $H_{\mathcal{H}}$ are path-connected, so is the subspace

$$\mu_{\mathcal{H}}^{-1}(0) = H_{\mathcal{H}}^{-1}(\iota_{\mathcal{H}}^{*-1}(0)) \subset \mathbb{C}^{\mathcal{H}}.$$

If $\langle \emptyset \rangle_{\mathcal{H}}^\partial \neq \emptyset$, (6.6) implies that there exists $z \equiv (z_v)_{v \in \mathcal{H}} \in \mu_{\mathcal{H}}^{-1}(0)$ such that $H_{\mathcal{H}}(z) \in L_{\mathcal{H}}^*(\langle \emptyset \rangle_{\mathcal{H}}^\partial)$ and thus $z_v \neq 0$ for any $v \in \mathcal{H}$. It follows that $\text{Stab}_z(\psi_{\mathcal{H}}) = \{\mathbb{1}\}$. \square

From now on, we assume that $\mathcal{H} \subset (T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a regular Delzant subcollection such that $\langle \mathcal{H} \rangle \neq \emptyset$. By Exercise 5.6(b), $\mathcal{K}_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}}$ is then a codimension n subtorus. By Exercise 6.3(a), $(\mathbb{C}^{\mathcal{H}}, \omega_{\mathcal{H}}, \psi_{\mathcal{H}}, \mu_{\mathcal{H}})$ is a Hamiltonian $\mathcal{K}_{\mathcal{H}}$ -manifold such that $\mathcal{K}_{\mathcal{H}}$ acts freely on $\mu_{\mathcal{H}}^{-1}(0)$. Let $(X, \omega) \equiv (X_0, \omega_0)$ be the quotient symplectic manifold provided by the first part of Theorem 4. By (SQ0) and (SQ1) in this theorem,

$$\dim X = \dim_{\mathbb{R}} \mathbb{C}^{\mathcal{H}} - 2 \dim \mathcal{K}_{\mathcal{H}} = 2|\mathcal{H}| - 2(|\mathcal{H}| - n) = 2n.$$

By Lemma 6.5, $X \equiv \mu_{\mathcal{H}}^{-1}(0)/\mathcal{K}_{\mathcal{H}}$ is connected.

The torus actions $\psi_{\mathcal{H}}|_{\mathcal{K}_{\mathcal{H}}}$ and $\psi_{\mathcal{H}}$ commute, the Hamiltonian $H_{\mathcal{H}}$ for $\psi_{\mathcal{H}}$ is $\psi_{\mathcal{H}}|_{\mathcal{K}_{\mathcal{H}}}$ -invariant and the moment map $\mu_{\mathcal{H}}$ for $\psi_{\mathcal{H}}|_{\mathcal{K}_{\mathcal{H}}}$ is $\psi_{\mathcal{H}}$ -invariant. Let

$$\psi'_0: \mathbb{T}^{\mathcal{H}} \times X \longrightarrow X \quad \text{and} \quad \mu'_0: X \longrightarrow T_{\mathbb{1}}^* \mathbb{T}^{\mathcal{H}}$$

be the $\mathbb{T}^{\mathcal{H}}$ -action on X induced by $\psi_{\mathcal{H}}$ and its moment map with respect to ω induced by $H_{\mathcal{H}}$, respectively, as provided by the last part of Theorem 4.

By Exercise 5.6(a), there exists a subset $\mathcal{H}_\bullet \subset \mathcal{H}$ so that $\langle \mathcal{H}_\bullet \rangle^\partial$ is a vertex of $\langle \mathcal{H} \rangle$, i.e. $\langle \mathcal{H}_\bullet \rangle^\partial \subset \langle \mathcal{H} \rangle$ consists of a single point. Since \mathcal{H} is Delzant, the Lie homomorphism group $\Phi_{\mathcal{H}_\bullet}$ as in (5.2) is an isomorphism and thus so is the Lie group homomorphism

$$\mathcal{K}_{\mathcal{H}} \times \mathbb{T}^{\mathcal{H}_\bullet} \longrightarrow \mathbb{T}^{\mathcal{H}}, \quad (u, u_\bullet) \longrightarrow uu_\bullet.$$

By the last statement of Lemma 6.5 above and Exercise 6.6 below, the composition ψ of the $\mathbb{T}^{\mathcal{H}}$ -action ψ'_0 on X with the homomorphism

$$\mathbb{T} \xrightarrow{\Phi_{\mathcal{H}_\bullet}^{-1}} \mathbb{T}^{\mathcal{H}_\bullet} \xrightarrow{\iota_{\mathcal{H}_\bullet; \mathcal{H}_\bullet}} \mathbb{T}^{\mathcal{H}}$$

is therefore an effective \mathbb{T} -action on X . The composition

$$\mu: X \xrightarrow{\mu'_0} T_{\mathbb{1}}^* \mathbb{T}^{\mathcal{H}} \xrightarrow{\iota_{\mathcal{H}; \mathcal{H}_\bullet}^*} T_{\mathbb{1}}^* \mathbb{T}^{\mathcal{H}_\bullet} \xrightarrow{L_{\mathcal{H}_\bullet}^{*-1}} T_{\mathbb{1}}^* \mathbb{T}$$

is its moment map with respect to ω . Thus, (X, ω, ψ, μ) is a symplectic toric manifold with moment polytope

$$\begin{aligned}\mu(X) &= L_{\mathcal{H}_\bullet}^{*-1}(\iota_{\mathcal{H}; \mathcal{H}_\bullet}^*(\mu'_0(X))) = L_{\mathcal{H}_\bullet}^{*-1}(\iota_{\mathcal{H}; \mathcal{H}_\bullet}^*(H_{\mathcal{H}}(\mu_{\mathcal{H}}^{-1}(0)))) \\ &= L_{\mathcal{H}_\bullet}^{*-1}(\iota_{\mathcal{H}; \mathcal{H}_\bullet}^*(L_{\mathcal{H}}^*(\langle \mathcal{H} \rangle))) = L_{\mathcal{H}_\bullet}^{*-1}(L_{\mathcal{H}_\bullet}^*(\langle \mathcal{H} \rangle)) = \langle \mathcal{H} \rangle;\end{aligned}$$

the second equality above holds by (6.6) with $\mathcal{H}' = \emptyset$ and (6.8).

Exercise 6.6. Suppose ψ_1 and ψ_2 are commuting actions of groups G_1 and G_2 on a set Z and $z \in Z$ is a point such that $\text{Stab}_z(\psi_1 \times \psi_2) = \{\mathbb{1}\}$. Let $\bar{\psi}_2$ be the induced G_2 -action on the quotient Z/G_1 and $G_1 z \in Z/G_1$ be the G_1 -orbit of z . Show that $\text{Stab}_{G_1 z}(\bar{\psi}_2) = \{\mathbb{1}\}$.

Exercise 6.7. Show that the Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) constructed above does not depend on the choice of subcollection $\mathcal{H}_\bullet \subset \mathcal{H}$ so that $\langle \mathcal{H}_\bullet \rangle^\partial$ is a vertex of $\langle \mathcal{H} \rangle$.

Exercise 6.8. Suppose $\mathcal{H}^\dagger \subset (T_1 \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant subcollection such that $\mathcal{H}^\dagger \supset \mathcal{H}$ and $\langle \mathcal{H}^\dagger \rangle = \langle \mathcal{H} \rangle$. Show that

(a) there exist (unique) linear functionals $\ell_{v'}: \text{Im } L_{\mathcal{H}}^* \rightarrow \mathbb{R}$ with $v' \in \mathcal{H}^\dagger - \mathcal{H}$ such that

$$L_{\mathcal{H}^\dagger}^*(\alpha) = (L_{\mathcal{H}}^*(\alpha), (\ell_{v'}(L_{\mathcal{H}}^*(\alpha)))_{v' \in \mathcal{H}^\dagger - \mathcal{H}}) \in \mathbb{R}^{\mathcal{H}} \times \mathbb{R}^{\mathcal{H}^\dagger - \mathcal{H}} = \mathbb{R}^{\mathcal{H}^\dagger} \quad \forall \alpha \in T_1^* \mathbb{T};$$

(b) $\ell_{v'}(s) > c_{v'}$ for all $v' \in \mathcal{H}^\dagger - \mathcal{H}$ and $s \in L_{\mathcal{H}}^*(\langle \mathcal{H} \rangle)$;

(c) the projection $\pi_{\mathcal{H}, \mathcal{H}^\dagger}: \mathbb{C}^{\mathcal{H}^\dagger} \rightarrow \mathbb{C}^{\mathcal{H}}$ restricts to a principal $\mathbb{T}^{\mathcal{H}^\dagger - \mathcal{H}}$ -bundle $\mu_{\mathcal{H}^\dagger}^{-1}(0) \rightarrow \mu_{\mathcal{H}}^{-1}(0)$ with a smooth $\mathbb{T}^{\mathcal{H}}$ -equivariant section

$$\begin{aligned}\tilde{s}: \mu_{\mathcal{H}}^{-1}(0) &\rightarrow \mu_{\mathcal{H}^\dagger}^{-1}(0) \subset \mathbb{C}^{\mathcal{H}} \times \mathbb{C}^{\mathcal{H}^\dagger - \mathcal{H}}, \\ \tilde{s}((z_v)_{v \in \mathcal{H}}) &= \left((z_v)_{v \in \mathcal{H}}, \left(\sqrt{(\ell_{v'}(H_{\mathcal{H}}((z_v)_{v \in \mathcal{H}})) - c_{v'}) / \pi} \right)_{v' \in \mathcal{H}^\dagger - \mathcal{H}} \right);\end{aligned}$$

(d) $\omega_{\mathcal{H}^\dagger} \big|_{T\mu_{\mathcal{H}^\dagger}^{-1}(0)} = \pi_{\mathcal{H}, \mathcal{H}^\dagger}^* \omega_{\mathcal{H}} \big|_{T\mu_{\mathcal{H}}^{-1}(0)}$ and

$$H_{\mathcal{H}} = \iota_{\mathcal{H}^\dagger; \mathcal{H}}^* \circ H_{\mathcal{H}^\dagger} \circ \tilde{s}: \mu_{\mathcal{H}}^{-1}(0) \rightarrow \text{Im } L_{\mathcal{H}}^*.$$

Conclude that the Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) constructed above does not depend on the choice of Delzant subcollection $\mathcal{H} \subset (T_1 \mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ with $\langle \mathcal{H} \rangle$ fixed. *Hint:* using Exercise 5.7(a), show that the section \tilde{s} descends to an identification of the Hamiltonian \mathbb{T} -manifolds determined by \mathcal{H} and \mathcal{H}^\dagger via the above construction.

Remark 6.9. The section \tilde{s} of Exercise 6.8(c) is equivariant with respect to the inclusion $\mathcal{K}_{\mathcal{H}} \rightarrow \mathcal{K}_{\mathcal{H}^\dagger}$ in the exact sequence of Exercise 5.7(a). This implies that \tilde{s} descends to a continuous map

$$X = \mu_{\mathcal{H}}^{-1}(0) / \mathcal{K}_{\mathcal{H}} \rightarrow X^\dagger \equiv \mu_{\mathcal{H}^\dagger}^{-1}(0) / \mathcal{K}_{\mathcal{H}^\dagger}.$$

In contrast, the projection $\pi_{\mathcal{H}, \mathcal{H}^\dagger}: \mu_{\mathcal{H}^\dagger}^{-1}(0) \rightarrow \mu_{\mathcal{H}}^{-1}(0)$ does not generally descend to a continuous map $X^\dagger \rightarrow X$, as $\mathbb{T}^{\mathcal{H}^\dagger - \mathcal{H}} \subset \mathbb{T}^{\mathcal{H}^\dagger}$ is generally not a subgroup of $\mathcal{K}_{\mathcal{H}^\dagger}$ and thus the homomorphisms in the exact sequence of Exercise 5.7(a) are not reversible.

Exercise 6.10. Let $\mathcal{H}_0 \subset \mathcal{H}$. Show that

- (a) $\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial) = \{ [(z_v)_{v \in \mathcal{H}}] \in X \equiv \mu_{\mathcal{H}}^{-1}(0)/\mathcal{K}_{\mathcal{H}} : z_v = 0 \ \forall v \in \mathcal{H}_0 \} \equiv (\mu_{\mathcal{H}}^{-1}(0) \cap \mathbb{C}^{\mathcal{H} - \mathcal{H}_0})/\mathcal{K}_{\mathcal{H}}$;
- (b) $\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial) \subset X$ is a connected ω -symplectic submanifold;
- (c) the subtorus $\mathbb{T}_{\mathcal{H}_0} \subset \mathbb{T}$ acts trivially on $\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial)$ and thus the \mathbb{T} -action ψ induces a $\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}$ -action $\psi_{\mathcal{H}_0}$ on $\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial)$;
- (d) if $\alpha_0 \in \langle \mathcal{H}_0 \rangle^\partial$, then

$$\mu_{\alpha_0} \equiv \mu - \alpha_0 : \mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial) \longrightarrow T_1^*(\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}) = \{ \alpha \in T_1^*\mathbb{T} : \alpha|_{T_1\mathbb{T}_{\mathcal{H}_0}} = 0 \}$$

is a moment map for the $\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}$ -action $\psi_{\mathcal{H}_0}$ with respect to $\omega|_{\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial)}$.

Exercise 6.11. Suppose $\mathcal{H}_0 \subset \mathcal{H}$, $\alpha_0 \in \langle \mathcal{H}_0 \rangle^\partial$, and q_{α_0} and $\mathcal{H}/\mathcal{H}_0$ are as in Exercise 5.8. Define

$$\begin{aligned} \tilde{i}_{\mathcal{H}; \mathcal{H}_0} &\equiv (\tilde{i}_{\mathcal{H}; \mathcal{H}_0; v})_{v \in \mathcal{H} - \mathcal{H}_0} : \mathbb{C}^{\mathcal{H}/\mathcal{H}_0} \longrightarrow \mathbb{C}^{\mathcal{H} - \mathcal{H}_0} \subset \mathbb{C}^{\mathcal{H}}, \\ \tilde{i}_{\mathcal{H}; \mathcal{H}_0; v} &((z_{v'})_{v' \in \mathcal{H}/\mathcal{H}_0}) = z_{q_{\alpha_0}(v)} \quad \forall v \in \mathcal{H} - \mathcal{H}_0; \\ \phi_{\mathcal{H}; \mathcal{H}_0}^{\mathbb{R}} &\equiv (\phi_{\mathcal{H}; \mathcal{H}_0; v}^{\mathbb{R}})_{v \in \mathcal{H}/\mathcal{H}_0} : \mathbb{R}^{\mathcal{H}} \longrightarrow \mathbb{R}^{\mathcal{H}/\mathcal{H}_0}, \quad \phi_{\mathcal{H}; \mathcal{H}_0; q_{\alpha_0}(v)}^{\mathbb{R}}((s_{v'})_{v' \in \mathcal{H}}) = s_v \quad \forall v \in \mathcal{H} - \mathcal{H}_0. \end{aligned}$$

Show that

- (a) the smooth map $\tilde{i}_{\mathcal{H}; \mathcal{H}_0}$ restricts to a $\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}$ -equivariant diffeomorphism between smooth submanifolds $\mu_{\mathcal{H}/\mathcal{H}_0}^{-1}(0)$ and $\mu_{\mathcal{H}}^{-1}(0) \cap \mathbb{C}^{\mathcal{H} - \mathcal{H}_0}$ of the domain and target, respectively;
- (b) $\omega_{\mathcal{H}/\mathcal{H}_0} = \tilde{i}_{\mathcal{H}; \mathcal{H}_0}^* \omega_{\mathcal{H}}$,

$$\begin{aligned} H_{\mathcal{H}} - L_{\mathcal{H}}^*(\alpha_0) &: \mu_{\mathcal{H}}^{-1}(0) \cap \mathbb{C}^{\mathcal{H} - \mathcal{H}_0} \longrightarrow \mathbb{R}^{\mathcal{H} - \mathcal{H}_0} \subset \mathbb{R}^{\mathcal{H}}, \quad \text{and} \\ H_{\mathcal{H}/\mathcal{H}_0} &= \phi_{\mathcal{H}; \mathcal{H}_0}^{\mathbb{R}} \circ \{ H_{\mathcal{H}} \circ \tilde{i}_{\mathcal{H}; \mathcal{H}_0} - L_{\mathcal{H}}^*(\alpha_0) \} : \mathbb{C}^{\mathcal{H}/\mathcal{H}_0} \longrightarrow T_1^*\mathbb{T}^{\mathcal{H}/\mathcal{H}_0}; \end{aligned}$$

- (c) if $\mathcal{H}_0 \subset \mathcal{H}_\bullet \subset \mathcal{H}$ and $\langle \mathcal{H}_\bullet \rangle^\partial$ is a vertex of $\langle \mathcal{H} \rangle$, then

$$\begin{aligned} L_{\mathcal{H}_\bullet}^{*-1}(\mathbb{R}^{\mathcal{H}_\bullet - \mathcal{H}_0}) &= T_1^*(\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}) = \{ \alpha \in T_1^*\mathbb{T} : \alpha|_{T_1\mathbb{T}_{\mathcal{H}_0}} = 0 \} \quad \text{and} \\ L_{\mathcal{H}_\bullet}^{*-1} \circ \iota_{\mathcal{H}; \mathcal{H}_\bullet}^* &= L_{\mathcal{H}_\bullet/\mathcal{H}_0}^{*-1} \circ \iota_{\mathcal{H}/\mathcal{H}_0; \mathcal{H}_\bullet/\mathcal{H}_0}^* \circ \phi_{\mathcal{H}; \mathcal{H}_0}^{\mathbb{R}} : \mathbb{R}^{\mathcal{H} - \mathcal{H}_0} \longrightarrow T_1^*(\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}). \end{aligned}$$

Conclude that $\tilde{i}_{\mathcal{H}; \mathcal{H}_0}$ induces an identification of the Hamiltonian $\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}$ -manifold determined by the regular Delzant subcollection $\mathcal{H}/\mathcal{H}_0 \subset (T_1(\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}))_{\mathbb{Z}} \times \mathbb{R}$ with the Hamiltonian $\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}$ -manifold $(\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial), \omega|_{\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial)}, \psi_{\mathcal{H}_0}, \mu_{\alpha_0})$ of Exercise 6.10. *Hint:* use Exercise 5.8(c).

Exercise 6.12. Suppose (X, ω, ψ, μ) is the Hamiltonian \mathbb{T} -manifold determined by a Delzant subcollection $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ and $\mathcal{H}^+ \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection. Show that

- (a) (X, ω, ψ, μ) is \mathcal{H}^+ -cuttable if and only if $\mathcal{H} \sqcup \mathcal{H}^+$ is a Delzant subcollection;
- (b) if $\mathcal{H} \sqcup \mathcal{H}^+$ is a Delzant subcollection, then the Hamiltonian \mathbb{T} -manifold determined by $\mathcal{H} \sqcup \mathcal{H}^+$ is the \mathcal{H}^+ -cut $(X, \omega, \psi, \mu)_{\mathcal{H}^+}$ of (X, ω, ψ, μ) provided by Theorem 5.

6.2 Complex quotient construction

As before, let $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a regular Delzant subcollection. In this section, we describe the quotient $X \equiv \mu_{\mathcal{H}}^{-1}(0)/\mathcal{K}_{\mathcal{H}}$ constructed in Section 6.1 as the quotient of the complement $\tilde{X}_{\mathcal{H}}$ of certain coordinate subspaces of $\mathbb{C}^{\mathcal{H}}$ by the complexification $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}} \subset (\mathbb{C}^*)^{\mathcal{H}}$ of $\mathcal{K}_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}}$. This description turns out to be more suitable for constructing (complex) coordinate charts on X , as demonstrated in Section 6.3. We continue with the notation introduced in Section 6.1.

Let $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}} \subset \mathbb{T}_{\mathbb{C}}^{\mathcal{H}}$ be the complexification of $\mathcal{K}_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}}$ and $(\mathcal{K}_{\mathcal{H}})_i \subset (\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$ be the purely imaginary subgroup. Under the identification

$$\mathbb{T}_{\mathbb{C}}^{\mathcal{H}} \equiv \mathbb{C}^{\mathcal{H}}/\mathbb{Z}^{\mathcal{H}} \longrightarrow (\mathbb{C}^*)^{\mathcal{H}}, \quad [v] \longrightarrow e^{2\pi i v},$$

the subgroup $(\mathcal{K}_{\mathcal{H}})_i$ corresponds to a subgroup of $(\mathbb{R}^*)^{\mathcal{H}} \subset (\mathbb{C}^*)^{\mathcal{H}}$. The group $\mathbb{T}_{\mathbb{C}}^{\mathcal{H}} \approx (\mathbb{C}^*)^{\mathcal{H}}$ acts on $\mathbb{C}^{\mathcal{H}}$ by the coordinate-wise multiplication in the usual way, i.e. as in (6.5); we denote this complexified action in the same way. Define

$$\begin{aligned} \text{Ver}(\mathcal{H}) &= \{ \mathcal{H}_{\bullet} \subset \mathcal{H} : \langle \mathcal{H}_{\bullet} \rangle^{\partial} \cap \langle \mathcal{H} \rangle \neq \emptyset, |\mathcal{H}_{\bullet}| = n \}, \\ \tilde{X}_{\mathcal{H}} &= \mathbb{C}^{\mathcal{H}} - \bigcup_{\substack{\mathcal{H}' \subset \mathcal{H} \\ \mathbb{C}^{\mathcal{H}'} \cap \mu_{\mathcal{H}}^{-1}(0) = \emptyset}} \mathbb{C}^{\mathcal{H}'}, \quad X_{\mathcal{H}} \equiv \tilde{X}_{\mathcal{H}} / (\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}. \end{aligned} \quad (6.9)$$

Thus, $\tilde{X}_{\mathcal{H}} \subset \mathbb{C}^{\mathcal{H}}$ is a $\mathbb{T}_{\mathbb{C}}^{\mathcal{H}}$ -invariant path-connected open subset containing $\mu_{\mathcal{H}}^{-1}(0)$. If $\mu_{\mathcal{H}}^{-1}(0) = \emptyset$, then $\tilde{X}_{\mathcal{H}} = \emptyset$. By Exercise 5.6(a), $\text{Ver}(\mathcal{H}) \neq \emptyset$ if $\mu_{\mathcal{H}}^{-1}(0) \neq \emptyset$. Furthermore, every subset $\mathcal{H}' \subset \mathcal{H}$ such that $\langle \mathcal{H}' \rangle^{\partial} \cap \langle \mathcal{H} \rangle \neq \emptyset$ is contained in some element \mathcal{H}_{\bullet} of $\text{Ver}(\mathcal{H})$. Along with (6.6), this implies that for every element $z \equiv (z_v)_{v \in \mathcal{H}}$ of $\tilde{X}_{\mathcal{H}}$ the set

$$\mathcal{H}_z \equiv \{ v \in \mathcal{H} : z_v = 0 \}$$

is contained in some element \mathcal{H}_{\bullet} of $\text{Ver}(\mathcal{H})$.

Exercise 6.13. Show that

- (a) $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$ acts freely on the subspace $\tilde{X}_{\mathcal{H}} \subset \mathbb{C}^{\mathcal{H}}$;
- (b) the subspace $\tilde{X}_{\mathcal{H}} \subset \mathbb{C}^{\mathcal{H}}$ is simply connected if \mathcal{H} is minimal.

Hint: proceed similarly to the paragraph containing (5.11).

Lemma 6.14. Suppose $v_k \equiv (t_{k;v})_{v \in \mathcal{H}}$ with $k \in \mathbb{Z}^+$ is a sequence in $T_1\mathcal{K}_{\mathcal{H}} \subset \mathbb{R}^{\mathcal{H}}$ and $\mathcal{H}_1, \mathcal{H}_2$ are elements of $\text{Ver}(\mathcal{H})$ such that

$$\inf\{t_{k;v} : k \in \mathbb{Z}^+, v \in \mathcal{H} - \mathcal{H}_1\} \in \mathbb{R} \quad \text{and} \quad \sup\{t_{k;v} : k \in \mathbb{Z}^+, v \in \mathcal{H} - \mathcal{H}_2\} = +\infty. \quad (6.10)$$

Then, $\sup\{t_{k;v} : k \in \mathbb{Z}^+, v \in \mathcal{H} - \mathcal{H}_2\} = +\infty$.

Proof. Since \mathcal{H} is Delzant, the projection

$$\pi_{\mathcal{H}; \mathcal{H}_{\bullet}} : T_1\mathcal{K}_{\mathcal{H}} \longrightarrow \mathbb{R}^{\mathcal{H} - \mathcal{H}_{\bullet}}, \quad \pi_{\mathcal{H}; \mathcal{H}_{\bullet}}((t_v)_{v \in \mathcal{H}}) = (t_v)_{v \in \mathcal{H} - \mathcal{H}_{\bullet}},$$

is an isomorphism for every $\mathcal{H}_\bullet \in \text{Ver}(\mathcal{H})$. Thus, so is its dual,

$$\pi_{\mathcal{H};\mathcal{H}_\bullet}^* = \iota_{\mathcal{H}}^* \big|_{\mathbb{R}^{\mathcal{H}-\mathcal{H}_\bullet}} : \mathbb{R}^{\mathcal{H}-\mathcal{H}_\bullet} \longrightarrow T_{\mathbb{1}}^* \mathcal{K}_{\mathcal{H}}.$$

In particular, there is a unique element $s_{\mathcal{H};\mathcal{H}_\bullet} \in \mathbb{R}^{\mathcal{H}-\mathcal{H}_\bullet}$ such that

$$\pi_{\mathcal{H};\mathcal{H}_\bullet}^*(s_{\mathcal{H};\mathcal{H}_\bullet}) = -\iota_{\mathcal{H}}^*((c_v)_{v \in \mathcal{H}}) \in T_{\mathbb{1}}^* \mathcal{K}_{\mathcal{H}}.$$

Since $L_{\mathcal{H}}^*(\langle \mathcal{H}_\bullet \rangle^\partial)$ is the single-element set $\{s + (c_v)_{v \in \mathcal{H}}\}$ for some $s \in (\mathbb{R}^+)^{\mathcal{H}-\mathcal{H}_\bullet}$ satisfying the above condition with $s_{\mathcal{H};\mathcal{H}_\bullet}$ replaced by s , $s_{\mathcal{H};\mathcal{H}_\bullet} \in (\mathbb{R}^+)^{\mathcal{H}-\mathcal{H}_\bullet}$ for every $\mathcal{H}_\bullet \in \text{Ver}(\mathcal{H})$.

Let $\pi_{\mathcal{H};\mathcal{H}_2,\mathcal{H}_1} = \pi_{\mathcal{H};\mathcal{H}_2} \circ \pi_{\mathcal{H};\mathcal{H}_1}^{-1} : \mathbb{R}^{\mathcal{H}-\mathcal{H}_1} \longrightarrow \mathbb{R}^{\mathcal{H}-\mathcal{H}_2}$. Thus,

$$\begin{aligned} \sup\{\langle s_{\mathcal{H};\mathcal{H}_2}, \pi_{\mathcal{H};\mathcal{H}_2}(v_k) \rangle : k \in \mathbb{Z}^+\} &= \sup\{\langle s_{\mathcal{H};\mathcal{H}_2}, \pi_{\mathcal{H};\mathcal{H}_2,\mathcal{H}_1}(\pi_{\mathcal{H};\mathcal{H}_1}(v_k)) \rangle : k \in \mathbb{Z}^+\} \\ &= \sup\{\langle \pi_{\mathcal{H};\mathcal{H}_2,\mathcal{H}_1}^*(s_{\mathcal{H};\mathcal{H}_2}), \pi_{\mathcal{H};\mathcal{H}_1}(v_k) \rangle : k \in \mathbb{Z}^+\} \\ &= \sup\{\langle s_{\mathcal{H};\mathcal{H}_1}, \pi_{\mathcal{H};\mathcal{H}_1}(v_k) \rangle : k \in \mathbb{Z}^+\}. \end{aligned}$$

Since $s_{\mathcal{H};\mathcal{H}_1} \in (\mathbb{R}^+)^{\mathcal{H}-\mathcal{H}_1}$, the two assumptions in (6.10) imply that the last supremum above is $+\infty$ and so

$$\sup\{\langle s_{\mathcal{H};\mathcal{H}_2}, \pi_{\mathcal{H};\mathcal{H}_2}(v_k) \rangle : k \in \mathbb{Z}^+\} = +\infty.$$

Since $s_{\mathcal{H};\mathcal{H}_2} \in (\mathbb{R}^+)^{\mathcal{H}-\mathcal{H}_2}$, this establishes the claim. \square

Proposition 6.15. *Suppose $(z_k)_{k \in \mathbb{Z}^+}$ is a sequence in $\mu_{\mathcal{H}}^{-1}(0) \subset \mathbb{C}^{\mathcal{H}}$ and $(v_k)_{k \in \mathbb{Z}^+}$ is a sequence in $T_{\mathbb{1}} \mathcal{K}_{\mathcal{H}} \subset \mathbb{R}^{\mathcal{H}}$.*

- (a) *If the sequence $\psi_{\mathcal{H};[iv_k]}(z_k)$ converges to some $y \in \mathbb{C}^{\mathcal{H}}$, then the sequence z_k is bounded.*
- (b) *If the sequence $\psi_{\mathcal{H};[iv_k]}(z_k)$ converges to some $y \in \tilde{X}_{\mathcal{H}}$, then subsequences of z_k and v_k converge to some $z \in \mu_{\mathcal{H}}^{-1}(0)$ and $v \in T_{\mathbb{1}} \mathcal{K}_{\mathcal{H}}$, respectively, with $\psi_{\mathcal{H};[iv]}(z) = y$.*

Proof. We denote by $\{e_v : v \in \mathcal{H}\}$ the standard basis for $T_{\mathbb{1}} \mathbb{T}^{\mathcal{H}} = \mathbb{R}^{\mathcal{H}}$, by $\{e_v^* : v \in \mathcal{H}\}$ the corresponding dual basis for $T_{\mathbb{1}}^* \mathbb{T}^{\mathcal{H}} \approx \mathbb{R}^{\mathcal{H}}$, by $\langle \cdot, \cdot \rangle$ the standard inner-products on $T_{\mathbb{1}} \mathbb{T}^{\mathcal{H}}$ and $T_{\mathbb{1}}^* \mathbb{T}^{\mathcal{H}}$, and by $|\cdot|$ the corresponding norms and the standard norm on $\mathbb{C}^{\mathcal{H}}$. If $\mathcal{H}_\bullet \in \text{Ver}(\mathcal{H})$, the elements $v \in T_{\mathbb{1}} \mathbb{T}$ with $(v, c) \in \mathcal{H}_\bullet$ form an \mathbb{R} -basis for $T_{\mathbb{1}} \mathbb{T}$ (in fact, \mathbb{Z} -basis for the lattice $(T_{\mathbb{1}} \mathbb{T})_{\mathbb{Z}}$). Thus, there exists a matrix $A \equiv (a_{vv'})_{v \in \mathcal{H}-\mathcal{H}_\bullet, v' \in \mathcal{H}_\bullet}$ with real (in fact, integer) coefficients so that

$$v = \sum_{(v',c') \in \mathcal{H}_\bullet} a_{(v,c)(v',c')} v' \quad \forall (v,c) \in \mathcal{H}-\mathcal{H}_\bullet. \quad (6.11)$$

From (5.2) and the exactness of (6.1), we then obtain

$$\begin{aligned} T_{\mathbb{1}} \mathcal{K}_{\mathcal{H}} &\equiv \ker L_{\mathcal{H}} = \text{Span}_{\mathbb{R}} \left(\left\{ e_v - \sum_{v' \in \mathcal{H}_\bullet} a_{vv'} e_{v'} : v \in \mathcal{H}-\mathcal{H}_\bullet \right\} \right), \\ \ker \iota_{\mathcal{H}}^* &= \text{Im } L_{\mathcal{H}}^* = \text{Span}_{\mathbb{R}} \left(\left\{ e_{v'}^* + \sum_{v \in \mathcal{H}-\mathcal{H}_\bullet} a_{vv'} e_v^* : v' \in \mathcal{H}_\bullet \right\} \right). \end{aligned} \quad (6.12)$$

For each $k \in \mathbb{Z}^+$, let $z_k = (z_{k;v})_{v \in \mathcal{H}} \in \mathbb{C}^{\mathcal{H}}$ and $v_k = (t_{k;v})_{v \in \mathcal{H}} \in \mathbb{R}^{\mathcal{H}}$. By passing to subsequences if necessary, we can assume that

- (i) there is a subset $\mathcal{H}_z^\infty \subset \mathcal{H}$ such that $|z_{k;v}| \rightarrow \infty$ for every $v \in \mathcal{H}_z^\infty$ and $z_{k;v}$ converges to some $z_v \in \mathbb{C}$ for every $v \in \mathcal{H} - \mathcal{H}_z^\infty$;
- (ii) there are (disjoint) subsets $\mathcal{H}_v^-, \mathcal{H}_v^+ \subset \mathcal{H}$ such that $t_{k;v} \rightarrow \pm\infty$ for every $v \in \mathcal{H}_v^\pm$ and $t_{k;v}$ converges to some $t_v \in \mathbb{C}$ for every $v \in \mathcal{H} - \mathcal{H}_v^- \cup \mathcal{H}_v^+$.

Let $y = (y_v)_{v \in \mathcal{H}}$. By the assumption in either (a) or (b),

$$\lim_{k \rightarrow \infty} e^{-2\pi t_{k;v}} z_{k;v} = y_v \quad \forall v \in \mathcal{H}. \quad (6.13)$$

In particular, $\mathcal{H}_z^\infty \subset \mathcal{H}_v^+$.

(a) Suppose $\mathcal{H}_z^\infty \neq \emptyset$ and thus $|H_{\mathcal{H}}(z_k)| \rightarrow \infty$. After passing to further subsequences if necessary, we can assume that the sequence $H_{\mathcal{H}}(z_k)/|H_{\mathcal{H}}(z_k)|$ converges to some

$$s \equiv (s_v)_{v \in \mathcal{H}} \in (\ker \iota_{\mathcal{H}}^*) \cap (\mathbb{R}^{\geq 0})^{\mathcal{H}} \subset \mathbb{R}^{\mathcal{H}}.$$

Let $\mathcal{H}_s^+ = \{v \in \mathcal{H} : s_v > 0\}$. We note that

$$\emptyset \neq \mathcal{H}_s^+ \subset \mathcal{H}_z^\infty \subset \mathcal{H}_v^+. \quad (6.14)$$

Choose $\mathcal{H}_\bullet \in \text{Ver}(\mathcal{H})$. With $A \equiv (a_{vv'})_{v,v'}$ as in (6.11), define

$$A_1 \equiv (a_{vv'})_{v \in \mathcal{H}_s^+ - \mathcal{H}_\bullet, v' \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet} \quad \text{and} \quad A_2 \equiv (a_{vv'})_{v \in \mathcal{H} - \mathcal{H}_\bullet - \mathcal{H}_s^+, v' \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet}.$$

By the second equation in (6.12) and the definition of \mathcal{H}_s^+ ,

$$A_1(s_v)_{v \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet} = (s_v)_{v \in \mathcal{H}_s^+ - \mathcal{H}_\bullet} \in (\mathbb{R}^+)^{\mathcal{H}_s^+ - \mathcal{H}_\bullet}, \quad A_2(s_v)_{v \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet} = 0 \in \mathbb{R}^{\mathcal{H} - \mathcal{H}_\bullet - \mathcal{H}_s^+}.$$

By the first equation in (6.12),

$$A_1^{\text{tr}}(t_{k;v})_{v \in \mathcal{H}_s^+ - \mathcal{H}_\bullet} + A_2^{\text{tr}}(t_{k;v})_{v \in \mathcal{H} - \mathcal{H}_\bullet - \mathcal{H}_s^+} = -(t_{k;v})_{v \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet}$$

for all $k \in \mathbb{Z}^+$. By (6.14), $t_{k;v} \in \mathbb{R}^+$ for all $v \in \mathcal{H}_s^+$ and $k \in \mathbb{Z}^+$ sufficiently large. Thus,

$$\begin{aligned} 0 &< \langle (s_v)_{v \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet}, (t_{k;v})_{v \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet} \rangle = -\langle (s_v)_{v \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet}, A_1^{\text{tr}}(t_{k;v})_{v \in \mathcal{H}_s^+ - \mathcal{H}_\bullet} + A_2^{\text{tr}}(t_{k;v})_{v \in \mathcal{H} - \mathcal{H}_\bullet - \mathcal{H}_s^+} \rangle \\ &= -\langle A_1(s_v)_{v \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet}, (t_{k;v})_{v \in \mathcal{H}_s^+ - \mathcal{H}_\bullet} \rangle - \langle A_2(s_v)_{v \in \mathcal{H}_s^+ \cap \mathcal{H}_\bullet}, (t_{k;v})_{v \in \mathcal{H} - \mathcal{H}_\bullet - \mathcal{H}_s^+} \rangle \\ &= -\langle (s_v)_{v \in \mathcal{H}_s^+ - \mathcal{H}_\bullet}, (t_{k;v})_{v \in \mathcal{H}_s^+ - \mathcal{H}_\bullet} \rangle - 0 < 0 \end{aligned}$$

for all $k \in \mathbb{Z}^+$ large. Since this is a contradiction, $\mathcal{H}_z^\infty = \emptyset$.

(b) After passing to further subsequences if necessary, we can thus assume that the sequence z_k converges to some element $z \equiv (z_v)_{v \in \mathcal{H}}$ of $\mu_{\mathcal{H}}^{-1}(0) \subset \mathbb{C}^{\mathcal{H}}$. By (6.13),

$$\mathcal{H}_v^+ \subset \mathcal{H}_y \equiv \{v \in \mathcal{H} : y_v = 0\} \quad \text{and} \quad \mathcal{H}_v^- \subset \mathcal{H}_z \equiv \{v \in \mathcal{H} : z_v = 0\}.$$

Let $\mathcal{H}_1, \mathcal{H}_2 \in \text{Ver}(\mathcal{H})$ be such that $\mathcal{H}_y \subset \mathcal{H}_1$ and $\mathcal{H}_z \subset \mathcal{H}_2$. By the definition of \mathcal{H}_v^+ and \mathcal{H}_v^- ,

$$\sup\{t_{k;v} : k \in \mathbb{Z}^+, v \in \mathcal{H} - \mathcal{H}_1\} \in \mathbb{R} \quad \text{and} \quad \inf\{t_{k;v} : k \in \mathbb{Z}^+, v \in \mathcal{H} - \mathcal{H}_2\} \in \mathbb{R}. \quad (6.15)$$

By Lemma 6.14 with \mathcal{H}_1 and \mathcal{H}_2 interchanged, this implies that

$$\sup\{t_{k,v} : k \in \mathbb{Z}^+, v \in \mathcal{H} - \mathcal{H}_2\} \in \mathbb{R}.$$

Along with the first statement in (6.12) and the second statement (6.15), this gives

$$\sup\{|t_{k,v}| : k \in \mathbb{Z}^+, v \in \mathcal{H}\} \in \mathbb{R}.$$

After passing to further subsequences if necessary, we can thus assume that the sequence v_k converges to some element $v \equiv (t_v)_{v \in \mathcal{H}}$ of $T_{\mathbb{1}}\mathcal{K}_{\mathcal{H}} \subset \mathbb{R}^{\mathcal{H}}$. The last claim then follows from the continuity of the complexified action $\psi_{\mathcal{H}}$. \square

Corollary 6.16. *The smooth map*

$$\Psi_{\mathcal{H}} : (\mathcal{K}_{\mathcal{H}})_i \times \mu_{\mathcal{H}}^{-1}(0) \longrightarrow \tilde{X}_{\mathcal{H}}, \quad \Psi_{\mathcal{H}}(u, z) = \psi_{\mathcal{H};u}(z),$$

is a diffeomorphism. The map

$$\iota : X \equiv \mu_{\mathcal{H}}^{-1}(0)/\mathcal{K}_{\mathcal{H}} \longrightarrow X_{\mathcal{H}} \equiv \tilde{X}_{\mathcal{H}}/(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}} \quad (6.16)$$

induced by the inclusions $\tilde{\iota} : \mu_{\mathcal{H}}^{-1}(0) \longrightarrow \tilde{X}_{\mathcal{H}}$ and $\mathcal{K}_{\mathcal{H}} \longrightarrow (\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$ is a homeomorphism. For every $\mathcal{H}_0 \subset \mathcal{H}$,

$$\iota(\mu^{-1}(\langle \mathcal{H}_0 \rangle^{\partial})) = \{[(z_v)_{v \in \mathcal{H}}] \in X_{\mathcal{H}} : z_v = 0 \ \forall v \in \mathcal{H}_0\}. \quad (6.17)$$

Proof. By the definition of the action $\psi_{\mathcal{H}}$, the image of the map $\Psi_{\mathcal{H}}$ is contained in $\tilde{X}_{\mathcal{H}}$. By Corollary 4.3, $\Psi_{\mathcal{H}}$ is a diffeomorphism onto an open subset of $\mathbb{C}^{\mathcal{H}}$ and thus of $\tilde{X}_{\mathcal{H}}$. By Proposition 6.15(b), the image of $\Psi_{\mathcal{H}}$ is closed in the connected space $\tilde{X}_{\mathcal{H}}$. Since $\tilde{X}_{\mathcal{H}} \neq \emptyset$ if and only if $\mu_{\mathcal{H}}^{-1}(0) \neq \emptyset$, it follows that the map $\Psi_{\mathcal{H}}$ is a diffeomorphism onto the entire space $\tilde{X}_{\mathcal{H}}$.

Let $\Psi_{\mathcal{H}}^{-1} = (\phi, \tilde{\mathcal{R}}) : \tilde{X}_{\mathcal{H}} \longrightarrow (\mathcal{K}_{\mathcal{H}})_i \times \mu_{\mathcal{H}}^{-1}(0)$. Since the map $\tilde{\mathcal{R}}$ is $\mathcal{K}_{\mathcal{H}}$ -equivariant and $(\mathcal{K}_{\mathcal{H}})_i$ -invariant with respect to the action $\psi_{\mathcal{H}}$, it descends to a continuous map

$$\mathcal{R} : \tilde{X}_{\mathcal{H}}/(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}} \longrightarrow \mu_{\mathcal{H}}^{-1}(0)/\mathcal{K}_{\mathcal{H}}.$$

Since the map $\tilde{\iota}$ is equivariant with respect to the inclusion $\mathcal{K}_{\mathcal{H}} \longrightarrow (\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$, it similarly descends to a continuous map ι as in (6.16). Since $\mathcal{R} \circ \iota = \text{id}$ and $\iota \circ \mathcal{R} = \text{id}$, the continuous maps \mathcal{R} and ι are homeomorphisms. Since the action of $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}} \subset (\mathbb{C}^*)^{\mathcal{H}}$ on $\tilde{X}_{\mathcal{H}} \subset \mathbb{C}^{\mathcal{H}}$ preserves the sets $\{v \in \mathcal{H} : z_v = 0\}$ with $(z_v)_v \in \tilde{X}_{\mathcal{H}}$, (6.17) follows from Exercise 6.10. \square

Exercise 6.17. Show that the manifold X determined by a regular Delzant subcollection \mathcal{H} of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ as in Section 6.1 is simply connected. *Hint:* use Exercises 6.8 and 6.13, along with the homotopy exact sequence for fibration.

Exercise 6.18. Let

$$\Psi_{\mathcal{H}}^{-1} \equiv ((\phi, (\tilde{\mathcal{R}}_v)_{v \in \mathcal{H}})) : \tilde{X}_{\mathcal{H}} \longrightarrow (\mathcal{K}_{\mathcal{H}})_i \times \mu_{\mathcal{H}}^{-1}(0) \subset \mathbb{R}^+ \times \mathbb{C}^{\mathcal{H}}$$

be the inverse of the diffeomorphism $\Psi_{\mathcal{H}}$ of Corollary 6.16. Show that the maps

$$X_{\mathcal{H}} \equiv \tilde{X}_{\mathcal{H}}/(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}} \longrightarrow \mathbb{R}^{\geq 0}, \quad [(z_{v'})_{v' \in \mathcal{H}}] \longrightarrow |\tilde{\mathcal{R}}_v((z_{v'})_{v' \in \mathcal{H}})|^2, \quad v \in \mathcal{H},$$

are well-defined and smooth.

Exercise 6.19. Suppose $n \in \mathbb{Z}^+$ and $\mathcal{H} \equiv \mathcal{H}_{n;n}()$ is the regular minimal Delzant subset of $(T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}} \times \mathbb{R}$ as in Exercise 5.18(b). Show that the inverse of the diffeomorphism $\Psi_{\mathcal{H}}$ of Corollary 6.16 is given by

$$\begin{aligned} \Psi_{\mathcal{H}}^{-1} \equiv (\phi, \tilde{\mathcal{R}}) : \tilde{X}_{\mathcal{H}} = \mathbb{C}^{\mathcal{H}} - \{0\} &\longrightarrow (\mathcal{K}_{\mathcal{H}})_i \times \mu_{\mathcal{H}}^{-1}(0) = \mathbb{R}^+ \times \{z \in \mathbb{C}^{\mathcal{H}} : |z|^2 = 1/\pi\}, \\ \Psi_{\mathcal{H}}^{-1}(z) &= (\sqrt{\pi}|z|, z/\sqrt{\pi}|z|). \end{aligned}$$

6.3 Kähler structure

We now show that the Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) constructed in Section 6.1 admits a compatible (integrable) complex structure J , i.e. J is compatible with the symplectic form ω and is preserved by the effective \mathbb{T} -action ψ on X . From now on, we identify the two quotients in (6.16) via ι .

Proposition 6.20. *The Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) determined by \mathcal{H} as in Section 6.1 admits a unique compatible complex structure $J_{\mathcal{H}}$ so that the quotient projection*

$$q : \tilde{X}_{\mathcal{H}} \longrightarrow X = X_{\mathcal{H}} \tag{6.18}$$

is a holomorphic submersion. For every $\mathcal{H}_0 \subset \mathcal{H}$, $\mu^{-1}(\langle \mathcal{H}_0 \rangle^{\partial}) \subset X$ is a complex submanifold of codimension $|\mathcal{H}_0|$ with respect to this complex structure.

Proof. Let $\mathcal{H}_{\bullet} \in \text{Ver}(\mathcal{H})$ and $(a_{vv'})_{v,v'}$ be as in (6.11). Define

$$\tilde{\mathcal{U}}_{\mathcal{H}_{\bullet}} = \{(z_v)_{v \in \mathcal{H}} \in \tilde{X}_{\mathcal{H}} : z_v \neq 0 \ \forall v \in \mathcal{H} - \mathcal{H}_{\bullet}\}, \quad \mathcal{U}_{\mathcal{H}_{\bullet}} = q(\tilde{\mathcal{U}}_{\mathcal{H}_{\bullet}}).$$

Since the open subset $\tilde{\mathcal{U}}_{\mathcal{H}_{\bullet}} \subset \tilde{X}_{\mathcal{H}}$ is $\mathcal{K}_{\mathcal{H}}$ -invariant (in fact, $\mathbb{T}^{\mathcal{H}}$ -invariant) the subset $\mathcal{U}_{\mathcal{H}_{\bullet}} \subset X_{\mathcal{H}}$ is also open. By the definition of $\tilde{\mathcal{U}}_{\mathcal{H}_{\bullet}}$ and the first equation in (6.12),

$$\begin{aligned} \tilde{\phi}_{\mathcal{H}_{\bullet}} &\equiv (\tilde{\phi}_{\mathcal{H}_{\bullet};v'})_{v' \in \mathcal{H}} : \tilde{\mathcal{U}}_{\mathcal{H}_{\bullet}} \longrightarrow (\mathcal{K}_{\mathcal{H}})_{\mathbb{C}} \subset (\mathbb{C}^*)^{\mathcal{H}}, \\ \tilde{\phi}_{\mathcal{H}_{\bullet};v'}((z_v)_{v \in \mathcal{H}}) &= \begin{cases} z_{v'}, & \text{if } v' \in \mathcal{H} - \mathcal{H}_{\bullet}; \\ \prod_{v \in \mathcal{H} - \mathcal{H}_{\bullet}} z_v^{-a_{vv'}}, & \text{if } v' \in \mathcal{H}_{\bullet}; \end{cases} \end{aligned}$$

is a well-defined map to $(\mathbb{C}^*)^{\mathcal{H}}$ with values in $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$. Since this map is $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$ -equivariant, the smooth map

$$\tilde{\psi}_{\mathcal{H}_{\bullet}} : \tilde{\mathcal{U}}_{\mathcal{H}_{\bullet}} \longrightarrow \mathbb{C}^{\mathcal{H}_{\bullet}}, \quad \tilde{\psi}_{\mathcal{H}_{\bullet}}((z_v)_{v \in \mathcal{H}}) = ((z_{v'}/\tilde{\phi}_{\mathcal{H}_{\bullet};v'}((z_v)_{v \in \mathcal{H}}))_{v' \in \mathcal{H}_{\bullet}}),$$

is $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$ -invariant and thus descends to a continuous map

$$\psi_{\mathcal{H}_{\bullet}} : \mathcal{U}_{\mathcal{H}_{\bullet}} \longrightarrow \mathbb{C}^{\mathcal{H}_{\bullet}}.$$

The inverse of this map is $q \circ \tilde{\vartheta}_{\mathcal{H}_{\bullet}}$, where

$$\tilde{\vartheta}_{\mathcal{H}_{\bullet}} \equiv (\tilde{\vartheta}_v)_{v \in \mathcal{H}} : \mathbb{C}^{\mathcal{H}_{\bullet}} \longrightarrow \tilde{\mathcal{U}}_{\mathcal{H}_{\bullet}}, \quad \tilde{\vartheta}_{\mathcal{H}_{\bullet};v}((z_{v'})_{v' \in \mathcal{H}_{\bullet}}) = \begin{cases} 1, & \text{if } v \in \mathcal{H} - \mathcal{H}_{\bullet}; \\ z_v, & \text{if } v \in \mathcal{H}_{\bullet}. \end{cases}$$

Since this map is continuous (in fact, smooth), $q \circ \tilde{\vartheta}$ is also continuous and thus $\psi_{\mathcal{H}_\bullet}$ is a homeomorphism.

The restriction of the map $\psi_{\mathcal{H}_\bullet} \circ q$ to the slice

$$\tilde{\mathcal{U}}_{\mathcal{H}_\bullet;1} \equiv \{(z_v)_{v \in \mathcal{H}} \in \mathbb{C}^{\mathcal{H}} : z_v = 1 \ \forall v \in \mathcal{H} - \mathcal{H}_\bullet\} \subset \tilde{\mathcal{U}}_{\mathcal{H}_\bullet}$$

is the projection onto $\mathbb{C}^{\mathcal{H}_\bullet}$. Since the image of $\tilde{\mathcal{U}}_{\mathcal{H}_\bullet;1}$ under $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$ is $\tilde{\mathcal{U}}_{\mathcal{H}_\bullet}$ and the map

$$\psi_{\mathcal{H}_\bullet} \circ q : \tilde{\mathcal{U}}_{\mathcal{H}_\bullet} \longrightarrow \mathbb{C}^{\mathcal{H}_\bullet}$$

is $(\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}$ -invariant, it follows that this map is a holomorphic submersion. Thus,

$$q : \tilde{\mathcal{U}}_{\mathcal{H}_\bullet} = q^{-1}(\mathcal{U}_{\mathcal{H}_\bullet}) \longrightarrow \mathcal{U}_{\mathcal{H}_\bullet}$$

is also a holomorphic submersion with respect to the complex structure $J_{\mathcal{H}_\bullet}$ on $\mathcal{U}_{\mathcal{H}_\bullet}$ induced from the standard complex structure on $\mathbb{C}^{\mathcal{H}_\bullet}$ by $\psi_{\mathcal{H}_\bullet}$. Since the standard complex structure $J_{\mathbb{C}^{\mathcal{H}}}$ on $\mathbb{C}^{\mathcal{H}}$ is $\mathbb{T}^{\mathcal{H}}$ -invariant and the \mathbb{T} -action on $\mathcal{U}_{\mathcal{H}_\bullet}$ is induced by the $\mathbb{T}^{\mathcal{H}}$ -action on $\mathbb{C}^{\mathcal{H}}$, it follows that the complex structure $J_{\mathcal{H}_\bullet}$ is \mathbb{T} -invariant.

Let $\mathcal{H}_0 \subset \mathcal{H}$. By (6.17),

$$\psi_{\mathcal{H}_\bullet}(\mu^{-1}((\mathcal{H}_0)^\partial) \cap \mathcal{U}_{\mathcal{H}_\bullet}) = \begin{cases} \emptyset, & \text{if } \mathcal{H}_0 \not\subset \mathcal{H}_\bullet; \\ \{(z_v)_{v \in \mathcal{H}_\bullet} : z_v = 0 \ \forall v \in \mathcal{H}_0\}, & \text{if } \mathcal{H}_0 \subset \mathcal{H}_\bullet. \end{cases}$$

Thus, $\mu^{-1}((\mathcal{H}_0)^\partial) \cap \mathcal{U}_{\mathcal{H}_\bullet}$ is a coordinate slice with respect to $\psi_{\mathcal{H}_\bullet}$.

Suppose $\mathcal{H}_1, \mathcal{H}_2 \in \text{Ver}(\mathcal{H})$ and thus

$$\psi_{\mathcal{H}_1}(\mathcal{U}_{\mathcal{H}_1} \cap \mathcal{U}_{\mathcal{H}_2}) = \{(z_v)_{v \in \mathcal{H}_1} \in \mathbb{C}^{\mathcal{H}_1} : z_v \neq 0 \ \forall v \in \mathcal{H}_1 - \mathcal{H}_2\}.$$

The overlap map between the charts $\psi_{\mathcal{H}_1}$ and $\psi_{\mathcal{H}_2}$ as above,

$$\begin{aligned} \psi_{\mathcal{H}_2} \circ \psi_{\mathcal{H}_1}^{-1} &\equiv (\psi_{\mathcal{H}_2; \mathcal{H}_1; v_2})_{v_2 \in \mathcal{H}_2} : \psi_{\mathcal{H}_1}(\mathcal{U}_{\mathcal{H}_1} \cap \mathcal{U}_{\mathcal{H}_2}) \longrightarrow \psi_{\mathcal{H}_2}(\mathcal{U}_{\mathcal{H}_1} \cap \mathcal{U}_{\mathcal{H}_2}) \subset \mathbb{C}^{\mathcal{H}_2}, \\ \psi_{\mathcal{H}_2; \mathcal{H}_1; v_2}((z_{v_1})_{v_1 \in \mathcal{H}_1}) &= \begin{cases} 1/\tilde{\phi}_{\mathcal{H}_2; v_2}(\tilde{\vartheta}_{\mathcal{H}_1}((z_{v_1})_{v_1 \in \mathcal{H}_1})), & \text{if } v_2 \in \mathcal{H}_2 - \mathcal{H}_1; \\ z_{v_2}/\tilde{\phi}_{\mathcal{H}_2; v_2}(\tilde{\vartheta}_{\mathcal{H}_1}((z_{v_1})_{v_1 \in \mathcal{H}_1})), & \text{if } v_2 \in \mathcal{H}_2 \cap \mathcal{H}_1; \end{cases} \end{aligned}$$

is a holomorphic function. Thus, the charts $\tilde{\psi}_{\mathcal{H}_\bullet} : \tilde{\mathcal{U}}_{\mathcal{H}_\bullet} \longrightarrow \mathbb{C}^{\mathcal{H}_\bullet}$ with $\mathcal{H}_\bullet \in \text{Ver}(\mathcal{H})$ determine a ψ -invariant complex structure $J_{\mathcal{H}}$ on $X = X_{\mathcal{H}}$ so that the quotient projection q in (6.18) is a holomorphic submersion. Since the standard complex structure $J_{\mathbb{C}^{\mathcal{H}}}$ on $\mathbb{C}^{\mathcal{H}}$ is compatible with the standard symplectic form $\omega_{\mathcal{H}}$ on $\mathbb{C}^{\mathcal{H}}$ and $q^*\omega = \omega_{\mathcal{H}}|_{T\mu_{\mathcal{H}}^{-1}(0)}$, it follows that $J_{\mathcal{H}}$ is compatible with ω . The map q being a holomorphic submersion implies the uniqueness of $J_{\mathcal{H}}$. By the previous paragraph, $\mu^{-1}((\mathcal{H}_0)^\partial) \subset X$ is a codimension $|\mathcal{H}_0|$ complex submanifold with respect to $J_{\mathcal{H}}$ for every $\mathcal{H}_0 \subset \mathcal{H}$. \square

Exercise 6.21. Let $[[\mathcal{H}]]^* \subset [[\mathcal{H}]]^* \subset [[\mathcal{H}]] \approx \mathbb{R}^{\mathcal{H}}$ be as in Exercise 6.4 and $\mathcal{H}^\dagger \in [[\mathcal{H}]]^*$. Show that

- (a) $\text{Ver}(\mathcal{H}^\dagger) = \text{Ver}(\mathcal{H})$ and $\tilde{X}_{\mathcal{H}^\dagger} = \tilde{X}_{\mathcal{H}}$, with the identification induced by a bijection $\Psi_{\mathcal{H}^\dagger, \mathcal{H}}$ as in Exercise 6.4;

(b) $(\mathcal{K}_{\mathcal{H}^\dagger})_{\mathbb{C}}$ acts freely on the subspace $\tilde{X}_{\mathcal{H}^\dagger} \subset \mathbb{C}^{\mathcal{H}^\dagger}$ and

$$(X_{\mathcal{H}^\dagger}, J_{\mathcal{H}^\dagger}, \psi_{\mathcal{H}^\dagger}) = (X_{\mathcal{H}}, J_{\mathcal{H}}, \psi_{\mathcal{H}}),$$

with the identification induced by a bijection $\Psi_{\mathcal{H}^\dagger, \mathcal{H}}$ as in Exercise 6.4.

Exercise 6.22. Suppose $\mathcal{H}^\dagger \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant subcollection such that $\mathcal{H}^\dagger \supset \mathcal{H}$ and $\langle \mathcal{H}^\dagger \rangle = \langle \mathcal{H} \rangle$. Show that

(a) $\text{Ver}(\mathcal{H}^\dagger) = \text{Ver}(\mathcal{H})$ and $\tilde{X}_{\mathcal{H}^\dagger} = \tilde{X}_{\mathcal{H}} \times (\mathbb{C}^*)^{\mathcal{H}^\dagger - \mathcal{H}}$;

(b) the $\mathbb{T}^{\mathcal{H}}$ -equivariant holomorphic immersion

$$\tilde{s}: \tilde{X}_{\mathcal{H}} \longrightarrow \tilde{X}_{\mathcal{H}^\dagger} \subset \mathbb{C}^{\mathcal{H}} \times \mathbb{C}^{\mathcal{H}^\dagger - \mathcal{H}}, \quad \tilde{s}(z) = (z, 1^{\mathcal{H}^\dagger - \mathcal{H}}),$$

descends to a \mathbb{T} -equivariant biholomorphism $s: (X_{\mathcal{H}}, J_{\mathcal{H}}) \longrightarrow (X_{\mathcal{H}^\dagger}, J_{\mathcal{H}^\dagger})$.

Hint: use Exercise 5.7(a).

Exercise 6.23. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The regular Delzant subcollections

$$\mathcal{H} \equiv \{(1, 0)\} \quad \text{and} \quad \mathcal{H}^\dagger \equiv \{(1, 0), (1, -\pi)\}$$

of $(T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ satisfy $\mathcal{H} \subset \mathcal{H}^\dagger$ and $\langle \mathcal{H} \rangle = \langle \mathcal{H}^\dagger \rangle$. Let $(X_{\mathcal{H}}, \omega_{\mathcal{H}})$ and $(X_{\mathcal{H}^\dagger}, \omega_{\mathcal{H}^\dagger})$ be the symplectic manifolds determined by \mathcal{H} and \mathcal{H}^\dagger , respectively, as in Section 6.1, $s: X_{\mathcal{H}} \longrightarrow X_{\mathcal{H}^\dagger}$ be the identification as in Exercise 6.22(b), and

$$f: \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}^+, \quad f(t) = \sqrt{\frac{2}{1 + \sqrt{1 + 4t}}}.$$

Show that

(a) the Hamiltonian Kähler \mathbb{T} -manifold $(X_{\mathcal{H}}, \omega_{\mathcal{H}}, J_{\mathcal{H}}, \psi_{\mathcal{H}}, \mu_{\mathcal{H}})$ determined by \mathcal{H} is the symplectic manifold $(\mathbb{C}, \omega_{\mathbb{C}})$ with the standard complex structure $J_{\mathbb{C}}$, the standard S^1 -action by multiplication, and the Hamiltonian given by

$$H: \mathbb{C} \longrightarrow \mathbb{R}, \quad H(z) = \pi|z|^2;$$

(b) the biholomorphism s is the composition of the map

$$X_{\mathcal{H}} = \mu_{\mathcal{H}}^{-1}(0) \equiv \mathbb{C} \longrightarrow \mu_{\mathcal{H}^\dagger}^{-1}(0) \equiv \{(z_1, z_2) \in \mathbb{C}^2: |z_2|^2 = |z_1|^2 + 1\}, \quad z \longrightarrow (f(|z|^2)z, 1/f(|z|^2)),$$

with the quotient projection $\mu_{\mathcal{H}^\dagger}^{-1}(0) \longrightarrow X_{\mathcal{H}^\dagger}$;

(c) $s^*\omega_{\mathcal{H}^\dagger} = (f(z)^2 + 2f(|z|^2)f'(|z|^2)|z|^2)\omega_{\mathbb{C}}$.

Conclude that the Hamiltonian Kähler \mathbb{T} -manifolds determined by \mathcal{H} and \mathcal{H}^\dagger are not isomorphic (there is no \mathbb{T} -equivariant identification of $X_{\mathcal{H}}$ with $X_{\mathcal{H}^\dagger}$ which intertwines the symplectic forms and complex structures at the same time).

Remark 6.24. By Exercises 6.8 and 6.22, the Hamiltonian \mathbb{T} -manifold (X, ω, ψ, μ) and the complex \mathbb{T} -manifold (X, J, ψ) obtained via the constructions of Section 6.1 and Proposition 6.20 do not depend on the choice of Delzant subcollection $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ with $\langle \mathcal{H} \rangle$ fixed. However, the identifications of Exercises 6.8 and 6.22 generally do not provide identifications of the Hamiltonian Kähler \mathbb{T} -manifolds determined by regular Delzant subcollections $\mathcal{H}, \mathcal{H}^\dagger \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ with $\langle \mathcal{H}^\dagger \rangle = \langle \mathcal{H} \rangle$. By Exercise 6.23, these Hamiltonian Kähler \mathbb{T} -manifolds need not be isomorphic at all.

Exercise 6.25. Suppose $\mathcal{H}_0 \subset \mathcal{H}$, $\alpha_0 \in \langle \mathcal{H}_0 \rangle^\partial$, and q_{α_0} and $\mathcal{H}/\mathcal{H}_0$ are as in Exercise 5.8. Show that the identification of the Hamiltonian $\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}$ -manifold determined by $\mathcal{H}/\mathcal{H}_0$ with the Hamiltonian $\mathbb{T}/\mathbb{T}_{\mathcal{H}_0}$ -manifold $(\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial), \omega|_{\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial)}, \psi_{\mathcal{H}_0}, \mu_{\alpha_0})$ in Exercise 6.11 intertwines the complex structure $J_{\mathcal{H}/\mathcal{H}_0}$ on the former and the restriction of the complex structure $J_{\mathcal{H}}$ to the latter.

Exercise 6.26. Suppose \mathbb{T} is a torus, $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a regular Delzant subcollection, and $(X, \omega, J, \psi, \mu)$ is the Hamiltonian Kähler \mathbb{T} -manifold determined by \mathcal{H} as above.

(a) Let $\lambda \in \mathbb{R}^+$ and $\lambda\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be as in Exercise 5.15. Show that the biholomorphism

$$\tilde{X}_{\mathcal{H}} \longrightarrow \tilde{X}_{\lambda\mathcal{H}} \subset \mathbb{C}^{\lambda\mathcal{H}}, \quad (z_{v,c})_{(v,c) \in \mathcal{H}} \longrightarrow (\sqrt{\lambda}z_{v,c})_{(v,\lambda c) \in \lambda\mathcal{H}},$$

restricts to a diffeomorphism $\mu_{\mathcal{H}}^{-1}(0) \longrightarrow \mu_{\lambda\mathcal{H}}^{-1}(0)$ and descends to an identification of $(X, \lambda\omega, J, \psi, \lambda\mu)$ with the Hamiltonian Kähler \mathbb{T} -manifold determined by $\lambda\mathcal{H}$ as above.

(b) Let $\alpha_0 \in T_1^*\mathbb{T}$ and $\mathcal{H} + \alpha_0 \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be as in Exercise 5.16. Show that the biholomorphism

$$\tilde{X}_{\mathcal{H}} \longrightarrow \tilde{X}_{\mathcal{H} + \alpha_0} \subset \mathbb{C}^{\mathcal{H} + \alpha_0}, \quad (z_{v,c})_{(v,c) \in \mathcal{H}} \longrightarrow (z_{v,c})_{(v,c + \alpha_0(v)) \in \mathcal{H} + \alpha_0},$$

restricts to a diffeomorphism $\mu_{\mathcal{H}}^{-1}(0) \longrightarrow \mu_{\mathcal{H} + \alpha_0}^{-1}(0)$ and descends to an identification of $(X, \omega, J, \psi, \mu + \alpha_0)$ with the Hamiltonian Kähler \mathbb{T} -manifold determined by $\mathcal{H} + \alpha_0$ as above.

(c) Let $\Theta \in \mathrm{GL}_{\mathbb{Z}}((T_1\mathbb{T})_{\mathbb{Z}})$ and $\Theta^* \in \mathrm{GL}_{\mathbb{Z}}((T_1^*\mathbb{T})_{\mathbb{Z}})$, ψ_{Θ} , and $\Theta^{-1}\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be as in Exercise 5.17. Show that the biholomorphism

$$\tilde{X}_{\mathcal{H}} \longrightarrow \tilde{X}_{\Theta^{-1}\mathcal{H}} \subset \mathbb{C}^{\Theta^{-1}\mathcal{H}}, \quad (z_{v,c})_{(v,c) \in \mathcal{H}} \longrightarrow (z_{v,c})_{(\Theta^{-1}(v),c) \in \Theta^{-1}\mathcal{H}},$$

restricts to a diffeomorphism $\mu_{\mathcal{H}}^{-1}(0) \longrightarrow \mu_{\Theta^{-1}\mathcal{H}}^{-1}(0)$ and descends to an identification of $(X, \omega, J, \psi_{\Theta}, \Theta^* \circ \mu)$ with the Hamiltonian Kähler \mathbb{T} -manifold determined by $\Theta^{-1}\mathcal{H}$ as above.

In light of Exercise 6.26, it is sufficient to consider the Hamiltonian Kähler \mathbb{T} -manifolds determined by representatives for the equivalence classes of regular Delzant subcollections of $(T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ as defined above Exercise 5.18. In light of Exercises 6.8 and 6.22, it is usually sufficient to consider the Hamiltonian Kähler \mathbb{T} -manifolds determined by minimal regular subsets, up to the nuance pointed out in Remark 6.24. A regular subcollection of $(T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ has at least $\dim \mathbb{T}$ elements. Exercise 5.18 provides specific representatives \mathcal{H} for the equivalence classes of regular minimal Delzant subsets of $(T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ of small cardinality relative to the dimension. The associated subsets $\langle \mathcal{H} \rangle \subset T_1^*\mathbb{T}$ in the $\dim \mathbb{T} = 2$ case are shown in Figures 2 and 3.

Exercise 6.27. Suppose $n \in \mathbb{Z}^+$ and $\mathcal{H}_n^* \subset (T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}} \times \mathbb{R}$ is the regular minimal Delzant subset of Exercise 5.18(a). Show that the Hamiltonian Kähler \mathbb{T} -manifold determined by \mathcal{H}_n^* is $(\mathbb{C}^n, \omega_{\mathbb{C}^n}, J_{\mathbb{C}^n}, \psi_{\mathbb{C}^n}, H_{\mathbb{C}^n})$, where $\omega_{\mathbb{C}^n}$ is the standard symplectic form on \mathbb{C}^n as in (2.11), $J_{\mathbb{C}^n}$ is the standard complex structure on \mathbb{C}^n , $\psi_{\mathbb{C}^n}$ is the standard coordinate-wise \mathbb{T}^n -action on \mathbb{C}^n , i.e. as in (6.5) with $\mathcal{H} = [n]$, and

$$H_{\mathbb{C}^n} : \mathbb{C}^n \longrightarrow \mathbb{R}^n = T_{\mathbb{1}}\mathbb{T}^n, \quad H_{\mathbb{C}^n}(z_1, \dots, z_n) = \pi(|z_1|^2, \dots, |z_n|^2).$$

Exercise 6.28. Suppose $n \in \mathbb{Z}^+$, $k \in [n]$, $a_{k+1}, \dots, a_n \in \mathbb{Z}^{\geq 0}$,

$$\mathcal{H} \equiv \mathcal{H}_{n;k}(a_{k+1}, \dots, a_n) \subset (T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}} \times \mathbb{R}$$

is the regular minimal Delzant subset of Exercise 5.18(b), and $(X, \omega, J, \psi, \mu)$ is the Hamiltonian Kähler \mathbb{T} -manifold determined by \mathcal{H} . Let

$$v_k^+ = (-e_1 - \dots - e_k, -1) \in \mathcal{H}_{k;k}(), \quad v_n^+ = (-e_1 - \dots - e_k + a_{k+1}e_{k+1} + \dots + a_n e_n, -1) \in \mathcal{H},$$

and

$$\pi_{\gamma_k} : \gamma_k \equiv \{(\ell, v) \in \mathbb{C}P^k \times \mathbb{C}^{k+1} : v \in \ell\} \longrightarrow \mathbb{C}P^k, \quad \pi_{\gamma_k}(\ell, v) = \ell,$$

be the tautological holomorphic line bundle.

(a) Show that the projection

$$\tilde{\pi} \equiv (\tilde{\pi}_v)_{v \in \mathcal{H}_{k;k}()} : \tilde{X}_{\mathcal{H}} \longrightarrow \tilde{X}_{\mathcal{H}_{k;k}()}, \quad \tilde{\pi}_v((z_{v'})_{v' \in \mathcal{H}}) = \begin{cases} z_v, & \text{if } v \in \mathcal{H}_k^*; \\ z_{v_n^+}, & \text{if } v = v_k^+; \end{cases}$$

is well-defined and descends to a holomorphic vector bundle $\pi : X \longrightarrow X_{\mathcal{H}_{k;k}()}$.

(b) Construct an identification of $(X, \omega, J, \psi, \mu, \pi)$ with the holomorphic vector bundle

$$\pi_E : E \equiv \bigoplus_{j=k+1}^n \gamma^{\otimes a_j} \longrightarrow \mathbb{C}P^k$$

with a symplectic form ω_E such that $\omega_E|_{\mathbb{C}P^k} = \omega_{\text{FS};k}$, with the \mathbb{T}^n -action ψ_E given by

$$\begin{aligned} \psi_{E;(u_i)_{i \in [n]}} & \left(\left[(z_i)_{i \in [k+1]}, (z_j ((z_i)_{i \in [k+1]})^{\otimes a_j})_{j \in [n]-[k]} \right] \right) \\ & = \left(\left[((u_i z_i)_{i \in [k]}, z_{k+1}), (u_j z_j ((u_i z_i)_{i \in [k]}, z_{k+1})^{\otimes a_j})_{j \in [n]-[k]} \right] \right), \end{aligned}$$

and with a Hamiltonian H_E given by

$$\begin{aligned} H_E & \left(\left[(z_i)_{i \in [k+1]}, (z_j ((z_i)_{i \in [k+1]})^{\otimes a_j})_{j \in [n]-[k]} \right] \right) \\ & = \left(\pi | \tilde{\mathcal{R}}_{(e_j, 0)}((z_i)_{(e_i, 0) \in \mathcal{H}_n^*}, z_{n+1})|^2 \right)_{j \in [n]} \in \mathbb{R}^n = T_{\mathbb{1}}^* \mathbb{T}^n, \end{aligned}$$

with $\tilde{\mathcal{R}}_{(e_j, 0)}$ denoting the $v = (e_j, 0)$ component of the inverse of the diffeomorphism $\Psi_{\mathcal{H}}$ of Corollary 6.16 as in Exercise 6.18 and z_{n+1} denoting the v_n^+ -component of the input \mathcal{H} -tuple.

- (c) Conclude that the Hamiltonian Kähler \mathbb{T} -manifold $(X, \omega, J, \psi, \mu)$ determined by $\mathcal{H}_{n;n}()$ is the complex projective space $\mathbb{C}P^n$ with the Fubini-Study symplectic form $\omega_{\text{FS};n}$, its standard complex structure, the \mathbb{T}^n -action given by (2.14) with n replaced by $n+1$, and the Hamiltonian given by

$$H_{\mathbb{C}P^n} : \mathbb{C}P^n \longrightarrow \mathbb{R}^n = T_{\mathbb{1}}^* \mathbb{T}^n, \quad H_{\mathbb{C}P^n}([z_1, \dots, z_{n+1}]) = \frac{(|z_1|^2, \dots, |z_n|^2)}{|z_1|^2 + \dots + |z_{n+1}|^2}.$$

Exercise 6.29. Suppose $n, k, a_{k+1}, \dots, a_n \in \mathbb{Z}^{\geq 0}$, \mathcal{H} , $(X, \omega, J, \psi, \mu)$, E , and ψ_E are as in Exercise 6.28 and $k < n$. Let $c \in \mathbb{R}^+$ and

$$\mathcal{H}' = \{(-e_{k+1} - \dots - e_n, -c)\} \subset (T_{\mathbb{1}} \mathbb{T}^n)_{\mathbb{Z}} \times \mathbb{R}.$$

Show that

- (a) $\mathcal{H} \sqcup \mathcal{H}'$ is a Delzant subset of $(T_{\mathbb{1}} \mathbb{T}^n)_{\mathbb{Z}} \times \mathbb{R}$ and thus (X, ω, ψ, μ) is \mathcal{H}' -cuttable;
(b) if $(X', \omega', J', \psi', \mu')$ is the Hamiltonian Kähler \mathbb{T} -manifold determined by $\mathcal{H} \sqcup \mathcal{H}'$, then

$$(X', \omega', \psi', \mu') = (X, \omega, \psi, \mu)_{\mathcal{H}'}$$

and there is an identification of (X', J') with the projectivization $\mathbb{P}(E \oplus \tau_{\mathbb{C}P^k}^1)$ of the direct sum of the holomorphic vector bundle E with the trivial complex line bundle $\tau_{\mathbb{C}P^k}^1$ over $\mathbb{C}P^k$ so that ψ' is the \mathbb{T}^n -action on $\mathbb{P}(E \oplus \tau_{\mathbb{C}P^k}^1)$ induced by the trivial extension of the \mathbb{T}^n -action ψ_E to $E \oplus \tau_{\mathbb{C}P^k}$,

$$\int_{\mathbb{C}P^1} \omega' = c$$

for any $\mathbb{C}P^1 \subset \mathbb{P}(E \oplus \tau_{\mathbb{C}P^k}^1)$ linearly embedded in a fiber of the $\mathbb{C}P^{n-k}$ -bundle $\mathbb{P}(E \oplus \tau_{\mathbb{C}P^k}) \longrightarrow \mathbb{C}P^k$, and $s_0^* \omega' = \omega_{\text{FS};k}$, where s_0 is the section

$$s_0 : \mathbb{C}P^k \longrightarrow \mathbb{P}(E \oplus \tau_{\mathbb{C}P^k}^1), \quad s_0(x) = [0, 1] \in \mathbb{P}(E_x \oplus \mathbb{C}).$$

Exercise 6.30. Suppose $a \in \mathbb{Z}^{\geq 0}$, $c \in \mathbb{R}^+$, $\mathcal{H}_{2;1}(a; 0, c) \subset (T_{\mathbb{1}} \mathbb{T}^2)_{\mathbb{Z}} \times \mathbb{R}$ is the regular minimal Delzant subset of Exercise 5.18(c), and $(X, \omega, J, \psi, \mu)$ is the Hamiltonian Kähler \mathbb{T} -manifold determined by $\mathcal{H}_{2;1}(a; 0, c)$. Let $\gamma_1, \tau_{\mathbb{C}P^1}^1 \longrightarrow \mathbb{C}P^1$ be the complex tautological line bundle and the trivial complex line bundle, respectively. Construct an identification of $(X, \omega, \psi, \mu, J)$ with the a -th Hirzebruch surface,

$$\mathbb{F}_a \equiv \mathbb{P}(\gamma_1^{\otimes a} \oplus \tau_{\mathbb{C}P^1}^1) \longrightarrow \mathbb{C}P^1,$$

with a symplectic form $\omega_{a;c}$ so that $\omega_{a;c}|_{\mathbb{P}(0 \oplus \tau_{\mathbb{C}P^1}^1)} = \omega_{\text{FS};1}$ and $\int_F \omega_{a;c} = c$ for every fiber $F \subset \mathbb{F}_a$, with \mathbb{T}^2 -action $\psi_{a;c}$ given by

$$\psi_{a;c;(u_1, u_2)}([([z_1, w_1], z_2(z_1, w_1)^{\otimes a}, w_2)]) = ([([u_1 z_1, w_1], u_2 z_2(u_1 z_1, w_1)^{\otimes a}, w_2)]),$$

and with a Hamiltonian $H_{a;c} : \mathbb{F}_a \longrightarrow \mathbb{R}^2$,

$$H_{a;c}([([z_1, w_1], z_2(z_1, w_1)^{\otimes a}, w_2)]) = \left(\pi \lambda(z_1, w_1, z_2, w_2) |z_1|^2, \frac{c \lambda(z_1, w_1, z_2, w_2)^{-a} |z_2|^2}{\lambda(z_1, w_1, z_2, w_2)^{-a} |z_2|^2 + |w_2|^2} \right),$$

where $\lambda(z_1, w_1, z_2, w_2) \in \mathbb{R}^+$ is defined by

$$\pi \lambda(z_1, w_1, z_2, w_2) (|z_1|^2 + |w_1|^2) = (1 + ac) - \frac{ac |w_2|^2}{\lambda(z_1, w_1, z_2, w_2)^{-a} |z_2|^2 + |w_2|^2}.$$

6.4 Line bundles and projectivity

As before, let $\mathcal{H} \subset (T_1\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a regular Delzant subcollection, $\mathcal{K}_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}}$ be the kernel of the homomorphism $\Phi_{\mathcal{H}}$ in (5.2), and $\tilde{X}_{\mathcal{H}} \subset \mathbb{C}^{\mathcal{H}}$ be the dense open subset given by (6.9). For $\mathbf{c}' \equiv (c'_v)_{v \in \mathcal{H}} \in \mathbb{Z}^{\mathcal{H}}$, define

$$L_{\mathcal{H}\mathbf{c}'} = (\tilde{X}_{\mathcal{H}} \times \mathbb{C}) / \sim, \quad \text{where}$$

$$((z_v)_{v \in \mathcal{H}}, z') \sim \left((u_v^{-1} z_v)_{v \in \mathcal{H}}, \left(\prod_{v \in \mathcal{H}} u_v^{c'_v} \right) z' \right) \quad \forall (u_v)_{v \in \mathcal{H}} \in (\mathcal{K}_{\mathcal{H}})_{\mathbb{C}} \subset (\mathbb{C}^*)^{\mathcal{H}}.$$

By Exercise 6.31 below, the map

$$\pi_{\mathcal{H}\mathbf{c}'} : L_{\mathcal{H}\mathbf{c}'} \longrightarrow X_{\mathcal{H}} \equiv \tilde{X}_{\mathcal{H}} / (\mathcal{K}_{\mathcal{H}})_{\mathbb{C}}, \quad \pi_{\mathcal{H}\mathbf{c}'}([(z_v)_{v \in \mathcal{H}}, z']) = [(z_v)_{v \in \mathcal{H}}], \quad (6.19)$$

is a holomorphic line bundle with respect to the complex structure $J_{\mathcal{H}}$ on $X_{\mathcal{H}}$ provided by Proposition 6.20. By definition, this holomorphic line bundle depends only on the restriction $\iota_{\mathcal{H}}^*(\mathbf{c}') \in T_1^* \mathcal{K}_{\mathcal{H}}$ of $\mathbf{c}' \in \mathbb{R}^{\mathcal{H}} \approx T_1^* \mathbb{T}^{\mathcal{H}}$ to $T_1 \mathcal{K}_{\mathcal{H}}$.

By Lemma 6.33 below, $L_{\mathcal{H}\mathbf{c}'}$ is a positive line bundle in the sense of [15, p148] if the subcollection \mathcal{H}' obtained from \mathcal{H} by replacing the second input in each element $v \equiv (v, c)$ of \mathcal{H} with c'_v lies in $[\mathcal{H}]^*$; see Exercise 6.4. In light of the Kodaira Embedding Theorem [15, p181], this implies that $(X_{\mathcal{H}}, J_{\mathcal{H}})$ can be holomorphically embedded into a complex projective space $\mathbb{C}P^N$ if $\langle \mathcal{H} \rangle$ is compact (or equivalently $X_{\mathcal{H}}$ is compact).

Exercise 6.31. Show that the complex charts on $(X_{\mathcal{H}}, J_{\mathcal{H}})$ provided by the proof of Proposition 6.20 lift to a holomorphic atlas of trivializations of the complex line bundle (6.19).

Exercise 6.32. Let $\mu_{\mathcal{H}}^{-1}(0) \subset \tilde{X}_{\mathcal{H}}$ be as before. For $\mathbf{c}' \equiv (c'_v)_{v \in \mathcal{H}} \in \mathbb{Z}^{\mathcal{H}}$, define

$$S_{\mathcal{H}\mathbf{c}'} = (\mu_{\mathcal{H}}^{-1}(0) \times S^1) / \sim, \quad \text{where}$$

$$((z_v)_{v \in \mathcal{H}}, z') \sim \left((u_v^{-1} z_v)_{v \in \mathcal{H}}, \left(\prod_{v \in \mathcal{H}} u_v^{c'_v} \right) z' \right) \quad \forall (u_v)_{v \in \mathcal{H}} \in \mathcal{K}_{\mathcal{H}} \subset \mathbb{T}^{\mathcal{H}}. \quad (6.20)$$

(a) Show that the map

$$\pi_{\mathcal{H}\mathbf{c}'} : S_{\mathcal{H}\mathbf{c}'} \longrightarrow X_{\mathcal{H}} \equiv \mu_{\mathcal{H}}^{-1}(0) / \mathcal{K}_{\mathcal{H}}, \quad \pi_{\mathcal{H}\mathbf{c}'}([(z_v)_{v \in \mathcal{H}}, z']) = [(z_v)_{v \in \mathcal{H}}], \quad (6.21)$$

is a principal S^1 -bundle.

(b) Let $L'_{\mathcal{H}\mathbf{c}'} \equiv S_{\mathcal{H}\mathbf{c}'} \times_{S^1} \mathbb{C} \longrightarrow X_{\mathcal{H}}$ be the associated complex line bundle as in Appendix B.3. Show that the map

$$L'_{\mathcal{H}\mathbf{c}'} \longrightarrow L_{\mathcal{H}\mathbf{c}'}, \quad [[(z_v)_{v \in \mathcal{H}}, z'], z''] \longrightarrow [(z_v)_{v \in \mathcal{H}}, z' z''],$$

is a well-defined identification of smooth complex line bundles over $X_{\mathcal{H}}$.

Lemma 6.33. Suppose $\mathbf{c} = (c_v)_{v \in \mathcal{H}} \in \pi\mathbb{Z}$. There exists a connection ∇ in the complex line bundle $L_{\mathcal{H}(-\mathbf{c}/\pi)} \longrightarrow X_{\mathcal{H}}$ with curvature $\kappa_{\nabla} = -2i\omega_{\mathcal{H}}$, where $\omega_{\mathcal{H}}$ is the symplectic form on $X_{\mathcal{H}}$ constructed in Section 6.1.

Proof. By Exercise B.9, it is sufficient to construct a connection 1-form λ on the principal S^1 -bundle (6.21) with $\mathbf{c}' \equiv \mathbf{c}/\pi$ so that its curvature κ_λ is $-2\omega_{\mathcal{H}}$. Let

$$p_{\mathcal{H}}: \mu_{\mathcal{H}}^{-1}(0) \longrightarrow X_{\mathcal{H}} \quad \text{and} \quad p_{\mathcal{H}\mathbf{c}'}: \mu_{\mathcal{H}}^{-1}(0) \times S^1 \longrightarrow S_{\mathcal{H}\mathbf{c}'}$$

be the quotient projections. With $z = x + iy$ as usual, let

$$\zeta_{\mathbb{C}} = 2\pi \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) = \frac{d}{dt} (e^{2\pi i t} z) \Big|_{t=0} \in \Gamma(\mathbb{C}; T\mathbb{C})$$

be the vector field generating the standard action of S^1 on \mathbb{C} and

$$\lambda_{\mathbb{C}} = -y dx + x dy \in \Gamma(\mathbb{C}; T^*\mathbb{C}).$$

With $z_v = x_v + iy_v$, define

$$\begin{aligned} \lambda_{\mathcal{H}} &= \sum_{v \in \mathcal{H}} (-y_v dx_v + x_v dy_v) \in \Gamma(\mathbb{C}^{\mathcal{H}}; T^*\mathbb{C}^{\mathcal{H}}) \quad \text{and} \\ \tilde{\lambda} &= (-\lambda_{\mathcal{H}}) \oplus \lambda_{\mathbb{C}} \in \Gamma(\mathbb{C}^{\mathcal{H}} \times \mathbb{C}; T^*(\mathbb{C}^{\mathcal{H}} \times \mathbb{C})). \end{aligned}$$

For $\mathbf{r} \equiv (r_v)_{v \in \mathcal{H}} \in \mathbb{R}^{\mathcal{H}}$, define

$$\mathbf{c}' \cdot \mathbf{r} = \sum_{v \in \mathcal{H}} c'_v r_v \quad \text{and} \quad \zeta_{\mathbf{r}} = (r_v \zeta_{\mathbb{C}})_{v \in \mathcal{H}} = \frac{d}{dt} (e^{2\pi i r_v t} z_v)_{v \in \mathcal{H}} \Big|_{t=0} \in \Gamma(\mathbb{C}^{\mathcal{H}}; T\mathbb{C}^{\mathcal{H}}).$$

In particular,

$$\begin{aligned} \{\tilde{\lambda}(0, \zeta_{\mathbb{C}})\}((z_v)_{v \in \mathcal{H}}, z') &= 2\pi |z'|^2, \\ \{\tilde{\lambda}(\zeta_{-\mathbf{r}}, (\mathbf{c}' \cdot \mathbf{r}) \zeta_{\mathbb{C}})\}((z_v)_{v \in \mathcal{H}}, z') &= 2 \sum_{v \in \mathcal{H}} (\pi |z_v|^2 + c_v |z'|^2) r_v. \end{aligned} \quad (6.22)$$

The right-hand side of the last equation above vanishes if $((z_v)_{v \in \mathcal{H}}, z') \in \mu_{\mathcal{H}}^{-1}(0) \times S^1$ and $\mathbf{r} \in T_{\mathbb{1}} \mathcal{K}_{\mathcal{H}}$.

Since the 1-form $\tilde{\lambda}$ is invariant under the standard action of $\mathbb{T}^{\mathcal{H}} \times S^1$ on $\mathbb{C}^{\mathcal{H}} \times \mathbb{C}$ by coordinate-wise multiplication, $\tilde{\lambda}$ is invariant under the action of $\mathcal{K}_{\mathcal{H}}$ on $\mu_{\mathcal{H}}^{-1}(0) \times S^1$ as in (6.20). Along with the second equation in (6.22) and the following sentence, this implies that $\tilde{\lambda}$ descends to a 1-form λ on $S_{\mathcal{H}\mathbf{c}'}$, i.e. $p_{\mathcal{H}\mathbf{c}'}^* \lambda = \tilde{\lambda}$. Since $\tilde{\lambda}$ is invariant under the S^1 -action on the second factor in $\mathbb{C}^{\mathcal{H}} \times \mathbb{C}$, λ is invariant under the S^1 -action on the principal S^1 -bundle (6.21) with $\mathbf{c}' = \mathbf{c}/\pi$. Along with the first equation in (6.22), this implies that λ is a connection 1-form on this principal S^1 -bundle. Let κ_λ be its curvature. Since

$$p_{\mathcal{H}\mathbf{c}'}^* \pi_{\mathcal{H}\mathbf{c}'}^* \kappa_\lambda = p_{\mathcal{H}\mathbf{c}'}^* d\lambda = -2\omega_{\mathbb{C}^{\mathcal{H}}} \in \Gamma(\mu_{\mathcal{H}}^{-1}(0) \times S^1; \Lambda^2 T^*(\mu_{\mathcal{H}}^{-1}(0) \times S^1)),$$

it follows that $p_{\mathcal{H}\mathbf{c}'}^* \kappa_\lambda = -2\omega_{\mathbb{C}^{\mathcal{H}}} |_{T\mu_{\mathcal{H}}^{-1}(0)}$ and so $\kappa_\lambda = -2\omega_{\mathcal{H}}$. This establishes the claim. \square

Exercise 6.34. Show that the map

$$\Psi_{\mathcal{H}}: \iota_{\mathcal{H}}^* (\mathbb{Z}^{\mathcal{H}}) \longrightarrow \text{Pic}(X_{\mathcal{H}}, J_{\mathcal{H}}), \quad \Psi_{\mathcal{H}}(\iota_{\mathcal{H}}^*(\mathbf{c}')) = [\pi_{\mathcal{H}\mathbf{c}'}: L_{\mathcal{H}\mathbf{c}'} \longrightarrow X_{\mathcal{H}}],$$

where $\text{Pic}(X_{\mathcal{H}}, J_{\mathcal{H}})$ is the Picard group of isomorphism classes of holomorphic line bundles on $(X_{\mathcal{H}}, J_{\mathcal{H}})$, is a well-defined group homomorphism.

Exercise 6.35. Let $v_0 \in \mathcal{H}$ and $e_{v_0} \in \mathbb{Z}^{\mathcal{H}}$ be the corresponding coordinate unit vector. Show that the map

$$s_{v_0}: X_{\mathcal{H}} \longrightarrow L_{\mathcal{H}(-e_{v_0})}, \quad s_{v_0}([(z_v)_{v \in \mathcal{H}}]) = [(z_v)_{v \in \mathcal{H}}, z_{v_0}],$$

is a well-defined holomorphic section of the holomorphic line bundle $L_{\mathcal{H}(-e_{v_0})}$, it is transverse to the zero set, and

$$s_{v_0}^{-1}(0) = \mu_{\mathcal{H}}^{-1}(0) \subset X_{\mathcal{H}}.$$

Exercise 6.36. Suppose $\mathcal{H}^\dagger \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant subcollection such that $\mathcal{H}^\dagger \supset \mathcal{H}$ and $\langle \mathcal{H}^\dagger \rangle = \langle \mathcal{H} \rangle$. Let

$$\iota_{\mathcal{H}^\dagger; \mathcal{H}}^*: T_{\mathbb{1}}^* \mathbb{T}^{\mathcal{H}^\dagger} = \mathbb{R}^{\mathcal{H}^\dagger} \longrightarrow T_{\mathbb{1}}^* \mathbb{T}^{\mathcal{H}} = \mathbb{R}^{\mathcal{H}}$$

be the restriction/projection homomorphism as before and $\mathbf{c}' \in \mathbb{R}^{\mathcal{H}^\dagger}$. Show that the identification of the complex manifolds $(X_{\mathcal{H}}, J_{\mathcal{H}})$ and $(X_{\mathcal{H}^\dagger}, J_{\mathcal{H}^\dagger})$ provided by Exercise 6.22 lifts to an identification of the holomorphic line bundles

$$L_{\mathcal{H} \iota_{\mathcal{H}^\dagger; \mathcal{H}}^*}(\mathbf{c}') \longrightarrow X_{\mathcal{H}} \quad \text{and} \quad L_{\mathcal{H}^\dagger \mathbf{c}'} \longrightarrow X_{\mathcal{H}^\dagger}.$$

Exercise 6.37. Suppose $\mathcal{H}_0 \subset \mathcal{H}$, $\alpha_0 \in \langle \mathcal{H}_0 \rangle^\partial$, and q_{α_0} and $\mathcal{H}/\mathcal{H}_0$ are as in Exercise 5.8. Let

$$\phi_{\mathcal{H}; \mathcal{H}_0}^*: T_{\mathbb{1}}^* \mathcal{K}_{\mathcal{H}/\mathcal{H}_0} \longrightarrow T_{\mathbb{1}}^* \mathcal{K}_{\mathcal{H}}$$

be the isomorphism induced by the isomorphism of Exercise 5.8(c).

(a) Show that $\phi_{\mathcal{H}; \mathcal{H}_0}^*$ restricts to an isomorphism from $\iota_{\mathcal{H}/\mathcal{H}_0}^*(\mathbb{Z}^{\mathcal{H}/\mathcal{H}_0})$ to $\iota_{\mathcal{H}}^*(\mathbb{Z}^{\mathcal{H}})$.

(b) Let $\alpha \in \iota_{\mathcal{H}/\mathcal{H}_0}^*(\mathbb{Z}^{\mathcal{H}/\mathcal{H}_0})$. Show that the identification of the complex manifolds $(X_{\mathcal{H}/\mathcal{H}_0}, J_{\mathcal{H}/\mathcal{H}_0})$ and $(\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial), J_{\mathcal{H}}|_{\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial)})$ provided by Exercises 6.11 and 6.25 lifts to an identification of the isomorphic classes of holomorphic line bundles

$$\Psi_{\mathcal{H}/\mathcal{H}_0}(\alpha) \in \text{Pic}(X_{\mathcal{H}/\mathcal{H}_0}, J_{\mathcal{H}/\mathcal{H}_0}) \quad \text{and} \\ \Psi_{\mathcal{H}}(\phi_{\mathcal{H}; \mathcal{H}_0}^*(\alpha)|_{\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial)}) \in \text{Pic}(\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial), J_{\mathcal{H}}|_{\mu^{-1}(\langle \mathcal{H}_0 \rangle^\partial)}),$$

with Ψ as in Exercise 6.34.

Exercise 6.38. Suppose $n \in \mathbb{Z}^+$, $k \in [n]$, $a_{k+1}, \dots, a_n \in \mathbb{Z}^{\geq 0}$, $\mathcal{H} \subset (T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}} \times \mathbb{R}$,

$$\pi_{\gamma_k}: \gamma_k \longrightarrow \mathbb{C}P^k, \quad \text{and} \quad \pi_E: E \longrightarrow \mathbb{C}P^k$$

are as in Exercise 6.28. Let $\mathbf{c} = (c_v)_{v \in \mathcal{H}}$ as above.

(a) Show that $\iota_{\mathcal{H}}^*(\mathbb{Z}^{\mathcal{H}}) = \iota_{\mathcal{H}}^*(\mathbb{Z}\mathbf{c}) \subset T_{\mathbb{1}}^* \mathcal{K}_{\mathcal{H}}$.

(b) Let $a \in \mathbb{Z}$. Show that the identification of $(X_{\mathcal{H}}, J_{\mathcal{H}})$ with the total space of the holomorphic vector bundle E provided by Exercise 6.28 lifts to an identification of the holomorphic line bundles

$$L_{\mathcal{H}(\mathbf{ac})} \longrightarrow X_{\mathcal{H}} \quad \text{and} \quad \pi_E^* \mathcal{O}_{\mathbb{C}P^k}(a) \equiv \pi_E^* \gamma_k^{*\otimes a} \longrightarrow \mathbb{C}P^k.$$

Exercise 6.39. Suppose $a \in \mathbb{Z}^{\geq 0}$, $c \in \mathbb{R}^+$, $\mathcal{H} \equiv \mathcal{H}_{2;1}(a; 0, c) \subset (T_{\mathbb{1}}\mathbb{T}^2)_{\mathbb{Z}} \times \mathbb{R}$,

$$\pi_{\gamma_1} : \gamma_1 \longrightarrow \mathbb{C}P^1, \quad \text{and} \quad \pi_{\mathbb{F}_a} : \mathbb{F}_a \equiv \mathbb{P}(\gamma_1^{\otimes a} \oplus \tau_{\mathbb{C}P^1}^1) \longrightarrow \mathbb{C}P^1$$

are as in Exercise 6.30. Let $\mathbf{c}_1 = (0, 0, -1, 0)$, $\mathbf{c}_2 = (0, 0, 0, -1)$, and

$$\pi_{\gamma_{\mathbb{F}_a}} : \gamma_{\mathbb{F}_a} \equiv \{(\ell, v) \in \mathbb{F}_a \times (\gamma_1^{\otimes a} \oplus \tau_{\mathbb{C}P^1}^1) : v \in \ell\} \longrightarrow \mathbb{F}_a, \quad \pi_{\gamma_{\mathbb{F}_a}}(\ell, v) = \ell,$$

be the tautological holomorphic line bundle.

(a) Show that $\iota_{\mathcal{H}}^*(\mathbb{Z}^{\mathcal{H}}) = \iota_{\mathcal{H}}^*(\mathbb{Z}\mathbf{c}_1) \oplus \iota_{\mathcal{H}}^*(\mathbb{Z}\mathbf{c}_2) \subset T_{\mathbb{1}}^*\mathcal{K}_{\mathcal{H}}$.

(b) Let $b \in \mathbb{Z}$. Show that the identification of $(X_{\mathcal{H}}, J_{\mathcal{H}})$ with \mathbb{F}_a provided by Exercise 6.30 lifts to identifications of the holomorphic line bundles

$$L_{\mathcal{H}(b\mathbf{c}_1)}, L_{\mathcal{H}(b\mathbf{c}_2)} \longrightarrow X_{\mathcal{H}} \quad \text{and} \quad \pi_{\mathbb{F}_a}^* \mathcal{O}_{\mathbb{C}P^1}(b), \mathcal{O}_{\mathbb{F}_a}(b) \longrightarrow \mathbb{F}_a,$$

where $\mathcal{O}_{\mathbb{C}P^1}(b) = \gamma_1^{*\otimes b}$ and $\mathcal{O}_{\mathbb{F}_a}(b) = \gamma_{\mathbb{F}_a}^{*\otimes b}$.

A Morse-Bott Theory

A.1 Definitions and notation

Let X be a smooth manifold and $H : X \longrightarrow \mathbb{R}$ be a smooth function. If $x \in \text{Crit}(H)$, then the gradient $\nabla^g H$ of H with respect to any Riemannian metric g vanishes at x and the Hessian

$$\nabla^2 H|_x \equiv \nabla(\nabla^g H)|_x : T_x X \longrightarrow T_x X, \quad \nabla^2 H|_x(w) = \nabla_w(\nabla^g H), \quad (\text{A.1})$$

of H at x does not depend on the choice of a connection ∇ in TX (it does depend on the metric g though). If in addition ξ, ξ' are vector fields on X and ∇ is the Levi-Civita connection of g , then

$$\begin{aligned} g(\nabla^2 H|_x(\xi(x)), \xi'(x)) &\equiv g(\nabla_{\xi(x)} \nabla^g H, \xi'(x)) = \{\xi(x)\}(g(\nabla^g H, \xi')) - g(\nabla^g H|_x, \nabla_{\xi(x)} \xi') \\ &= \{\xi(x)\}(dH(\xi')) - d_x H(\nabla_{\xi(x)} \xi') = \{\xi(x)\}(\xi'(H)) - \{\nabla_{\xi(x)} \xi'\}(H) \\ &= \{\xi'(x)\}(\xi(H)) + \{[\xi, \xi'](x)\}(H) - \{\nabla_{\xi(x)} \xi'\}(H) \\ &= \{\xi'(x)\}(\xi(H)) - \{\nabla_{\xi'(x)} \xi\}(H) = g(\nabla^2 H|_x(\xi'(x)), \xi(x)). \end{aligned}$$

Thus, the linear automorphism (A.1) is symmetric with respect to the metric g and therefore diagonalizable. We denote by

$$\mathbb{E}_x^0(H), \mathbb{E}_x^-(H), \mathbb{E}_x^+(H) \subset T_x X \quad \text{and} \quad n_x^0(H), n_x^-(H), n_x^+(H) \in \mathbb{Z}^{\geq 0}$$

the nullspace of $\nabla^2 H|_x$, the negative eigenspace of $\nabla^2 H|_x$, the positive eigenspace of $\nabla^2 H|_x$, and their respective dimensions. In particular,

$$T_x X = \mathbb{E}_x^0(H) \oplus \mathbb{E}_x^-(H) \oplus \mathbb{E}_x^+(H) \quad \text{and} \quad \dim X = n_x^0(H) + n_x^-(H) + n_x^+(H).$$

Exercise A.1. Let X be a smooth manifold, $H : X \longrightarrow \mathbb{R}$ be a smooth function, and $x \in \text{Crit}(H)$. Show that

(a) the negative and positive eigenspaces $\mathbb{E}_x^-(H), \mathbb{E}_x^+(H) \subset T_x X$ of $\nabla^2 H|_x$ depend on the choice of a Riemannian metric g on X , but

(b) their dimensions $n_x^-(H)$, $n_x^+(H)$ and the nullspace $E_x^0(H) \subset T_x X$ of $\nabla^2 H|_x$ do not.

Definition A.2. Let X be a smooth manifold. A smooth function $H : X \rightarrow \mathbb{R}$ is **Morse-Bott** if $\text{Crit}(H) \subset X$ is a closed submanifold of X with $T_x Y = E_x^0(H)$ for all $Y \in \pi_0(\text{Crit}(H))$ and $x \in Y$.

If $H : X \rightarrow \mathbb{R}$ is a Morse-Bott function and $Y \in \pi_0(\text{Crit}(H))$, $H|_Y$ is constant. Furthermore, the numbers $n_x^-(H)$, $n_x^+(H)$ do not depend on $x \in Y$; we denote them by $n_Y^-(H)$, $n_Y^+(H)$, respectively. The subspaces $E_x^-(H)$, $E_x^+(H)$ of $T_x X$ form subbundles $E_Y^-(H)$, $E_Y^+(H)$ of $TX|_Y$ so that

$$TX|_Y = TY \oplus E_Y^-(H) \oplus E_Y^+(H).$$

Exercise A.3. Suppose X is a smooth manifold, $H : X \rightarrow \mathbb{R}$ is a Morse-Bott function, and $Y \in \pi_0(\text{Crit}(H))$. Show that H reaches a local minimum (resp. maximum) on Y if and only if $n_Y^-(H) = 0$ (resp. $n_Y^+(H) = 0$).

Exercise A.4. Suppose $H : X \rightarrow \mathbb{R}$ and $Y \subset \text{Crit}(H)$ are in Exercise A.3 and $Z \subset X$ is a smooth submanifold transverse to the closed submanifold $Y \subset X$. Show that $Y \cap Z$ is a closed submanifold of Z , is an open subset of $\text{Crit}(H|_Z)$, and

$$T_x(Y \cap Z) = T_x Y \cap T_x Z = E_x^0(H|_Z), \quad n_x^\pm(H|_Z) = n_x^\pm(H) \quad \forall x \in Y \cap Z.$$

Proposition A.5. Let ψ be an almost periodic \mathbb{R} -action on a symplectic manifold (X, ω) . If $H : X \rightarrow \mathbb{R}$ is a Hamiltonian for ψ , then $\text{Crit}(H) \subset X$ is a closed symplectic submanifold and H is a Morse-Bott function with $n_x^\pm(H) \in 2\mathbb{Z}^{\geq 0}$ for every $x \in \text{Crit}(H)$. If in addition X is compact and connected, then

(2) H has a unique local minimum and a unique local maximum;

(3) $H^{-1}(c) \subset X$ is connected for every $c \in \mathbb{R}$.

Proof. Let $\rho : \mathbb{R} \rightarrow \mathbb{T}$ and ψ' be as in (1.5) and $\zeta \in \Gamma(X; TX)$ be the generating vector field for the ψ -action as in (3.4). We can assume that the image of ρ is dense in \mathbb{T} and so $X^\psi = X^{\psi'}$. By Exercise 3.14, there exist a ψ -invariant ω -compatible almost complex structure J on X . Let $g(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$ be the Riemannian metric on X determined by ω and J . By (2.1) and (2.5), the gradient of H with respect to g is then given by

$$\nabla^g H = -J\zeta \in \Gamma(X; TX). \quad (\text{A.2})$$

By (2.5) and Proposition 3.2(1),

$$\text{Crit}(H) \equiv \{x \in X : d_x H = 0\} = \{x \in X : \zeta(x) = 0\} = X^\psi = X^{\psi'}. \quad (\text{A.3})$$

Along with (A.1), (A.2), and (3.7), this implies that the Hessian $\nabla^2 H$ of H satisfies

$$\nabla^2 H|_x(w) = -J\nabla_w \zeta, \quad \nabla^2 H|_x(Jw) = J\nabla^2 H|_x(w) \quad \forall w \in T_x X, x \in \text{Crit}(H). \quad (\text{A.4})$$

By (A.3), Propositions 3.16(1) and 3.2(2), and the first equation in (A.4), $\text{Crit}(H) \subset X$ is thus a closed symplectic submanifold of (X, ω) with

$$T_x Y = (T_x X)^{d\psi'} = (T_x X)^{d\psi} = \{w \in T_x X : \nabla_w \zeta = 0\} = E_x^0(H) \quad \forall Y \in \pi_0(\text{Crit}(H)), x \in Y.$$

By the second equation in (A.4), the subspaces $E_x^\pm(H) \subset T_x X$ are preserved by J for every $x \in \text{Crit}(H)$ and thus $n_x^\pm(H) \in 2\mathbb{Z}^{\geq 0}$ for every $x \in \text{Crit}(H)$. The remaining claims of the proposition now follow immediately from Proposition A.8. \square

A.2 Gradient flows

The second proof of Proposition A.8 is based on standard properties of gradient flows of Morse-Bott functions. As these properties are also used in the proof of Theorem 2(2) in Section 4.1, we collect them in Proposition A.7 below and justify at the end of this section.

Exercise A.6. Suppose (X, g) is a compact Riemannian manifold and $H : X \rightarrow \mathbb{R}$ is a smooth function. Since X is compact, the negative gradient flow of H ,

$$\psi_{H;t} : X \rightarrow X, \quad \psi_{H;0} = \text{id}_X, \quad \frac{d}{dt}\psi_{H;t} = -\nabla^g H|_{\psi_{H;t}},$$

is defined for all $t \in \mathbb{R}$. Show that the limits

$$x_H^\pm \equiv \lim_{t \rightarrow \pm\infty} \psi_{H;t}(x)$$

exist for every $x \in X$.

Let (X, g) be a compact Riemannian manifold and $H : X \rightarrow \mathbb{R}$ be a Morse-Bott function. For $Y \in \pi_0(\text{Crit}(H))$, we denote by

$$\pi_{H;Y}^\pm : E_Y^\pm(H) \rightarrow Y$$

the bundle projections and by $S(E_Y^\pm(H)) \subset E_Y^\pm(H)$ the sphere bundle of $E_Y^\pm(H)$. Let

$$X_Y^\pm(H) \equiv \{x \in X : x_H^\pm \in Y\} \supset Y \tag{A.5}$$

be the H -stable and unstable manifolds of Y . For $A \subset X_Y^\pm(H)$, let

$$A_H^\pm = \{x_H^\pm : x \in A\} \subset Y.$$

Proposition A.7. *Suppose (X, g) is a compact Riemannian manifold, $H : X \rightarrow \mathbb{R}$ is a Morse-Bott function, and $Y \in \pi_0(\text{Crit}(H))$.*

(1) *The subspaces $X_Y^\pm(H) \subset X$ are smooth submanifolds with*

$$T(X_Y^\pm(H))|_Y = TY \oplus E_Y^\pm(H) \subset TX|_Y. \tag{A.6}$$

(2) *There exist diffeomorphisms $\Phi_{H;Y}^\pm : E_Y^\pm(H) \rightarrow X_Y^\pm(H)$ such that*

$$\begin{aligned} (\Phi_{H;Y}^\pm(w))_H^\pm &= \pi_{H;Y}^\pm(w) \quad \forall w \in E_Y^\pm(H), \quad \Phi_{H;Y}^\pm(y) = y \quad \forall y \in Y, \\ d_y \Phi_{H;Y}^\pm(w) &= w \quad \forall y \in Y, w \in E_Y^\pm(H) \subset T_y(E_Y^\pm(H)). \end{aligned} \tag{A.7}$$

(3) *For every $c \in \mathbb{R}$, the submanifolds $X_Y^\pm(H)$ and $H^{-1}(c) - \text{Crit}(H)$ of X are transverse.*

(4) *For every $\epsilon \in \mathbb{R}^+$ such that $H(Y)$ is the only critical value of H in $[H(Y) - \epsilon, H(Y) + \epsilon]$, there exist diffeomorphisms*

$$\Phi_{H;Y;\epsilon}^\pm : S(E_Y^\pm(H)) \rightarrow X_Y^\pm(H) \cap H^{-1}(H(Y) \pm \epsilon) \subset X$$

satisfying the first property in (A.7) with $(\Phi_{H;Y}^\pm, E_Y^\pm(H))$ replaced by $(\Phi_{H;Y;\epsilon}^\pm, S(E_Y^\pm(H)))$.

(5) The intersection of the closure $\overline{X_Y^\pm(H)} \subset X$ of $X_Y^\pm(H)$ with the level set $H^{-1}(H(Y))$ is Y .

(6) If $A \subset X_Y^\pm(H)$ and $\overline{A} \subset X$ is the closure of A , $\overline{A} \cap H^{-1}(H(Y)) \subset \overline{A_H^\pm}$. If in addition A is preserved by the gradient flow of H , i.e. $\psi_{H;t}(A) = A$ for all $t \in \mathbb{R}$, then $\overline{A} \cap H^{-1}(H(Y)) = \overline{A_H^\pm}$.

Proof. By the Tubular Neighborhood Theorem, there are neighborhoods $\mathcal{U} \subset E_Y^-(H) \oplus E_Y^+(H)$ and $U \subset X$ of Y and a diffeomorphism $\Phi: \mathcal{U} \rightarrow U$ such that

$$\Phi(y) = y \quad \forall y \in Y, \quad d_y \Phi(w) = w \quad \forall y \in Y, w \in E_Y^-(H) \oplus E_Y^+(H) \subset T_y(E_Y^-(H) \oplus E_Y^+(H)).$$

By the statement and proof of [3, Theorem A.9], there are then neighborhoods $\mathcal{U}' \subset \mathcal{U}$ and $U' \subset U$ of Y and smooth embeddings

$$\Phi_{H;Y}^\pm: \mathcal{U}_{H;Y}^\pm \equiv \mathcal{U}' \cap E_Y^\pm(H) \rightarrow U'$$

so that $\Phi_{H;Y}^\pm(\mathcal{U}_{H;Y}^\pm) = U' \cap X_Y^\pm(H)$, the first identity in (A.7) holds whenever $w \in \mathcal{U}_{H;Y}^\pm$, and the other two identities in (A.7) hold as stated. For all $\delta \in \mathbb{R}^+$ sufficiently small, \mathcal{U}' contains the closed disk bundle of $E_Y^-(H) \oplus E_Y^+(H)$ of radius δ and

$$\pm \frac{d}{dt} H(\Phi_{H;Y}^\pm(tw)) \Big|_{t=1} > 0 \quad \forall w \in E_Y^\pm(H), |w| = \delta. \quad (\text{A.8})$$

The smooth maps

$$\Psi_{H;Y}^\pm: E_Y^\pm(H) - Y \rightarrow X, \quad \Psi_{H;Y}^\pm(w) = \psi_{H;\mp \ln(|w|/\delta)}(\Phi_{H;Y}^\pm(\delta w/|w|)),$$

are then smooth embeddings onto $X_Y^\pm(H) - Y$ that agree with $\Phi_{H;Y}^\pm$ on the sphere bundle $S_\delta(E_Y^\pm(H))$ in $E_Y^\pm(H)$ of radius δ , satisfy the first identity in (A.7) whenever $w \neq 0$, and satisfy (A.8) with $\Phi_{H;Y}^\pm$ replaced by $\Psi_{H;Y}^\pm$. We can thus paste $\Phi_{H;Y}^\pm$ and $\Psi_{H;Y}^\pm$ together on a neighborhood of $S_\delta(E_Y^\pm(H))$ in $E_Y^\pm(H)$ to obtain smooth embeddings $\Phi_{H;Y}^\pm$ of $E_Y^\pm(H)$ into X with image $X_Y^\pm(H)$ which satisfy (A.7). This establishes (1) and (2).

Let $c \in \mathbb{R}$ and $x \in X_Y^\pm(H) \cap (H^{-1}(c) - \text{Crit}(H))$. Thus, $d_x H \neq 0$, $H^{-1}(c) - \text{Crit}(H) \subset X$ is a smooth submanifold with

$$T_x(H^{-1}(c) - \text{Crit}(H)) = \ker d_x H,$$

and $\psi_{H;t}(x)$ is a curve in $X_Y^\pm(H)$ with

$$\frac{d}{dt} H(\psi_{H;t}(x)) \Big|_{t=0} = d_x H(-\nabla^g H) = -g(\nabla^g H, \nabla^g H) \neq 0.$$

This gives (3).

Let $\epsilon, \delta \in \mathbb{R}^+$ be as in (4) and above, respectively, with

$$\Psi_{H;Y}^\pm(S_\delta(E_Y^\pm(H))) \subset H^{-1}((H(Y) - \epsilon, H(Y) + \epsilon)).$$

Since the norm of $\nabla^g H$ is bounded below on $H^{-1}((H(Y) - \epsilon, H(Y) - \epsilon'))$ and $H^{-1}((H(Y) + \epsilon', H(Y) + \epsilon))$ for every $\epsilon' \in (0, \epsilon)$, (A.8) and the smoothness of the negative gradient flow $\psi_{H;t}$ imply that there is a smooth function

$$\rho: S_\delta(E_Y^\pm(H)) \rightarrow \mathbb{R} \quad \text{s.t.} \quad H(\psi_{H;\rho(w)}(\Phi_{H;Y}^\pm(w))) = H(Y) \pm \epsilon.$$

Along with (A.8) again, the map

$$\Phi_{H;Y;\epsilon}^{\pm} : S(E_Y^{\pm}(H)) \longrightarrow X_Y^{\pm}(H) \cap H^{-1}(H(Y) \pm \epsilon), \quad \Phi_{H;Y;\epsilon}^{\pm} = \psi_{H;\rho(w)}(\Phi_{H;Y}^{\pm}(w)),$$

is then a diffeomorphism satisfying the first property in (A.7) with $(\Phi_{H;Y}^{\pm}, E_Y^{\pm}(H))$ replaced by $(\Phi_{H;Y;\epsilon}^{\pm}, S(E_Y^{\pm}(H)))$.

Suppose $x' \in H^{-1}(H(Y)) - Y$. Choose disjoint open neighborhoods $U, U' \subset X$ of Y and x' , respectively. By (A.8), there exists $\epsilon \in \mathbb{R}^+$ so that

$$|H(x) - H(Y)| \geq \epsilon \quad \forall x \in X_Y^{\pm}(H) - U.$$

By shrinking U' if necessary, we can assume that

$$|H(x) - H(Y)| < \epsilon \quad \forall x \in U'.$$

It then follows that U' is disjoint from $X_Y^{\pm}(H)$ and thus from $\overline{X_Y^{\pm}(H)}$. This establishes (5).

Let $A \subset X_Y^{\pm}(H)$. By ((5)), $\overline{A} \cap H^{-1}(H(Y)) = \overline{A} \cap Y$. Suppose $y \in \overline{A} \cap Y$ and $U \subset X$ is a neighborhood of y . By (2), there exists a neighborhood U' of y in U so that $x_H^{\pm} \in U'$ for every $x \in U'$. Since $y \in \overline{A}$, U' contains some $x \in A$ and thus $x_H^{\pm} \in A_H^{\pm}$. We conclude that $y \in \overline{A_H^{\pm}}$.

Suppose A is preserved by the gradient flow of H and $y \in \overline{A_H^{\pm}}$. Let $U \subset X$ be a neighborhood of y and $x \in A$ a point such that $x_H^{\pm} \in U$. Thus, $\psi_{H;\pm}(x) \in A \cap U$ for all $t \in \mathbb{R}$ sufficiently large. Therefore, $y \in \overline{A}$. \square

A.3 Fiber connectedness

The next proposition is the main point-set topology input in the proof of (A_k^*) on page 5 in [1] and [23].

Proposition A.8 ([1, Lemma 2.1], [23, Lemma 5.5.5]). *Suppose M is a compact connected smooth manifold and $H : X \rightarrow \mathbb{R}$ is a Morse-Bott function. If $n_x^{\pm}(H) \neq 1$ for every $x \in \text{Crit}(H)$, then*

- (1) H has a unique local minimum and a unique local maximum;
- (2) $H^{-1}(c) \subset X$ is connected for every $c \in \mathbb{R}$.

We give two proofs of this proposition, which are essentially two different formulations of the same reasoning. The first one is in the style of classical Morse theory, as in [27]. It is based on describing the changes in the homotopy type of $H^{-1}((-\infty, c])$ as c passes through critical values as adding “handles” of various kinds; see (A.9) below. The second proof is in the style of the modern take on Morse theory originating in [35]. It is based on partitioning X into stable or unstable manifolds of the negative gradient flow; see (A.10) below. In both cases, we first show that there are unique connected critical submanifolds $Y_-, Y_+ \subset X$ with $n_{Y_-}^-(Y_-) = 0$ and $n_{Y_+}^+(Y_+) = 0$. The function H reaches its global minimum along Y_- and maximum along Y_+ ; there are no other local minima or maxima. We then show that $H^{-1}(c)$ is connected whenever $c \in (\min H, \max H)$ is a regular value of H . The claim for arbitrary $c \in \mathbb{R}$ then follows from Lemma A.9 below.

Proof 1 of Proposition A.8 ([1]). For $Y \in \pi_0(\text{Crit}(H))$, we denote by $D(\mathbb{E}_Y^-(H)) \subset \mathbb{E}_Y^-(H)$ the disk bundle of $\mathbb{E}_Y^-(H)$. For $c \in \mathbb{R}$, let

$$X_c(H) = H^{-1}((-\infty, c]) \subset X.$$

If c is a regular value of H , i.e. $c \notin H(\text{Crit}(H))$, then $X_c(H)$ is a smooth manifold with boundary $\partial X_c(H) = H^{-1}(c)$. If $c_-, c_+ \in \mathbb{R}$ are regular values of H with $c_- < c_+$, then

$$X_{c_+}(H) \sim X_{c_-}(H) \cup \bigcup_{\substack{Y \in \pi_0(\text{Crit}(H)) \\ H(Y) \in (c_-, c_+)}} D(\mathbb{E}_Y^-(H)), \quad (\text{A.9})$$

with \sim denoting homotopy equivalence; see [7, Section 1]. The boundaries of the disk bundles on the right-hand side above are attached to $\partial X_{c_-}(H)$; the right-hand side is then a deformation retract of the left-hand side.

If $Y \in \pi_0(\text{Crit}(H))$ and $n_Y^-(H) = 0$ (i.e. H has a local minimum along Y), then adding $D(\mathbb{E}_Y^-(H))$ as in (A.9) adds a topological component to $X_{c_-}(H)$. If $Y \in \pi_0(\text{Crit}(H))$ and $n_Y^-(H) \geq 2$, then attaching $D(\mathbb{E}_Y^-(H))$ as in (A.9) has no impact on the topological components of $X_{c_-}(H)$ vs $X_{c_+}(H)$. Since $n_Y^-(H) \neq 1$ for any $Y \in \pi_0(\text{Crit}(H))$ and X is connected, it follows that there is a unique $Y_- \in \pi_0(\text{Crit}(H))$ with $n_{Y_-}^-(H) = 0$; thus, $H(Y_-) = \min H$. Since $n_{Y_-}^-(-H) = n_{Y_-}^+(H)$, the same reasoning shows that there is a unique $Y_+ \in \pi_0(\text{Crit}(H))$ with $n_{Y_+}^+(H) = 0$; thus, $H(Y_+) = \max H$. Furthermore, $X_c(H)$ is connected for every $c \in \mathbb{R}$.

We can assume that H is not a constant function. Let $c \in (\min H, \max H)$ be a regular value. By (A.9), $X_c(H)$ is a homotopy equivalent to a CW complex with cells of dimension at most the maximum of the numbers

$$\dim D(\mathbb{E}_Y^-(H)) = n_Y^0(H) + n_Y^-(H) = \dim X - n_Y^+(H) < \dim X - 1 = \dim \partial X_c(H)$$

taken over $Y \in \pi_0(\text{Crit}(H))$ with $H(Y) < c$. The inequality above holds because $n_Y^+(H) \neq 0$ for $Y \neq Y_+$ and $n_Y^+(H) \neq 1$ for any $Y \in \pi_0(\text{Crit}(H))$. Thus, $H_k(X_c(H); \mathbb{Z}_2) = 0$ for $k \geq \dim \partial X_c(H)$ and the boundary homomorphism

$$\partial: H_{\dim X}(X_c(H); \mathbb{Z}_2) \longrightarrow H_{\dim \partial X_c(H)}(\partial X_c(H); \mathbb{Z}_2)$$

in the homology exact sequence for $(X_c(H), \partial X_c(H))$ is surjective. Since $X_c(H)$ is connected, the domain of this homomorphism is isomorphic to \mathbb{Z}_2 . It follows that $\partial X_c(H) = H^{-1}(c)$ is connected.

Thus, $H^{-1}(c) \subset X$ is connected for every $c \in \mathbb{R} - H(\text{Crit}(H))$. Since $H(\text{Crit}(H)) \subset \mathbb{R}$ is a finite subset, Lemma A.9 below implies that $H^{-1}(c) \subset X$ is connected for every $c \in \mathbb{R}$. \square

Proof 2 of Proposition A.8 (modification of [23, pp233,4]). By definition,

$$X = \bigsqcup_{Y \in \pi_0(\text{Crit}(H))} X_Y^+(H) = \bigsqcup_{Y \in \pi_0(\text{Crit}(H))} X_Y^-(H). \quad (\text{A.10})$$

Since $n_Y^-(H) \neq 1$ for any $Y \in \pi_0(\text{Crit}(H))$,

$$\bigsqcup_{\substack{Y \in \pi_0(\text{Crit}(H)) \\ n_Y^-(H) = 0}} X_Y^+(H) = X - \bigsqcup_{\substack{Y \in \pi_0(\text{Crit}(H)) \\ n_Y^-(H) \geq 2}} X_Y^+(H).$$

Since X is connected and each submanifold $X_Y^+(H) \subset X$ on the right-hand side above is of codimension $n_Y^-(H) \geq 2$ by Proposition A.7(1), the union on the left-hand side is connected. Since each submanifold $X_Y^+(H) \subset X$ on the left-hand side is open, it follows that there is a unique $Y_- \in \pi_0(\text{Crit}(H))$ with $n_{Y_-}^-(H) = 0$; thus, $H(Y_-) = \min H$. By the same reasoning with the second partition in (A.10), there is a unique $Y_+ \in \pi_0(\text{Crit}(H))$ with $n_{Y_+}^+(H) = 0$; thus, $H(Y_+) = \max H$.

We can assume that H is not a constant function. Let ϵ be as in Proposition A.7(4) with $Y = Y_-$ and $c \in (\min H, \max H)$ be a regular value. By (A.10),

$$\begin{aligned} H^{-1}(c) - \bigcup_{\substack{Y \in \pi_0(\text{Crit}(H)) \\ n_Y^-(H) \geq 2}} X_Y^+(H) &= H^{-1}(c) \cap X_{Y_-}^+(H) \\ &\approx H^{-1}(\min H + \epsilon) - \bigcup_{\substack{Y \in \pi_0(\text{Crit}(H)) \\ H(Y) \in (\min H, c)}} H^{-1}(\min H + \epsilon) \cap X_Y^-(H); \end{aligned} \quad (\text{A.11})$$

the diffeomorphism \approx above is obtained via the gradient flow. Since $H^{-1}(\min H + \epsilon)$ is transverse to each $X_Y^-(H)$, the codimension of the last intersection above in $H^{-1}(\min H + \epsilon)$ is the codimension of $X_Y^-(H)$ in X , i.e. $n_Y^-(H) \geq 2$. The last inequality holds because $n_Y^+(H) \neq 0$ for $Y \neq Y_+$ and $n_Y^+(H) \neq 1$ for any $Y \in \pi_0(\text{Crit}(H))$. Since $H^{-1}(\min H + \epsilon)$ is diffeomorphic to the connected manifold $S(E_{Y_-}^+(H))$ by Proposition A.7(2), the right-hand side in (A.11) is connected as well. Since the codimension of $H^{-1}(c) \cap X_{Y_-}^+(H)$ in $H^{-1}(c)$ is $n_{Y_-}^-(H)$ and $n_{Y_-}^-(H) > 0$ whenever $Y \neq Y_-$, it then follows from (A.11) that $H^{-1}(c)$ is also connected.

By the above $H^{-1}(c) \subset X$ is connected for every $c \in \mathbb{R} - H(\text{Crit}(H))$. Since $H(\text{Crit}(H)) \subset \mathbb{R}$ is a finite subset, Lemma A.9 below implies that $H^{-1}(c) \subset X$ is connected for every $c \in \mathbb{R}$. \square

Lemma A.9. *Let X be a compact connected manifold (or more generally a topological space which is sequentially compact, connected, locally connected, and normal). Suppose $f : X \rightarrow \mathbb{R}$ is a continuous function and $P^* \subset \mathbb{R}$ is a dense subset. If $f^{-1}(c) \subset X$ is connected for every $c \in P^*$, then $f^{-1}(c) \subset X$ is connected for every $c \in \mathbb{R}$.*

Proof. Suppose $c \in \mathbb{R} - P^*$ and $f^{-1}(c) = A \cup B$ for some disjoint nonempty subsets A, B that are closed in $f^{-1}(c)$ and thus in X . Let $U_A, U_B \subset X$ be disjoint open subsets such that $A \subset U_A$ and $B \subset U_B$. Let $W \subset \mathbb{R}$ be a neighborhood of c such that $f^{-1}(W) \subset U_A \cup U_B$ (its existence follows from the first countability of \mathbb{R} , sequential compactness of X , and the continuity of f). For each $x \in A \cup B$, choose a connected neighborhood V_x of x in $f^{-1}(W)$. The subsets

$$V_A \equiv \bigcup_{x \in A} V_x \quad \text{and} \quad V_B \equiv \bigcup_{x \in B} V_x$$

of X are then open disjoint neighborhoods of A and B , respectively, in X . If $f^{-1}(c_A) \cap V_A \neq \emptyset$ for some $c_A < c$, then

$$f^{-1}(c') \cap V_A \neq \emptyset \quad \forall c' \in (c_A, c).$$

If in addition $f(x) < c$ for some $x \in V_B$, then there exists $c^* \in (c_A, c)$ such that

$$c^* \in P^*, \quad f^{-1}(c^*) \cap U_A \neq \emptyset, \quad \text{and} \quad f^{-1}(c^*) \cap U_B \neq \emptyset.$$

Since $f^{-1}(c^*) \subset U_A \cap U_B$, this would contradict the assumption that $f^{-1}(c^*) \subset M$ is connected for every $c^* \in P^*$. We can thus assume that $f(x) \leq c$ for all $x \in V_A$ and $f(x) \geq c$ for all $x \in V_B$. Then

$$\tilde{U}_A \equiv f^{-1}((-\infty, c)) \cup V_A \quad \text{and} \quad \tilde{U}_B \equiv f^{-1}((c, \infty)) \cup V_B$$

are disjoint nonempty open subsets of X that cover X . However, this contradicts the assumption that X is connected. \square

Exercise A.10. Let X be a compact connected manifold (or more generally a topological space which is sequentially compact, connected, and normal). Suppose P is a first countable topological space, $f: X \rightarrow P$ is a continuous open surjective map, and $P^* \subset P$ is a dense subset. Show that if $f^{-1}(c) \subset X$ is connected for every $c \in P^*$, then $f^{-1}(c) \subset X$ is connected for every $c \in P$.

B Bundle Connections

B.1 Connections and splittings

Suppose X is a smooth manifold and $\pi_E: E \rightarrow X$ is a (smooth) real vector bundle. We identify X with the zero section of E . Denote by

$$\mathbf{a}: E \oplus E \rightarrow E \quad \text{and} \quad \pi_{E \oplus E}: E \oplus E \rightarrow X$$

the associated addition map and the induced projection map, respectively. For $f \in C^\infty(X; \mathbb{R})$, define

$$m_f: E \rightarrow E \quad \text{by} \quad m_f(v) = f(\pi_E(v)) \cdot v \quad \forall v \in E. \quad (\text{B.1})$$

In particular,

$$\pi_{E \oplus E} = \pi_E \circ \mathbf{a}, \quad \pi_E = \pi_E \circ m_f \quad \forall f \in C^\infty(X; \mathbb{R}).$$

The total spaces of the vector bundles

$$\pi_{E \oplus E}: E \oplus E \rightarrow X \quad \text{and} \quad \pi_E^* E \rightarrow E$$

consist of the pairs (v, w) in $E \times E$ such that $\pi_E(v) = \pi_E(w)$.

Define a smooth bundle homomorphism

$$\iota_E: \pi_E^* E \rightarrow TE, \quad \iota_E(v, w) = \left. \frac{d}{dt}(v + tw) \right|_{t=0}. \quad (\text{B.2})$$

Since the restriction of ι_E to the fiber over $v \in E$ is the composition of the isomorphism

$$E_{\pi_E(v)} \rightarrow T_v E_{\pi_E(v)}, \quad w \rightarrow \left. \frac{d}{dt}(v + tw) \right|_{t=0},$$

with the differential of the embedding of the fiber $E_{\pi_E(v)}$ into E , ι_E is an injective bundle homomorphism. Furthermore,

$$d\pi_E \circ \iota_E = 0: \pi_E^* E \rightarrow \pi_E^* TX, \quad m_f^* \iota_E \circ \pi_E^* m_f = dm_f \circ \iota_E: \pi_E^* E \rightarrow m_f^* TE, \quad (\text{B.3})$$

$$\begin{aligned} \mathbf{a}^* \iota_E \circ \pi_{E \oplus E}^* \mathbf{a} &= d\mathbf{a} \circ \iota_{E \oplus E}: \pi_{E \oplus E}^*(E \oplus E) \rightarrow \mathbf{a}^* TE, \\ TE|_X &= TX \oplus \iota_E(\pi_E^* E|_X) = TX \oplus \iota_E(E). \end{aligned} \quad (\text{B.4})$$

Let

$$\zeta_E \in \Gamma(E; TE), \quad \zeta_E(v) = \iota_E(v, v) \in T_v E, \quad (\text{B.5})$$

be the canonical vertical vector field on E .

Exercise B.1. Suppose $p \in \mathbb{Z}^+$, $\pi_E: E \rightarrow X$ is a real vector bundle, $\mathcal{U} \subset E$ is a tubular neighborhood of X in E , i.e. $tv \in \mathcal{U}$ whenever $v \in \mathcal{U}$ and $t \in [0, 1]$, and ϖ is a closed p -form on \mathcal{U} . Show that the family $(m_t^* \varpi)_{t \in [0, 1]}$ of p -forms on \mathcal{U} satisfies

$$\frac{d}{dt} m_t^* \varpi = m_t^* (\mathcal{L}_{t^{-1} \zeta_E} \varpi) = d(m_t^* (\iota_{t^{-1} \zeta_E} \varpi)) \quad \forall t \in [0, 1],$$

where \mathcal{L} is the Lie derivative.

By the first statement in (B.3), the injectivity of ι_E , and surjectivity of $d\pi_E$,

$$0 \rightarrow \pi_E^* E \xrightarrow{\iota_E} TE \xrightarrow{d\pi_E} \pi_E^* TX \rightarrow 0 \quad (\text{B.6})$$

is an exact sequence of real vector bundles over E . By the second statement in (B.3), the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_E^* E & \xrightarrow{\iota_E} & TE & \xrightarrow{d\pi_E} & \pi_E^* TX \longrightarrow 0 \\ & & \downarrow \pi_E^* m_f & & \downarrow dm_f & & \downarrow \pi_E^* \text{id}_{TX} \\ 0 & \longrightarrow & \pi_E^* E & \xrightarrow{m_f^* \iota_E} & m_f^* TE & \xrightarrow{m_f^* d\pi_E} & \pi_E^* TX \longrightarrow 0 \end{array} \quad (\text{B.7})$$

of real vector bundle homomorphisms over E commutes. By the third statement in (B.3), the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{E \oplus E}^* (E \oplus E) & \xrightarrow{\iota_{E \oplus E}} & T(E \oplus E) & \xrightarrow{d\pi_{E \oplus E}} & \pi_{E \oplus E}^* TX \longrightarrow 0 \\ & & \downarrow \pi_{E \oplus E}^* \alpha & & \downarrow d\alpha & & \downarrow \pi_{E \oplus E}^* \text{id}_{TX} \\ 0 & \longrightarrow & \pi_{E \oplus E}^* E & \xrightarrow{\alpha^* \iota_E} & \alpha^* TE & \xrightarrow{\alpha^* d\pi_E} & \pi_{E \oplus E}^* TX \longrightarrow 0 \end{array} \quad (\text{B.8})$$

of real vector bundle homomorphisms over $E \oplus E$ commutes.

A connection in E is an \mathbb{R} -linear map

$$\begin{aligned} \nabla: \Gamma(X; E) &\longrightarrow \Gamma(X; T^*X \otimes_{\mathbb{R}} E) \quad \text{s.t.} \\ \nabla(fs) &= df \otimes s + f \nabla s \quad \forall f \in C^\infty(X), s \in \Gamma(X; E). \end{aligned} \quad (\text{B.9})$$

The Leibnitz property (B.9) implies that any two connections in E differ by a 1-form on X . In other words, if ∇ and ∇' are connections in E there exists

$$\begin{aligned} \theta &\in \Gamma(X; T^*X \otimes_{\mathbb{R}} \text{Hom}_{\mathbb{R}}(E, E)) \quad \text{s.t.} \\ \nabla'_v s &= \nabla_v s + \{\theta(v)\}(s(x)) \quad \forall s \in \Gamma(X; E), v \in T_x X, x \in X. \end{aligned} \quad (\text{B.10})$$

If U is a neighborhood of $x \in X$ and f is a smooth function on X supported in U such that $f(x) = 1$, then

$$\nabla s|_x = \nabla(fs)|_x - d_x f \otimes s(x) \quad (\text{B.11})$$

by (B.9). The right-hand side of (B.11) depends only on $s|_U$. Thus, a connection ∇ in E is a local operator, i.e. the value of $\nabla \xi$ at a point $x \in X$ depends only on the restriction of s to any neighborhood U of x .

Exercise B.2. Suppose ∇, ∇' are connections in real vector bundles $E, E' \rightarrow X$, respectively. Show that the map

$$\nabla \oplus \nabla' : \Gamma(X; E \oplus E') \rightarrow \Gamma(X; T^*X \otimes_{\mathbb{R}} (E \oplus E')), \quad \nabla \oplus \nabla'(s, s') = (\nabla s, \nabla' s'),$$

is a connection in the real vector bundle $E \oplus E' \rightarrow X$.

Exercise B.3. Suppose ∇ is a connection in a real vector bundle $\pi_E: E \rightarrow X$. Show that

(a) the linear map ∇ extends to a linear map on the E -valued p -forms by

$$\nabla : \Gamma(X; \Lambda^p(T^*X) \otimes_{\mathbb{R}} E) \rightarrow \Gamma(X; \Lambda^{p+1}(T^*X) \otimes_{\mathbb{R}} E), \quad \nabla(\eta \otimes s) = (d\eta) \otimes s + (-1)^p \eta \otimes (\nabla s);$$

(b) there exists $\kappa_{\nabla} \in \Gamma(X; \Lambda^2(T^*X) \otimes_{\mathbb{R}} \text{End}_{\mathbb{R}}(E))$ so that

$$\nabla(\nabla \tilde{\eta}) = \kappa_{\nabla} \wedge \tilde{\eta} \quad \forall \tilde{\eta} \in \Gamma(X; \Lambda^p(T^*X) \otimes_{\mathbb{R}} E), \quad p \in \mathbb{Z}^{\geq 0}.$$

Note: the bundle section κ_{∇} above is called the **curvature** of ∇ .

Suppose U is an open subset of X and $s_1, \dots, s_n \in \Gamma(U; E)$ is a frame for E on U , i.e.

$$s_1(x), \dots, s_n(x) \in E_x$$

is a basis for E_x for all $x \in U$. By definition of ∇ , there exist

$$\theta_{\ell}^k \in \Gamma(U; T^*U) \quad \text{s.t.} \quad \nabla s_{\ell} = \sum_{k=1}^{k=n} s_k \theta_{\ell}^k \equiv \sum_{k=1}^{k=n} \theta_{\ell}^k \otimes s_k \quad \forall \ell = 1, \dots, n.$$

We call

$$\theta \equiv (\theta_{\ell}^k)_{k, \ell=1, \dots, n} \in \Gamma(U; T^*U \otimes_{\mathbb{R}} \text{Mat}_n \mathbb{R})$$

the connection 1-form of ∇ with respect to the frame $(s_k)_k$.

For an arbitrary section

$$s \equiv \sum_{\ell=1}^{\ell=n} f^{\ell} s_{\ell} \in \Gamma(U; E),$$

by (B.9) we have

$$\nabla s = \sum_{k=1}^{k=n} s_k \left(df^k + \sum_{\ell=1}^{\ell=n} \theta_{\ell}^k f^{\ell} \right), \quad \text{i.e.} \quad \nabla(\underline{s} \cdot \underline{f}^t) = \underline{s} \cdot \{d + \theta\} \underline{f}^t, \quad (\text{B.12})$$

$$\text{where} \quad \underline{s} = (s_1, \dots, s_n), \quad \underline{f} = (f^1, \dots, f^n). \quad (\text{B.13})$$

This implies that

$$\nabla s|_x = \pi_2|_x \circ d_x s : T_x X \rightarrow E_x \quad \forall x \in X, \quad s \in \Gamma(X; E) \text{ s.t. } s(x) = 0, \quad (\text{B.14})$$

where $\pi_2|_x : T_x E \rightarrow E_x$ is the projection to the second component in (B.4).

Lemma B.4. *Suppose X is a smooth manifold and $\pi_E : E \rightarrow X$ is a real vector bundle. A connection ∇ in E induces a splitting*

$$TE \approx \pi_E^*TX \oplus \pi_E^*E \quad (\text{B.15})$$

of the exact sequence (B.6) extending the splitting (B.4) such that

$$\nabla s|_x = \pi_{\nabla} \circ d_x s : T_x X \rightarrow E_x \quad \forall s \in \Gamma(X; E), x \in X, \quad (\text{B.16})$$

where $\pi_{\nabla} : TE \rightarrow \pi_E^*E$ is the projection onto the second component in (B.15). Furthermore,

$$dm_t \approx \pi_E^* \text{id} \oplus \pi_E^* m_t \quad \forall t \in \mathbb{R} \quad \text{and} \quad \mathbf{a} \approx \pi_{E \oplus E}^* \text{id} \oplus \pi_{E \oplus E}^* \mathbf{a}, \quad (\text{B.17})$$

with respect to the splitting (B.15) and the corresponding splitting for the connection $\nabla \oplus \nabla$ in the real vector bundle $E \oplus E \rightarrow X$, i.e. these splittings are consistent with the commutative diagrams (B.7) and (B.8).

Proof. Given $x \in X$ and $v \in E_x$, choose $s \in \Gamma(X; E)$ such that $s(x) = v$ and let

$$T_v E^{\text{h}} = \{d_x s(w) - \iota_E(\nabla_w s) : w \in T_x X\} \subset T_v E.$$

Since $\pi_E \circ s = \text{id}_X$ and $d\pi_E \circ \iota_E = 0$,

$$d_v \pi_E \circ \{ds - \iota_E \circ \nabla s\}|_x = \text{id}_{T_x X} \quad \implies \quad T_v E \approx T_v E^{\text{h}} \oplus E_x \approx T_x X \oplus E_x.$$

This splitting of $T_v E$ satisfies (B.16) at $v = s(x)$.

With the notation as in (B.12),

$$\{ds - \iota_E \circ \nabla s\}|_x = \left(d_x \text{id}_X, - \sum_{\ell=1}^{\ell=n} f^\ell(x) \theta_\ell^1|_x, \dots, - \sum_{\ell=1}^{\ell=n} f^\ell(x) \theta_\ell^n|_x \right) : T_x X \rightarrow T_x X \oplus \mathbb{R}^n \quad (\text{B.18})$$

with respect to the identification $E|_U \approx U \times \mathbb{R}^n$ determined by the frame $(s_k)_k$. Thus, $T_v E^{\text{h}}$ is independent of the choice of s . Since $T_x E^{\text{h}} = T_x X$ for every $x \in X$, the resulting splitting (B.15) of (B.6) extends (B.4). By (B.18), it also satisfies (B.17). \square

Exercise B.5. Suppose $p \in \mathbb{Z}^+$, $\pi_E : E \rightarrow X$ is a real vector bundle, Ω is a fiberwise p -form on E , and ∇ is a connection in E with the associated projection $\pi_{\nabla} : TE \rightarrow \pi_E^*E$ as in Lemma B.4. Thus, $\Omega_{\nabla} \equiv \pi_{\nabla}^* \Omega$ is a p -form on the total space of E . Let $\zeta_E \in \Gamma(E; TE)$ be the canonical vertical vector field on E as in (B.5). Show that

$$\iota_E^* (d(\iota_{\zeta_E} \Omega_{\nabla})) = p\Omega \quad \text{and} \quad (d(\iota_{\zeta_E} \Omega_{\nabla}))|_{T_x X} = 0 \quad \forall x \in X. \quad (\text{B.19})$$

Suppose g is a metric on a real vector bundle $E \rightarrow X$, i.e.

$$g \in \Gamma(X; E^* \otimes_{\mathbb{R}} E^*) \quad \text{s.t.} \quad g(v, w) = g(w, v), \quad g(v, v) > 0 \quad \forall v, w \in E_x, v \neq 0, x \in X.$$

A connection ∇ in E is g -compatible if

$$d(g(s, s')) = g(\nabla s, s') + g(s, \nabla s') \in \Gamma(X; T^*X) \quad \forall s, s' \in \Gamma(X; E).$$

Suppose U is an open subset of X and $s_1, \dots, s_n \in \Gamma(U; E)$ is a frame for E on U . For $i, j = 1, \dots, n$, let

$$g_{ij} = g(s_i, s_j) \in C^\infty(U).$$

If ∇ is a connection in E and θ_{kl} is the connection 1-form for ∇ with respect to the frame $\{s_k\}_k$, then ∇ is g -compatible on U if and only if

$$\sum_{k=1}^{k=n} (g_{ik}\theta_j^k + g_{jk}\theta_i^k) = dg_{ij} \quad \forall i, j = 1, 2, \dots, n. \quad (\text{B.20})$$

B.2 Complex vector bundles

Suppose X is a smooth manifold and $\pi_E : E \rightarrow X$ is a complex vector bundle. Similarly to Section B.1, there is an exact sequence

$$0 \longrightarrow \pi_E^* E \xrightarrow{\iota_E} TE \xrightarrow{d\pi_E} \pi_E^* TX \longrightarrow 0 \quad (\text{B.21})$$

of complex vector bundles over E . The homomorphism ι_E is now \mathbb{C} -linear. If $f \in C^\infty(X; \mathbb{C})$ and $m_f : E \rightarrow E$ is defined as in (B.1), there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_E^* E & \xrightarrow{\iota_E} & TE & \xrightarrow{d\pi_E} & \pi_E^* TX \longrightarrow 0 \\ & & \downarrow \pi_E^* m_f & & \downarrow dm_f & & \downarrow \pi_E^* \text{id}_{TX} \\ 0 & \longrightarrow & \pi_E^* E & \xrightarrow{m_f \iota_E} & m_f^* TE & \xrightarrow{m_f^* d\pi_E} & \pi_E^* TX \longrightarrow 0 \end{array} \quad (\text{B.22})$$

of complex vector bundle maps over E .

Suppose ∇ is a (\mathbb{C} -linear) connection in the complex vector bundle $\pi_E : E \rightarrow X$, i.e.

$$\nabla_v(\mathbf{i}s) = \mathbf{i}(\nabla_v s) \quad \forall s \in \Gamma(X; E), v \in TX.$$

If U is an open subset of X and $s_1, \dots, s_n \in \Gamma(U; E)$ is a \mathbb{C} -frame for E on U , then there exist

$$\theta_\ell^k \in \Gamma(U; T^*U \times_{\mathbb{R}} \mathbb{C}) \quad \text{s.t.} \quad \nabla \xi_\ell = \sum_{k=1}^{k=n} s_k \theta_\ell^k \equiv \sum_{k=1}^{k=n} \theta_\ell^k \otimes s_k \quad \forall \ell = 1, \dots, n.$$

For an arbitrary section

$$s = \sum_{\ell=1}^{\ell=n} f^\ell s_\ell \in \Gamma(U; E),$$

by (B.9) and \mathbb{C} -linearity of ∇ we have

$$\nabla \xi = \sum_{k=1}^{k=n} s_k \left(df^k + \sum_{\ell=1}^{\ell=n} \theta_\ell^k f^\ell \right), \quad \text{i.e.} \quad \nabla(\underline{s} \cdot \underline{f}^t) = \underline{\xi} \cdot \{d + \theta\} \underline{f}^t, \quad (\text{B.23})$$

where \underline{s} and \underline{f} are as (B.13).

Let g be a Hermitian metric on E , i.e.

$$g \in \Gamma(X; \text{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{C}} \bar{E}, \mathbb{C})) \quad \text{s.t.} \quad g(v, w) = \overline{g(w, v)}, \quad g(v, v) > 0 \quad \forall v, w \in E_x, \quad v \neq 0, \quad x \in X.$$

A (\mathbb{C} -linear) connection ∇ in E is g -compatible if

$$d(g(s, s')) = g(\nabla s, s') + g(s, \nabla s') \in \Gamma(X; T^*X \otimes_{\mathbb{R}} \mathbb{C}) \quad \forall s, s' \in \Gamma(X; E).$$

With the notation as in the previous paragraph, let

$$g_{ij} = g(s_i, s_j) \in C^\infty(U; \mathbb{C}) \quad \forall i, j = 1, \dots, n.$$

Then ∇ is g -compatible on U if and only if

$$\sum_{k=1}^{k=n} (g_{ik} \theta_j^k + \bar{g}_{jk} \bar{\theta}_i^k) = dg_{ij} \quad \forall i, j = 1, 2, \dots, n. \quad (\text{B.24})$$

Exercise B.6. Suppose ∇ is a connection in a complex vector bundle $\pi_E : E \rightarrow X$. Let $\kappa_{\nabla} \in \Gamma(X; \Lambda^2(T^*X) \otimes_{\mathbb{R}} \text{End}_{\mathbb{R}}(E))$ be the curvature of ∇ as in Exercise B.3 and $TE^h \subset TE$ be the complement of $\iota_E(\pi_E^*E) \subset TE$ determined by ∇ as in the proof of Lemma B.4.

(a) Show that the splitting (B.15) satisfies the first property in (B.17) for all $t \in \mathbb{C}$ and that $\kappa_{\nabla} \in \Gamma(X; \Lambda^2(T^*X) \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E))$;

(b) Suppose ∇ is compatible with a Hermitian metric g on E and

$$v \in S_{E,g} \equiv \{w \in E : g(w, w) = 1\}.$$

Show that $T_v E^h \subset T_v S_{E,g}$.

(c) Suppose in addition that $\text{rk}_{\mathbb{C}} E = 1$. Show that κ_{∇} is a 2-form on X with values in $i\mathbb{R}$.

B.3 Principal S^1 -bundles

Suppose X is a smooth manifold and $\pi_S : S \rightarrow X$ is a (smooth) principal S^1 -bundle. Let

$$\zeta_S \in \Gamma(S; TS), \quad \zeta_S(v) = \left. \frac{d}{dt} (e^{2\pi i t} \cdot v) \right|_{t=0}, \quad (\text{B.25})$$

be the vector field generating the S^1 -action. This vector field generates the vertical tangent bundle of π_S , i.e.

$$TS^{\text{ver}} \equiv \ker d\pi_S = \{t\zeta_S(v) : v \in S, t \in \mathbb{R}\} \rightarrow S.$$

A connection 1-form on S is an S^1 -invariant 1-form λ on (the total space of) S such that $\lambda(\zeta_S) = 2\pi$. Such a form determines an S^1 -equivariant splitting of the exact sequence

$$0 \rightarrow TS^{\text{ver}} \rightarrow TS \xrightarrow{d\pi_S} \pi_S^* TX \rightarrow 0$$

of real vector bundles over S with

$$TS = TS^{\text{ver}} \oplus (\ker \lambda).$$

Exercise B.7. Suppose $\pi_S: S \rightarrow X$ is a principal S^1 -bundle.

- (a) Show that the S^1 -invariance condition on 1-form λ being a connection 1-form can be equivalently replaced by the condition $\iota_{\zeta_S} d\lambda = 0$;
- (b) Let λ be a connection 1-form on S . Show that there exists a 2-form κ_λ on X so that $d\lambda = \pi_S^* \kappa_\lambda$.

Note: the 2-form κ_λ above is called the *curvature* of λ .

A principal S^1 -bundle $\pi_S: S \rightarrow X$ determines a complex line bundle

$$\begin{aligned} \pi_{L_S}: L_S &\equiv S \times_{S^1} \mathbb{C} \rightarrow X, \\ (v, z) &\sim (u \cdot v, u^{-1} \cdot z) \quad \forall (v, z) \in S \times \mathbb{C}, u \in S^1, \quad \pi_{L_S}([v, z]) = \pi_S(v), \end{aligned}$$

with a Hermitian metric specified by

$$g_S([v, z], [v, z']) = z\bar{z}'.$$

Conversely, a complex line bundle $\pi_L: L \rightarrow X$ with a Hermitian metric g determines a principal S^1 -bundle, the unit circle bundle of L ,

$$\pi_{S_{L,g}}: S_{L,g} \equiv \{v \in L: g(v, v) = 1\} \rightarrow X, \quad \pi_{S_{L,g}}(v) = \pi_L(v).$$

With S and (L, g) as above, the maps

$$S \rightarrow S_{L_S, g_S}, \quad v \rightarrow [v, 1], \quad \text{and} \quad L_{S_{L_S, g_S}} \rightarrow L, \quad [v, z] \rightarrow zv, \quad (\text{B.26})$$

are isomorphism of principal S^1 -bundles over X and of complex line bundles with Hermitian metrics over X . Thus, we have constructed a bijective correspondence between the isomorphism classes of principal S^1 -bundles over X and the isomorphism classes of complex line bundles with Hermitian metrics over X .

Exercise B.8. Suppose $\pi_S: S \rightarrow X$ is a principal S^1 -bundle and λ is a connection 1-form on S . Let $p: S \times \mathbb{C} \rightarrow L_S$ be the quotient projection and $\iota_{L_S}: \pi_{L_S}^* L_S \rightarrow TL_S$ be as in (B.2) with $E = L_S$. Show that

- (a) there is a unique 1-form λ_S on L_S so that

$$p^* \lambda_S|_{(v,z)} = |z|^2 \lambda_v + \frac{i}{2} (z d\bar{z} - \bar{z} dz) \quad \forall (v, z) \in L \times \mathbb{C};$$

- (b) $\iota_{L_S}^*(d\lambda_S) = 2 \operatorname{Re} g_S(i \cdot, \cdot)$ and $\iota_v(d\lambda_S) = 0$ for all $v \in T_x X \subset T_x L_S$, $x \in X$;
- (c) there is a unique (\mathbb{C} -linear) connection ∇^λ in the complex line bundle $\pi_S: L_S \rightarrow X$ compatible with the Hermitian metric g_S so that the 1-form λ_S vanishes on the complement $TL_S^h \subset TL_S$ of $\iota_{L_S}(\pi_{L_S}^* L_S) \subset TL_S$ determined by ∇^λ as in the proof of Lemma B.4.

Thus, a connection 1-form λ in a principal S^1 -bundle $\pi_S: S \rightarrow X$ determines a connection ∇^λ in the associated complex line bundle $\pi_{L_S}: L_S \rightarrow X$ compatible with the Hermitian metric g_S

on L_S . Suppose $\pi_L: L \rightarrow X$ is a complex line bundle with a Hermitian metric g and ∇ is a (\mathbb{C} -linear) connection in L compatible with g . Let $\zeta_L \in \Gamma(L; TL)$ be the canonical vertical vector field as in (B.5) and $TL^h \subset TL$ be the complement of $\iota_L(\pi_L^*L) \subset TL$ determined by ∇ as in the proof of Lemma B.4. In particular, $2\pi i \zeta_L \in \Gamma(L; TL)$ is the vector field generating the S^1 -action on L by scalar multiplication. By Exercise B.6, the S^1 -action on L preserves the subbundle $TL^h|_{S_{L,g}}$ of $TS_{L,g}$. Thus, the 1-form λ_∇ on $S_{L,g}$ defined by

$$\lambda_\nabla(2\pi i \zeta_L(v)) = 2\pi, \quad \lambda_\nabla|_{T_v L^h} = 0 \quad \forall v \in S_{L,g}$$

is a connection 1-form on the principal S^1 -bundle $S_{L,g} \rightarrow X$. By Exercise B.9 below, we have constructed a bijective correspondence between the isomorphism classes of principal S^1 -bundles over X with connection 1-forms and the isomorphism classes of complex line bundles with Hermitian metrics over X and compatible connections.

Exercise B.9. Suppose λ is a connection 1-form on a principal S^1 -bundle $\pi_S: S \rightarrow X$ and ∇ is a connection in a complex line bundle $\pi_L: L \rightarrow X$ compatible with a Hermitian metric g . Show that

$$\lambda = \lambda_{\nabla\lambda} \quad \text{and} \quad \nabla^{\lambda_\nabla} = \nabla$$

under the isomorphisms (B.26) and that $\kappa_{\nabla\lambda} = i\kappa_\lambda$.

Remark B.10. By a Čech cohomology computation [15, p141] and Exercise B.9,

$$c_1(L) = \frac{i}{2\pi} [\kappa_\nabla] = -\frac{1}{2\pi} [\kappa_\lambda] \in H_{\text{deR}}^2(X),$$

if ∇ is a connection in a complex line bundle $L \rightarrow X$ and λ is a connection 1-form in an associated principal S^1 -bundle.

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